PRECALCULUS
&
CALCULUS

REVISED EDITION

LARSON • HOSTETLER • EDWARDS

FORMULAS FROM GEOMETRY

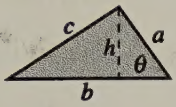
Triangle

$h = a \sin \theta$

Area $= \frac{1}{2}bh$

Laws of Cosines:

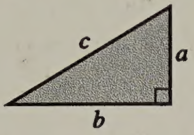
$c^2 = a^2 + b^2 - 2ab \cos \theta$



Right Triangle

Pythagorean Theorem:

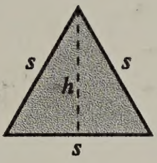
$c^2 = a^2 + b^2$



Equilateral Triangle

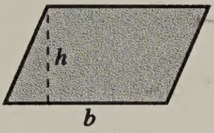
$h = \frac{\sqrt{3}s}{2}$

Area $= \frac{\sqrt{3}s^2}{4}$



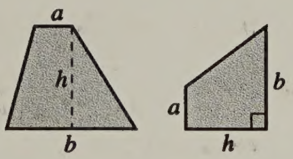
Parallelogram

Area $= bh$



Trapezoid

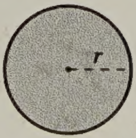
Area $= \frac{h}{2}(a + b)$



Circle

Area $= \pi r^2$

Circumference $= 2\pi r$

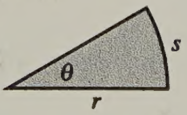


Sector of Circle

Area $= \frac{\theta r^2}{2}$

$s = r\theta$

(θ in radians)



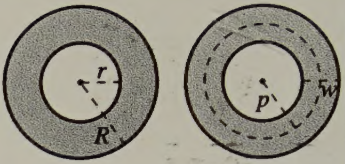
Circular Ring

Area $= \pi(R^2 - r^2)$

$= 2\pi pw$

(p = average radius,

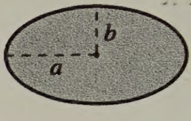
w = width of ring)



Ellipse

Area $= \pi ab$

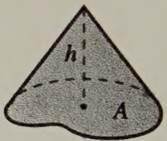
Circumference $\approx 2\pi\sqrt{\frac{a^2 + b^2}{2}}$



Cone

(A = area of base)

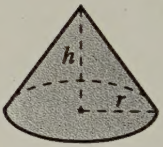
Volume $= \frac{Ah}{3}$



Right Circular Cone

Volume $= \frac{\pi r^2 h}{3}$

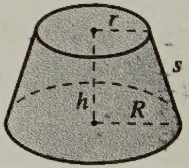
Lateral Surface Area $= \pi r\sqrt{r^2 + h^2}$



Frustum of Right Circular Cone

Volume $= \frac{\pi(r^2 + rR + R^2)h}{3}$

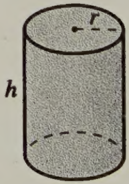
Lateral Surface Area $= \pi s(R + r)$



Right Circular Cylinder

Volume $= \pi r^2 h$

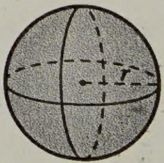
Lateral Surface Area $= 2\pi rh$



Sphere

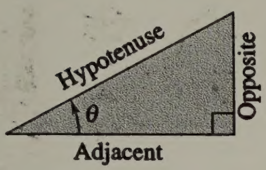
Volume $= \frac{4}{3}\pi r^3$

Surface Area $= 4\pi r^2$



Definition of the Six Trigonometric Functions

Right triangle definitions, where $0 < \theta < \pi/2$.

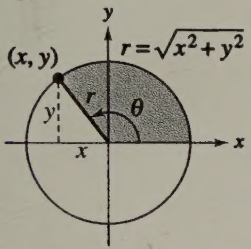


$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} \qquad \csc\theta = \frac{\text{hyp.}}{\text{opp.}}$$

$$\cos\theta = \frac{\text{adj.}}{\text{hyp.}} \qquad \sec\theta = \frac{\text{hyp.}}{\text{adj.}}$$

$$\tan\theta = \frac{\text{opp.}}{\text{adj.}} \qquad \cot\theta = \frac{\text{adj.}}{\text{opp.}}$$

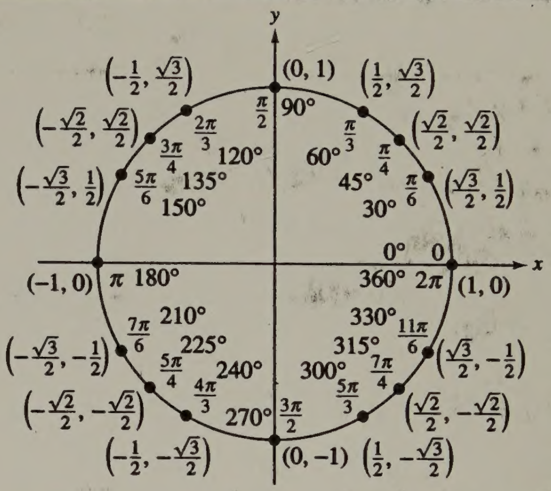
Circular function definitions, where θ *is any angle.*



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$



y
x
$(0, 1)$
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{2}$ 90°
$\frac{2\pi}{3}$ 120°
$\frac{3\pi}{4}$ 135°
$\frac{5\pi}{6}$ 150°
$\frac{\pi}{3}$ 60°
$\frac{\pi}{4}$ 45°
$\frac{\pi}{6}$ 30°
0° 0
360° 2π
$(1, 0)$
$(-1, 0)$
π 180°
$\frac{7\pi}{6}$ 210°
$\frac{5\pi}{4}$ 225°
$\frac{4\pi}{3}$ 240°
$\frac{3\pi}{2}$ 270°
$\frac{5\pi}{3}$ 300°
$\frac{7\pi}{4}$ 315°
$\frac{11\pi}{6}$ 330°
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$(0, -1)$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Reciprocal Identities

$$\sin u = \frac{1}{\csc u} \qquad \cos u = \frac{1}{\sec u} \qquad \tan u = \frac{1}{\cot u}$$

$$\csc u = \frac{1}{\sin u} \qquad \sec u = \frac{1}{\cos u} \qquad \cot u = \frac{1}{\tan u}$$

Quotient Identities

$$\tan u = \frac{\sin u}{\cos u} \qquad \cot u = \frac{\cos u}{\sin u}$$

Pythagorean Identities

$$\sin^2 u + \cos^2 u = 1$$

$$1 + \tan^2 u = \sec^2 u \qquad 1 + \cot^2 u = \csc^2 u$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u$$

$$\cos\left(\frac{\pi}{2} - u\right) = \sin u \qquad \sec\left(\frac{\pi}{2} - u\right) = \csc u$$

$$\tan\left(\frac{\pi}{2} - u\right) = \cot u \qquad \csc\left(\frac{\pi}{2} - u\right) = \sec u$$

Even/Odd Identities

$$\sin(-u) = -\sin u \qquad \cot(-u) = -\cot u$$

$$\cos(-u) = \cos u \qquad \sec(-u) = \sec u$$

$$\tan(-u) = -\tan u \qquad \csc(-u) = -\csc u$$

Sum and Difference Formulas

$$\sin(u \pm v) = \sin u \cos v + \cos u \sin v$$

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

Double-Angle Formulas

$$\sin 2u = 2 \sin u \cos u$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

Power-Reducing Formulas

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

Sum-to-Product Formulas

$$\sin u + \sin v = 2 \sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin u - \sin v = 2 \cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

$$\cos u + \cos v = 2 \cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2 \sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$

$$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \frac{1}{2}[\cos(u + v) - \sin(u - v)]$$

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Precalculus
A Graphing Approach

Third Edition

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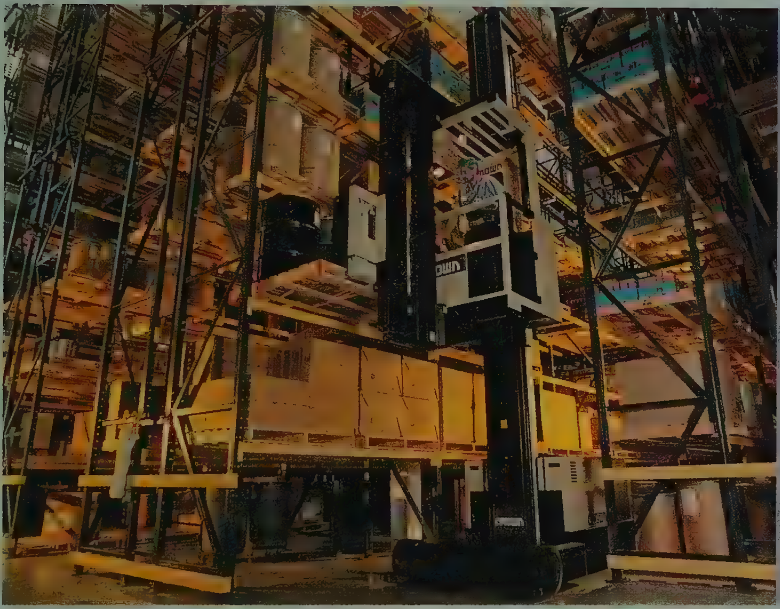
Prerequisites

P.1 Real Numbers
P.2 Exponents and Radicals
P.3 Polynomials and Factoring
P.4 Rational Expressions
P.5 Graphical Representation of Data


Superstock

In 1997, over $24 billion worth of corrugated and solid fiber boxes were produced to create shipping containers. (Source: U.S. Bureau of the Census)

The Big Picture

In this chapter you will learn how to

☐ represent, classify, and order real numbers and use inequalities.
☐ evaluate algebraic expressions using the basic rules of algebra.
☐ use properties of exponents and radicals to simplify and evaluate expressions.
☐ add, subtract, and multiply polynomials.
☐ factor expressions completely.
☐ determine the domains of algebraic expressions and simplify rational expressions.
☐ avoid common algebraic errors and use algebraic techniques common in calculus.
☐ plot points in the coordinate plane and use the Distance and Midpoint Formulas.

Important Vocabulary

As you encounter each new vocabulary term in this chapter, add the term and its definition to your notebook glossary.

- real numbers (p. 2)
- rational numbers (p. 2)
- irrational numbers (p. 2)
- real number line (p. 2)
- origin (p. 2)
- coordinate (p. 2)
- inequality (p. 3)
- positive infinity (p. 4)
- negative infinity (p. 4)
- absolute value (p. 5)
- distance between two points (p. 5)
- variable (p. 6)
- constant (p. 6)
- algebraic expressions (p. 6)
- evaluate an expression (p. 6)
- additive inverse (p. 6)
- multiplicative inverse (p. 6)
- factors (p. 8)
- prime number (p. 8)
- composite number (p. 8)
- exponential form (p. 12)
- exponent (p. 12)
- base (p. 12)
- scientific notation (p. 14)
- square root (p. 15)
- cube root (p. 15)
- principal *n*th root (p. 15)
- simplest form (p. 17)
- conjugate (p. 18)
- rational exponent (p. 19)
- polynomial (p. 24)
- FOIL Method (p. 25)
- factoring (p. 27)
- factoring by grouping (p. 31)
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- complex fractions (p. 41)
- Cartesian plane (p. 47)
- rectangular coordinate system (p. 47)
- Distance Formula (p. 50)
- Midpoint Formula (p. 52)

Additional Resources Text-specific additional resources are available to help you do well in this course. See page xvi for details.

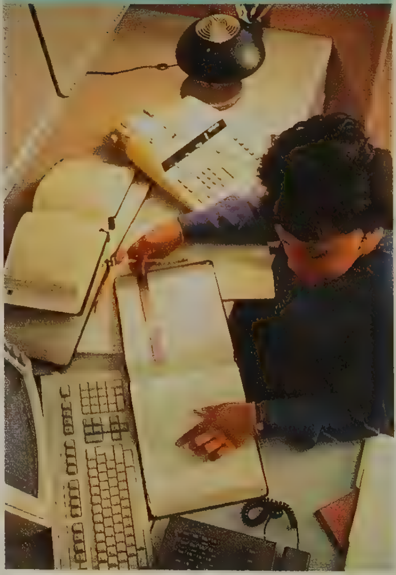
P.1 Real Numbers

What You Should Learn:

- How to represent and classify real numbers
- How to order real numbers and use inequalities
- How to find the absolute values of real numbers and the distance between two real numbers
- How to evaluate algebraic expressions
- How to use the basic rules and properties of algebra

Why You Should Learn It:

Real numbers are used in every aspect of our daily lives, such as the variance of a budget in Exercises 81–84 on page 10.



Superstock

Real Numbers

Real numbers are used in everyday life to describe quantities such as age, miles per gallon, container size, and population. Real numbers are represented by symbols such as

$$-5, 9, 0, \tfrac{4}{3}, 0.666\ldots, 28.21, \sqrt{2}, \pi, \text{ and } \sqrt[3]{-32}.$$

Here are some important subsets of the set of real numbers.

$\{1, 2, 3, 4, \ldots\}$	Set of natural numbers
$\{0, 1, 2, 3, 4, \ldots\}$	Set of whole numbers
$\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$	Set of integers

A real number is **rational** if it can be written as the ratio p/q of two integers, where $q \neq 0$. For instance, the numbers

$$\frac{1}{3} = 0.3333\ldots, \quad \frac{1}{8} = 0.125, \quad \text{and} \quad \frac{125}{111} = 1.126126\ldots = 1.\overline{126}$$

are rational. The decimal representation of a rational number either *repeats* (as in $\frac{173}{55} = 3.1\overline{45}$) or *terminates* (as in $\frac{1}{2} = 0.5$). A real number that cannot be written as the ratio of two integers is called **irrational.** Irrational numbers have infinite nonrepeating decimal representations. For instance, the numbers

$$\sqrt{2} \approx 1.4142136 \quad \text{and} \quad \pi \approx 3.1415927$$

are irrational. (The symbol $\approx$ means "is approximately equal to.")

Real numbers are represented graphically by a **real number line.** The point 0 on the real number line is the **origin.** Numbers to the right of 0 are positive, and numbers to the left of 0 are negative, as shown in Figure P.1. The term **nonnegative** describes a number that is either positive or zero.

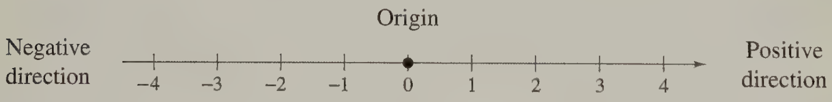
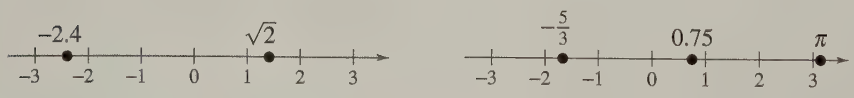


Figure P.1 *The Real Number Line*

There is a *one-to-one correspondence* between real numbers and points on the real number line. That is, every point on the real number line corresponds to exactly one real number called its **coordinate,** and every real number corresponds to exactly one point on the real number line, as shown in Figure P.2.



Every point on the real number line corresponds to exactly one real number.

Every real number corresponds to exactly one point on the real number line.

Figure P.2 *One-to-One Correspondence*

Ordering Real Numbers

One important property of real numbers is that they are **ordered.**

Definition of Order on the Real Number Line

If a and b are real numbers, a is **less than** b if $b - a$ is positive. This order is denoted by the **inequality**

$$a < b.$$

This relationship can also be described by saying that b is **greater than** a and writing $b > a$. The inequality $a \leq b$ means that a is **less than or equal to** b, and the inequality $b \geq a$ means that b is **greater than or equal to** a. The symbols $<$, $>$, $\leq$, and $\geq$ are **inequality symbols.**

Geometrically, this definition implies that $a < b$ if and only if a lies to the *left* of b on the real number line, as shown in Figure P.3.

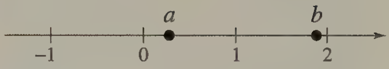


Figure P.3 *$a < b$ if and only if a lies to the left of b.*

EXAMPLE 1 Interpreting Inequalities

Describe the subset of real numbers represented by each inequality.

a. $x \leq 2$ **b.** $x > -1$ **c.** $-2 \leq x \leq 3$

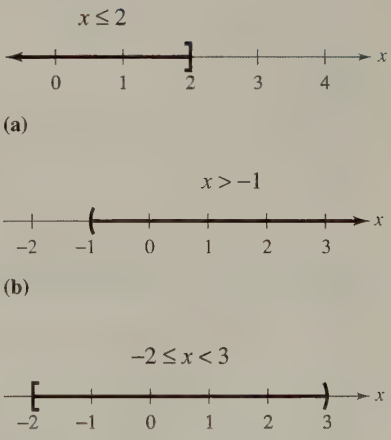
A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Solution

a. The inequality $x \leq 2$ denotes all real numbers less than or equal to 2, as shown in Figure P.4(a).

b. The inequality $x > -1$ denotes all real numbers greater than -1, as shown in Figure P.4(b).

c. The inequality $-2 \leq x < 3$ means that $x \geq -2$ *and* $x < 3$. The "double inequality" denotes all real numbers between -2 and 3, including -2 but not including 3, as shown in Figure P.4(c).



(a)

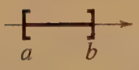
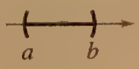
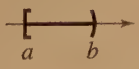
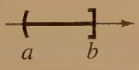
(b)

(c)

Figure P.4

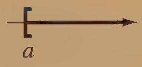
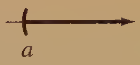
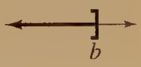
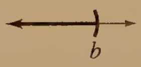
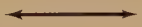
Inequalities can be used to describe subsets of real numbers called **intervals.** In the bounded intervals below, the real numbers a and b are the **endpoints** of each interval.

Bounded Intervals on the Real Number Line

Notation	*Interval Type*	*Inequality*	*Graph*
$[a, b]$	Closed	$a \leq x \leq b$	
(a, b)	Open	$a < x < b$	
$[a, b)$		$a \leq x < b$	
$(a, b]$		$a < x \leq b$	

The symbols ∞, **positive infinity,** and $-\infty$, **negative infinity,** do not represent real numbers. They are simply convenient symbols used to describe the unboundedness of an interval such as $(1, \infty)$ or $(-\infty, 3]$.

Unbounded Intervals on the Real Number Line

Notation	*Interval Type*	*Inequality*	*Graph*
$[a, \infty)$		$x \geq a$	
(a, ∞)	Open	$x > a$	
$(-\infty, b]$		$x \leq b$	
$(-\infty, b)$	Open	$x < b$	
$(-\infty, \infty)$	Entire real line		

The *Interactive* CD-ROM and *Internet* versions of this text show every example with its solution; clicking on the *Try It!* button brings up similar problems. Guided Examples and Integrated Examples show step-by-step solutions to additional examples. Integrated Examples are related to several concepts in the section.

EXAMPLE 2 Using Inequalities to Represent Intervals

Use inequality notation to describe each of the following.

a. c is at most 2.

b. All x in the interval $(-3, 5]$

Solution

a. The statement "c is at most 2" can be represented by $c \leq 2$.

b. "All x in the interval $(-3, 5]$" can be represented by $-3 < x \leq 5$.

EXAMPLE 3 Interpreting Intervals

Give a verbal description of each interval.

a. $(-1, 0)$ **b.** $[2, \infty)$ **c.** $(-\infty, 0)$

Solution

a. This interval consists of all real numbers that are greater than -1 and less than 0.

b. This interval consists of all real numbers that are greater than or equal to 2.

c. This interval consists of all negative real numbers.

The **Law of Trichotomy** states that for any two real numbers a and b, precisely one of three relationships is possible:

$$a = b, \quad a < b, \quad \text{or} \quad a > b.$$

Law of Trichotomy

Absolute Value and Distance

The **absolute value** of a real number is its distance from the origin on the real number line.

Definition of Absolute Value

If a is a real number, the **absolute value** of a is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}.$$

Notice from this definition that the absolute value of a real number is never negative. For instance, if $a = -5$, then $|-5| = -(-5) = 5$. The absolute value of a real number is either positive or zero. Moreover, 0 is the only real number whose absolute value is 0. So, $|0| = 0$.

Exploration

Absolute value expressions can be evaluated on a graphing utility. When evaluating an expression such as $|3 - 8|$, parentheses should surround the expression as in *abs* $(3 - 8)$. Evaluate each expression below. What can you conclude?

a. $|6|$ **b.** $|-1|$

c. $|5 - 2|$ **d.** $|2 - 5|$

EXAMPLE 4 Evaluating the Absolute Value of a Number

Evaluate $\dfrac{|x|}{x}$ for (a) $x > 0$ and (b) $x < 0$.

Solution

a. If $x > 0$, then $|x| = x$ and $\dfrac{|x|}{x} = \dfrac{x}{x} = 1$.

b. If $x < 0$, then $|x| = -x$ and $\dfrac{|x|}{x} = \dfrac{-x}{x} = -1$.

Properties of Absolute Value

1. $|a| \geq 0$ **2.** $|-a| = |a|$

3. $|ab| = |a||b|$ **4.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, \quad b \neq 0$

Absolute value can be used to define the distance between two points on the real number line. For instance, the distance between -3 and 4 is

$$|-3 - 4| = |-7| = 7$$

as shown in Figure P.5.

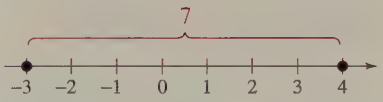


Figure P.5 *The distance between −3 and 4 is 7.*

Distance Between Two Points on the Real Line

Let a and b be real numbers. The **distance between a and b** is

$$d(a, b) = |b - a| = |a - b|.$$

Algebraic Expressions

One characteristic of algebra is the use of letters to represent numbers. The letters are **variables,** and combinations of letters and numbers are **algebraic expressions.** Here are a few examples of algebraic expressions.

$$5x, \qquad 2x - 3, \qquad \frac{4}{x^2 + 2}, \qquad 7x + y$$

Definition of Algebraic Expression

A collection of letters (**variables**) and real numbers (**constants**) combined using the operations of addition, subtraction, multiplication, division, and exponentiation is an **algebraic expression.**

The **terms** of an algebraic expression are those parts that are separated by *addition.* For example,

$$x^2 - 5x + 8 = x^2 + (-5x) + 8$$

has three terms: x^2 and $-5x$ are the **variable terms** and 8 is the **constant term.** The numerical factor of a variable term is the **coefficient** of the variable term. For instance, the coefficient of $-5x$ is -5, and the coefficient of x^2 is 1.

To **evaluate** an algebraic expression, substitute numerical values for each of the variables in the expression. Here are two examples.

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Expression	*Value of Variable*	*Substitute*	*Value of Expression*
$-3x + 5$	$x = 3$	$-3(3) + 5$	$-9 + 5 = -4$
$3x^2 + 2x - 1$	$x = -1$	$3(-1)^2 + 2(-1) - 1$	$3 - 2 - 1 = 0$

When an algebraic expression is evaluated, the **Substitution Principle** is used. It states, "If $a = b$, then a can be replaced by b in any expression involving a." In the first evaluation shown above, for instance, 3 is *substituted* for x in the expression $-3x + 5$.

Basic Rules of Algebra

There are four arithmetic operations with real numbers: *addition, multiplication, subtraction,* and *division,* denoted by the symbols $+$, $\times$ or $\cdot$, $-$, and $\div$ or $/$. Of these, addition and multiplication are the two primary operations. Subtraction and division are the inverse operations of addition and multiplication, respectively.

Subtraction: Add the opposite.

$$a - b = a + (-b)$$

Division: Multiply by the reciprocal.

If $b \neq 0$, then $a/b = a\left(\frac{1}{b}\right) = \frac{a}{b}$.

In these definitions, $-b$ is the **additive inverse** (or opposite) of b, and $1/b$ is the **multiplicative inverse** (or reciprocal) of b. In the fractional form a/b, a is the **numerator** of the fraction and b is the **denominator.**

Because the properties of real numbers below are true for variables and algebraic expressions, as well as for real numbers, they are often called the **Basic Rules of Algebra.** Try to formulate a verbal description of each property. For instance, the first property states that *the order in which two real numbers are added does not affect their sum.*

Basic Rules of Algebra

Let a, b, and c be real numbers, variables, or algebraic expressions.

Property		*Example*
Commutative Property of Addition:	$a + b = b + a$	$4x + x^2 = x^2 + 4x$
Commutative Property of Multiplication:	$ab = ba$	$(4 - x)x^2 = x^2(4 - x)$
Associative Property of Addition:	$(a + b) + c = a + (b + c)$	$(x + 5) + x^2 = x + (5 + x^2)$
Associative Property of Multiplication:	$(ab)c = a(bc)$	$(2x \cdot 3y)(8) = (2x)(3y \cdot 8)$
Distributive Properties:	$a(b + c) = ab + ac$ $(a + b)c = ac + bc$	$3x(5 + 2x) = 3x \cdot 5 + 3x \cdot 2x$ $(y + 8)y = y \cdot y + 8 \cdot y$
Additive Identity Property:	$a + 0 = a$	$5y^2 + 0 = 5y^2$
Multiplicative Identity Property:	$a \cdot 1 = a$	$(4x^2)(1) = 4x^2$
Additive Inverse Property:	$a + (-a) = 0$	$5x^3 + (-5x^3) = 0$
Multiplicative Inverse Property:	$a \cdot \frac{1}{a} = 1, \quad a \neq 0$	$(x^2 + 4)\left(\frac{1}{x^2 + 4}\right) = 1$

Because subtraction is defined as "adding the opposite," the Distributive Properties are also true for subtraction. For instance, the "subtraction form" of $a(b + c) = ab + ac$ is

$$a(b - c) = ab - ac.$$

Properties of Negation and Equality

Let a, b, and c be real numbers, variables, or algebraic expressions.

Property	*Example*
1. $(-1)a = -a$	$(-1)7 = -7$
2. $-(-a) = a$	$-(-6) = 6$
3. $(-a)b = -(ab) = a(-b)$	$(-5)3 = -(5 \cdot 3) = 5(-3)$
4. $(-a)(-b) = ab$	$(-2)(-x) = 2x$
5. $-(a + b) = (-a) + (-b)$	$-(x + 8) = (-x) + (-8) = -x - 8$
6. If $a = b$, then $a + c = b + c$.	$\frac{1}{2} + 3 = 0.5 + 3$
7. If $a = b$, then $ac = bc$.	$4^2(2) = 16(2)$
8. If $a + c = b + c$, then $a = b$.	$1.4 - 1 = \frac{7}{5} - 1$
9. If $ac = bc$ and $c \neq 0$, then $a = b$.	$\frac{3}{4} = \frac{\sqrt{9}}{4}$

STUDY T!P

Be sure you see the difference between the opposite of a number and a negative number. If a is already negative, then its opposite, $-a$, is positive. For instance, if $a = -5$, then $-a = -(-5) = 5$.

Properties of Zero

Let a and b be real numbers, variables, or algebraic expressions.

1. $a + 0 = a$ and $a - 0 = a$
2. $a \cdot 0 = 0$
3. $\frac{0}{a} = 0, \quad a \neq 0$
4. $\frac{a}{0}$ is undefined.
5. **Zero-Factor Property:** If $ab = 0$, then $a = 0$ or $b = 0$.

STUDY T!P

The "or" in the Zero-Factor Property includes the possibility that either or both factors may be zero. This is an **inclusive or,** and it is the way the word "or" is generally used in mathematics.

Properties and Operations of Fractions

Let a, b, c, and d be real numbers, variables, or algebraic expressions such that $b \neq 0$ and $d \neq 0$.

1. **Equivalent Fractions:** $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.
2. **Rules of Signs:** $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$ and $\frac{-a}{-b} = \frac{a}{b}$
3. **Generate Equivalent Fractions:** $\frac{a}{b} = \frac{ac}{bc}, \quad c \neq 0$
4. **Add or Subtract with Like Denominators:** $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$
5. **Add or Subtract with Unlike Denominators:** $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$
6. **Multiply Fractions:** $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
7. **Divide Fractions:** $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}, \quad c \neq 0$

STUDY T!P

In Property 1, the phrase "if and only if" implies two statements. One statement is: If $a/b = c/d$, then $ad = bc$. The other statement is: If $ad = bc$, where $b \neq 0$ and $d \neq 0$, then $a/b = c/d$.

EXAMPLE 5 Properties and Operations of Fractions

a. $\frac{x}{5} = \frac{3 \cdot x}{3 \cdot 5} = \frac{3x}{15}$ Write equivalent fractions.

b. $\frac{x}{3} + \frac{2x}{5} = \frac{5 \cdot x + 3 \cdot 2x}{15} = \frac{11x}{15}$ Add fractions with unlike denominators.

c. $\frac{7}{x} \div \frac{3}{2} = \frac{7}{x} \cdot \frac{2}{3} = \frac{14}{3x}$ Divide fractions.

If a, b, and c are integers such that $ab = c$, then a and b are **factors** or **divisors** of c. A **prime number** is an integer that has exactly two positive factors: itself and 1. For example, 2, 3, 5, 7, and 11 are prime numbers. The numbers 4, 6, 8, 9, and 10 are **composite** because they can be written as the product of two or more prime numbers. The number 1 is neither prime nor composite. The **Fundamental Theorem of Arithmetic** states that every positive integer greater than 1 can be written as the product of prime numbers in precisely one way (disregarding order). For instance, the *prime factorization* of 24 is $24 = 2 \cdot 2 \cdot 2 \cdot 3$.

P.1 Exercises

In Exercises 1–6, determine which numbers are (a) natural numbers, (b) integers, (c) rational numbers, and (d) irrational numbers.

1. $-9, -\frac{7}{2}, 5, \frac{2}{3}, \sqrt{2}, 0, 1, -4, -1$

2. $\sqrt{5}, -7, -\frac{7}{3}, 0, 3.12, \frac{5}{4}, -2, -8, 3$

3. $2.01, 0.666 \ldots, -13, 0.010110111 \ldots, 1, -10, 20$

4. $2.3030030003 \ldots, 0.7575, -4.63, \sqrt{10}, -2, 0.03, -10$

5. $-\pi, -\frac{1}{3}, \frac{6}{3}, \frac{1}{2}\sqrt{2}, -7.5, -2, 3, -3$

6. $25, -17, -\frac{12}{5}, \sqrt{9}, 3.12, \frac{1}{2}\pi, 6, -4, 18$

In Exercises 7–10, use a calculator to find the decimal form of the rational number. If it is a nonterminating decimal, write the repeating pattern.

7. $\frac{5}{8}$

8. $\frac{1}{3}$

9. $\frac{41}{333}$

10. $\frac{6}{11}$

In Exercises 11–16, use a graphing utility to rewrite the rational number as the ratio of two integers.

11. 4.6

12. 12.3

13. 6.5

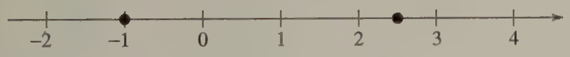
14. 3.81

15. -1.83

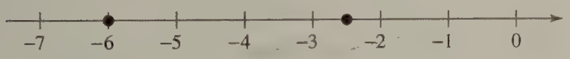
16. -2.490

In Exercises 17 and 18, approximate the numbers and place the correct inequality symbol (< or >) between them.

17.



18.



In Exercises 19–24, plot the two real numbers on the real number line. Then place the correct inequality symbol (< or >) between them.

19. $-4, -8$

20. $-3.5, 1$

21. $\frac{3}{2}, 7$

22. $1, \frac{16}{3}$

23. $\frac{5}{6}, \frac{2}{3}$

24. $-\frac{8}{7}, -\frac{3}{7}$

In Exercises 25–34, (a) verbally describe the subset of real numbers represented by the inequality, (b) sketch the subset on the real number line, and (c) state whether the interval is bounded or unbounded.

25. $x \le 5$

26. $x \ge -2$

27. $x < 0$

28. $x > 3$

29. $x \ge 4$

30. $x < 2$

31. $-2 < x < 2$

32. $0 \le x \le 5$

33. $-1 \le x < 0$

34. $0 < x \le 6$

In Exercises 35 and 36, use a calculator to order the numbers from least to greatest.

35. $\frac{7071}{5000}, \frac{584}{413}, \sqrt{2}, \frac{47}{33}, \frac{127}{90}$

36. $\frac{26}{15}, \sqrt{3}, 1.732, \frac{381}{220}, \frac{2103}{1214}$

In Exercises 37–46, use inequality and interval notation to describe the set.

37. x is negative.

38. z is at least 10.

39. y is nonnegative.

40. y is no more than 25.

41. c is at least 12 and at most 32.

42. p is less than 8 but no less than -1.

43. The dog's weight W is more than 45 pounds.

44. The annual rate of inflation r is expected to be at least 2.5%, but no more than 5%.

In Exercises 45 and 46, give a verbal description of the interval.

45. $(-6, \infty)$

46. $(-\infty, 4]$

In Exercises 47–56, evaluate the expression.

47. $|-10|$

48. $|0|$

49. $|3 - \pi|$

50. $|4 - \pi|$

51. $\dfrac{-5}{|-5|}$

52. $-3 - |-3|$

53. $-3|-3|$

54. $|-1| - |-2|$

55. $\dfrac{|x + 2|}{x + 2}$

56. $\dfrac{|x - 1|}{x - 1}$

The *Interactive* CD-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered Section and Review Exercises. They also provide Tutorial Exercises, which link to Guided Examples for additional help.

In Exercises 57–62, place the correct symbol (<, >, or =) between the pair of real numbers.

57. $|-3| \quad\quad -|-3|$

58. $|-4| \quad\quad |4|$

59. $-5 \quad\quad -|5|$

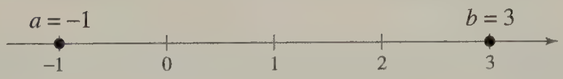
60. $-|-6| \quad\quad |-6|$

61. $-|-2| \quad\quad -|2|$

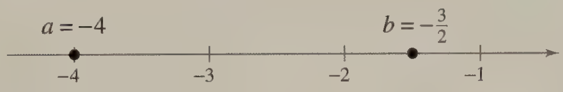
62. $-(-2) \quad\quad -2$

In Exercises 63–70, find the distance between *a* and *b*.

63.



64.



65. $a = 126, b = 75$

66. $a = -126, b = -75$

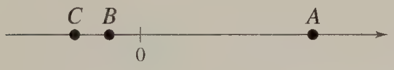
67. $a = -\frac{5}{2}, b = 0$

68. $a = \frac{1}{4}, b = \frac{11}{4}$

69. $a = \frac{16}{5}, b = \frac{112}{75}$

70. $a = 9.34, b = -5.65$

In Exercises 71 and 72, use the real numbers *A*, *B*, and *C* shown on the number line. Determine the sign of each expression.



71. (a) $-A$
(b) $B - A$

72. (a) $-C$
(b) $A - C$

In Exercises 73–80, use absolute value notation to describe the situation.

73. The distance between x and 5 is no more than 3.

74. The distance between x and -10 is at least 6.

75. y is at least six units from 0.

76. y is at most two units from a.

77. ***Distance*** While traveling, you pass milepost 7, then milepost 18. How far do you travel during that time period?

78. ***Distance*** While traveling, you pass milepost 103, then milepost 86. How far do you travel during that time period?

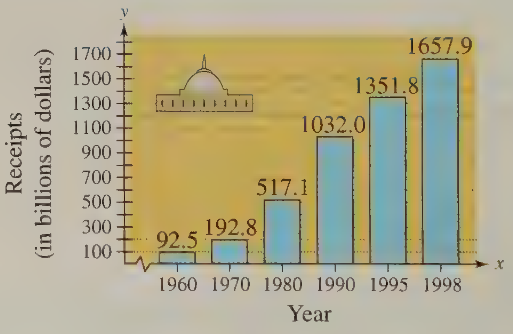
79. ***Temperature*** The temperature was 50° at noon, then 27° at midnight. What was the change in temperature over the twelve-hour period?

80. ***Temperature*** The temperature was 38° last night at midnight, then 78° at noon today. What was the change in temperature over the twelve-hour period?

Budget Variance **In Exercises 81–84, the accounting department of a company is checking to see whether the actual expenses of a department differ from the budgeted expenses by more than $500 or by more than 5%. Fill in the missing parts of the table, and determine whether the actual expense passes the "budget variance test."**

	Budgeted Expense, b	*Actual Expense, a*	$\lvert a - b \rvert$	*0.05b*
81. Wages	$112,700	$113,356		
82. Utilities	$9400	$9772		
83. Taxes	$37,640	$37,335		
84. Insurance	$2575	$2613		

Federal Deficit **In Exercises 85–90, use the bar graph, which shows the receipts of the federal government (in billions of dollars) for selected years from 1960 through 1998. In each exercise you are given the outlay of the federal government. Find the magnitude of the surplus or deficit for the year.** (Source: U.S. Office of Management and Budget)



	Receipts, y	*Outlay, x*	$\lvert y - x \rvert$
85. 1960		$92.2 billion	
86. 1970		$195.6 billion	
87. 1980		$590.9 billion	
88. 1990		$1253.2 billion	
89. 1995		$1515.7 billion	
90. 1998		$1667.8 billion	

In Exercises 91–96, identify the terms. Then identify the coefficients of the variable terms of the expression.

91. $7x + 4$

92. $2x - 9$

93. $3x^2 - 8x - 11$

94. $7\sqrt{5}x^2 + 3$

95. $4x^3 + \frac{x}{2} - 5$

96. $3x^4 + 3x^3$

In Exercises 97–102, evaluate the expression for each value of x. (If not possible, state the reason.)

Expression	*Values*	
97. $4x - 6$	(a) $x = -1$	(b) $x = 0$
98. $9 - 7x$	(a) $x = -3$	(b) $x = 3$
99. $x^2 - 3x + 4$	(a) $x = -2$	(b) $x = 2$
100. $-x^2 + 5x - 4$	(a) $x = -1$	(b) $x = 1$
101. $\dfrac{x+1}{x-1}$	(a) $x = 1$	(b) $x = -1$
102. $\dfrac{x}{x+2}$	(a) $x = 2$	(b) $x = -2$

In Exercises 103–112, identify the rule(s) of algebra illustrated by the equation.

103. $x + 9 = 9 + x$

104. $2\left(\frac{1}{2}\right) = 1$

105. $\dfrac{1}{h+6}(h + 6) = 1, \quad h \neq -6$

106. $(x + 3) - (x + 3) = 0$

107. $2(x + 3) = 2x + 6$

108. $(z - 2) + 0 = z - 2$

109. $1 \cdot (1 + x) = 1 + x$

110. $x + (y + 10) = (x + y) + 10$

111. $x(3y) = (x \cdot 3)y = (3x)y$

112. $\frac{1}{7}(7 \cdot 12) = \left(\frac{1}{7} \cdot 7\right)12 = 1 \cdot 12 = 12$

In Exercises 113–116, evaluate the expression. (If not possible, state the reason.)

113. $\dfrac{81 - (90 - 9)}{5}$

114. $10(23 - 30 + 7)$

115. $\dfrac{8 - 8}{-9 + (6 + 3)}$

116. $15 - \dfrac{3 - 3}{5}$

In Exercises 117–124, perform the operations. (Write fractional answers in simplest form.)

117. $\frac{3}{16} + \frac{5}{16}$

118. $\frac{6}{7} - \frac{4}{7}$

119. $\frac{5}{8} - \frac{5}{12} + \frac{1}{6}$

120. $\frac{10}{11} + \frac{6}{33} - \frac{13}{66}$

121. $\dfrac{x}{6} + \dfrac{3x}{4}$

122. $\dfrac{11}{x} \div \dfrac{3}{4}$

123. $12 \div \frac{1}{4}$

124. $\left(\frac{3}{5} \div 3\right) - \left(6 \cdot \frac{4}{8}\right)$

In Exercises 125–128, use a calculator to evaluate the expression. (Round your answer to two decimal places.)

125. $-3 + \frac{3}{7}$

126. $3\left(-\frac{5}{12} + \frac{3}{8}\right)$

127. $\dfrac{11.46 - 5.37}{3.91}$

128. $\dfrac{\frac{1}{5}(-8 - 9)}{-\frac{1}{3}}$

129. (a) Use a calculator to complete the table.

n	1	0.5	0.01	0.0001	0.000001
$5/n$					

(b) Use the result from part (a) to make a conjecture about the value of $\dfrac{5}{n}$ as n approaches 0.

130. (a) Use a calculator to complete the table.

n	1	10	100	10,000	100,000
$5/n$					

(b) Use the result from part (a) to make a conjecture about the value of $\dfrac{5}{n}$ as n increases without bound.

Synthesis

131. ***Exploration*** Consider $|u + v|$ and $|u| + |v|$.

(a) Are the values of the expressions always equal? If not, under what conditions are they unequal?

(b) If the two expressions are not equal for certain values of u and v, is one of the expressions always greater than the other? Explain.

132. ***Think About It*** Is there a difference between saying that a real number is positive and saying that a real number is nonnegative? Explain.

133. ***Writing*** Describe the differences among the sets of natural numbers, integers, rational numbers, and irrational numbers.

True or False? **In Exercises 134 and 135, determine whether the statement is true or false. Justify your answer.**

134. Let $a > b$, then $\dfrac{1}{a} > \dfrac{1}{b}$, where $a \neq 0$ and $b \neq 0$.

135. Because $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$, then $\dfrac{c}{a+b} = \dfrac{c}{a} + \dfrac{c}{b}$.

P.2 Exponents and Radicals

Exponents

Repeated *multiplication* can be written in **exponential form.** Here are some examples.

Repeated Multiplication	*Exponential Form*
$a \cdot a \cdot a \cdot a \cdot a$	a^5
$(-4)(-4)(-4)$	$(-4)^3$
$(2x)(2x)(2x)(2x)$	$(2x)^4$

In general, if a is a real number, variable, or algebraic expression and n is a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}$$

where n is the **exponent** and a is the **base.** The expression a^n is read "a to the nth **power.**" An exponent can be negative. Property 3 below shows how to use a negative exponent.

Properties of Exponents

Let a and b be real numbers, variables, or algebraic expressions, and let m and n be integers. (All denominators and bases are nonzero.)

Property	*Example*
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^4 = 3^{2+4} = 3^6 = 729$
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{x^7}{x^4} = x^{7-4} = x^3$
3. $a^{-n} = \dfrac{1}{a^n} = \left(\dfrac{1}{a}\right)^n$	$y^{-4} = \dfrac{1}{y^4} = \left(\dfrac{1}{y}\right)^4$
4. $a^0 = 1, \quad a \neq 0$	$(x^2 + 1)^0 = 1$
5. $(ab)^m = a^m b^m$	$(5x)^3 = 5^3x^3 = 125x^3$
6. $(a^m)^n = a^{mn}$	$(y^3)^{-4} = y^{3(-4)} = y^{-12} = \dfrac{1}{y^{12}}$
7. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\left(\dfrac{2}{x}\right)^3 = \dfrac{2^3}{x^3} = \dfrac{8}{x^3}$
8. $\lvert a^2 \rvert = \lvert a \rvert^2 = a^2$	$\lvert(-2)^2\rvert = \lvert -2 \rvert^2 = (-2)^2 = 4$

It is important to recognize the difference between expressions such as $(-2)^4$ and -2^4. In $(-2)^4$, the parentheses indicate that the exponent applies to the negative sign as well as to the 2, but in $-2^4 = -(2^4)$, the exponent applies only to the 2. So, $(-2)^4 = 16$, whereas $-2^4 = -16$.

What You Should Learn:

- How to use properties of exponents
- How to use scientific notation to represent real numbers
- How to use properties of radicals
- How to simplify and combine radicals
- How to rationalize denominators and numerators
- How to use properties of rational exponents

Why You Should Learn It:

Real numbers and algebraic expressions are often written with exponents and radicals. For instance, in Exercise 100 on page 23, you will use an expression involving a radical to find the size of a particle that can be carried by a stream moving at a certain velocity.

Superstock

The properties of exponents listed on the previous page apply to *all* integers m and n, not just positive integers. For instance, by Property 2, you can write

$$\frac{3^4}{3^{-5}} = 3^{4-(-5)} = 3^{4+5} = 3^9.$$

EXAMPLE 1 Using Properties of Exponents

a. $(-3ab^4)(4ab^{-3}) = -12(a)(a)(b^4)(b^{-3}) = -12a^2b$

b. $(2xy^2)^3 = 2^3(x)^3(y^2)^3 = 8x^3y^6$

c. $3a(-4a^2)^0 = 3a(1) = 3a, \qquad a \neq 0$

d. $\left(\dfrac{5x^3}{y}\right)^2 = \dfrac{5^2(x^3)^2}{y^2} = \dfrac{25x^6}{y^2}$

EXAMPLE 2 Rewriting with Positive Exponents

a. $x^{-1} = \dfrac{1}{x}$ — Property 3

b. $\dfrac{1}{3x^{-2}} = \dfrac{1(x^2)}{3} = \dfrac{x^2}{3}$ — The exponent -2 does not apply to 3.

c. $\dfrac{1}{(3x)^{-2}} = (3x)^2 = 9x^2$ — The exponent -2 does apply to 3.

d. $\dfrac{12a^3b^{-4}}{4a^{-2}b} = \dfrac{12a^3 \cdot a^2}{4b \cdot b^4} = \dfrac{3a^5}{b^5}$

e. $\left(\dfrac{3x^2}{y}\right)^{-2} = \dfrac{3^{-2}(x^2)^{-2}}{y^{-2}}$ — Properties 5 and 7

$= \dfrac{3^{-2}x^{-4}}{y^{-2}}$ — Property 6

$= \dfrac{y^2}{3^2x^4}$ — Property 3

$= \dfrac{y^2}{9x^4}$ — Simplify.

EXAMPLE 3 Calculators and Exponents

	Expression	*Graphing Calculator Keystrokes*	*Display*
a.	$13^4 + 5$	13 $\boxed{\wedge}$ 4 $\boxed{+}$ 5 $\boxed{\text{ENTER}}$	28566
b.	$3^{-2} + 4^{-1}$	3 $\boxed{\wedge}$ $\boxed{(-)}$ 2 $\boxed{+}$ 4 $\boxed{\wedge}$ $\boxed{(-)}$ 1 $\boxed{\text{ENTER}}$	.3611111111
c.	$\dfrac{3^5 + 1}{3^5 - 1}$	$\boxed{(}$ 3 $\boxed{\wedge}$ 5 $\boxed{+}$ 1 $\boxed{)}$ $\boxed{\div}$ $\boxed{(}$ 3 $\boxed{\wedge}$ 5 $\boxed{-}$ 1 $\boxed{)}$ $\boxed{\text{ENTER}}$	1.008264463

The graphing calculator keystrokes given in this text may not be the same as the keystrokes for your graphing calculator. Be sure you are familiar with the use of the keys on your own calculator.

The French mathematician Nicolas Chuquet (ca. 1500) wrote *Triparty en la science des nombres*, in which a form of exponent notation was used. Our expressions $6x^3$ and $10x^2$ were written as $.6.^3$ and $.10.^2$. Zero and negative exponents were also represented, so x^0 would be written as $.1.^0$ and $3x^{-2}$ as $.3.^{2m}$. Chuquet wrote that $.72.^1$ divided by $.8.^3$ is $.9.^{2m}$. That is, $72x \div 8x^3 = 9x^{-2}$.

STUDY TIP

Rarely in algebra is there only one way to solve a problem. Don't be concerned if the steps you use to solve a problem are not exactly the same as the steps presented in this text. The important thing is to use steps that you understand *and*, of course, that are justified by the rules of algebra. For instance, you might prefer the following steps for Example 2(e).

$$\left(\frac{3x^2}{y}\right)^{-2} = \left(\frac{y}{3x^2}\right)^2 = \frac{y^2}{9x^4}$$

Scientific Notation

Exponents provide an efficient way of writing and computing with very large (or very small) numbers. For instance, there are about 326 billion billion gallons of water on Earth—that is, 326 followed by 18 zeros.

$$326{,}000{,}000{,}000{,}000{,}000{,}000$$

It is convenient to write such numbers in **scientific notation.** This notation has the form $\pm c \times 10^n$, where $1 \leq c < 10$ and n is an integer. So, the number of gallons of water on Earth can be written in scientific notation as

$$3.26 \times 100{,}000{,}000{,}000{,}000{,}000{,}000 = 3.26 \times 10^{20}.$$

The *positive* exponent 20 indicates that the number is *large* (10 or more) and that the decimal point has been moved 20 places. A *negative* exponent indicates that the number is *small* (less than 1). For instance, the mass (in grams) of one electron is approximately

$$9.0 \times 10^{-28} = 0.\underbrace{000000000000000000000000000 9}_{\text{28 decimal places}}.$$

EXAMPLE 4 Scientific Notation

a. $1.345 \times 10^2 = 134.5$

b. $0.0000782 = 7.82 \times 10^{-5}$

c. $-9.36 \times 10^{-6} = -0.00000936$

d. $836{,}100{,}000 = 8.361 \times 10^8$

Most calculators switch to scientific notation when they are showing large or small numbers that exceed the display range. Try evaluating $86{,}500{,}000 \times 6000$. If your calculator follows standard conventions, its display should be

$\boxed{5.19 \quad 11}$ or $\boxed{5.19 \quad \text{E} \quad 11}$

which is 5.19×10^{11}.

EXAMPLE 5 Using Scientific Notation with a Calculator

Use a calculator to evaluate $65{,}000 \times 3{,}400{,}000{,}000$.

Solution

Because $65{,}000 = 6.5 \times 10^4$ and $3{,}400{,}000{,}000 = 3.4 \times 10^9$, you can multiply the two numbers using the following graphing calculator steps.

$6.5 \ \boxed{\text{EE}} \ 4 \ \boxed{\times} \ 3.4 \ \boxed{\text{EE}} \ 9 \ \boxed{\text{ENTER}}$

After entering these keystrokes, the calculator display should read $\boxed{2.21 \quad \text{E} \quad 14}$. Therefore, the product of the two numbers is

$$\begin{aligned}(6.5 \times 10^4)(3.4 \times 10^9) &= 2.21 \times 10^{14} \\ &= 221{,}000{,}000{,}000{,}000.\end{aligned}$$

Radicals and Their Properties

A **square root** of a number is one of its two equal factors. For example, 5 is a square root of 25 because 5 is one of the two equal factors of $25 = 5 \cdot 5$. In a similar way, a **cube root** of a number is one of its three equal factors, as in $125 = 5^3$.

Definition of the *n*th Root of a Number

Let a and b be real numbers and let $n \geq 2$ be a positive integer. If

$$a = b^n$$

then b is an ***n*th root of *a*.** If $n = 2$, the root is a **square root.** If $n = 3$, the root is a **cube root.**

Some numbers have more than one nth root. For example, both 5 and -5 are square roots of 25. The *principal square root* of 25, written as $\sqrt{25}$, is the positive root, 5. The **principal *n*th root** of a number is defined as follows.

Principal *n*th Root of a Number

Let a be a real number that has at least one nth root. The **principal *n*th root of *a*** is the nth root that has the same sign as a. It is denoted by a **radical symbol**

$$\sqrt[n]{a}. \qquad \text{Principal } n\text{th root}$$

The positive integer n is the **index** of the radical, and the number a is the **radicand.** If $n = 2$, omit the index and write $\sqrt{a}$ rather than $\sqrt[2]{a}$. (The plural of index is *indices*.)

A common misunderstanding is that the square root sign implies both negative and positive roots. This is not correct. The square root sign implies only a positive root. When a negative root is needed, you must use the negative sign with the square root sign.

Incorrect: $\cancel{\sqrt{4} = \pm 2}$ Correct: $-\sqrt{4} = -2$ and $\sqrt{4} = 2$

EXAMPLE 6 Evaluating Expressions Involving Radicals

a. $\sqrt{36} = 6$ because $6^2 = 36$.

b. $-\sqrt{36} = -6$ because $-\left(\sqrt{36}\right) = -(6) = -6$.

c. $\sqrt[3]{\dfrac{125}{64}} = \dfrac{5}{4}$ because $\left(\dfrac{5}{4}\right)^3 = \dfrac{5^3}{4^3} = \dfrac{125}{64}$.

d. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$.

e. $\sqrt[4]{-81}$ is not a real number because there is no real number that can be raised to the fourth power to produce -81.

Here are some generalizations about the nth roots of a real number.

Generalizations About nth Roots of Real Numbers

Real Number a	Integer n	Root(s) of a	Example
$a > 0$	$n > 0, n$ is even	$\sqrt[n]{a}, -\sqrt[n]{a}$	$\sqrt[4]{81} = 3, -\sqrt[4]{81} = -3$
$a > 0$ or $a < 0$	n is odd	$\sqrt[n]{a}$	$\sqrt[3]{-8} = -2$
$a < 0$	n is even	No real roots	$\sqrt{-4}$ is not real
$a = 0$	n is even or odd	$\sqrt[n]{0} = 0$	$\sqrt[5]{0} = 0$

Integers such as 1, 4, 9, 16, 25, and 36 are called **perfect squares** because they have integer square roots. Similarly, integers such as 1, 8, 27, 64, and 125 are called **perfect cubes** because they have integer cube roots.

Properties of Radicals

Let a and b be real numbers, variables, or algebraic expressions such that the indicated roots are real numbers, and let m and n be positive integers.

Property	*Example*
1. $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$	$\sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$
2. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	$\sqrt{5} \cdot \sqrt{7} = \sqrt{5 \cdot 7} = \sqrt{35}$
3. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}, \quad b \neq 0$	$\dfrac{\sqrt[4]{27}}{\sqrt[4]{9}} = \sqrt[4]{\dfrac{27}{9}} = \sqrt[4]{3}$
4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt[3]{\sqrt{10}} = \sqrt[6]{10}$
5. $\left(\sqrt[n]{a}\right)^n = a$	$\left(\sqrt{3}\right)^2 = 3$
6. For n even, $\sqrt[n]{a^n} = \lvert a \rvert$.	$\sqrt{(-12)^2} = \lvert -12 \rvert = 12$
For n odd, $\sqrt[n]{a^n} = a$.	$\sqrt[3]{(-12)^3} = -12$

A common special case of Property 6 is $\sqrt{a^2} = \lvert a \rvert$.

EXAMPLE 7 Using Properties of Radicals

Use the properties of radicals to simplify each expression.

a. $\sqrt{8} \cdot \sqrt{2}$ **b.** $\left(\sqrt[3]{5}\right)^3$

c. $\sqrt[3]{x^3}$ **d.** $\sqrt[6]{y^6}$

Solution

a. $\sqrt{8} \cdot \sqrt{2} = \sqrt{8 \cdot 2} = \sqrt{16} = 4$

b. $\left(\sqrt[3]{5}\right)^3 = 5$

c. $\sqrt[3]{x^3} = x$

d. $\sqrt[6]{y^6} = \lvert y \rvert$

STUDY T!P

There are four methods of evaluating radicals on most graphing calculators. For square roots, you can use the *square root key* [√]. For cube roots, you can use the *cube root key* [∛] (or menu choice). For other roots, you can first convert the radical to exponential form and then use the *exponential key* [^] or you can use the *nth root key* [ⁿ√].

Simplifying Radicals

An expression involving radicals is in **simplest form** when the following conditions are satisfied.

1. All possible factors have been removed from the radical.
2. All fractions have radical-free denominators (accomplished by a process called *rationalizing the denominator*).
3. The index of the radical is reduced.

To simplify a radical, factor the radicand into factors whose exponents are multiples of the index. The roots of these factors are written outside the radical, and the "leftover" factors make up the new radicand.

EXAMPLE 8 Simplifying Even Roots

a. $\sqrt[4]{48} = \sqrt[4]{\overset{\text{Perfect 4th power}}{\downarrow \atop 16} \cdot \overset{\text{Leftover factor}}{\downarrow \atop 3}} = \sqrt[4]{2^4 \cdot 3} = 2\sqrt[4]{3}$

b.
$$\begin{aligned} \sqrt{75x^3} &= \sqrt{\overset{\text{Perfect square}}{\downarrow \atop 25x^2} \cdot \overset{\text{Leftover factor}}{\downarrow \atop 3x}} && \text{Find largest square factor.} \\ &= \sqrt{(5x)^2 \cdot 3x} \\ &= 5x\sqrt{3x} && \text{Find root of perfect square.} \end{aligned}$$

c. $\sqrt[4]{(5x)^4} = |5x| = 5|x|$

STUDY T!P

When you simplify a radical, it is important that both expressions are defined for the same values of the variable. For instance, in Example 8(b), $\sqrt{75x^3}$ and $5x\sqrt{3x}$ are both defined only for nonnegative values of x. Similarly, in Example 8(c), $\sqrt[4]{(5x)^4}$ and $5|x|$ are both defined for all real values of x.

In Example 8(b), the expression $\sqrt{75x^3}$ makes sense only for nonnegative values of x.

EXAMPLE 9 Simplifying Odd Roots

a. $\sqrt[3]{24} = \sqrt[3]{\overset{\text{Perfect cube}}{\downarrow \atop 8} \cdot \overset{\text{Leftover factor}}{\downarrow \atop 3}} = \sqrt[3]{2^3 \cdot 3} = 2\sqrt[3]{3}$

b.
$$\begin{aligned} \sqrt[3]{24a^4} &= \sqrt[3]{\overset{\text{Perfect cube}}{\downarrow \atop 8a^3} \cdot \overset{\text{Leftover factor}}{\downarrow \atop 3a}} && \text{Find largest cube factor.} \\ &= \sqrt[3]{(2a)^3 \cdot 3a} \\ &= 2a\sqrt[3]{3a} && \text{Find root of perfect cube.} \end{aligned}$$

c.
$$\begin{aligned} \sqrt[3]{-40x^6} &= \sqrt[3]{(-8x^6) \cdot 5} && \text{Find largest cube factor.} \\ &= \sqrt[3]{(-2x^2)^3 \cdot 5} \\ &= -2x^2\sqrt[3]{5} && \text{Find root of perfect cube.} \end{aligned}$$

Radical expressions can be combined (added or subtracted) if they are **like radicals**—that is, if they have the same index and radicand. For instance, $\sqrt{2}$, $3\sqrt{2}$, and $\frac{1}{2}\sqrt{2}$ are like radicals, but $\sqrt{3}$ and $\sqrt{2}$ are unlike radicals. To determine whether two radicals can be combined, you should first simplify each radical.

EXAMPLE 10 Combining Radicals

a. $2\sqrt{48} - 3\sqrt{27} = 2\sqrt{16 \cdot 3} - 3\sqrt{9 \cdot 3}$ Find square factors.

$= 8\sqrt{3} - 9\sqrt{3}$ Find square roots.

$= (8 - 9)\sqrt{3}$ Combine like terms.

$= -\sqrt{3}$

b. $\sqrt[3]{16x} - \sqrt[3]{54x^4} = \sqrt[3]{8 \cdot 2x} - \sqrt[3]{27 \cdot x^3 \cdot 2x}$ Find cube factors.

$= 2\sqrt[3]{2x} - 3x\sqrt[3]{2x}$ Find cube roots.

$= (2 - 3x)\sqrt[3]{2x}$ Combine like terms.

Try using your calculator to check the result of Example 10(a). You should obtain -1.732050808, which is the same as the calculator's approximation for $-\sqrt{3}$.

Rationalizing Denominators and Numerators

To rationalize a denominator or numerator of the form $a - b\sqrt{m}$ or $a + b\sqrt{m}$, multiply both numerator and denominator by a **conjugate:** $a + b\sqrt{m}$ and $a - b\sqrt{m}$ are conjugates of each other. If $a = 0$, the rationalizing factor for $\sqrt{m}$ is itself, $\sqrt{m}$.

EXAMPLE 11 Rationalizing Denominators

Rationalize the denominator of

a. $\dfrac{5}{2\sqrt{3}}$ **b.** $\dfrac{2}{\sqrt[3]{5}}$.

Solution

a. $$\frac{5}{2\sqrt{3}} = \frac{5}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$ $\sqrt{3}$ is rationalizing factor.

$$= \frac{5\sqrt{3}}{2(3)}$$

$$= \frac{5\sqrt{3}}{6}$$

b. $$\frac{2}{\sqrt[3]{5}} = \frac{2}{\sqrt[3]{5}} \cdot \frac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}}$$ $\sqrt[3]{5^2}$ is rationalizing factor.

$$= \frac{2\sqrt[3]{5^2}}{\sqrt[3]{5^3}}$$

$$= \frac{2\sqrt[3]{25}}{5}$$

STUDY TIP

Notice in Example 11(b) that the numerator and denominator are multiplied by $\sqrt[3]{5^2}$ to produce a perfect cube radicand.

EXAMPLE 12 Rationalizing a Denominator with Two Terms

Rationalize the denominator of $\dfrac{2}{3+\sqrt{7}}$.

Solution

$$\frac{2}{3+\sqrt{7}} = \frac{2}{3+\sqrt{7}} \cdot \frac{3-\sqrt{7}}{3-\sqrt{7}}$$

Multiply numerator and denominator by conjugate of denominator.

$$= \frac{2\left(3-\sqrt{7}\right)}{(3)^2-\left(\sqrt{7}\right)^2}$$

Find products. In denominator, $(a+b)(a-b)=a^2-b^2$.

$$= \frac{2\left(3-\sqrt{7}\right)}{2} = 3-\sqrt{7}$$

Simplify and divide out like factors.

Sometimes it is necessary to rationalize the numerator of an expression. For instance, in Section P.4 you will use the technique shown in the next example to rationalize the numerator of an expression from calculus.

EXAMPLE 13 Rationalizing a Numerator

Rationalize the numerator of $\dfrac{\sqrt{5}-\sqrt{7}}{2}$.

Solution

$$\frac{\sqrt{5}-\sqrt{7}}{2} = \frac{\sqrt{5}-\sqrt{7}}{2} \cdot \frac{\sqrt{5}+\sqrt{7}}{\sqrt{5}+\sqrt{7}}$$

Multiply numerator and denominator by conjugate of numerator.

$$= \frac{\left(\sqrt{5}\right)^2-\left(\sqrt{7}\right)^2}{2\left(\sqrt{5}+\sqrt{7}\right)}$$

Find products. In numerator, $(a+b)(a-b)=a^2-b^2$.

$$= \frac{-2}{2\left(\sqrt{5}+\sqrt{7}\right)} = \frac{-1}{\sqrt{5}+\sqrt{7}}$$

Simplify and divide out like terms.

STUDY TIP

Do not confuse the expression $\sqrt{5}+\sqrt{7}$ with the expression $\sqrt{5+7}$. In general, $\sqrt{x+y}$ does not equal $\sqrt{x}+\sqrt{y}$. Similarly, $\sqrt{x^2+y^2}$ does not equal $x+y$.

Rational Exponents

Definition of Rational Exponents

If a is a real number and n is a positive integer such that the principal nth root of a exists, then $a^{1/n}$ is defined as

$a^{1/n} = \sqrt[n]{a}$ where $1/n$ is the **rational exponent** of a.

Moreover, if m is a positive integer that has no common factor with n, then

$$a^{m/n} = \left(a^{1/n}\right)^m = \left(\sqrt[n]{a}\right)^m \quad \text{and} \quad a^{m/n} = \left(a^m\right)^{1/n} = \sqrt[n]{a^m}.$$

The symbol indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

The numerator of a rational exponent denotes the *power* to which the base is raised, and the denominator denotes the *index* or the *root* to be taken.

Power

Index

$$b^{m/n} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$$

When you are working with rational exponents, the properties of integer exponents still apply. For instance,

$$2^{1/2}2^{1/3} = 2^{(1/2)+(1/3)} = 2^{5/6}.$$

STUDY TIP

Rational exponents can be tricky, and you must remember that the expression $b^{m/n}$ is not defined unless $\sqrt[n]{b}$ is a real number. This restriction produces some unusual-looking results. For instance, the number $(-8)^{1/3}$ is defined because $\sqrt[3]{-8} = -2$, but the number $(-8)^{2/6}$ is undefined because $\sqrt[6]{-8}$ is not a real number.

EXAMPLE 14 Changing from Radical to Exponential Form

a. $\sqrt{3} = 3^{1/2}$

b. $\sqrt{(3xy)^5} = \sqrt[2]{(3xy)^5} = (3xy)^{(5/2)}$

c. $2x\sqrt[4]{x^3} = (2x)(x^{3/4}) = 2x^{1+(3/4)} = 2x^{7/4}$

EXAMPLE 15 Changing from Exponential to Radical Form

a. $(x^2 + y^2)^{3/2} = \left(\sqrt{x^2 + y^2}\right)^3 = \sqrt{(x^2 + y^2)^3}$

b. $2y^{3/4}z^{1/4} = 2(y^3z)^{1/4} = 2\sqrt[4]{y^3z}$

c. $a^{-3/2} = \dfrac{1}{a^{3/2}} = \dfrac{1}{\sqrt{a^3}}$

d. $x^{0.2} = x^{1/5} = \sqrt[5]{x}$

Rational exponents are useful for evaluating roots of numbers on a calculator, reducing the index of a radical, and simplifying calculus expressions.

EXAMPLE 16 Simplifying with Rational Exponents

a. $(-32)^{-4/5} = \left(\sqrt[5]{-32}\right)^{-4} = (-2)^{-4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$

b. $(-5x^{5/3})(3x^{-3/4}) = -15x^{(5/3)-(3/4)} = -15x^{11/12}, \qquad x \neq 0$

c. $\sqrt[9]{a^3} = a^{3/9} = a^{1/3} = \sqrt[3]{a}$ Reduce index.

d. $\sqrt[3]{\sqrt{125}} = \sqrt[6]{125} = \sqrt[6]{(5)^3} = 5^{3/6} = 5^{1/2} = \sqrt{5}$

e. $(2x - 1)^{4/3}(2x - 1)^{-1/3} = (2x - 1)^{(4/3)-(1/3)} = 2x - 1, \qquad x \neq \dfrac{1}{2}$

f.
$$\begin{aligned}\frac{x-1}{(x-1)^{-1/2}} &= \frac{x-1}{(x-1)^{-1/2}} \cdot \frac{(x-1)^{1/2}}{(x-1)^{1/2}} \\ &= \frac{(x-1)^{3/2}}{(x-1)^0} \\ &= (x-1)^{3/2}, \qquad x \neq 1\end{aligned}$$

P.2 Exercises

In Exercises 1–8, evaluate each expression.

1. (a) $4^2 \cdot 3$ (b) $3 \cdot 3^3$

2. (a) $\dfrac{5^5}{5^2}$ (b) $\dfrac{3^2}{3^4}$

3. (a) $(3^3)^2$ (b) -3^2

4. (a) $(2^3 \cdot 3^2)^2$ (b) $\left(-\frac{3}{5}\right)^3\left(\frac{5}{3}\right)^2$

5. (a) $\dfrac{3}{3^{-4}}$ (b) $24(-2)^{-5}$

6. (a) $\dfrac{4 \cdot 3^{-2}}{2^{-2} \cdot 3^{-1}}$ (b) $(-2)^0$

7. (a) $2^{-1} + 3^{-1}$ (b) $(2^{-1})^{-2}$

8. (a) $3^{-1} + 2^{-2}$ (b) $(3^{-2})^2$

In Exercises 9–12, use a calculator to evaluate the expression. (Round your answer to three decimal places.)

9. $(-4)^3(5^2)$

10. $(8^{-4})(10^3)$

11. $\dfrac{3^6}{7^3}$

12. $\dfrac{4^3}{3^{-4}}$

In Exercises 13–18, evaluate the expression for the value of x.

	Expression	*Value*
13.	$7x^{-2}$	4
14.	$6x^0 - (6x)^0$	10
15.	$4x^3$	-3
16.	$-5x^4$	-2
17.	$6x^2$	$-\frac{1}{2}$
18.	$4(-x)^3$	$\frac{1}{3}$

In Exercises 19–26, simplify each expression.

19. (a) $(-5z)^3$ (b) $5x^4(x^2)$

20. (a) $(3x)^2$ (b) $(4x^3)^2$

21. (a) $\dfrac{7x^2}{x^3}$ (b) $\dfrac{12(x+y)^3}{9(x+y)}$

22. (a) $\dfrac{r^4}{r^6}$ (b) $\left(\dfrac{4}{y}\right)^3\left(\dfrac{3}{y}\right)^4$

23. (a) $(x+5)^0, \quad x \neq -5$ (b) $(2x^2)^{-2}$

24. (a) $(2x^5)^0, \quad x \neq 0$ (b) $(z+2)^{-3}(z+2)^{-1}$

25. (a) $[(x^2y^{-2})^{-1}]^{-1}$ (b) $(5x^2z^6)^3(5x^2z^6)^{-3}$

26. (a) $3^n \cdot 3^{2n}$ (b) $\left(\dfrac{a^{-2}}{b^{-2}}\right)\left(\dfrac{b}{a}\right)^3$

In Exercises 27–34, fill in the missing description.

	Radical Form	*Rational Exponent Form*
27.	$\sqrt[3]{64} = 4$	
28.		$-(144^{1/2}) = -12$
29.		$196^{1/2} = 14$
30.	$\sqrt[3]{614.125} = 8.5$	
31.	$\sqrt[3]{-216} = -6$	
32.		$(-243)^{1/5} = -3$
33.	$\sqrt[4]{81^3} = 27$	
34.		$16^{5/4} = 32$

In Exercises 35–44, evaluate each expression without using a calculator.

35. $\sqrt{9}$

36. $\sqrt{49}$

37. $-\sqrt[3]{-27}$

38. $\dfrac{\sqrt[4]{81}}{3}$

39. $\left(\sqrt[3]{-125}\right)^3$

40. $\sqrt[4]{562^4}$

41. $32^{-3/5}$

42. $\left(\frac{9}{4}\right)^{-1/2}$

43. $\left(-\dfrac{1}{64}\right)^{-1/3}$

44. $-\left(\dfrac{1}{125}\right)^{-4/3}$

In Exercises 45–50, use a calculator to approximate the number. (Round your answer to three decimal places.)

45. $\sqrt[5]{-27^3}$

46. $\sqrt[3]{45^2}$

47. $(3.4)^{2.5}$

48. $(6.1)^{-2.9}$

49. $(1.2^{-2})\sqrt{75} + 3\sqrt{8}$

50. $\dfrac{-5 + \sqrt{33}}{5}$

In Exercises 51–54, simplify by removing all possible factors from the radical.

51. (a) $\sqrt{8}$ (b) $\sqrt[3]{24}$

52. (a) $\sqrt[3]{\frac{16}{27}}$ (b) $\sqrt{\frac{75}{4}}$

53. (a) $\sqrt{54xy^4}$ (b) $\sqrt{\dfrac{32a^4}{b^2}}$

54. (a) $\sqrt[4]{(3x^2)^4}$ (b) $\sqrt[5]{96x^5}$

In Exercises 55–60, perform the operations and simplify.

55. $5^{4/3} \cdot 5^{8/3}$

56. $\dfrac{8^{12/5}}{8^{2/5}}$

57. $\dfrac{(2x^2)^{3/2}}{2^{1/2}x^4}$

58. $\dfrac{x^{4/3}y^{2/3}}{(xy)^{1/3}}$

59. $\dfrac{x^{-3} \cdot x^{1/2}}{x^{3/2} \cdot x^{-1}}$

60. $\dfrac{5^{-1/2} \cdot 5x^{5/2}}{(5x)^{3/2}}$

In Exercises 61–64, rationalize the denominator of each expression. Then simplify your answer.

61. (a) $\dfrac{1}{\sqrt{3}}$ (b) $\dfrac{8}{\sqrt[3]{2}}$

62. (a) $\dfrac{5}{\sqrt{10}}$ (b) $\dfrac{5}{\sqrt[3]{(5x)^2}}$

63. (a) $\dfrac{2x}{5 - \sqrt{3}}$ (b) $\dfrac{3}{\sqrt{5} + \sqrt{6}}$

64. (a) $\dfrac{5}{\sqrt{14} - 2}$ (b) $\dfrac{5}{2\sqrt{10} - 5}$

In Exercises 65–68, rationalize the numerator of each expression. Then simplify your answer.

65. $\dfrac{\sqrt{8}}{2}$

66. $\dfrac{\sqrt{2}}{3}$

67. $\dfrac{\sqrt{5} + \sqrt{3}}{3}$

68. $\dfrac{\sqrt{3} - \sqrt{2}}{2}$

In Exercises 69 and 70, reduce the index of each radical.

69. (a) $\sqrt[4]{3^2}$ (b) $\sqrt[6]{(x+1)^4}$

70. (a) $\sqrt[6]{x^3}$ (b) $\sqrt[4]{(3x^2)^4}$

In Exercises 71 and 72, write each expression as a single radical. Then simplify your answer.

71. (a) $\sqrt{\sqrt{32}}$ (b) $\sqrt{\sqrt[4]{2x}}$

72. (a) $\sqrt{\sqrt{243(x+1)}}$ (b) $\sqrt{\sqrt[3]{10a^7b}}$

In Exercises 73–76, simplify each expression.

73. (a) $2\sqrt{50} + 12\sqrt{8}$ (b) $10\sqrt{32} - 6\sqrt{18}$

74. (a) $5\sqrt{x} - 3\sqrt{x}$ (b) $-2\sqrt{9y} + 10\sqrt{y}$

75. (a) $3\sqrt{x+1} + 10\sqrt{x+1}$
(b) $7\sqrt{80x} - 2\sqrt{125x}$

76. (a) $5\sqrt{10x^2} - \sqrt{90x^2}$
(b) $8\sqrt[3]{27x} - \frac{1}{2}\sqrt[3]{64x}$

In Exercises 77–80, fill in the blank with <, =, or >.

77. $\sqrt{5} + \sqrt{3}$ ▨ $\sqrt{5+3}$

78. $\sqrt{\dfrac{3}{11}}$ ▨ $\dfrac{\sqrt{3}}{\sqrt{11}}$

79. 5 ▨ $\sqrt{3^2 + 2^2}$

80. 5 ▨ $\sqrt{3^2 + 4^2}$

In Exercises 81–84, write the number in scientific notation.

81. Land Area of Earth: 57,300,000 square miles

82. Light Year: 9,460,000,000,000,000 kilometers

83. Relative Density of Hydrogen: 0.0000899 gram per cm^3

84. One Micron (Millionth of a Meter): 0.00003937 inch

In Exercises 85–88, write the number in decimal form.

85. U.S. Daily Coca-Cola Consumption: 6.048×10^8 servings (Source: The World of Coca-Cola Pavilion)

86. Interior Temperature of Sun: 1.5×10^7 degrees Celsius

87. Charge of Electron: 1.602×10^{-19} coulomb

88. Width of Human Hair: 9.0×10^{-5} meter

In Exercises 89–92, use a calculator to evaluate each expression. (Round your answer to three decimal places.)

89. (a) $750\left(1 + \dfrac{0.11}{365}\right)^{800}$ (b) $\dfrac{67{,}000{,}000 + 93{,}000{,}000}{0.0052}$

The symbol indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

90. (a) $(9.3 \times 10^6)^3(6.1 \times 10^{-4})$

(b) $\dfrac{(2.414 \times 10^4)^6}{(1.68 \times 10^5)^5}$

91. (a) $\sqrt{4.5 \times 10^9}$ (b) $\sqrt[3]{6.3 \times 10^4}$

92. (a) $(2.65 \times 10^{-4})^{1/3}$ (b) $\sqrt{9 \times 10^{-4}}$

In Exercises 93 and 94, evaluate each expression without using a calculator.

93. (a) $\sqrt{25 \times 10^8}$ (b) $\sqrt[3]{8 \times 10^{15}}$

94. (a) $(1.2 \times 10^7)(5 \times 10^{-3})$

(b) $\dfrac{(6.0 \times 10^8)}{(3.0 \times 10^{-3})}$

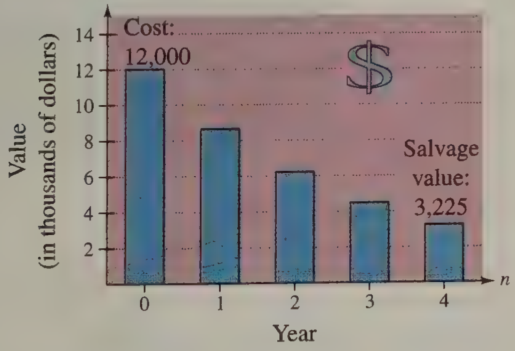
95. ***Exploration*** List all possible digits that occur in the units place of the square of a positive integer. Use that list to determine whether $\sqrt{5233}$ is an integer.

96. ***Think About It*** Square the real number $2/\sqrt{5}$ and note that the radical is eliminated from the denominator. Is this equivalent to rationalizing the denominator? Why or why not?

97. ***Declining Balances Depreciation*** For the data in the graph, find the annual depreciation rate r by the declining balances method, which uses the formula

$$r = 1 - \left(\frac{S}{C}\right)^{1/n}$$

where n is the useful life of the item (in years), S is the salvage value (in dollars), and C is the original cost (in dollars).



98. ***Period of a Pendulum*** The period T (in seconds) of a pendulum is

$$T = 2\pi\sqrt{\frac{L}{32}}$$

where L is the length of the pendulum (in feet). Find the period of a pendulum of length 2 feet.

99. ***Mathematical Modeling*** A funnel is filled with water to a height of h centimeters. The time t (in seconds) for the funnel to empty is

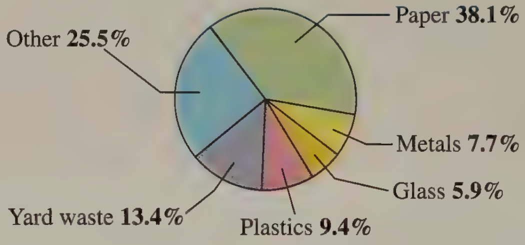
$$t = 0.03[12^{5/2} - (12 - h)^{5/2}], \quad 0 \le h \le 12.$$

Find t for $h = 7$ centimeters.

100. ***Erosion*** A stream of water moving at the rate of v feet per second can carry particles of size $0.03\sqrt{v}$ inches. Find the size of the particle that can be carried by a stream flowing at the rate of $\frac{3}{4}$ foot per second.

101. ***Speed of Light*** The speed of light is 11,160,000 miles per minute. The distance from the sun to the earth is 93,000,000 miles. Find the time for light to travel from the sun to the earth.

102. ***Environment*** There were 2.097×10^8 million tons of municipal waste generated in 1996. Find the number of tons for each of the categories in the graph. (Source: Franklin Associates)



Synthesis

True or False? **In Exercises 103 and 104, determine whether the statement is true or false. Justify your answer.**

103. $\dfrac{x^{k+1}}{x} = x^k$ **104.** $(a^n)^k = a^{(n^k)}$

105. ***Think About It*** Verify that $a^0 = 1, a \neq 0$. (*Hint:* Use the property of exponents $\dfrac{a^m}{a^n} = a^{m-n}$.)

106. ***Think About It*** Is the real number 52.7×10^5 written in scientific notation? Explain.

P.3 Polynomials and Factoring

What You Should Learn:

- How to write polynomials in standard form
- How to add, subtract, and multiply polynomials
- How to use special products to multiply polynomials
- How to remove common factors from polynomials
- How to factor special polynomial forms
- How to factor trinomials as the product of two binomials
- How to factor by grouping

Why You Should Learn It:

Polynomials can be used to model and solve real-life problems. For instance, in Exercise 159 on page 34, a polynomial is used to model the stopping distance of an automobile.

Allen Russell/Index Stock

Polynomials

An algebraic expression is a collection of variables and real numbers. The most common type of algebraic expression is the **polynomial.** Some examples are

$$2x + 5, \quad 3x^4 - 7x^2 + 2x + 4, \quad \text{and} \quad 5x^2y^2 - xy + 3.$$

The first two are *polynomials in x* and the third is a *polynomial in x and y*. The terms of a polynomial in x have the form ax^k, where a is the **coefficient** and k is the **degree** of the term. For instance, the third-degree polynomial

$$2x^3 - 5x^2 + 1 = 2x^3 + (-5)x^2 + (0)x + 1$$

has coefficients 2, -5, 0, and 1.

Definition of a Polynomial in x

Let $a_0, a_1, a_2, \ldots, a_n$ be *real numbers* and let n be a *nonnegative integer*. A **polynomial in x** is an expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

where $a_n \neq 0$. The polynomial is of **degree** n, a_n is the **leading coefficient,** and a_0 is the **constant term.**

In **standard form,** a polynomial in x is written with descending powers of x. Polynomials with one, two, and three terms are called **monomials, binomials,** and **trinomials,** respectively.

A polynomial that has all zero coefficients is called the **zero polynomial,** denoted by 0. No degree is assigned to this particular polynomial. For polynomials in more than one variable, the degree of a *term* is the sum of the exponents of the variables in the term. The degree of the *polynomial* is the highest degree of its terms. Expressions such as the following are not polynomials.

$x^3 - \sqrt{3x} = x^3 - (3x)^{1/2}$ Exponent in $\sqrt{3x}$ is not an integer.

$x^2 + 5x^{-1}$ Exponent in $5x^{-1}$ is not a nonnegative integer.

EXAMPLE 1 Writing Polynomials in Standard Form

	Polynomial	*Standard Form*	*Degree*
a.	$4x^2 - 5x^7 - 2 + 3x$	$-5x^7 + 4x^2 + 3x - 2$	7
b.	$4 - 9x^2$	$-9x^2 + 4$	2
c.	8	$8 \; (8 = 8x^0)$	0

Operations with Polynomials

You can add and subtract polynomials in much the same way you add and subtract real numbers. Simply add or subtract the *like terms* (terms having the same variables to the same powers) by adding their coefficients. For instance, $-3xy^2$ and $5xy^2$ are like terms and their sum is

$$-3xy^2 + 5xy^2 = (-3 + 5)xy^2 = 2xy^2.$$

STUDY TIP

A common mistake is to fail to change the sign of each term inside parentheses preceded by a negative sign. For instance, note that

$$-(x^2 - x + 3) = -x^2 + x - 3$$

and

$$-(x^2 - x + 3) \neq -x^2 - x + 3.$$

EXAMPLE 2 Sums and Differences of Polynomials

Perform the indicated operation.

a. $(5x^3 - 7x^2 - 3) + (x^3 + 2x^2 - x + 8)$

b. $(7x^4 - x^2 - 4x + 2) - (3x^4 - 4x^2 + 3x)$

Solution

a. $(5x^3 - 7x^2 - 3) + (x^3 + 2x^2 - x + 8)$

$= (5x^3 + x^3) + (2x^2 - 7x^2) - x + (8 - 3)$ Group like terms.

$= 6x^3 - 5x^2 - x + 5$ Combine like terms.

b. $(7x^4 - x^2 - 4x + 2) - (3x^4 - 4x^2 + 3x)$

$= 7x^4 - x^2 - 4x + 2 - 3x^4 + 4x^2 - 3x$ Distributive Property

$= (7x^4 - 3x^4) + (4x^2 - x^2) + (-3x - 4x) + 2$ Group like terms.

$= 4x^4 + 3x^2 - 7x + 2$ Combine like terms.

To find the product of two polynomials, use the left and right Distributive Properties.

EXAMPLE 3 Multiplying Polynomials: The FOIL Method

Multiply $(3x - 2)$ by $(5x + 7)$.

Solution

$$\begin{aligned}(3x - 2)(5x + 7) &= 3x(5x + 7) - 2(5x + 7)\\ &= (3x)(5x) + (3x)(7) - (2)(5x) - (2)(7)\\ &= \underbrace{15x^2}_{\substack{\text{Product of}\\ \text{First terms}}} + \underbrace{21x}_{\substack{\text{Product of}\\ \text{Outer terms}}} - \underbrace{10x}_{\substack{\text{Product of}\\ \text{Inner terms}}} - \underbrace{14}_{\substack{\text{Product of}\\ \text{Last terms}}}\\ &= 15x^2 + 11x - 14\end{aligned}$$

Note in this **FOIL Method** that for binomials the outer (O) and inner (I) terms are alike and can be combined into one term.

A computer simulation to accompany this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 4 The Product of Two Trinomials

Find the product of $(4x^2 + x - 2)$ and $(-x^2 + 3x + 5)$.

Solution

When multiplying two polynomials, be sure to multiply *each* term of one polynomial by *each* term of the other. A vertical format is helpful.

$$\begin{array}{rl}
4x^2 + x - 2 & \text{Standard form}\\
\times \qquad -x^2 + 3x + 5 & \text{Standard form}\\
\hline
20x^2 + 5x - 10 & \Leftarrow \quad 5(4x^2 + x - 2)\\
12x^3 + 3x^2 - 6x & \Leftarrow \quad 3x(4x^2 + x - 2)\\
-4x^4 - x^3 + 2x^2 & \Leftarrow \quad -x^2(4x^2 + x - 2)\\
\hline
-4x^4 + 11x^3 + 25x^2 - x - 10 &
\end{array}$$

The *Interactive* CD-ROM and *Internet* versions of this text offer a built-in calculator, which can be used with the Examples, Explorations, and Exercises.

Special Products

Some binomial products have special forms that occur frequently in algebra.

Special Products

Let u and v be real numbers, variables, or algebraic expressions.

Special Product	*Example*
Sum and Difference of Same Terms	
$(u + v)(u - v) = u^2 - v^2$	$(x + 4)(x - 4) = x^2 - 4^2 = x^2 - 16$
Square of a Binomial	
$(u + v)^2 = u^2 + 2uv + v^2$	$(x + 3)^2 = x^2 + 2(x)(3) + 3^2 = x^2 + 6x + 9$
$(u - v)^2 = u^2 - 2uv + v^2$	$(3x - 2)^2 = (3x)^2 - 2(3x)(2) + 2^2 = 9x^2 - 12x + 4$
Cube of a Binomial	
$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$	$(x + 2)^3 = x^3 + 3x^2(2) + 3x(2^2) + 2^3 = x^3 + 6x^2 + 12x + 8$
$(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3$	$(x - 1)^3 = x^3 - 3x^2(1) + 3x(1^2) - 1^3 = x^3 - 3x^2 + 3x - 1$

EXAMPLE 5 The Product of Two Trinomials

Find the product of $(x + y - 2)$ and $(x + y + 2)$.

Solution

By grouping $x + y$ in parentheses, you can write the product of the trinomials as a special product.

$$\begin{aligned}
(x + y - 2)(x + y + 2) &= [(x + y) - 2][(x + y) + 2]\\
&= (x + y)^2 - 2^2\\
&= x^2 + 2xy + y^2 - 4
\end{aligned}$$

Factoring

The process of writing a polynomial as a product is called **factoring.** It is an important tool for solving equations and for simplifying fractional expressions.

Unless noted otherwise, when you are asked to factor a polynomial, you can assume that you are looking for factors with integer coefficients. If a polynomial cannot be factored using integer coefficients, it is **prime** or **irreducible over the integers.** For instance, the polynomial $x^2 - 3$ is irreducible over the integers. Over the real numbers, this polynomial can be factored as

$$x^2 - 3 = \left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right).$$

A polynomial is **completely factored** when each of its factors is prime.

EXAMPLE 6 Recognizing Completely Factored Polynomials

a. $x^3 - x^2 + 4x - 4 = (x - 1)(x^2 + 4)$

is completely factored.

b. $x^3 - x^2 - 4x + 4 = (x - 1)(x^2 - 4)$

is not completely factored. Its complete factorization would be

$$x^3 - x^2 - 4x + 4 = (x - 1)(x + 2)(x - 2).$$

The simplest type of factoring involves a polynomial that can be written as the product of a monomial and another polynomial. The technique used here is the Distributive Property, $a(b + c) = ab + ac$, in the *reverse* direction.

$$ab + ac = a(b + c)$$ a is a common factor.

For instance, the polynomial $5x^2 + 15x$ can be factored as follows.

$$\begin{aligned} 5x^2 + 15x &= 5x(x) + 5x(3) \\ &= 5x(x + 3) \end{aligned}$$

The first step in completely factoring a polynomial is to remove (factor out) any common factors, as shown in the next example.

EXAMPLE 7 Removing Common Factors

Factor each polynomial.

a. $6x^3 - 4x$ **b.** $3x^4 + 9x^3 + 6x^2$ **c.** $(x - 2)(2x) + (x - 2)(3)$

Solution

a. $$\begin{aligned} 6x^3 - 4x &= 2x(3x^2) - 2x(2) \\ &= 2x(3x^2 - 2) \end{aligned}$$ $2x$ is a common factor.

b. $$\begin{aligned} 3x^4 + 9x^3 + 6x^2 &= 3x^2(x^2) + 3x^2(3x) + 3x^2(2) \\ &= 3x^2(x^2 + 3x + 2) \end{aligned}$$ $3x^2$ is a common factor.

c. $(x - 2)(2x) + (x - 2)(3) = (x - 2)(2x + 3)$ $x - 2$ is a common factor.

Factoring Special Polynomial Forms

Some polynomials have special forms you should learn to recognize so that they can be factored easily.

Factoring Special Polynomial Forms

Factored Form	Example
Difference of Two Squares	
$u^2 - v^2 = (u + v)(u - v)$	$9x^2 - 4 = (3x)^2 - 2^2 = (3x + 2)(3x - 2)$
Perfect Square Trinomial	
$u^2 + 2uv + v^2 = (u + v)^2$	$x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2 = (x + 3)^2$
$u^2 - 2uv + v^2 = (u - v)^2$	$x^2 - 6x + 9 = x^2 - 2(x)(3) + 3^2 = (x - 3)^2$
Sum or Difference of Two Cubes	
$u^3 + v^3 = (u + v)(u^2 - uv + v^2)$	$x^3 + 8 = x^3 + 2^3 = (x + 2)(x^2 - 2x + 4)$
$u^3 - v^3 = (u - v)(u^2 + uv + v^2)$	$27x^3 - 1 = (3x)^3 - 1^3 = (3x - 1)(9x^2 + 3x + 1)$

One of the easiest special polynomial forms to factor is the difference of two squares. Think of this form as follows.

$$u^2 \underset{\substack{\uparrow \\ \text{Difference}}}{-} v^2 = (u \underbrace{+ v)(u -}_{\text{Opposite signs}} v)$$

To recognize perfect square terms, look for coefficients that are squares of integers and variables raised to *even powers.*

STUDY T!P

In Example 8, note that the first step in factoring a polynomial is to check for a common factor. Once the common factor is removed, it is often possible to recognize patterns that were not immediately obvious.

EXAMPLE 8 Removing a Common Factor First

$$\begin{aligned} 3 - 12x^2 &= 3(1 - 4x^2) && \text{3 is a common factor.} \\ &= 3[1^2 - (2x)^2] \\ &= 3(1 + 2x)(1 - 2x) && \text{Difference of two squares} \end{aligned}$$

EXAMPLE 9 Factoring the Difference of Two Squares

a. $$\begin{aligned} (x + 2)^2 - y^2 &= [(x + 2) + y][(x + 2) - y] \\ &= (x + 2 + y)(x + 2 - y) \end{aligned}$$

b. $$\begin{aligned} 16x^4 - 81 &= (4x^2)^2 - 9^2 \\ &= (4x^2 + 9)(4x^2 - 9) && \text{Difference of two squares} \\ &= (4x^2 + 9)[(2x)^2 - 3^2] \\ &= (4x^2 + 9)(2x + 3)(2x - 3) && \text{Difference of two squares} \end{aligned}$$

A perfect square trinomial is the square of a binomial, and it has the following form.

$$u^2 + 2uv + v^2 = (u + v)^2 \qquad \text{or} \qquad u^2 - 2uv + v^2 = (u - v)^2$$

Like signs

Like signs

Note that the first and last terms are squares and the middle term is twice the product of u and v.

EXAMPLE 10 Factoring Perfect Square Trinomials

Factor each trinomial.

a. $x^2 - 10x + 25$ **b.** $16x^2 + 8x + 1$

Solution

a. $x^2 - 10x + 25 = x^2 - 2(x)(5) + 5^2$ Rewrite in $u^2 - 2uv + v^2$ form.

$= (x - 5)^2$

b. $16x^2 + 8x + 1 = (4x)^2 + 2(4x)(1) + 1^2$ Rewrite in $u^2 + 2uv + v^2$ form.

$= (4x + 1)^2$

The next two formulas show the sums and differences of cubes. Pay special attention to the signs of the terms.

Like signs

Like signs

$$u^3 + v^3 = (u + v)(u^2 - uv + v^2) \qquad u^3 - v^3 = (u - v)(u^2 + uv + v^2)$$

Unlike signs

Unlike signs

Exploration

Rewrite $u^6 - v^6$ as the difference of two squares. Then find a formula for completely factoring $u^6 - v^6$. Use your formula to completely factor $x^6 - 1$ and $x^6 - 64$.

EXAMPLE 11 Factoring the Difference of Cubes

Factor $x^3 - 27$.

Solution

$x^3 - 27 = x^3 - 3^3$ Rewrite 27 as 3^3.

$= (x - 3)(x^2 + 3x + 9)$ Factor.

EXAMPLE 12 Factoring the Sum of Cubes

a. $y^3 + 8 = y^3 + 2^3$ Rewrite 8 as 2^3.

$= (y + 2)(y^2 - 2y + 4)$ Factor.

b. $3(x^3 + 64) = 3(x^3 + 4^3)$ Rewrite 64 as 4^3.

$= 3(x + 4)(x^2 - 4x + 16)$ Factor.

Trinomials with Binomial Factors

To factor a trinomial of the form $ax^2 + bx + c$, use the following pattern.

Factors of a

$$ax^2 + bx + c = (\square x + \square)(\square x + \square)$$

Factors of c

The goal is to find a combination of factors of a and c so that the outer and inner products add up to the middle term bx. For instance, in the trinomial $6x^2 + 17x + 5$, you can write all possible factorizations and determine which one has outer and inner products that add up to $17x$.

$$(6x + 5)(x + 1) \qquad (2x + 1)(3x + 5)$$

$$(6x + 1)(x + 5) \qquad (2x + 5)(3x + 1)$$

You can see that $(2x + 5)(3x + 1)$ is the correct factorization because the outer (O) and inner (I) products add up to $17x$.

$$(2x + 5)(3x + 1) = \overset{\substack{\text{F}\\ \downarrow}}{6x^2} + \overset{\substack{\text{O}\\ \downarrow}}{2x} + \overset{\substack{\text{I}\\ \downarrow}}{15x} + \overset{\substack{\text{L}\\ \downarrow}}{5} = 6x^2 + \overset{\substack{\text{O + I}\\ \downarrow}}{17x} + 5.$$

EXAMPLE 13 Factoring a Trinomial: Leading Coefficient Is 1

Factor $x^2 - 7x + 12$.

Solution

The possible factorizations are

$$(x - 2)(x - 6), \quad (x - 1)(x - 12), \quad \text{and} \quad (x - 3)(x - 4).$$

Testing the middle term, you will find the correct factorization to be

$$x^2 - 7x + 12 = (x - 3)(x - 4). \qquad \text{O + I} = -4x + (-3x) = -7x$$

EXAMPLE 14 Factoring a Trinomial: Leading Coefficient Is Not 1

Factor $2x^2 + x - 15$.

Solution

The eight possible factorizations are as follows.

$$(2x - 1)(x + 15) \qquad (2x + 1)(x - 15)$$

$$(2x - 3)(x + 5) \qquad (2x + 3)(x - 5)$$

$$(2x - 5)(x + 3) \qquad (2x + 5)(x - 3)$$

$$(2x - 15)(x + 1) \qquad (2x + 15)(x - 1)$$

Testing the middle term, you will find the correct factorization to be

$$2x^2 + x - 15 = (2x - 5)(x + 3). \qquad \text{O + I} = 6x - 5x = x$$

STUDY TIP

Factoring a trinomial can involve trial and error. However, once you have produced the factored form, it is an easy matter to check your answer. For instance, you can verify the factorization in Example 13 by multiplying out the expression $(x - 3)(x - 4)$ to see that you obtain the original trinomial, $x^2 - 7x + 12$.

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Factoring by Grouping

Sometimes polynomials with more than three terms can be factored by a method called **factoring by grouping.** It is not always obvious which terms to group, and sometimes several different groupings will work.

EXAMPLE 15 Factoring by Grouping

Use factoring by grouping to factor

$x^3 - 2x^2 - 3x + 6.$

Solution

$$\begin{aligned} x^3 - 2x^2 - 3x + 6 &= (x^3 - 2x^2) - (3x - 6) && \text{Group terms.} \\ &= x^2(x - 2) - 3(x - 2) && \text{Factor groups.} \\ &= (x - 2)(x^2 - 3) && \text{Distributive Property} \end{aligned}$$

Factoring a trinomial can involve quite a bit of trial and error. Some of this trial and error can be lessened by using factoring by grouping.

EXAMPLE 16 Factoring a Trinomial by Grouping

Use factoring by grouping to factor

$2x^2 + 5x - 3.$

Solution

In the trinomial $2x^2 + 5x - 3$, $a = 2$ and $c = -3$, which implies that the product ac is -6. Now, because -6 factors as $(6)(-1)$ and $6 - 1 = 5 = b$, rewrite the middle term as $5x = 6x - x$. This produces the following.

$$\begin{aligned} 2x^2 + 5x - 3 &= 2x^2 + 6x - x - 3 && \text{Rewrite middle term.} \\ &= (2x^2 + 6x) - (x + 3) && \text{Group terms.} \\ &= 2x(x + 3) - (x + 3) && \text{Factor groups.} \\ &= (x + 3)(2x - 1) && \text{Distributive Property} \end{aligned}$$

So, the trinomial factors as

$2x^2 + 5x - 3 = (x + 3)(2x - 1).$

Guidelines for Factoring Polynomials

1. Factor out any common factors using the Distributive Property.
2. Factor according to one of the special polynomial forms.
3. Factor as $ax^2 + bx + c = (mx + r)(nx + s)$.
4. Factor by grouping.

P.3 Exercises

In Exercises 1–6, match the polynomial with its description. [The polynomials are labeled (a), (b), (c), (d), (e), and (f).]

(a) $5x^2$

(b) $1 - 4x^3$

(c) $x^3 + 2x^2 - 4x + 1$

(d) 18

(e) $-3x^5 + 2x^3 + x$

(f) $\frac{3}{4}x^4 + x^2 + 14$

1. A polynomial of degree zero

2. A trinomial of degree five

3. A binomial with leading coefficient -4

4. A monomial of positive degree

5. A trinomial with leading coefficient $\frac{3}{4}$

6. A third-degree polynomial with leading coefficient 1

In Exercises 7–10, write a polynomial for the description. (There are many correct answers.)

7. A third-degree polynomial with leading coefficient -2

8. A fifth-degree polynomial with leading coefficient 8

9. A fourth-degree polynomial with a negative leading coefficient

10. A third-degree trinomial with an even leading coefficient

In Exercises 11–16, find the degree and leading coefficient of the polynomial.

11. $5x^2 - x + 1$

12. $-3x^4 + x^2 - 4$

13. $x^7 - 1$

14. $9x$

15. $1 - x + 6x^4 - 2x^5$

16. $7 + 3x$

In Exercises 17–22, determine whether the expression is a polynomial. If so, write the polynomial in standard form.

17. $7x - 2x^3 + 10$

18. $4x^3 + x - x^{-1}$

19. $\dfrac{9x + 4}{x}$

20. $\dfrac{x^2 + 2x - 3}{6}$

21. $x^2 - x^4 + x^3 - x$

22. $\sqrt{x^2 - x^4}$

In Exercises 23–38, perform the operations and write the result in standard form.

23. $(6x + 5) - (8x + 15)$

24. $(2x^2 + 1) - (x^2 - 2x + 1)$

25. $-(x^3 - 2) + (4x^3 - 2x)$

26. $-(5x^2 - 1) - (-3x^2 + 5)$

27. $(15x^2 - 6) - (-8.1x^3 - 14.7x^2 - 17)$

28. $(15.6x^4 - 18x - 19.4) - (13.9x^4 - 9.2x + 15)$

29. $3x(x^2 - 2x + 1)$

30. $y^2(4y^2 + 2y - 3)$

31. $-5z(3z - 1)$

32. $(-3x)(5x + 2)$

33. $(1 - x^3)(4x)$

34. $-4x(3 - x^3)$

35. $(2.5x^2 + 5)(3x)$

36. $(2 - 3.5y)(4y^3)$

37. $-2x\left(\frac{1}{8}x + 3\right)$

38. $6y\left(4 - \frac{3}{8}y\right)$

In Exercises 39–70, multiply or find the special product.

39. $(x + 3)(x + 4)$

40. $(x - 5)(x + 10)$

41. $(3x - 5)(2x + 1)$

42. $(7x - 2)(4x - 3)$

43. $(2x - 5y)^2$

44. $(5 - 8x)^2$

45. $(x + 10)(x - 10)$

46. $(2x + 3)(2x - 3)$

47. $(x + 2y)(x - 2y)$

48. $(2x + 3y)(2x - 3y)$

49. $[(m - 3) + n][(m - 3) - n]$

50. $[(x + y) + 1][(x + y) - 1]$

51. $[(x - 3) + y]^2$

52. $[(x + 1) - y]^2$

53. $(2r^2 - 5)(2r^2 + 5)$

54. $(3a^3 - 4b^2)(3a^3 + 4b^2)$

55. $(x + 1)^3$

56. $(x - 2)^3$

57. $(2x - y)^3$

58. $(3x + 2y)^3$

59. $\left(\frac{1}{2}x - 5\right)^2$

60. $\left(\frac{3}{5}t + 4\right)^2$

61. $\left(\frac{1}{4}x - 3\right)\left(\frac{1}{4}x + 3\right)$

62. $\left(2x + \frac{1}{6}\right)\left(2x - \frac{1}{6}\right)$

63. $(2.4x + 3)^2$

64. $(1.8y - 5)^2$

65. $(1.5x - 4)(1.5x + 4)$

66. $(3.3y + 1)(3.3y - 1)$

67. $5x(x + 1) - 3x(x + 1)$

68. $(2x - 1)(x + 3) + 3(x + 3)$

69. $(u + 2)(u - 2)(u^2 + 4)$

70. $(x + y)(x - y)(x^2 + y^2)$

In Exercises 71–76, factor out the common factor.

71. $3x + 6$

72. $5y - 30$

73. $2x^3 - 6x$

74. $4x^3 - 6x^2 + 12x$

75. $3x(x - 5) + 8(x - 5)$

76. $(3x - 1)^2 + (3x - 1)$

In Exercises 77–84, factor the difference of two squares.

77. $x^2 - 36$
78. $x^2 - \frac{1}{4}$
79. $16y^2 - 9$
80. $49 - 9y^2$
81. $4x^2 - \frac{1}{9}$
82. $\frac{25}{36}y^2 - 49$
83. $(x - 1)^2 - 4$
84. $25 - (z + 5)^2$

In Exercises 85–92, factor the perfect square trinomial.

85. $x^2 - 4x + 4$
86. $x^2 + 10x + 25$
87. $x^2 + x + \frac{1}{4}$
88. $x^2 - \frac{4}{3}x + \frac{4}{9}$
89. $4t^2 + 4t + 1$
90. $9x^2 - 12x + 4$
91. $9t^2 + \frac{3}{2}t + \frac{1}{16}$
92. $4t^2 + \frac{8}{5}t + \frac{4}{25}$

In Exercises 93–106, factor the trinomial.

93. $x^2 + x - 2$
94. $x^2 + 5x + 6$
95. $s^2 - 5s + 6$
96. $t^2 - t - 6$
97. $20 - y - y^2$
98. $24 + 5z - z^2$
99. $3x^2 - 5x + 2$
100. $3x^2 + 13x - 10$
101. $2x^2 - x - 1$
102. $2x^2 - x - 21$
103. $5x^2 + 26x + 5$
104. $8x^2 - 45x - 18$
105. $-5u^2 - 13u + 6$
106. $-6x^2 + 23x + 4$

In Exercises 107–116, factor the sum or difference of cubes.

107. $x^3 - 8$
108. $x^3 - 27$
109. $y^3 + 64$
110. $z^3 + 125$
111. $x^3 - \frac{8}{27}$
112. $x^3 + \frac{8}{125}$
113. $8x^3 - 1$
114. $27x^3 + 8$
115. $\frac{1}{8}x^3 + 1$
116. $\frac{27}{64}x^3 - 1$

In Exercises 117–120, factor by grouping.

117. $x^3 - x^2 + 2x - 2$
118. $x^3 + 5x^2 - 5x - 25$
119. $2x^3 - x^2 - 6x + 3$
120. $5x^3 - 10x^2 + 3x - 6$

In Exercises 121–154, completely factor the expression.

121. $x^3 - 9x$
122. $12x^2 - 48$
123. $x^3 - 4x^2$
124. $6x^2 - 54$
125. $x^2 - 2x + 1$
126. $16 + 6x - x^2$
127. $1 - 4x + 4x^2$
128. $-9x^2 + 6x - 1$
129. $2x^2 + 4x - 2x^3$
130. $2y^3 - 7y^2 - 15y$
131. $9x^2 + 10x + 1$
132. $13x + 6 + 5x^2$
133. $\frac{1}{8}x^2 - \frac{1}{96}x - \frac{1}{16}$
134. $\frac{1}{81}x^2 + \frac{2}{9}x - 8$
135. $3x^3 + x^2 + 15x + 5$
136. $5 - x + 5x^2 - x^3$
137. $x^4 - 4x^3 + x^2 - 4x$
138. $3u - 2u^2 + 6 - u^3$
139. $25 - (z + 5)^2$
140. $(t - 1)^2 - 49$
141. $(x^2 + 1)^2 - 4x^2$
142. $(x^2 + 8)^2 - 36x^2$
143. $2t^3 - 16$
144. $5x^3 + 40$
145. $4x(2x - 1) + (2x - 1)^2$
146. $5(3 - 4x)^2 - 8(3 - 4x)(5x - 1)$
147. $2(x + 1)(x - 3)^2 - 3(x + 1)^2(x - 3)$
148. $7(3x + 2)^2(1 - x)^2 + (3x + 2)(1 - x)^3$
149. $7x(2)(x^2 + 1)(2x) - (x^2 + 1)^2(7)$
150. $3(x - 2)^2(x + 1)^4 + (x - 2)^3(4)(x + 1)^3$
151. $2x(x - 5)^4 - x^2(4)(x - 5)^3$
152. $5(x^6 + 1)^4(6x^5)(3x + 2)^3 + 3(3x + 2)^2(3)(x^6 + 1)^5$
153. $\frac{x^2}{2}(x^2 + 1)^4 - (x^2 + 1)^5$
154. $5w^3(9w + 1)^4(9) + (2w + 1)^5(3w^2)$

155. ***Finance*** After 2 years, an investment of \$500 compounded annually at an interest rate r will yield an amount of

$500(1 + r)^2$.

(a) Write this polynomial in standard form.

(b) Use a calculator to evaluate the polynomial for the values of r in the table.

r	$2\frac{1}{2}\%$	3%	4%	$4\frac{1}{2}\%$	5%
$500(1 + r)^2$					

(c) What conclusion can you make from the table?

156. ***Finance*** After 3 years, an investment of \$1200 compounded annually at an interest rate r will yield an amount of

$$1200(1 + r)^3.$$

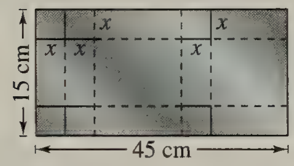
(a) Write this polynomial in standard form.

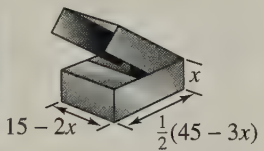
(b) Use a calculator to evaluate the polynomial for the values of r in the table.

r	2%	3%	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$
$1200(1 + r)^3$					

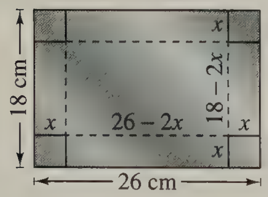
(c) What conclusion can you make from the table?

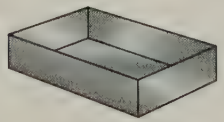
157. ***Volume of a Box*** A closed box is constructed by cutting along the solid lines and folding along the broken lines on the rectangular piece of metal shown in the figure. The length and width of the rectangle are 45 centimeters and 15 centimeters, respectively. Find the volume of the box in terms of x. Find the volume when $x = 3$, $x = 5$, and $x = 7$.





158. ***Volume of a Box*** An open box is made by cutting squares out of the corners of a piece of metal that is 18 centimeters by 26 centimeters. If the edge of each cut-out square is x inches, find the volume when $x = 1$, $x = 2$, and $x = 3$.





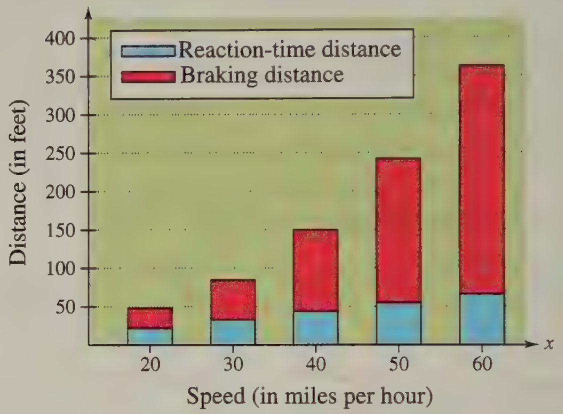
159. ***Stopping Distance*** The stopping distance of an automobile is the distance traveled during the driver's reaction time plus the distance traveled after the brakes are applied. In an experiment, these distances were measured (in feet) when the automobile was traveling at a speed of x miles per hour. The distance traveled during the reaction time was $R = 1.1x$, and the braking distance was

$$B = 0.14x^2 - 4.43x + 58.40.$$

(a) Determine the polynomial that represents the total stopping distance.

(b) Use the result of part (a) to estimate the total stopping distance when $x = 30$, $x = 40$, and $x = 55$.

(c) Use the bar graph below to make a statement about the total stopping distance required for increasing speeds.



160. ***Engineering*** A uniformly distributed load is placed on a 1-inch-wide steel beam. When the span of the beam is x feet and its depth is 6 inches, the safe load S is approximated by

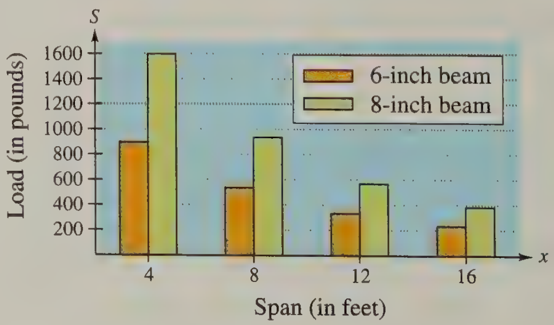
$$S_6 = (0.06x^2 - 2.42x + 38.71)^2.$$

When the depth is 8 inches, the safe load is approximated by

$$S_8 = (0.08x^2 - 3.30x + 51.93)^2.$$

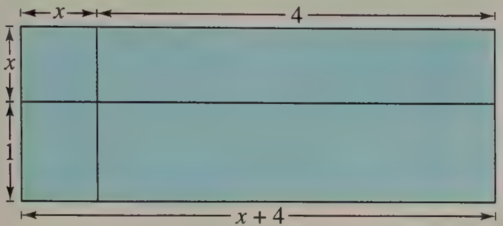
(a) Use the bar graph below to estimate the difference in the safe loads of these two beams when the span is 12 feet.

(b) How does the difference in safe load change as the span increases?

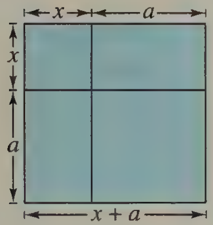


Geometry **In Exercises 161 and 162, use the area model to write two different expressions for the area. Then equate the two expressions and name the algebraic property that is illustrated.**

161.



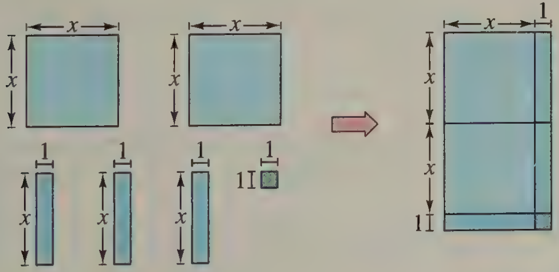
162.



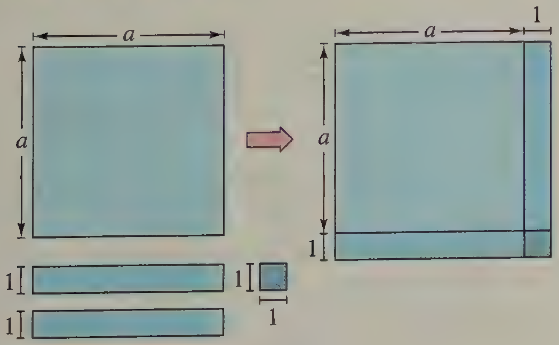
Geometry **In Exercises 163–166, match the factoring formula with the correct geometric factoring model. [The models are labeled (a), (b), (c), and (d).] For instance, a factoring model for**

$2x^2 + 3x + 1 = (2x + 1)(x + 1)$

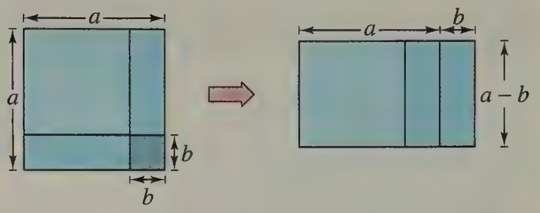
is shown in the figure.



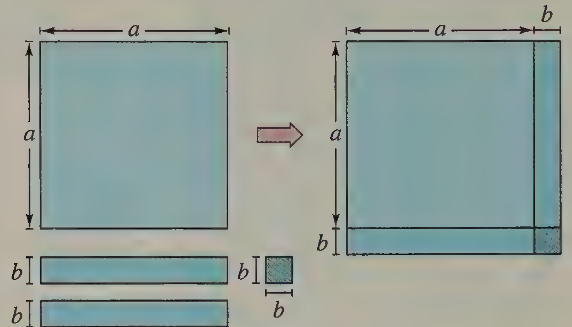
(a)



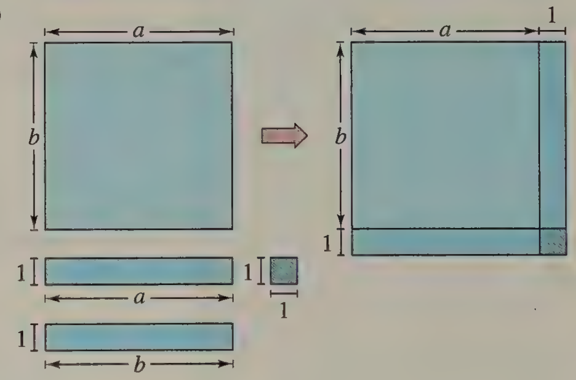
(b)



(c)



(d)



163. $a^2 - b^2 = (a + b)(a - b)$

164. $a^2 + 2ab + b^2 = (a + b)^2$

165. $a^2 + 2a + 1 = (a + 1)^2$

166. $ab + a + b + 1 = (a + 1)(b + 1)$

Geometry **In Exercises 167–170, draw a geometric factoring model to represent the factorization.**

167. $3x^2 + 7x + 2 = (3x + 1)(x + 2)$

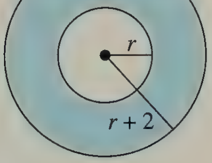
168. $x^2 + 4x + 3 = (x + 3)(x + 1)$

169. $2x^2 + 7x + 3 = (2x + 1)(x + 3)$

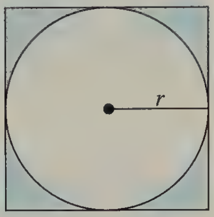
170. $x^2 + 3x + 2 = (x + 2)(x + 1)$

Geometry **In Exercises 171–174, write, in factored form, an expression for the shaded portion of the figure.**

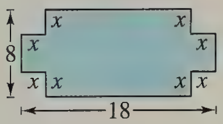
171.



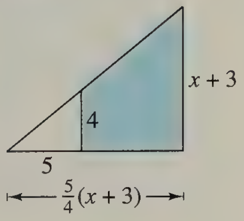
172.



173.



174.



In Exercises 175–178, find all values of b for which the trinomial can be factored with integer coefficients.

175. $x^2 + bx - 15$

176. $x^2 + bx - 12$

177. $x^2 + bx + 50$

178. $x^2 + bx + 24$

In Exercises 179–182, find two integers for c such that the trinomial can be factored. (There are many correct answers.)

179. $2x^2 + 5x + c$

180. $3x^2 - x + c$

181. $3x^2 - 10x + c$

182. $2x^2 + 9x + c$

183. ***Error Analysis*** Describe the error.

$$\begin{aligned} 9x^2 - 9x - 54 &= (3x + 6)(3x - 9) \\ &= 3(x + 2)(x - 3) \end{aligned}$$

184. ***Think About It*** Is $(3x - 6)(x + 1)$ completely factored? Explain.

185. ***Chemistry*** The rate of change of an autocatalytic chemical reaction is $kQx - kx^2$, where Q is the amount of the original substance, x is the amount of substance formed, and k is a constant of proportionality. Factor the expression.

186. ***Geometry*** The cylindrical shell shown in the figure has a volume of

$$V = \pi R^2h - \pi r^2h.$$

(a) Factor the expression for the volume.

(b) From the result of part (a), show that the volume is 2π (average radius)(thickness of the shell)h.

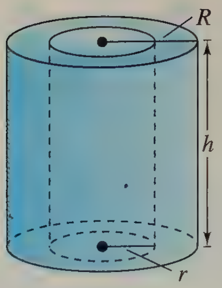


FIGURE FOR 186

Synthesis

True or False? **In Exercises 187–189, determine whether the statement is true or false. Justify your answer.**

187. The product of two binomials is always a second-degree polynomial.

188. The difference of two perfect squares can be factored as the product of conjugate pairs.

189. The sum of two perfect squares can be factored as the binomial sum squared.

190. ***Exploration*** Find the degree of the product of two polynomials of degrees m and n.

191. ***Exploration*** Find the degree of the sum of two polynomials of degrees m and n if $m < n$.

192. ***Writing*** A student's homework paper included the following.

$$(x - 3)^2 = x^2 + 9$$

Write a paragraph fully explaining the error and give the correct method for squaring a binomial.

193. ***Think About It*** A third-degree polynomial and a fourth-degree polynomial are added.

(a) Can the sum be a fourth-degree polynomial? Explain or give an example.

(b) Can the sum be a second-degree polynomial? Explain or give an example.

(c) Can the sum be a seventh-degree polynomial? Explain or give an example.

194. ***Think About It*** Must the sum of two second-degree polynomials be a second-degree polynomial? If not, give an example.

195. ***Writing*** Explain what is meant when it is said that a polynomial is in factored form.

P.4 Rational Expressions

Domain of an Algebraic Expression

The set of real numbers for which an algebraic expression is defined is the **domain** of the expression. Two algebraic expressions are **equivalent** if they have the same domain and yield the same values for all numbers in their domain. For instance, the expressions $(x + 1) + (x + 2)$ and $2x + 3$ are equivalent because

$$\begin{aligned}(x + 1) + (x + 2) &= x + 1 + x + 2\\ &= x + x + 1 + 2\\ &= 2x + 3.\end{aligned}$$

EXAMPLE 1 Finding the Domain of an Algebraic Expression

a. The domain of the polynomial

$$2x^3 + 3x + 4$$ x>0

is the set of all real numbers. In fact, the domain of any polynomial is the set of all real numbers, un~~less the domain~~ is specifically restricted.

b. The domain of the radical expression

$$\sqrt{x - 2}$$

is the set of real numbers greater than or ~~equal to~~ 2, because the square root of a negative number is not a real number.

c. The domain of the expression rational

$$\frac{x + 2}{x - 3}$$

is the set of all real numbers except $x = 3$, which would produce an undefined division by zero.

The quotient of two algebraic expressions is a **fractional expression.** Moreover, the quotient of two *polynomials* such as

$$\frac{1}{x}, \quad \frac{2x - 1}{x + 1}, \quad \text{or} \quad \frac{x^2 - 1}{x^2 + 1}$$

is a **rational expression.**

Simplifying Rational Expressions

Recall that a fraction is in simplest form if its numerator and denominator have no factors in common aside from ±1. To write a fraction in simplest form, divide out common factors.

$$\frac{a \cdot \cancel{c}}{b \cdot \cancel{c}} = \frac{a}{b}, \quad c \neq 0.$$

What You Should Learn:

- How to find domains of algebraic expressions
- How to simplify rational expressions
- How to add, subtract, multiply, and divide rational expressions
- How to simplify complex fractions

Why You Should Learn It:

Rational expressions are useful in estimating prices and determining behavioral trends over time. For instance, a rational expression is used in Exercise 82 on page 46 to model costs of precious metals from 1992 to 1997.

Louise Gubb/The Image Works

The key to success in simplifying rational expressions lies in your ability to *factor* polynomials.

EXAMPLE 2 Simplifying a Rational Expression

Write $\dfrac{x^2 + 4x - 12}{3x - 6}$ in simplest form.

Solution

$$\frac{x^2 + 4x - 12}{3x - 6} = \frac{(x + 6)\cancel{(x - 2)}}{3\cancel{(x - 2)}} \qquad \text{Factor completely.}$$

$$= \frac{x + 6}{3}, \quad x \neq 2 \qquad \text{Divide out common factors.}$$

Note that the original expression is undefined when $x = 2$ (because division by zero is undefined). To make sure that the simplified expression is *equivalent* to the original expression, you must restrict the domain of the simplified expression by excluding the value $x = 2$.

When simplifying rational expressions, be sure to factor each polynomial completely before concluding that the numerator and denominator have no factors in common. Moreover, changing the sign of a factor may allow further simplification, as shown in part (b) of the next example.

EXAMPLE 3 Simplifying Rational Expressions

Write each expression in simplest form.

a. $\dfrac{x^3 - 4x}{x^2 + x - 2}$ **b.** $\dfrac{12 + x - x^2}{2x^2 - 9x + 4}$

Solution

a. $$\frac{x^3 - 4x}{x^2 + x - 2} = \frac{x(x^2 - 4)}{(x + 2)(x - 1)}$$

$$= \frac{x\cancel{(x + 2)}(x - 2)}{\cancel{(x + 2)}(x - 1)} \qquad \text{Factor completely.}$$

$$= \frac{x(x - 2)}{x - 1}, \quad x \neq -2 \qquad \text{Divide out common factors.}$$

b. $$\frac{12 + x - x^2}{2x^2 - 9x + 4} = \frac{(4 - x)(3 + x)}{(2x - 1)(x - 4)} \qquad \text{Factor completely.}$$

$$= \frac{-\cancel{(x - 4)}(3 + x)}{(2x - 1)\cancel{(x - 4)}} \qquad (4 - x) = -(x - 4)$$

$$= -\frac{3 + x}{2x - 1}, \quad x \neq 4 \qquad \text{Divide out common factors.}$$

STUDY TIP

In Example 2, do not make the mistake of trying to simplify further by dividing *terms*.

$$\frac{x + 6}{3} \neq \frac{x + \cancel{6}^{2}}{\cancel{3}} \neq x + 2$$

Remember that to simplify fractions, divide out common *factors*, not terms.

Operations with Rational Expressions

To multiply or divide rational expressions, you can use the properties of fractions discussed in Section P.1. Recall that to divide fractions you invert the divisor and multiply.

EXAMPLE 4 Multiplying Rational Expressions

$$\frac{2x^2+x-6}{x^2+4x-5}\cdot\frac{x^3-3x^2+2x}{4x^2-6x}=\frac{\cancel{(2x-3)}(x+2)}{(x+5)\cancel{(x-1)}}\cdot\frac{\cancel{x}(x-2)\cancel{(x-1)}}{2\cancel{x}\cancel{(2x-3)}}$$

$$=\frac{(x+2)(x-2)}{2(x+5)},\qquad x\neq 0,\ x\neq 1,\ x\neq\frac{3}{2}$$

EXAMPLE 5 Dividing Rational Expressions

Divide $\dfrac{x^3-8}{x^2-4}$ by $\dfrac{x^2+2x+4}{x^3+8}$.

Solution

$$\frac{x^3-8}{x^2-4}\div\frac{x^2+2x+4}{x^3+8}=\frac{x^3-8}{x^2-4}\cdot\frac{x^3+8}{x^2+2x+4}$$ Invert and multiply.

$$=\frac{\cancel{(x-2)}\cancel{(x^2+2x+4)}}{\cancel{(x+2)}\cancel{(x-2)}}\cdot\frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{x^2+2x+4}}$$

$$=x^2-2x+4,\qquad x\neq\pm 2$$

To add or subtract rational expressions, you can use the LCD (least common denominator) method or the basic definition

$$\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm bc}{bd},\qquad b\neq 0 \text{ and } d\neq 0.$$ Basic definition

This definition provides an efficient way of adding or subtracting *two* fractions that have no common factors in their denominators.

EXAMPLE 6 Subtracting Rational Expressions

Subtract $\dfrac{2}{3x+4}$ from $\dfrac{x}{x-3}$.

Solution

$$\frac{x}{x-3}-\frac{2}{3x+4}=\frac{x(3x+4)-2(x-3)}{(x-3)(3x+4)}$$ Basic definition

$$=\frac{3x^2+4x-2x+6}{(x-3)(3x+4)}$$ Distributive Property

$$=\frac{3x^2+2x+6}{(x-3)(3x+4)}$$ Combine like terms.

For three or more fractions, or for fractions with a repeated factor in the denominators, the LCD method works well. Recall that the least common denominator of several fractions consists of the product of all prime factors in the denominators, with each factor given the highest power of its occurrence in any denominator. Here is a numerical example.

$$\begin{aligned}\frac{1}{6}+\frac{3}{4}-\frac{2}{3} &= \frac{1\cdot 2}{6\cdot 2}+\frac{3\cdot 3}{4\cdot 3}-\frac{2\cdot 4}{3\cdot 4} && \text{The LCD is 12.}\\ &= \frac{2}{12}+\frac{9}{12}-\frac{8}{12}\\ &= \frac{3}{12}\\ &= \frac{1}{4}\end{aligned}$$

Sometimes the numerator of the answer has a factor in common with the denominator. In such cases the answer should be simplified. For instance, in the example above, $\frac{3}{12}$ was simplified to $\frac{1}{4}$.

EXAMPLE 7 Combining Rational Expressions: The LCD Method

Perform the operations and simplify.

$$\frac{3}{x-1}-\frac{2}{x}+\frac{x+3}{x^2-1}$$

Solution

Using the factored denominators $(x-1)$, x, and $(x+1)(x-1)$, you can see that the LCD is $x(x+1)(x-1)$.

$$\begin{aligned}\frac{3}{x-1}-\frac{2}{x}+\frac{x+3}{(x+1)(x-1)}\\ &= \frac{3(x)(x+1)}{x(x+1)(x-1)}-\frac{2(x+1)(x-1)}{x(x+1)(x-1)}+\frac{(x+3)(x)}{x(x+1)(x-1)}\\ &= \frac{3(x)(x+1)-2(x+1)(x-1)+(x+3)(x)}{x(x+1)(x-1)}\\ &= \frac{3x^2+3x-2x^2+2+x^2+3x}{x(x+1)(x-1)} && \text{Distributive Property}\\ &= \frac{3x^2-2x^2+x^2+3x+3x+2}{x(x+1)(x-1)} && \text{Group like terms.}\\ &= \frac{2x^2+6x+2}{x(x+1)(x-1)} && \text{Combine like terms.}\\ &= \frac{2(x^2+3x+1)}{x(x+1)(x-1)} && \text{Factor.}\end{aligned}$$

Complex Fractions

Fractional expressions with separate fractions in the numerator, denominator, or both, are called **complex fractions.** Here are two examples.

$$\frac{\left(\dfrac{1}{x}\right)}{x^2+1} \quad \text{and} \quad \frac{\left(\dfrac{1}{x}\right)}{\left(\dfrac{1}{x^2+1}\right)}$$

A complex fraction can be simplified by first combining both its numerator and its denominator into single fractions, then inverting the denominator and multiplying.

EXAMPLE 8 Simplifying a Complex Fraction

$$\frac{\left(\dfrac{2}{x}-3\right)}{\left(1-\dfrac{1}{x-1}\right)} = \frac{\left[\dfrac{2-3(x)}{x}\right]}{\left[\dfrac{1(x-1)-1}{x-1}\right]} \qquad \text{Combine fractions.}$$

$$= \frac{\left(\dfrac{2-3x}{x}\right)}{\left(\dfrac{x-2}{x-1}\right)} \qquad \text{Simplify.}$$

$$= \frac{2-3x}{x} \cdot \frac{x-1}{x-2} \qquad \text{Invert and multiply.}$$

$$= \frac{(2-3x)(x-1)}{x(x-2)}, \quad x \neq 1$$

In Example 8, the disclaimer $x \neq 1$ is added to the final expression to make its domain agree with the domain of the original expression.

Another way to simplify a complex fraction is to multiply each term in its numerator and denominator by the LCD of all fractions in its numerator and denominator. This method is applied to the fraction in Example 8 as follows.

$$\frac{\left(\dfrac{2}{x}-3\right)}{\left(1-\dfrac{1}{x-1}\right)} = \frac{\left(\dfrac{2}{x}-3\right)}{\left(1-\dfrac{1}{x-1}\right)} \cdot \frac{x(x-1)}{x(x-1)} \qquad \text{LCD is } x(x-1).$$

$$= \frac{\left(\dfrac{2-3x}{\cancel{x}}\right) \cdot \cancel{x}(x-1)}{\left(\dfrac{x-2}{\cancel{x-1}}\right) \cdot x\cancel{(x-1)}} \qquad \text{Combine fractions.}$$

$$= \frac{(2-3x)(x-1)}{x(x-2)}, \quad x \neq 1 \qquad \text{Simplify.}$$

The next three examples illustrate some methods for simplifying rational expressions involving radicals and negative exponents. These types of expressions occur frequently in calculus.

To simplify an expression with negative exponents, begin by factoring out the common factor with the smaller exponent. Remember that when factoring, you subtract exponents. For instance, in $3x^{-5/2} + 2x^{-3/2}$ the smaller exponent is $-\frac{5}{2}$ and the common factor is $x^{-5/2}$.

$$\begin{aligned} 3x^{-5/2} + 2x^{-3/2} &= x^{-5/2}\left(3(1) + 2x^{-3/2-(-5/2)}\right) \\ &= x^{-5/2}(3 + 2x^{1}) \\ &= \frac{3 + 2x}{x^{5/2}} \end{aligned}$$

EXAMPLE 9 Simplifying an Expression with Negative Exponents

Simplify

$$x(1 - 2x)^{-3/2} + (1 - 2x)^{-1/2}.$$

Solution

Begin by factoring out the common factor with the *smaller exponent.*

$$\begin{aligned} x(1 - 2x)^{-3/2} + (1 - 2x)^{-1/2} &= (1 - 2x)^{-3/2}\left[x + (1 - 2x)^{(-1/2)-(-3/2)}\right] \\ &= (1 - 2x)^{-3/2}\left[x + (1 - 2x)^{1}\right] \\ &= \frac{1 - x}{(1 - 2x)^{3/2}} \end{aligned}$$

A second method for simplifying an expression with negative exponents is shown in the next example. This method involves multiplying the numerator and denominator by a term to eliminate the negative exponent.

EXAMPLE 10 Simplifying a Complex Fraction

Simplify

$$\frac{(4 - x^2)^{1/2} + x^2(4 - x^2)^{-1/2}}{4 - x^2}.$$

Solution

$$\begin{aligned} &\frac{(4 - x^2)^{1/2} + x^2(4 - x^2)^{-1/2}}{4 - x^2} \\ &= \frac{(4 - x^2)^{1/2} + x^2(4 - x^2)^{-1/2}}{4 - x^2} \cdot \frac{(4 - x^2)^{1/2}}{(4 - x^2)^{1/2}} \\ &= \frac{(4 - x^2)^{1} + x^2(4 - x^2)^{0}}{(4 - x^2)^{3/2}} \\ &= \frac{4 - x^2 + x^2}{(4 - x^2)^{3/2}} = \frac{4}{(4 - x^2)^{3/2}} \end{aligned}$$

EXAMPLE 11 Rewriting a Difference Quotient

The expression from calculus

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

is an example of a *difference quotient.* Rewrite this expression by rationalizing its numerator.

Solution

$$\begin{aligned}\frac{\sqrt{x+h}-\sqrt{x}}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\ &= \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}, \qquad h \neq 0\end{aligned}$$

Notice that the original expression is undefined when $h = 0$. So, you must exclude $h = 0$ from the domain of the simplified expressions so that the expressions are equivalent.

Difference quotients, like that in Example 11, occur frequently in calculus. Often, they need to be rewritten in an equivalent form that can be evaluated when $h = 0$. Note that the equivalent form is not simpler than the original form, but it has the advantage in that it is defined when $h = 0$.

Writing About Math *Comparing the Domains of Two Expressions*

Complete the table by evaluating the expressions

$$\frac{x^2-3x+2}{x-2} \quad \text{and} \quad x-1$$

for the values of x. If your graphing utility has a *table* feature, use it to help create the table. Write a short paragraph describing the equivalence or non-equivalence of the two expressions.

x	-3	-2	-1	0	1	2	3
$\frac{x^2-3x+2}{x-2}$							
$x-1$							

P.4 Exercises

In Exercises 1–8, find the domain of the expression.

1. $3x^2 - 4x + 7$

2. $2x^2 + 5x - 2$

3. $4x^3 + 3, \quad x \geq 0$

4. $6x^2 - 9, \quad x > 0$

5. $\dfrac{1}{x - 2}$

6. $\dfrac{x + 1}{2x + 1}$

7. $\sqrt{x + 1}$

8. $\sqrt{6 - x}$

In Exercises 9 and 10, find the missing factor in the numerator so that the two fractions are equivalent.

9. $\dfrac{5}{2x} = \dfrac{5(\boxed{})}{6x^2}$

10. $\dfrac{3}{4} = \dfrac{3(\boxed{})}{4(x + 1)}$

In Exercises 11–28, write the rational expression in simplest form.

11. $\dfrac{15x^2}{10x}$

12. $\dfrac{18y^2}{60y^5}$

13. $\dfrac{3xy}{xy + x}$

14. $\dfrac{2x^2y}{xy - y}$

15. $\dfrac{4y - 8y^2}{10y - 5}$

16. $\dfrac{9x^2 + 9x}{2x + 2}$

17. $\dfrac{x - 5}{10 - 2x}$

18. $\dfrac{12 - 4x}{x - 3}$

19. $\dfrac{y^2 - 16}{y + 4}$

20. $\dfrac{x^2 - 25}{5 - x}$

21. $\dfrac{x^3 + 5x^2 + 6x}{x^2 - 4}$

22. $\dfrac{x^2 + 8x - 20}{x^2 + 11x + 10}$

23. $\dfrac{y^2 - 7y + 12}{y^2 + 3y - 18}$

24. $\dfrac{3 - x}{x^2 + 11x + 10}$

25. $\dfrac{2 - x + 2x^2 - x^3}{x - 2}$

26. $\dfrac{x^2 - 9}{x^3 + x^2 - 9x - 9}$

27. $\dfrac{z^3 - 8}{z^2 + 2z + 4}$

28. $\dfrac{y^3 - 2y^2 - 3y}{y^3 + 1}$

In Exercises 29 and 30, complete the table. What can you conclude?

29.

x	0	1	2	3	4	5	6
$\dfrac{x^2 - 2x - 3}{x - 3}$							
$x + 1$							

30.

x	0	1	2	3	4	5	6
$\dfrac{x - 3}{x^2 - x - 6}$							
$\dfrac{1}{x + 2}$							

31. ***Error Analysis*** Describe the error.

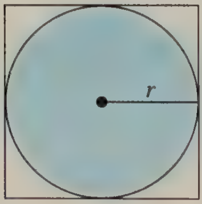
$$\frac{5x^3}{2x^3 + 4} = \frac{5\cancel{x^3}}{2\cancel{x^3} + 4} = \frac{5}{2 + 4} = \frac{5}{6}$$

32. ***Error Analysis*** Describe the error.

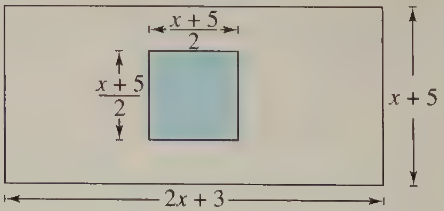
$$\begin{aligned}\frac{x^3 + 25x}{x^2 - 2x - 15} &= \frac{x(x^2 + 25)}{(x - 5)(x + 3)}\\ &= \frac{x\cancel{(x - 5)}(x + 5)}{\cancel{(x - 5)}(x + 3)}\\ &= \frac{x(x + 5)}{x + 3}\end{aligned}$$

In Exercises 33 and 34, find the ratio of the area of the shaded portion of the figure to the total area of the figure.

33.



34.



In Exercises 35–42, perform the multiplication or division and simplify.

35. $\dfrac{5}{x-1} \cdot \dfrac{x-1}{25(x-2)}$

36. $\dfrac{x+13}{x^3(3-x)} \cdot \dfrac{x(x-3)}{5}$

37. $\dfrac{r}{r-1} \cdot \dfrac{r^2-1}{r^2}$

38. $\dfrac{4y-16}{5y+15} \cdot \dfrac{2y+6}{4-y}$

2 **39.** $\dfrac{t^2-t-6}{t^2+6t+9} \cdot \dfrac{t+3}{t^2-4}$

40. $\dfrac{y^3-8}{2y^3} \cdot \dfrac{4y}{y^2-5y+6}$

41. $\dfrac{3(x+y)}{4} \div \dfrac{x+y}{2}$

42. $\dfrac{x+2}{5(x-3)} \div \dfrac{x-2}{5(x-3)}$

In Exercises 43–52, perform the addition or subtraction and simplify.

43. $\dfrac{5}{x-1} + \dfrac{x}{x-1}$

44. $\dfrac{2x-1}{x+3} + \dfrac{1-x}{x+3}$

2 **45.** $6 - \dfrac{5}{x+3}$

46. $\dfrac{3}{x-1} - 5$

2 **47.** $\dfrac{3}{x-2} + \dfrac{5}{2-x}$

48. $\dfrac{2x}{x-5} - \dfrac{5}{5-x}$

2 **49.** $\dfrac{1}{x^2-x-2} - \dfrac{x}{x^2-5x+6}$

50. $\dfrac{2}{x^2-x-2} + \dfrac{10}{x^2+2x-8}$

51. $-\dfrac{1}{x} + \dfrac{2}{x^2+1} + \dfrac{1}{x^3+x}$

52. $\dfrac{2}{x+1} + \dfrac{2}{x-1} + \dfrac{1}{x^2-1}$

In Exercises 53–60, simplify the expression by removing the common factor with the smaller exponent.

53. $x^5 - 2x^{-2}$

54. $x^5 - 5x^{-3}$

55. $3x^{3/2} - 2x^{-1/2}$

56. $5x^5 - 3x^{-3/2}$

57. $x^2(x^2+1)^{-5} - (x^2+1)^{-4}$

58. $2x(x-5)^{-3} - 4x^2(x-5)^{-4}$

59. $2x^2(x-1)^{1/2} - 5(x-1)^{-1/2}$

60. $4x^3(2x-1)^{3/2} - 2x(2x-1)^{-1/2}$

In Exercises 61–70, simplify the complex fraction.

61. $\dfrac{\left(\dfrac{x}{2}-1\right)}{(x-2)}$

62. $\dfrac{(x-4)}{\left(\dfrac{x}{4}-\dfrac{4}{x}\right)}$

63. $\dfrac{\left[\dfrac{x^2}{(x+1)^2}\right]}{\left[\dfrac{x}{(x+1)^3}\right]}$

64. $\dfrac{\left(\dfrac{x^2-1}{x}\right)}{\left[\dfrac{(x-1)^2}{x}\right]}$

2 **65.** $\dfrac{\left[\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}\right]}{h}$

66. $\dfrac{\left(\dfrac{x+h}{x+h+1}-\dfrac{x}{x+1}\right)}{h}$

67. $\dfrac{\left(\sqrt{x}-\dfrac{1}{2\sqrt{x}}\right)}{\sqrt{x}}$

68. $\dfrac{\left(\dfrac{t^2}{\sqrt{t^2+1}}-\sqrt{t^2+1}\right)}{t^2}$

69. $\dfrac{3x^{1/3}-x^{-2/3}}{3x^{-2/3}}$

70. $\dfrac{-x^3(1-x^2)^{-1/2}-2x(1-x^2)^{1/2}}{x^4}$

In Exercises 71 and 72, rationalize the numerator of the expression.

71. $\dfrac{\sqrt{x+2}-\sqrt{x}}{2}$

72. $\dfrac{\sqrt{z-3}-\sqrt{z}}{3}$

73. ***Rate*** A photocopier copies at a rate of 16 pages per minute.

(a) Find the time required to copy one page.

(b) Find the time required to copy x pages.

(c) Find the time required to copy 60 pages.

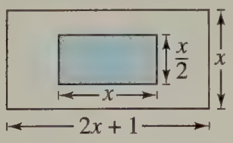
74. ***Rate*** After working together for t hours on a common task, two workers have done fractional parts of the job equal to $t/3$ and $t/5$, respectively. What fractional part of the task has been completed?

75. ***Average*** Determine the average of the two real numbers $x/3$ and $2x/5$.

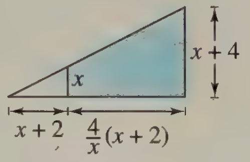
76. ***Partition into Equal Parts*** Find three real numbers that divide the real number line between $x/3$ and $3x/4$ into four equal parts.

Probability **In Exercises 77 and 78, consider an experiment in which a marble is tossed into a box whose base is shown in the figure. The probability that the marble will come to rest in the shaded portion of the box is equal to the ratio of the shaded area to the total area of the figure. Find the probability.**

77.



78.



Monthly Payment **In Exercises 79 and 80, use the formula that gives the approximate annual interest rate r of a monthly installment loan**

$$r = \frac{\left[\dfrac{24(NM - P)}{N}\right]}{\left(P + \dfrac{NM}{12}\right)}$$

where N is the total number of payments, M is the monthly payment, and P is the amount financed.

79. (a) Approximate the annual interest rate for a 4-year car loan of $16,000 that has monthly payments of $400.

(b) Simplify the expression for the annual interest rate r, and then rework part (a).

80. (a) Approximate the annual interest rate for a 5-year car loan of $20,000 that has monthly payments of $400.

(b) Simplify the expression for the annual interest rate r, and then rework part (a).

81. ***Refrigeration*** When food (at room temperature) is placed in the refrigerator, the time required for the food to cool depends on the amount of food, the air circulation in the refrigerator, the original temperature of the food, and the temperature of the refrigerator. Consider the model that gives the temperature of the food that is at 75°F and is placed in a 40°F refrigerator as

$$T = 10\left(\frac{4t^2 + 16t + 75}{t^2 + 4t + 10}\right)$$

where T is the temperature (in degrees Fahrenheit) and t is the time (in hours).

(a) Complete the table.

t	0	2	4	6	8	10
T						

t	12	14	16	18	20	22
T						

(b) What value of T does the mathematical model appear to be approaching?

82. ***Precious Metals*** The costs per fine ounce of gold and silver for the years 1992 through 1997 are given in the table. (Source: U.S. Bureau of Mines, U.S. Geological Survey)

Year	1992	1993	1994	1995	1996	1997
Gold	$345	$361	$385	$386	$389	$333
Silver	$3.94	$4.30	$5.29	$5.15	$5.19	$4.90

Mathematical models for this data are

$$\text{Cost of gold} = \frac{-38.5t + 310.1}{0.007t^2 - 0.176t + 1}$$

and

$$\text{Cost of silver} = \frac{0.42t^2 - 2.56t + 4.26}{0.09t^2 - 0.58t + 1}$$

where $t = 2$ corresponds to the year 1992.

(a) Create a table using the models to estimate the prices of each metal for the given years. Compare the estimates given by the models with the actual prices.

(b) Determine a model for the ratio of the price of gold to the price of silver. Use the model to find this ratio over the given years. Over this period of time, did the price of gold become more expensive or less expensive relative to the price of silver?

Synthesis

True or False? **In Exercises 83–85, determine whether the statement is true or false. Justify your answer.**

83. $\dfrac{x^{2n} - 1^{2n}}{x^n - 1^n} = x^n + 1^n$

84. $\dfrac{x^2 - 3x + 2}{x - 1} = x - 2$ for all values of x.

85. The least common denominator of two or more fractions is always the product of all the denominators of the fractions.

86. ***Think About It*** How do you determine whether a rational expression is in simplest form?

87. ***Think About It*** Is the following statement true for all nonzero real numbers a and b? Explain.

$$\frac{ax - b}{b - ax} = -1$$

P.5 Graphical Representation of Data

The Cartesian Plane

Just as you can represent real numbers by points on a real number line, you can represent ordered pairs of real numbers by points in a plane called the **rectangular coordinate system,** or the **Cartesian plane,** after the French mathematician René Descartes (1596–1650).

The Cartesian plane is formed by using two real number lines intersecting at right angles, as shown in Figure P.6. The horizontal real number line is usually called the **x-axis,** and the vertical real number line is usually called the **y-axis.** The point of intersection of these two axes is the **origin,** and the two axes divide the plane into four parts called **quadrants.**

What You Should Learn:

- How to plot points in the Cartesian plane
- How to represent data graphically using scatter plots, bar graphs, and line graphs
- How to use the Distance Formula to find the distance between two points
- How to use the Midpoint Formula to find the midpoint of a line segment
- How to find the equation of a circle

Why You Should Learn It:

The Cartesian plane can be used to represent relationships between two variables. For instance, Exercise 87 on page 58 shows how to graphically represent the number of recording artists elected to the Rock and Roll Hall of Fame from 1986 to 1999.

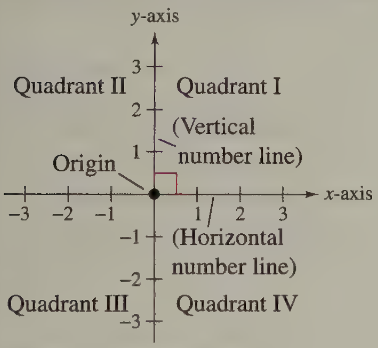


Figure P.6 *The Cartesian Plane*

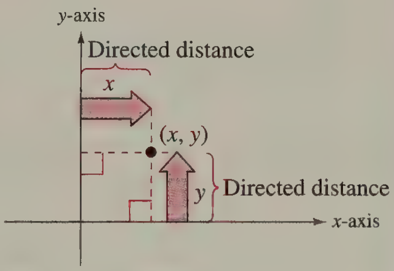


Figure P.7 *Ordered Pair (x, y)*

Each point in the plane corresponds to an **ordered pair** (x, y) of real numbers x and y, called **coordinates** of the point. The **x-coordinate** represents the directed distance from the y-axis to the point, and the **y-coordinate** represents the directed distance from the x-axis to the point, as shown in Figure P.7.

Directed distance from y-axis ↗ (x, y) ↖ Directed distance from x-axis

The notation (x, y) denotes both a point in the plane and an open interval on the real number line. The context will tell you which meaning is intended.

EXAMPLE 1 Plotting Points in the Cartesian Plane

Plot the points $(-1, 2)$, $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$.

Solution

To plot the point $(-1, 2)$, imagine a vertical line through -1 on the x-axis and a horizontal line through 2 on the y-axis. The intersection of these two lines is the point $(-1, 2)$. This point is 1 unit to the left of the y-axis and 2 units up from the x-axis. The other four points can be plotted in a similar way (see Figure P.8).

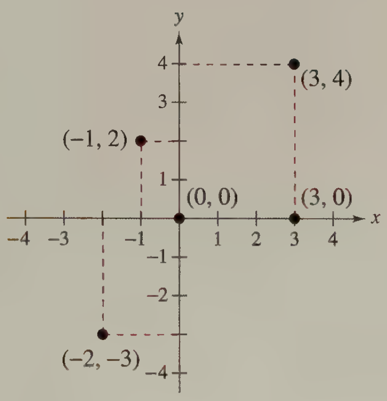
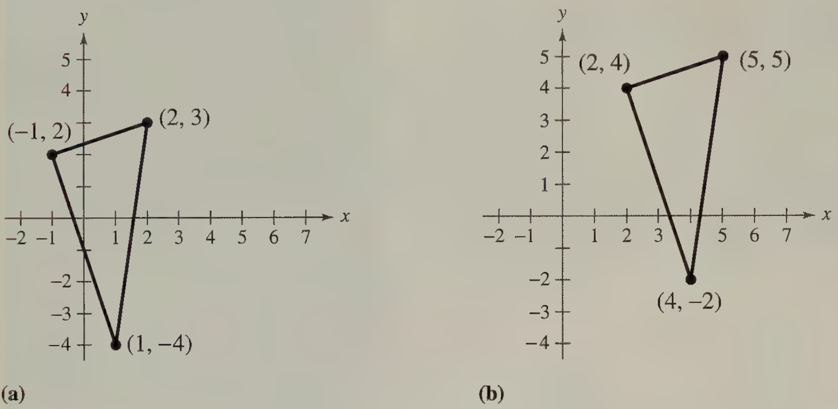


Figure P.8

EXAMPLE 2 Translating Points in the Plane

The triangle in Figure P.9(a) has vertices at the points $(-1, 2)$, $(1, -4)$, and $(2, 3)$. Shift the triangle three units to the right and two units up and find the vertices of the shifted triangle, as shown in Figure P.9(b).



(a) **(b)**

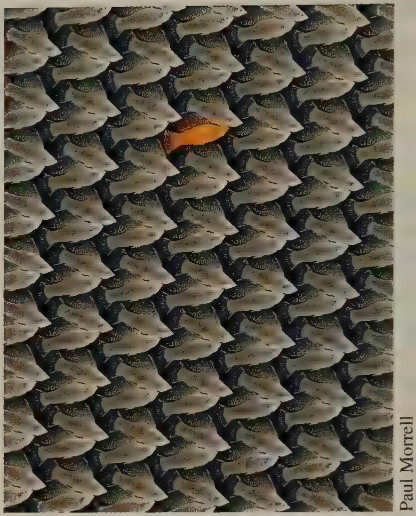
Figure P.9

Solution

To shift the vertices three units to the right, add 3 to each of the x-coordinates. To shift the vertices two units up, add 2 to each of the y-coordinates.

Original Point	*Translated Point*
$(-1, 2)$	$(-1 + 3, 2 + 2) = (2, 4)$
$(1, -4)$	$(1 + 3, -4 + 2) = (4, -2)$
$(2, 3)$	$(2 + 3, 3 + 2) = (5, 5)$

Plotting the translated points and sketching the line segments between them produces the shifted triangle shown in Figure P.9(b).



Paul Morrell

Much of computer graphics, including this computer-generated goldfish tessellation, consists of transformations of points in a coordinate plane. One type of transformation, a translation, is illustrated in Example 2. Other types include reflections, rotations, and stretches.

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Example 2 shows how to translate points in a coordinate plane. The following transformed points are related to the original points as follows.

Original Point	*Transformed Point*	
(x, y)	$(-x, y)$	$(-x, y)$ is a reflection of the original point in the y-axis.
(x, y)	$(x, -y)$	$(x, -y)$ is a reflection of the original point in the x-axis.
(x, y)	$(-x, -y)$	$(-x, -y)$ is a reflection of the original point through the origin.

Representing Data Graphically

The beauty of a rectangular coordinate system is that it allows you to see relationships between two variables. It would be difficult to overestimate the importance of Descartes's introduction of coordinates to the plane. Today, his ideas are in common use in virtually every scientific and business-related field.

EXAMPLE 3 Sketching a Scatter Plot, Bar Graph, and Line Graph

From 1988 through 1997, the amount A (in millions of dollars) spent on archery equipment in the United States is given in the table, where t represents the year. (a) Sketch a scatter plot of the data. (b) Sketch a bar graph of the data. (c) Sketch a line graph of the data. (Source: National Sporting Goods Association)

t	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997
A	235	261	265	270	334	285	306	287	272	273

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Solution

a. To sketch a *scatter plot* of the data given in the table, you simply represent each pair of values by an ordered pair (t, A) and plot the resulting points, as shown in Figure P.10. For instance, the first pair of values is represented by the ordered pair (1988, 235). Note that the break in the t-axis indicates that the numbers between 0 and 1988 have been omitted.

b. To create a *bar graph*, begin by drawing a vertical axis to represent the amount (in millions of dollars) and a horizontal axis to represent the year. Then for each value of t in the table, draw a vertical bar whose height is the corresponding value A. The bar graph is shown in Figure P.11.

c. To draw a *line graph*, begin by drawing a vertical axis to represent the amount (in millions of dollars). Then label the horizontal axis with years and plot the points given in the table. Finally, connect the points with line segments, as shown in Figure P.12.

STUDY TIP

You can use a graphing utility to sketch each of the graphs in Example 3. First, enter the data into the graphing utility. You can then use the *statistical plotting* feature to sketch a scatter plot, a bar graph, and a line graph.

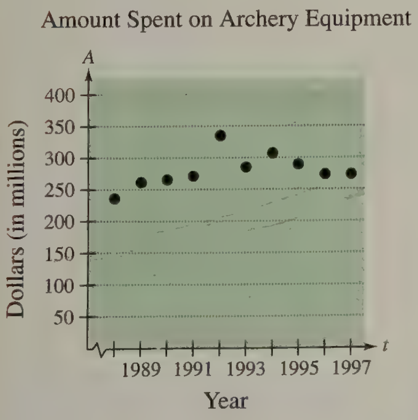


Figure P.10

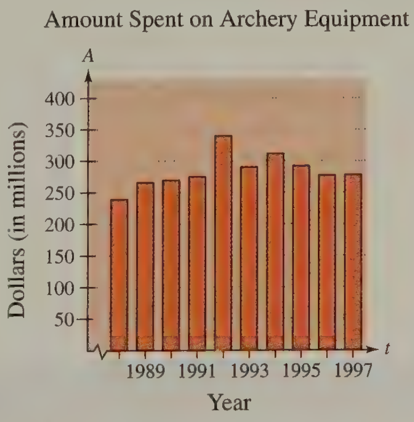


Figure P.11

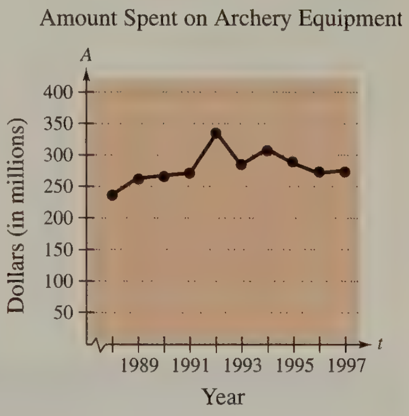


Figure P.12

In Example 3, you could have let $t = 1$ represent the year 1988. In that case, the horizontal axis of each graph would not have been broken, and the tick marks would have been labeled 1 through 10 (instead of 1988 through 1997).

EXAMPLE 4 Interpreting a Population Model

The population (in millions) of North Carolina from 1990 to 1997 can be modeled by $y = 0.11x + 6.63$, where x is the time in years, with $x = 0$ corresponding to 1990. During which year did the population exceed 7 million?

Graphical Solution

You can use the *statistical plotting* feature of a graphing utility to create a bar graph of the population values obtained using the model. On the same viewing window, graph the line $y = 7$.

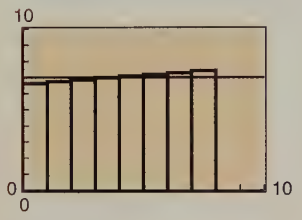
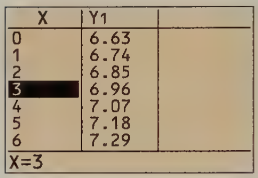


Figure P.13

From the graph in Figure P.13 you can see that the bars that correspond to $x = 0, 1, 2,$ and 3 are less than 7. The bar that corresponds to $x = 4$ is the first bar greater than 7, so the population exceeded 7 million sometime during 1993.

Numerical Solution

You can use the *table* feature of a graphing utility to evaluate the model $y = 0.11x + 6.63$ for different years. Enter the model into the equation editor. Then create a table such as the one below.



```
  X    |Y1     |
0      |6.63
1      |6.74
2      |6.85
3      |6.96
4      |7.07
5      |7.18
6      |7.29
X=3
```

Figure P.14

From the table in Figure P.14, you can see that the value of y when $x = 3$ is 6.96, and the value of y when $x = 4$ is 7.07. So, the population exceeded 7 million when x is greater than 3 and less than 4, or sometime during 1993.

The Distance Formula

Recall from the Pythagorean Theorem that for a right triangle with hypotenuse of length c and sides of lengths a and b, you have $a^2 + b^2 = c^2$ as shown in Figure P.15. (The converse is also true. That is, if $a^2 + b^2 = c^2$, the triangle is a right triangle.)

Suppose you want to determine the distance d between two points (x_1, y_1) and (x_2, y_2) in the plane. With these two points, a right triangle can be formed, as shown in Figure P.16. The length of the vertical side of the triangle is $|y_2 - y_1|$, and the length of the horizontal side is $|x_2 - x_1|$. By the Pythagorean Theorem,

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This result is the **Distance Formula.**

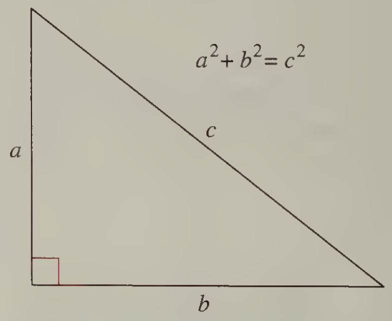


Figure P.15

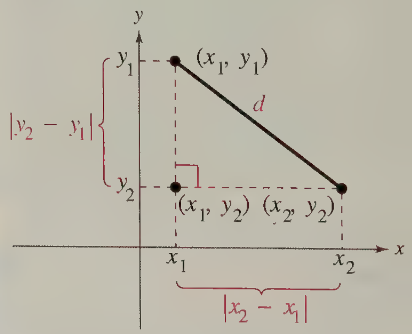


Figure P.16

The Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) in the plane is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 5 Finding a Distance

Find the distance between the points $(-2, 1)$ and $(3, 4)$.

Algebraic Solution

Let $(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (3, 4)$. Then apply the Distance Formula as follows.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \text{Distance Formula}$$

$$= \sqrt{[3 - (-2)]^2 + (4 - 1)^2} \qquad \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2.$$

$$= \sqrt{(5)^2 + (3)^2} \qquad \text{Simplify.}$$

$$= \sqrt{34}$$

$$\approx 5.83 \qquad \text{Use a calculator.}$$

So, the distance between the points is about 5.83 units.

You can use the Pythagorean Theorem to check that the distance is correct.

$$d^2 \overset{?}{=} 3^2 + 5^2 \qquad \text{Pythagorean Theorem}$$

$$\left(\sqrt{34}\right)^2 \overset{?}{=} 3^2 + 5^2 \qquad \text{Substitute for } d.$$

$$34 = 34 \qquad \text{Distance checks. } \checkmark$$

Graphical Solution

Use centimeter graph paper to plot the points A and B. Carefully sketch the line segment from A to B. Then use a centimeter ruler to measure the length of the segment.

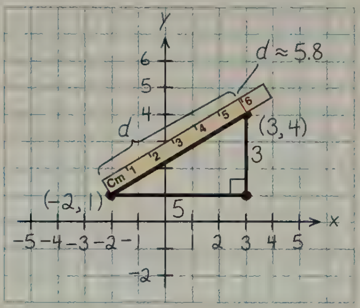


Figure P.17

The line segment measures about 5.8 centimeters as shown in Figure P.17. So, the distance between the points is about 5.8 units.

EXAMPLE 6 Verifying a Right Triangle

Show that the points $(2, 1)$, $(4, 0)$, and $(5, 7)$ are vertices of a right triangle.

Solution

The three points are plotted in Figure P.18. Using the Distance Formula, you can find the lengths of the three sides as follows.

$$d_1 = \sqrt{(5 - 2)^2 + (7 - 1)^2} = \sqrt{9 + 36} = \sqrt{45}$$

$$d_2 = \sqrt{(4 - 2)^2 + (0 - 1)^2} = \sqrt{4 + 1} = \sqrt{5}$$

$$d_3 = \sqrt{(5 - 4)^2 + (7 - 0)^2} = \sqrt{1 + 49} = \sqrt{50}$$

Because

$$d_1^2 + d_2^2 = 45 + 5 = 50 = d_3^2$$

you can conclude that the triangle must be a right triangle.

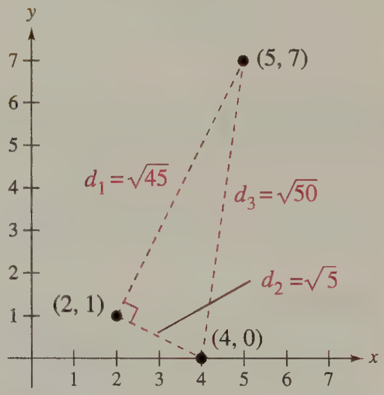


Figure P.18

The figure provided with Example 6 was not really essential to the solution. Nevertheless, it is strongly recommended that you develop the habit of including sketches with your solutions—even if they are not required.

The Midpoint Formula

To find the **midpoint** of the line segment that joins two points in a coordinate plane, you can simply find the average values of the respective coordinates of the two endpoints using the **Midpoint Formula.** (See Appendix A for a proof of the Midpoint Formula.)

The Midpoint Formula

The midpoint of the segment joining the points (x_1, y_1) and (x_2, y_2) is given by the Midpoint Formula

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

EXAMPLE 7 Finding a Line Segment's Midpoint

Find the midpoint of the line segment joining the points $(-5, -3)$ and $(9, 3)$, as shown in Figure P.19.

Solution

Let $(x_1, y_1) = (-5, -3)$ and $(x_2, y_2) = (9, 3)$.

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \qquad \text{Midpoint Formula}$$

$$= \left(\frac{-5 + 9}{2}, \frac{-3 + 3}{2}\right) \qquad \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2.$$

$$= (2, 0) \qquad \text{Simplify.}$$

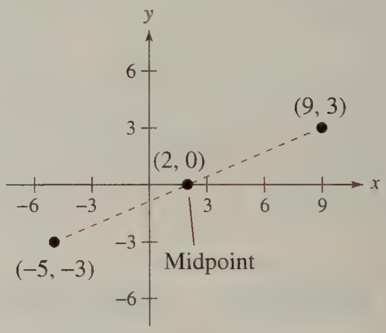


Figure P.19

EXAMPLE 8 Estimating Annual Sales

Winn-Dixie Stores had annual sales of \$1.30 billion in 1996 and \$1.36 billion in 1998. Without knowing any additional information, what would you estimate the 1997 sales to have been? (Source: Winn-Dixie Stores, Inc.)

Solution

One solution to the problem is to assume that sales followed a linear pattern. With this assumption, you can estimate the 1997 sales by finding the midpoint of the segment connecting the points (1996, 1.30) and (1998, 1.36).

$$\text{Midpoint} = \left(\frac{1996 + 1998}{2}, \frac{1.30 + 1.36}{2}\right)$$

$$= (1997, 1.33)$$

So, you would estimate the 1997 sales to have been about \$1.33 billion, as shown in Figure P.20. (The actual 1997 sales were \$1.32 billion.)

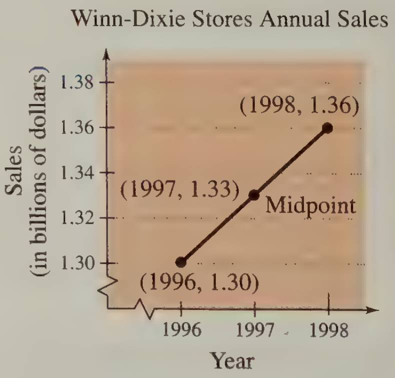


Figure P.20

The Equation of a Circle

The Distance Formula provides a convenient way to define circles. A **circle of radius r** with center at the point (h, k) is shown in Figure P.21. The point (x, y) is on this circle if and only if its distance from the center (h, k) is r. This means that a **circle** in the plane consists of all points (x, y) that are a given positive distance r from a fixed point (h, k). Using the Distance Formula, you can express this relationship by saying that the point (x, y) lies on the circle if and only if

$$\sqrt{(x-h)^2 + (y-k)^2} = r.$$

By squaring both sides of this equation, you can obtain the **standard form of the equation of a circle.**

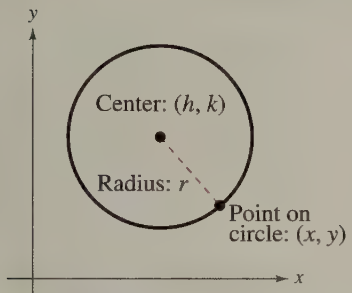


Figure P.21

Standard Form of the Equation of a Circle

The **standard form of the equation of a circle** is

$$(x-h)^2 + (y-k)^2 = r^2.$$

The point (h, k) is the **center** of the circle, and the positive number r is the **radius** of the circle. The standard form of the equation of a circle whose center is the origin is $x^2 + y^2 = r^2$.

EXAMPLE 9 Finding an Equation of a Circle

The point $(3, 4)$ lies on a circle whose center is at $(-1, 2)$, as shown in Figure P.22. Find an equation for the circle.

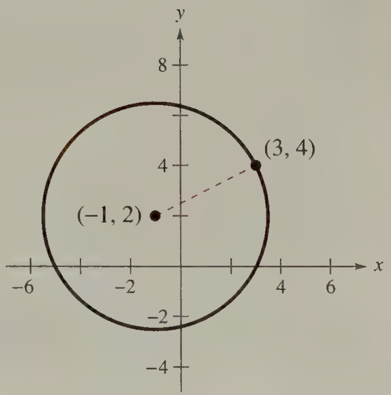


Figure P.22

Solution

The radius r of the circle is the distance between $(-1, 2)$ and $(3, 4)$.

$$r = \sqrt{[3-(-1)]^2 + (4-2)^2}$$

$$= \sqrt{16 + 4} = \sqrt{20}$$

So, the center of the circle is $(h, k) = (-1, 2)$ and the radius is $r = \sqrt{20}$, and you can write the standard form of the equation of the circle as follows.

$$(x-h)^2 + (y-k)^2 = r^2 \qquad \text{Standard form}$$

$$[x-(-1)]^2 + (y-2)^2 = \left(\sqrt{20}\right)^2 \qquad \text{Substitute for } h, k, \text{ and } r.$$

$$(x+1)^2 + (y-2)^2 = 20 \qquad \text{Equation of circle}$$

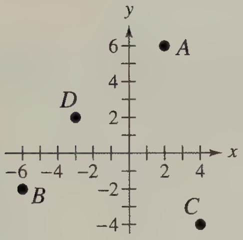
P.5 Exercises

In Exercises 1–4, sketch the polygon with the indicated vertices.

1. Triangle: $(-1, 1), (2, -1), (3, 4)$
2. Triangle: $(0, 3), (-1, -2), (4, 8)$
3. Square: $(2, 4), (5, 1), (2, -2), (-1, 1)$
4. Parallelogram: $(5, 2), (7, 0), (1, -2), (-1, 0)$

In Exercises 5–8, approximate the coordinates of the points.

5.



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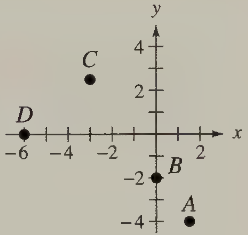

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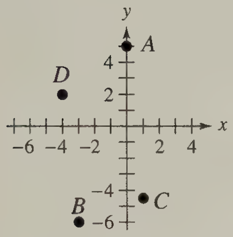
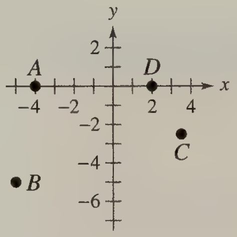


8.



In Exercises 9–12, find the coordinates of the point.

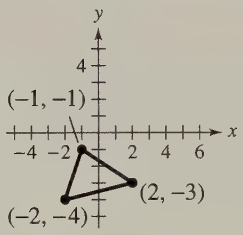
9. The point is located three units to the left of the y-axis and four units above the x-axis.
10. The point is located eight units below the x-axis and four units to the right of the y-axis.
11. The point is located five units below the x-axis and the coordinates of the point are equal.
12. The point is on the x-axis and twelve units to the left of the y-axis.

In Exercises 13–22, determine the quadrant(s) in which (x, y) is located so that the condition(s) is (are) satisfied.

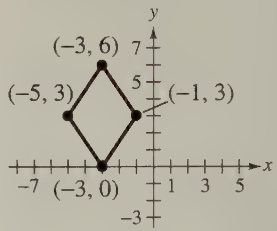
13. $x > 0$ and $y < 0$
14. $x < 0$ and $y < 0$
15. $x = -4$ and $y > 0$
16. $x > 2$ and $y = 3$
17. $y < -5$
18. $x > 4$
19. $(x, -y)$ is in the second quadrant.
20. $(-x, y)$ is in the fourth quadrant.
21. $xy > 0$
22. $xy < 0$

In Exercises 23 and 24, the polygon is shifted to a new position in the plane. Find the coordinates of the vertices of the polygon in its new position.

23. Shift: 5 units up, 2 units to the right



24. Shift: 3 units down, 6 units to the right



In Exercises 25–30, find the distance between the points algebraically and verify graphically by using centimeter graph paper and a centimeter ruler.

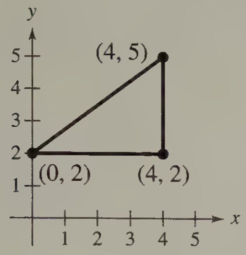
25. $(6, -3), (6, 5)$
26. $(1, 4), (8, 4)$
27. $(-3, -1), (2, -1)$
28. $(-3, -4), (-3, 6)$
29. $(-2, 6), (3, -6)$
30. $(8, 5), (0, 20)$

In Exercises 31–42, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

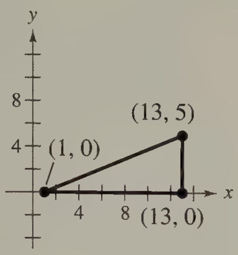
31. $(1, 1), (9, 7)$
32. $(1, 12), (6, 0)$
33. $(-4, 10), (4, -5)$
34. $(-7, -4), (2, 8)$
35. $(-1, 2), (5, 4)$
36. $(2, 10), (10, 2)$
37. $\left(\frac{1}{2}, 1\right), \left(-\frac{5}{2}, \frac{4}{3}\right)$
38. $\left(-\frac{1}{3}, -\frac{1}{3}\right), \left(-\frac{1}{6}, -\frac{1}{2}\right)$
39. $(6.2, 5.4), (-3.7, 1.8)$
40. $(-16.8, 12.3), (5.6, 4.9)$
41. $(-36, -18), (48, -72)$
42. $(1.451, 3.051), (5.906, 11.360)$

In Exercises 43–46, (a) find the length of each side of a right triangle and (b) show that these lengths satisfy the Pythagorean Theorem.

43.



44.



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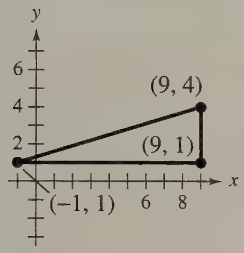
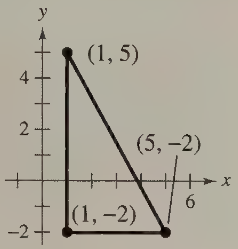


46.



Business **In Exercises 47 and 48, estimate the sales of a company in 1998, given the sales in 1996 and 2000. Assume the sales followed a linear pattern.**

47.

Year	1996	2000
Sales	\$520,000	\$740,000

48.

Year	1996	2000
Sales	\$4,200,000	\$5,650,000

In Exercises 49–52, show that the points form the vertices of the polygon.

49. Right triangle: $(4, 0), (2, 1), (-1, -5)$

50. Isosceles triangle: $(1, -3), (3, 2), (-2, 4)$

51. Parallelogram: $(2, 5), (0, 9), (-2, 0), (0, -4)$

52. Parallelogram: $(0, 1), (3, 7), (4, 4), (1, -2)$

53. ***Exploration*** A line segment has (x_1, y_1) as one endpoint and (x_m, y_m) as its midpoint. Find the other endpoint (x_2, y_2) of the line segment in terms of x_1, y_1, x_m, and y_m. Use the result to find the coordinates of the endpoint of a line segment if the coordinates of the other endpoint and midpoint are, respectively,

(a) $(1, -2), (4, -1)$ (b) $(-5, 11), (2, 4)$.

54. ***Exploration*** Use the Midpoint Formula three times to find the three points that divide the line segment joining (x_1, y_1) and (x_2, y_2) into four parts. Use the result to find the points that divide the line segment joining the given points into four equal parts.

(a) $(1, -2), (4, -1)$ (b) $(-2, -3), (0, 0)$

In Exercises 55–62, find the standard form of the equation of the specified circle.

55. Center: $(0, 0)$; radius: 3

56. Center: $(0, 0)$; radius: 5

57. Center: $(2, -1)$; radius: 4

58. Center: $\left(0, \frac{1}{3}\right)$; radius: $\frac{1}{3}$

59. Center: $(-1, 2)$; solution point: $(0, 0)$

60. Center: $(3, -2)$; solution point: $(-1, 1)$

61. Endpoints of a diameter: $(0, 0), (6, 8)$

62. Endpoints of a diameter: $(-4, -1), (4, 1)$

In Exercises 63–68, find the center and radius, and sketch the circle.

63. $x^2 + y^2 = 4$

64. $x^2 + y^2 = 16$

65. $(x - 1)^2 + (y + 3)^2 = 4$

66. $x^2 + (y - 1)^2 = 4$

67. $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{9}{4}$

68. $\left(x - \frac{2}{3}\right)^2 + \left(y + \frac{1}{4}\right)^2 = \frac{25}{9}$

In Exercises 69 and 70, sketch a scatter plot of the data given in the table.

69. ***Meteorology*** The table shows the lowest temperature of record y (in degrees Fahrenheit) in Duluth, Minnesota, for each month x, where $x = 1$ represents January. (Source: NOAA)

x	1	2	3	4	5	6
y	−39	−33	−29	−5	17	27

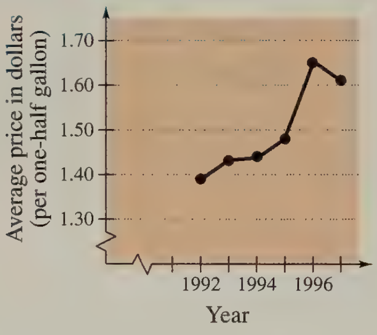
x	7	8	9	10	11	12
y	35	32	22	8	−23	−34

70. ***Business*** The table shows the number y of Wal-Mart stores for each year x from 1992 through 1999. (Source: Wal-Mart Stores, Inc.)

x	1992	1993	1994	1995
y	2136	2440	2759	2943

x	1996	1997	1998	1999
y	3054	3406	3630	3815

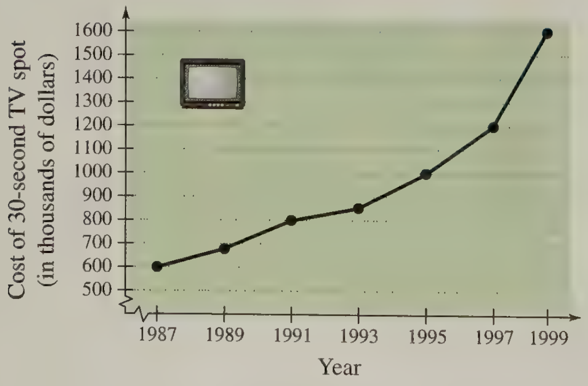
Milk Prices **In Exercises 71 and 72, use the graph below, which shows the average retail price of one-half gallon of milk from 1992 to 1997.** (Source: U.S. Bureau of Labor Statistics)



71. Approximate the highest price of one-half gallon of milk shown in the graph. When did this occur?

72. Approximate the difference in the price of milk from the highest price shown in the graph to the price in 1992.

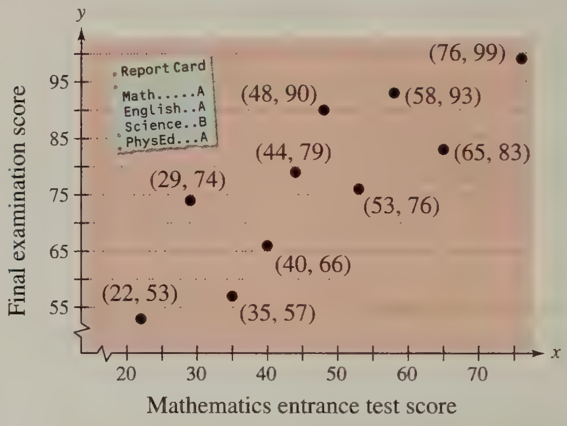
Advertising **In Exercises 73 and 74, use the graph below, which shows the cost of a 30-second television spot (in thousands of dollars) during the Super Bowl from 1987 to 1999.** (Source: USA Today Research)



73. Approximate the percent increase in cost of a 30-second spot from Super Bowl XXI in 1987 to Super Bowl XXXIII in 1999.

74. Estimate the increase in cost of a 30-second spot (a) from Super Bowl XXI to Super Bowl XXVII, and (b) from Super Bowl XXVII to Super Bowl XXXIII.

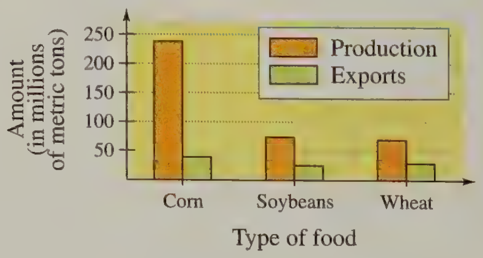
Analyzing Data **In Exercises 75 and 76, refer to the scatter plot, which shows the mathematics entrance test scores x and the final examination scores y in an algebra course for a sample of 10 students.**



75. Find the entrance exam score of any student with a final exam score in the 80s.

76. Does a higher entrance exam score necessarily imply a higher final exam score? Explain.

77. ***Food Production*** The double bar graph shows the production and exports (in millions of metric tons) of corn, soybeans, and wheat for the year 1997. Approximate the percent of each product that is exported. (Source: U.S. Department of Agriculture)



78. ***Fruit Crops*** The table shows farmers' cash receipts (in millions of dollars) from fruit crops in 1996. Construct a bar graph for the data. (Source: U.S. Department of Agriculture)

Fruit	Receipts	Fruit	Receipts
Apples	1846	Oranges	1798
Cherries	264	Peaches	380
Cranberries	246	Pears	292
Grapes	2334	Plums and Prunes	295
Lemons	228	Strawberries	770

79. ***Sports Participants*** The table shows the number of males and females (in millions) over the age of seven that participated in popular sports activities in 1996 in the United States. Construct a double bar graph for the data. (Source: National Sporting Goods Association)

Activity	Male	Female
Aerobic exercising	5.3	18.8
Basketball	22.4	10.9
Bicycling	28.6	24.7
Bowling	22.6	20.3
Camping	24.1	20.6
Exercise walking	26.7	46.6
Running	12.3	9.9
Swimming	29.1	31.1

80. ***Oil Imports*** The table shows the amount of crude oil imported into the United States (in millions of barrels) for the years 1988 through 1997. Construct a line graph for the data and state what information the graph reveals. (Source: Energy Information Administration)

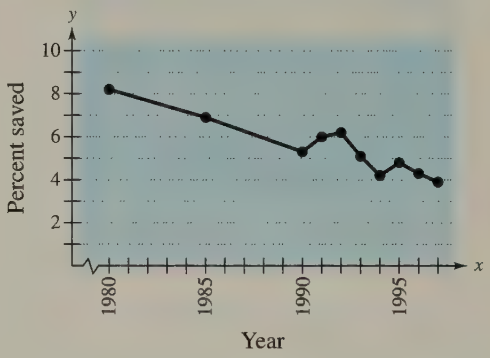
Year	1988	1989	1990	1991	1992
Imports	1864	2133	2151	2110	2220

Year	1993	1994	1995	1996	1997
Imports	2477	2578	2643	2748	2918

81. ***Personal Savings*** The line graph shows the percent of disposable income saved in the United States in selected years from 1980 to 1997. (Source: U.S. Bureau of Economic Analysis)

(a) Determine the percent decrease in the rate of saving from 1980 to 1997.

(b) Is the trend shown in the line graph good for the country? Explain your reasoning.



82. ***Population*** The population P (in millions) of Texas from 1990 to 1997 can be modeled by

$$P = 0.35x + 16.99$$

where x is the time in years, with $x = 0$ corresponding to 1990. (Source: U.S. Bureau of the Census)

(a) Use your graphing utility to create a bar graph of the model for the years 1990 to 1997. Use the graph to determine when the population of Texas exceeded 19 million.

(b) Determine algebraically when the population of Texas exceeded 19 million. Use your graph from part (a) to check your answer.

83. ***Health*** The average patient cost C to a community hospital per day in the United States from 1989 to 1996 can be modeled by

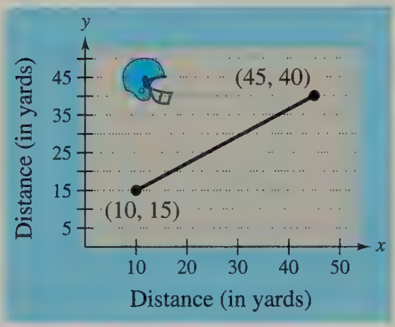
$$C = -2.37t^2 + 66.44t + 696.39$$

where t is the time in years, with $t = 0$ corresponding to 1990. (Source: American Hospital Association)

(a) Determine algebraically when the average cost exceeded $900 per day.

(b) Check your answer in part (a) by constructing a line graph of the model for the years 1989 to 1996. Use your graph to approximate when the average cost exceeded $900 per day.

84. ***Sports*** In a football game, a quarterback throws a pass from the 15-yard line, 10 yards from the sideline as shown in the figure. The pass is caught on the 40-yard line, 45 yards from the same sideline. How long is the pass?



85. ***Flying Distance*** A plane flies in a straight line to a city that is 100 kilometers east and 150 kilometers north of the point of departure. How far does it fly?

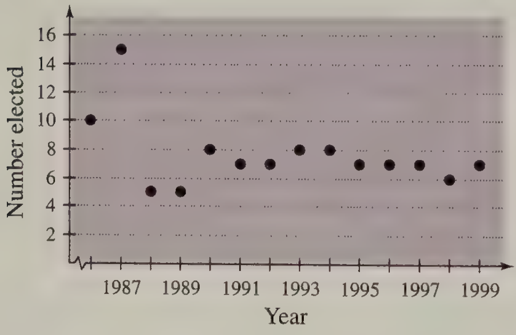
86. ***Make a Conjecture*** Plot the points (2, 1), (−3, 5), and (7, −3) on a rectangular coordinate system. Then change the sign of the indicated coordinate(s) of each point and plot the three new points on the same rectangular coordinate system. Make a conjecture about the location of a point when

(a) the sign of the x-coordinate is changed;

(b) the sign of the y-coordinate is changed;

(c) the sign of both the x- and y-coordinates are changed.

87. ***Rock and Roll Hall of Fame*** The graph below shows the number of recording artists who were elected to the Rock and Roll Hall of Fame from 1986 to 1999.



(a) Describe any trends in the data. From these trends, estimate the number of artists that will be elected in 2001.

(b) Why do you think the numbers elected in 1986 and 1987 were greater than in other years?

88. ***Business*** Starbucks Corporation had annual sales of \$696.5 million in 1996 and \$1308.7 million in 1998. Use the Midpoint Formula to estimate the 1997 sales. (Source: Starbucks Corporation)

89. ***Business*** Lands' End, Inc., had annual sales of \$1118.7 million in 1996 and \$1371.4 million in 1998. Use the Midpoint Formula to estimate the 1997 sales. (Source: Lands' End, Inc.)

Synthesis

True or False? **In Exercises 90–92, determine whether the statement is true or false. Justify your answer.**

90. In order to divide a line segment into 16 equal parts, you would have to use the Midpoint Formula 16 times.

91. The points (−8, 4), (2, 11), and (−5, 1) represent the vertices of an isosceles triangle.

92. If four points represent the vertices of a polygon, and the four sides are equal, then the polygon must be a square.

93. ***Think About It*** What is the y-coordinate of any point on the x-axis? What is the x-coordinate of any point on the y-axis?

94. ***Think About It*** When plotting points on the rectangular coordinate system, is it true that the scales on the x- and y-axes must be the same? Explain.

P Chapter Summary

What did you learn?

Section P.1

	Review Exercises
☐ How to represent and classify real numbers	1, 2
☐ How to order real numbers and use inequalities	3–6
☐ How to find the absolute values of real numbers and the distance between two real numbers	7–12
☐ How to evaluate algebraic expressions	13–16
☐ How to use the basic rules and properties of algebra	17–26

Section P.2

☐ How to use properties of exponents	27–30
☐ How to use scientific notation to represent real numbers	31–34
☐ How to use properties of radicals	35, 36
☐ How to simplify and combine radicals	37–49
☐ How to rationalize denominators and numerators	50–53
☐ How to use properties of rational exponents	54–59

Section P.3

☐ How to write polynomials in standard form	60, 61
☐ How to add, subtract, and multiply polynomials	62–69
☐ How to use special products to multiply polynomials	70–77
☐ How to remove common factors from polynomials	78–85
☐ How to factor special polynomial forms	86–89
☐ How to factor trinomials as the product of two binomials	90–93
☐ How to factor by grouping	94–97

Section P.4

☐ How to find domains of algebraic expressions	98–101
☐ How to simplify rational expressions	102–105
☐ How to add, subtract, multiply, and divide rational expressions	106–113
☐ How to simplify complex fractions	114, 115

Section P.5

☐ How to plot points in the Cartesian plane	116–121
☐ How to represent data graphically using scatter plots, bar graphs, and line graphs	122–124
☐ How to use the Distance Formula to find the distance between two points	125–128
☐ How to use the Midpoint Formula to find the midpoint of a line segment	129–131
☐ How to find the equation of a circle	132, 133

P Review Exercises

P.1 **In Exercises 1 and 2, determine which numbers are (a) natural numbers, (b) integers, (c) rational numbers, and (d) irrational numbers.**

1. $11, -14, -\frac{8}{9}, \frac{5}{2}, \sqrt{6}, 0.4$

2. $\sqrt{15}, -22, -\frac{10}{3}, 0, 5.2, \frac{3}{7}$

In Exercises 3 and 4, use a calculator to find the decimal form of the rational number. If it is a nonterminating decimal, write the repeating pattern. Then place the correct inequality symbol (< or >) between the numbers.

3. $\frac{5}{6}, \frac{7}{8}$

4. $\frac{9}{25}, \frac{5}{7}$

In Exercises 5 and 6, give a verbal description of the real numbers that are represented by the inequality. Then sketch the inequality on the real number line.

5. $x \leq 7$

6. $x > 1$

In Exercises 7 and 8, find the distance between *a* and *b*.

7. $a = -74, \quad b = 48$

8. $a = -123, \quad b = -9$

In Exercises 9–12, use absolute value notation to describe the expression.

9. The distance between x and 7 is at least 4.

10. The distance between x and 25 is no more than 10.

11. The distance between y and -30 is less than 5.

12. The distance between y and $\frac{1}{2}$ is more than 2.

In Exercises 13–16, evaluate the expression for each value of *x*. (If not possible, state the reason.)

Expression	*Values*	
13. $10x - 3$	(a) $x = -1$	(b) $x = 3$
14. $x^2 - 11x + 24$	(a) $x = -2$	(b) $x = 2$
15. $-2x^2 - x + 3$	(a) $x = 3$	(b) $x = -3$
16. $\frac{4x}{x - 1}$	(a) $x = -1$	(b) $x = 1$

In Exercises 17–20, identify the rule of algebra illustrated by the equation.

17. $2x + (3x - 10) = (2x + 3x) - 10$

18. $\frac{2}{y + 4} \cdot \frac{y + 4}{2} = 1, \quad y \neq -4$

19. $(t + 4)(2t) = (2t)(t + 4)$

20. $0 + (a - 5) = a - 5$

In Exercises 21–26, perform the operations. (Write fractional answers in simplest form.)

21. $\frac{2}{3} + \frac{8}{9}$

22. $\frac{3}{4} - \frac{1}{6} + \frac{1}{8}$

23. $\frac{3}{16} \div \frac{9}{2}$

24. $\frac{5}{8} \cdot \frac{2}{3}$

25. $\frac{x}{5} + \frac{7x}{12}$

26. $\frac{9}{x} \div \frac{1}{6}$

P.2 **In Exercises 27–30, simplify each expression.**

27. (a) $(-2z)^3$ (b) $(a^2b^4)(3ab^{-2})$

28. (a) $\frac{(8y)^0}{y^2}$ (b) $\frac{40(b - 3)^5}{75(b - 3)^2}$

29. (a) $\frac{6^2u^3v^{-3}}{12u^{-2}v}$ (b) $\frac{3^{-4}m^{-1}n^{-3}}{9^{-2}mn^{-3}}$

30. (a) $(x + y^{-1})^{-1}$ (b) $\left(\frac{x^{-3}}{y}\right)\left(\frac{x}{y}\right)^{-1}$

In Exercises 31 and 32, write the number in scientific notation.

31. *Sales of K-Mart Corporation in 1998:* \$33,674,000,000 (Source: K-Mart Corporation)

32. *Number of Meters in One Foot:* 0.3048

In Exercises 33 and 34, write the number in decimal form.

33. *Distance between Sun and Jupiter:* 4.833×10^8 miles

34. *Ratio of Day to Year:* 2.74×10^{-3}

In Exercises 35 and 36, use the properties of radicals to simplify the expression.

35. $\left(\sqrt[4]{78}\right)^4$

36. $\sqrt[3]{9} \cdot \sqrt[3]{3}$

In Exercises 37–42, simplify by removing all possible factors from the radical.

37. $\sqrt{4x^4}$

38. $\sqrt[5]{64x^6}$

39. $\sqrt{\frac{81}{144}}$

40. $\sqrt[3]{\frac{125}{216}}$

41. $\sqrt[3]{\dfrac{2x^3}{27}}$ **42.** $\sqrt{\dfrac{75x^2}{y^4}}$

In Exercises 43–48, simplify the expression.

43. $\sqrt{50} - \sqrt{18}$ **44.** $3\sqrt{32} + 4\sqrt{98}$

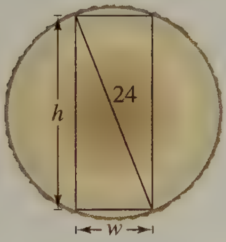
45. $8\sqrt{3x} - 5\sqrt{3x}$ **46.** $-11\sqrt{36y} - 6\sqrt{y}$

47. $\sqrt{8x^3} + \sqrt{2x}$ **48.** $3\sqrt{14x^2} - \sqrt{56x^2}$

49. ***Strength of a Wooden Beam*** The rectangular cross section of a wooden beam cut from a log of diameter 24 inches will have a maximum strength if its width w and height h are

$$w = 8\sqrt{3} \quad \text{and} \quad h = \sqrt{24^2 - \left(8\sqrt{3}\right)^2}.$$

Find the area of the rectangular cross section and express the answer in simplest form.



In Exercises 50 and 51, rewrite the expression by rationalizing the denominator. Simplify your answer.

50. $\dfrac{1}{2 - \sqrt{3}}$ **51.** $\dfrac{1}{\sqrt{x} - 1}$

In Exercises 52 and 53, rewrite the expression by rationalizing the numerator. Simplify your answer.

52. $\dfrac{\sqrt{20}}{4}$ **53.** $\dfrac{\sqrt{2} - \sqrt{11}}{3}$

In Exercises 54 and 55, fill in the missing description.

	Radical Form	*Rational Exponent Form*
54.	$\sqrt{16} = 4$	$= 4$
55.	$= 2$	$16^{1/4} = 2$

In Exercises 56–59, simplify the expression.

56. $81^{3/2}$ **57.** $64^{-2/3}$

58. $\left(-3x^{2/5}\right)\left(-2x^{1/2}\right)$ **59.** $(x-1)^{1/3}(x-1)^{-1/4}$

P.3 **In Exercises 60 and 61, write the polynomial in standard form.**

60. $15x^2 - 2x^5 + 3x^3 + 5 - x^4$

61. $-2x^4 + x^2 - 10 - x + x^3$

In Exercises 62–69, perform the operations and write the result in standard form.

62. $-(3x^2 + 2x) + (1 - 5x)$

63. $8y - [2y^2 - (3y - 8)]$

64. $(2x^3 - 5x^2 + 10x - 7) + (4x^2 - 7x - 2)$

65. $(6x^4 - 4x^3 - x + 3 - 20x^2) - (16 + 9x^4 - 11x^2)$

66. $(x^2 - 2x + 1)(x^3 - 1)$

67. $(x^3 - 3x)(2x^2 + 3x + 5)$

68. $(y^2 - y)(y^2 + 1)(y^2 + y + 1)$

69. $\left(x - \dfrac{1}{x}\right)(x + 2)$

In Exercises 70–75, find the product.

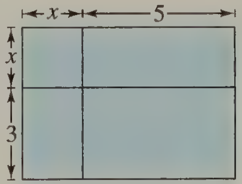
70. $(3x - 8)^2$ **71.** $(7x + 4)(7x - 4)$

72. $(x - 4)^3$ **73.** $(2x - 1)^3$

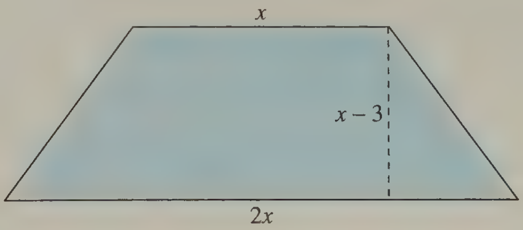
74. $\left(3\sqrt{5} + 2\right)\left(3\sqrt{5} - 2\right)$

75. $(x - y - 6)(x - y + 6)$

76. ***Geometry*** Use the area model to write two different expressions for the total area. Then equate the two expressions and name the algebraic property illustrated.



77. ***Geometry*** Write an expression for the area of the region and simplify the result.



In Exercises 78–83, factor out the common factor.

78. $7x + 35$

79. $10x - 2$

80. $x^3 - x$

81. $x(x - 3) + 4(x - 3)$

82. $2x^3 + 18x^2 - 4x$

83. $-6x^4 - 3x^3 + 12x$

84. ***Geometry*** The surface area of a right circular cylinder is $S = 2\pi r^2 + 2\pi rh$.

(a) Draw a right circular cylinder of radius r and height h. Use the figure to explain how the surface area formula was obtained.

(b) Factor the expression for surface area.

85. ***Business*** The revenue for selling x units of a product at a price of p dollars per unit is $R = xp$. For a particular product the revenue is

$R = 1600x - 0.50x^2$.

Factor the expression and determine an expression that gives the price in terms of x.

In Exercises 86–93, factor the expression.

86. $x^2 - 169$

87. $9x^2 - \frac{1}{25}$

88. $x^3 + 216$

89. $64x^3 - 27$

90. $x^2 - 6x - 27$

91. $x^2 - 9x + 14$

92. $2x^2 + 21x + 10$

93. $3x^2 + 14x + 8$

In Exercises 94–97, factor by grouping.

94. $x^3 - 4x^2 - 3x + 12$

95. $x^3 - 6x^2 - x + 6$

96. $4x^3 - 3x^2 + 20x - 15$

97. $2x^3 + 5x^2 - 14x - 35$

P.4 **In Exercises 98–101, find the domain of the expression.**

98. $-5x^2 - x - 1$

99. $9x^4 + 7, \quad x > 0$

100. $\dfrac{4}{2x - 3}$

101. $\dfrac{\sqrt{x + 12}}{4x}$

In Exercises 102–105, write the rational expression in simplest form.

102. $\dfrac{4x^2}{4x^3 + 28x}$

103. $\dfrac{6xy}{xy + 2x}$

104. $\dfrac{x^2 - x - 30}{x^2 - 25}$

105. $\dfrac{x^2 - 9x + 18}{8x - 48}$

In Exercises 106–113, perform the operation and simplify your answer.

106. $\dfrac{x^2 - 4}{x^4 - 2x^2 - 8} \cdot \dfrac{x^2 + 2}{x^2}$

107. $\dfrac{2x - 1}{x + 1} \cdot \dfrac{x^2 - 1}{2x^2 - 7x + 3}$

108. $\dfrac{x^2(5x - 6)}{2x + 3} \div \dfrac{5x}{2x + 3}$

109. $\dfrac{4x - 6}{(x - 1)^2} \div \dfrac{2x^2 - 3x}{x^2 + 2x - 3}$

110. $x - 1 + \dfrac{1}{x + 2} + \dfrac{1}{x - 1}$

111. $2x + \dfrac{3}{2(x - 4)} - \dfrac{1}{2(x + 2)}$

112. $\dfrac{1}{x} - \dfrac{x - 1}{x^2 + 1}$

113. $\dfrac{1}{x - 1} + \dfrac{1 - x}{x^2 + x + 1}$

In Exercises 114 and 115, simplify the complex fraction.

114. $\dfrac{\left(\dfrac{1}{x} - \dfrac{1}{y}\right)}{(x^2 - y^2)}$

115. $\dfrac{\left(\dfrac{1}{x} - \dfrac{1}{y}\right)}{\left(\dfrac{1}{x} + \dfrac{1}{y}\right)}$

P.5 **In Exercises 116–119, plot the point and determine the quadrant in which it is located.**

116. $(8, -3)$

117. $(-4, -9)$

118. $\left(-\frac{5}{2}, 10\right)$

119. $(-6.5, -0.5)$

In Exercises 120 and 121, determine the quadrant(s) in which (x, y) is located so that the conditions are satisfied.

120. $x > 0$ and $y = -2$

121. $(x, y), \quad xy = 4$

122. ***Patents*** The number of patents P (in thousands) issued in the United States from 1988 through 1996 is shown in the table. (Source: U.S. Patent and Trademark Office)

Year	1988	1989	1990	1991	1992
P	84.4	102.7	99.2	106.8	107.4

Year	1993	1994	1995	1996
P	109.7	113.6	113.8	121.7

(a) Sketch a scatter plot of the data.

(b) What statement can be made about the number of patents issued in the United States?

123. ***Weather*** The normal daily maximum and minimum temperatures for each month for the city of Chicago are shown in the table. Make a double line graph for the data. (Source: NOAA)

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.
Max.	29.0	33.5	45.8	58.6	70.1	79.6
Min.	12.9	17.2	28.5	38.6	47.7	57.5

Month	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
Max.	83.7	81.8	74.8	63.3	48.4	34.0
Min.	62.6	61.6	53.9	42.2	31.6	19.1

124. ***Business*** The net profits (in millions of dollars) for the Progressive Corporation for the years 1994 through 1998 are shown in the table. Create a bar graph for the data. (Source: Progressive Corporation)

Year	1994	1995	1996	1997	1998
Profits	228.1	250.5	316.6	400.0	456.7

In Exercises 125 and 126, plot the points and find the distance between the points.

125. $(-3, 8), (1, 5)$

126. $(5.6, 0), (0, 8.2)$

Geometry **In Exercises 127 and 128, plot the points and verify that the points form the polygon.**

127. *Right Triangle:* $(2, 3), (13, 11), (5, 22)$

128. *Parallelogram:* $(1, 2), (8, 3), (9, 6), (2, 5)$

In Exercises 129 and 130, plot the points and find the midpoint of the line segment joining the points.

129. $(-12, 5), (4, -7)$

130. $(1.8, 7.4), (-0.6, -14.5)$

131. ***Business*** The Sbarro, Inc., restaurant chain had revenues of \$329.5 million in 1996 and \$375.2 million in 1998. (Source: Sbarro, Inc.)

(a) Without any additional information, what would you estimate the 1997 revenues to have been?

(b) The actual revenue for 1997 was \$349.4 million. How accurate is your estimate?

In Exercises 132 and 133, find the standard form of the equation of the specified circle.

132. Center: $(3, -1)$; solution point: $(-5, 1)$

133. End points of a diameter: $(-4, 6), (10, -2)$

Synthesis

True or False? **In Exercises 134–136, determine whether the statement is true or false. Justify your answer.**

134. $\dfrac{x^3 - 1}{x - 1} = x^2 + x + 1$ for all values of x.

135. A binomial sum squared is equal to the sum of the terms squared.

136. $x^n - y^n$ factors as conjugates for all values of n.

In Exercises 137–142, describe the error and then make the necessary correction.

137. $\dfrac{x - 1}{1 - x} = 1$ ✗

138. $-x^2(-x^2 + 3) = x^4 + 3x^2$ ✗

139. $(2x)^4 = 2x^4$ ✗

140. $(-x)^6 = -x^6$ ✗

141. $\sqrt{3^2 + 4^2} = 3 + 4$ ✗

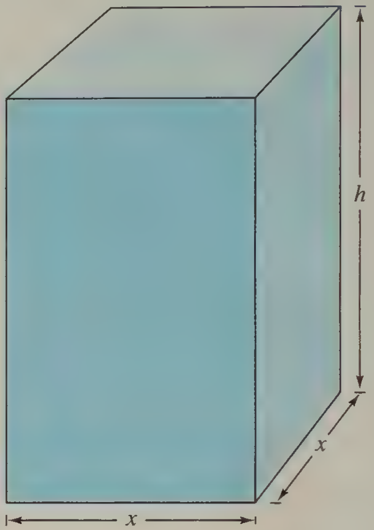
142. $\sqrt{10x} = 10\sqrt{x}$ ✗

143. ***Writing*** Explain why $\sqrt{5u} + \sqrt{3u} \neq 2\sqrt{2u}$.

Chapter Project *Modeling the Volume of a Box*

Many mathematical results are discovered experimentally by calculating examples and looking for patterns. Prior to the 1950s, this mode of discovery was very time-consuming because the calculations had to be done by hand. The introduction of computer and calculator technology has removed much of this drudgery. In the following project, you are asked to model a real-life situation and solve a problem by looking for patterns in the corresponding data.

Consider a rectangular box with a square base and a surface area of 216 square inches. Let x represent the length (in inches) of each side of the base and let h represent the height (in inches) of the box, as shown at the right. Your goal is to answer this question: "Of all rectangular boxes with square bases and surface area of 216 square inches, which has the greatest volume?"



a. Express the areas of the base, top, and sides in terms of x and h.

b. Find an expression in terms of x and h for the surface area of the box.

c. Use the fact that the surface area is 216 square inches to express the variable h in terms of x.

d. Find an expression for the volume of the box in terms of x alone.

e. Use the expression in part (d) and a graphing utility to complete the table. Then use the results to decide which box has the greatest volume.

Base, x	*Height, h*	*Surface Area*	*Volume*
1.0	53.5	216.0	53.5
1.5	35.3	216.0	79.3
2.0	26.0	216.0	104.0
⋮	⋮	⋮	⋮
10.0	0.4	216.0	40.0

A computer simulation to accompany this project appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Questions for Further Exploration

1. What happens to the height of the box as x gets closer and closer to 0? Of all boxes with square bases and a surface area of 216 square inches, is there a tallest? Explain your reasoning.
2. What is the maximum value of x? What happens to the height of the box as x gets closer and closer to this maximum value? Is there a shortest box that has a square base and a surface area of 216 square inches? Explain your reasoning.
3. Complete the table. Does it lend further support to your answer to part (e)? Explain.

x	5.9	5.99	5.999	6.001	6.01	6.1
V						

4. Of all rectangular boxes with surface area of 216 square inches and a base that is x inches by $2x$ inches, which has the maximum volume? Explain your reasoning.

P Chapter Test

Take this test as you would take a test in class. After you are done, check your work against the answers in the back of the book.

The *Interactive* CD-ROM and *Internet* versions of this text provide answers to the Chapter Tests and Cumulative Tests. They also offer Chapter Pre-Tests (that test key skills and concepts covered in previous chapters) and Chapter Post-Tests, both of which have randomly generated exercises with diagnostic capabilities.

1. Place the correct symbol (< or >) between $-\frac{10}{3}$ and $-|-4|$.
2. Find the distance between the real numbers -17 and 39.

In Exercises 3–6, evaluate each quantity without using a calculator.

3. (a) $27\left(-\frac{2}{3}\right)$ (b) $\frac{5}{18} \div \frac{15}{8}$
4. (a) $\left(-\frac{3}{5}\right)^3$ (b) $\left(\frac{3^2}{2}\right)^{-3}$
5. (a) $\sqrt{5} \cdot \sqrt{125}$ (b) $\frac{\sqrt{72}}{\sqrt{2}}$
6. (a) $\frac{5.4 \times 10^8}{3 \times 10^3}$ (b) $(3 \times 10^4)^3$

In Exercises 7 and 8, simplify each expression.

7. (a) $3z^2(2z^3)^2$ (b) $(u-2)^{-4}(u-2)^{-3}$ (c) $\left(\frac{x^{-2}y^2}{3}\right)^{-1}$
8. (a) $9z\sqrt{8z} - 3\sqrt{2z^3}$ (b) $-5\sqrt{16y} + 10\sqrt{y}$ (c) $\sqrt[3]{\frac{16}{v^5}}$

In Exercises 9–12, perform the operations and simplify.

9. $(x^2+3) - [3x + (8 - x^2)]$
10. $(x + \sqrt{5})(x - \sqrt{5})$
11. $\frac{8x}{x-3} + \frac{24}{3-x}$
12. $\left(\frac{2}{x} - \frac{2}{x+1}\right) \div \left(\frac{4}{x^2-1}\right)$

In Exercises 13–15, factor the expression completely.

13. $2x^4 - 3x^3 - 2x^2$
14. $x^3 + 2x^2 - 4x - 8$
15. $8x^3 - 27$
16. Rationalize the denominators of (a) $\frac{16}{\sqrt[3]{16}}$ and (b) $\frac{6}{1-\sqrt{3}}$.
17. Write an expression for the area of the shaded region in the figure at the right and simplify the result.
18. Plot the points $(-2, 5)$ and $(6, 0)$. Find the coordinates of the midpoint of the line segment joining the points and the distance between the points.
19. The area of a rectangle of length l is $45l - l^2$. Factor the expression to determine the width of the rectangle.
20. The numbers (in millions) of votes cast for the Democratic candidates for president in 1980, 1984, 1988, 1992, and 1996 were 35.5, 37.6, 41.8, 44.9, and 47.4, respectively. Create a bar graph for this data. (Source: Congressional Quarterly, Inc.)

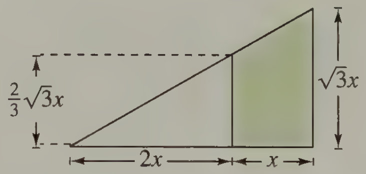


FIGURE FOR 17

LIBRARY OF FUNCTIONS

In Chapter 1, you will be introduced to the concept of a *function.* As you proceed through the text, you will see that functions play a primary role in modeling real-life situations.

Over the past few hundred years, many different types of functions have been introduced and studied. Those that have proven to be most important in modeling real life have come to be known as *elementary functions.* There are three basic types of elementary functions: algebraic functions, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions.

You will also encounter other types of functions in this text, such as functions defined by real-life data and piecewise-defined functions.

Library of Functions

Each time a new type of function is studied in detail in this text, it will be highlighted in a box like this one. For instance, a linear function is highlighted in the Library of Functions box in Section 1.2 because lines are discussed in that section.

In addition, there is a Library of Functions Summary inside the front cover which describes the functions listed below.

Algebraic Functions	
Polynomial Functions	
Linear or First-Degree Polynomial Functions	Section 1.2
Quadratic or Second-Degree Polynomial Functions	Section 3.1
Cubic or Third-Degree Polynomial Functions	Section 3.2
Fourth- and Higher-Degree Polynomial Functions	Section 3.2
Rational Functions	Section 3.5
Radical Functions	Section 1.3
Exponential and Logarithmic Functions	
Exponential Functions	Section 4.1
Logarithmic Functions	Section 4.2
Trigonometric and Inverse Trigonometric Functions	
Trigonometric Functions	Section 5.3
Inverse Trigonometric Functions	Section 5.6
Nonelementary Functions	
Piecewise-Defined Functions	Section 1.3
Greatest Integer Functions	Section 1.4
Absolute Value Functions	Section 1.4
Functions Defined by Real-Life Data	Section 1.3

Functions and Their Graphs

1

1.1 Graphs of Equations
1.2 Lines in the Plane
1.3 Functions
1.4 Graphs of Functions
1.5 Shifting, Reflecting, and Stretching Graphs
1.6 Combinations of Functions
1.7 Inverse Functions

Superstock

One and one-half percent of all households own a horse as a pet. Some owners board their horses on farms. (Source: American Veterinary Medical Association)

The Big Picture

In this chapter you will learn how to

- ☐ sketch graphs of equations by point plotting or using a graphing utility.
- ☐ find and use the slope of a line to write and graph linear equations.
- ☐ evaluate functions and find their domains.
- ☐ analyze graphs of functions.
- ☐ identify and graph shifts, reflections, and nonrigid transformations of functions.
- ☐ find arithmetic combinations and compositions of functions.
- ☐ find inverses of functions graphically and algebraically.

Important Vocabulary

As you encounter each new vocabulary term in this chapter, add the term and its definition to your notebook glossary.

- solution point (p. 68)
- graph of an equation (p. 68)
- intercepts (p. 69)
- slope (p. 79)
- point-slope form (p. 81)
- slope-intercept form (p. 83)
- general form (p. 84)
- parallel (p. 85)
- perpendicular (p. 85)
- function (p. 92)
- domain (p. 92)
- range (p. 92)
- independent variable (p. 94)
- dependent variable (p. 94)
- function notation (p. 94)
- implied domain (p. 96)
- graph of a function (p. 106)
- Vertical Line Test (p. 107)
- increasing (p. 108)
- decreasing (p. 108)
- constant (p. 108)
- relative minimum (p. 109)
- relative maximum (p. 109)
- greatest integer function (p. 111)
- step function (p. 111)
- even function (p. 112)
- odd function (p. 112)
- vertical and horizontal shifts (p. 119)
- reflection (p. 121)
- rigid transformations (p. 123)
- nonrigid transformations (p. 123)
- arithmetic combination (p. 127)
- composition (p. 129)
- inverse function (p. 138)
- one-to-one (p. 142)
- Horizontal Line Test (p. 142)

Additional Resources Text-specific additional resources are available to help you do well in this course. See page xvi for details.

1.1 Graphs of Equations

The Graph of an Equation

News magazines often show graphs comparing the rate of inflation, the federal deficit, wholesale prices, or the unemployment rate to the time of year. Industrial firms and businesses use graphs to report their monthly production and sales statistics. Such graphs provide geometric pictures of the way one quantity changes with respect to another. Frequently, the relationship between two quantities is expressed as an equation. This section introduces the basic procedure for determining the geometric picture associated with an equation.

For an equation in variables x and y, a point (a, b) is a **solution point** if the substitution of $x = a$ and $y = b$ satisfies the equation. Most equations have *infinitely* many solution points. For example, the equation

$$3x + y = 5$$

has solution points $(0, 5)$, $(1, 2)$, $(2, -1)$, $(3, -4)$, and so on. The set of all solution points of an equation is the **graph of the equation.**

What You Should Learn:

- How to sketch graphs of equations by point plotting
- How to sketch graphs of equations using a graphing utility
- How to use graphs of equations in real-life problems

Why You Should Learn It:

The graph of an equation can help you see relationships between real-life quantities. For example, Exercise 75 on page 78 shows how a graph can be used to understand the relationship between life expectancy and the year a child is born.

Bruce Ayres/Tony Stone Images

EXAMPLE 1 Determining Solution Points

Determine whether each point lies on the graph of $y = 10x - 7$.

a. $(2, 13)$ **b.** $(-1, -3)$

Solution

a. The point $(2, 13)$ lies on the graph of $y = 10x - 7$ because it is a solution point of the equation.

$$\begin{aligned} y &= 10x - 7 && \text{Write original equation.}\\ 13 &\overset{?}{=} 10(2) - 7 && \text{Substitute 2 for } x \text{ and 13 for } y.\\ 13 &= 13 && (2, 13) \text{ is a solution. } \checkmark \end{aligned}$$

b. The point $(-1, -3)$ does not lie on the graph of $y = 10x - 7$ because it is not a solution point of the equation.

$$\begin{aligned} y &= 10x - 7 && \text{Write original equation.}\\ -3 &\overset{?}{=} 10(-1) - 7 && \text{Substitute } -1 \text{ for } x \text{ and } -3 \text{ for } y.\\ -3 &\neq -17 && (-1, -3) \text{ is not a solution.} \end{aligned}$$

How to Sketch the Graph of an Equation by Point Plotting

1. If possible, rewrite the equation so that one of the variables is isolated on one side of the equation.
2. Make a table of several solution points.
3. Plot these points in the coordinate plane.
4. Connect the points with a smooth curve.

EXAMPLE 2 Sketching a Graph by Point Plotting

Use point plotting and graph paper to sketch the graph of

$$3x + y = 6.$$

Solution

In this case you can isolate the variable y.

$$y = 6 - 3x \qquad \text{Solve equation for } y.$$

Using negative, zero, and positive values for x, you can obtain the following table of values (solution points).

x	-1	0	1	2	3
$y = 6 - 3x$	9	6	3	0	-3

Next, plot these points and connect them, as shown in Figure 1.1. It appears that the graph is a straight line. You will study lines extensively in Section 1.2.

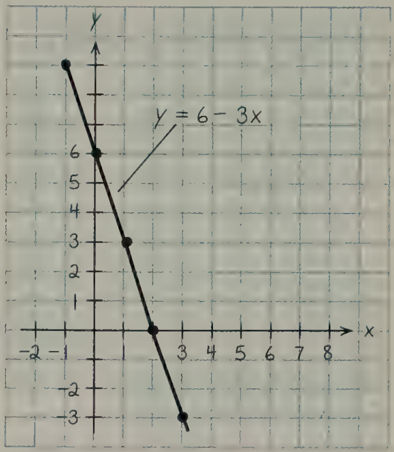


Figure 1.1

The points at which a graph touches or crosses an axis are the **intercepts** of the graph. For instance, in Example 2 the point $(0, 6)$ is the y-intercept of the graph because the graph crosses the y-axis at that point. The point $(2, 0)$ is the x-intercept of the graph because the graph crosses the x-axis at that point.

EXAMPLE 3 Sketching a Graph by Point Plotting

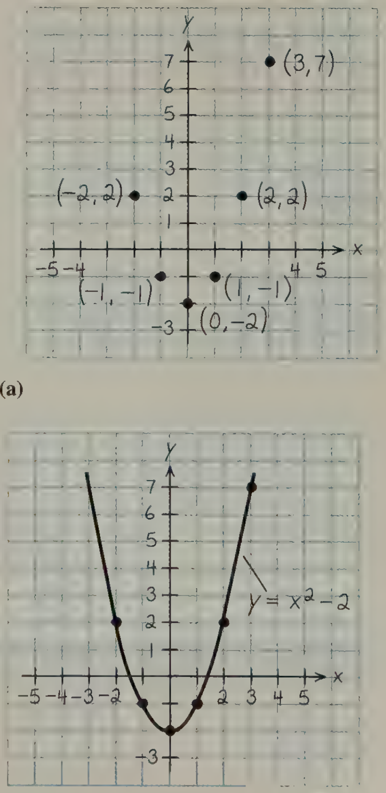
Use point plotting and graph paper to sketch the graph of $y = x^2 - 2$.

Solution

First, make a table of values by choosing several convenient values of x and calculating the corresponding values of y.

x	-2	-1	0	1	2	3
$y = x^2 - 2$	2	-1	-2	-1	2	7

Next, plot the corresponding solution points, as shown in Figure 1.2(a). Finally, connect the points with a smooth curve, as shown in Figure 1.2(b). This graph is called a *parabola*. You will study parabolas in Section 3.1.



(a)

(b)

Figure 1.2

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

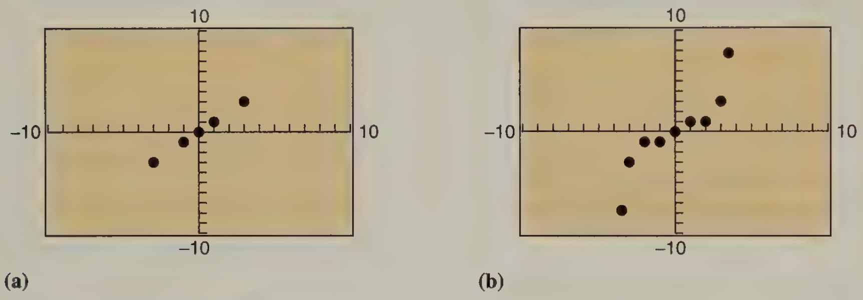
In this text, you will study two basic ways to create graphs: *by hand* and *using a graphing utility*. For instance, the graphs in Figures 1.1 and 1.2 were sketched by hand and the graph in Figure 1.4 was sketched using a graphing utility.

Using a Graphing Utility

One of the disadvantages of the point-plotting method is that to get a good idea about the shape of a graph you need to plot *many* points. With only a few points, you could badly misrepresent the graph. For instance, consider the equation

$$y = \frac{1}{30}x(x^4 - 10x^2 + 39).$$

Suppose you plotted only five points: $(-3, -3)$, $(-1, -1)$, $(0, 0)$, $(1, 1)$, and $(3, 3)$, as shown in Figure 1.3(a). From these five points, you might assume that the graph of the equation is a straight line. That, however, is not correct. By plotting several more points, you can see that the actual graph is not straight at all, as shown in Figure 1.3(b).



(a) (b)

Figure 1.3

STUDY T!P

This section presents a brief overview of how to use a graphing utility to graph an equation. For more extensive coverage on this topic, see "An Introduction to Graphing Utilities" on pages xvii–xxii, and the *Graphing Technology Guide.*

From this, you can see that the point-plotting method leaves you with a dilemma. On the one hand, the method can be very inaccurate if only a few points are plotted. But on the other hand, it is very time-consuming to plot a dozen (or more) points. Technology can help solve this dilemma. Plotting several (even several hundred) points on a rectangular coordinate system is something that a computer or calculator can do easily.

The point-plotting method is the method used by *all* graphing utilities. Each computer or calculator screen is made up of a grid of hundreds or thousands of small areas called *pixels.* Screens that have many pixels per square inch are said to have a higher *resolution* than screens with fewer pixels.

Using a Graphing Utility to Graph an Equation

To graph an equation involving x and y on a graphing utility, use the following procedure.

1. Rewrite the equation so that y is isolated on the left side.
2. Enter the equation into a graphing utility.
3. Determine a *viewing window* that shows all important features of the graph.
4. Graph the equation.

EXAMPLE 4 Using a Graphing Utility to Graph an Equation

Use a graphing utility to graph $2y + x^3 = 4x$.

Solution

To begin, solve the equation for y in terms of x.

$$2y + x^3 = 4x \qquad \text{Write original equation.}$$

$$2y = -x^3 + 4x \qquad \text{Subtract } x^3 \text{ from each side.}$$

$$y = -\frac{1}{2}x^3 + 2x \qquad \text{Divide each side by 2.}$$

Now, by entering this equation into a graphing utility (using a standard viewing window), you can obtain the graph shown in Figure 1.4.

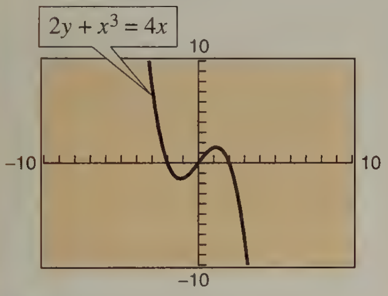


Figure 1.4

By choosing different viewing windows for a graph, it is possible to obtain very different impressions of the graph's shape. For instance, Figure 1.5 shows four different viewing windows for the graph of the equation in Example 4. None of these views shows *all* of the important features of the graph as Figure 1.4 does.

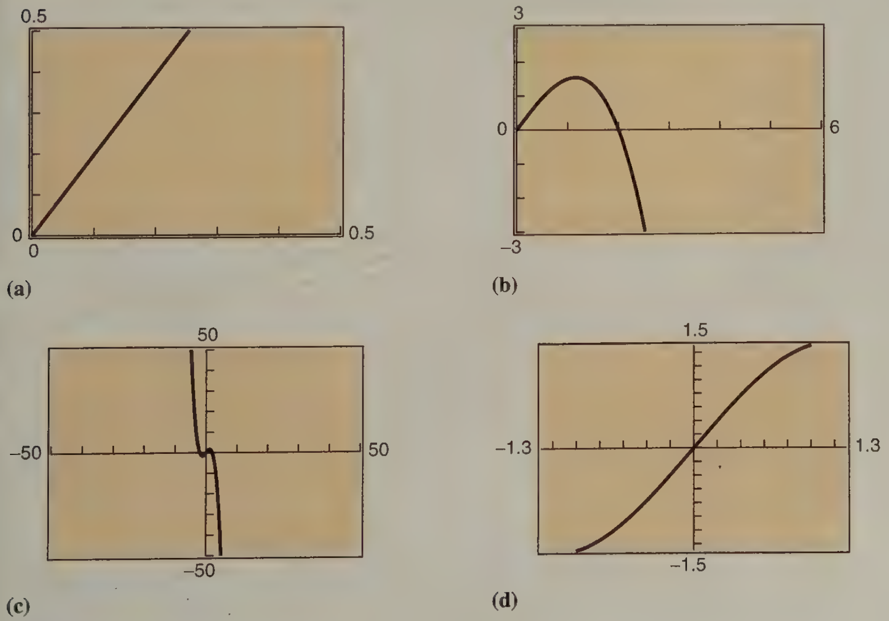
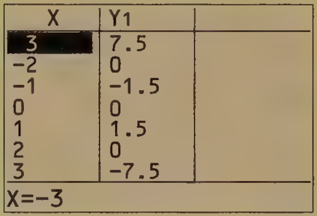


Figure 1.5

STUDY TIP

Many graphing utilities have the capability of creating a table, such as the following table, which shows some points of the graph in Figure 1.4.



```
    X    | Y1    |
  3      | 7.5   |
 -2      | 0     |
 -1      | -1.5  |
 0       | 0     |
 1       | 1.5   |
 2       | 0     |
 3       | -7.5  |
X=-3
```

The standard viewing window on many graphing utilities does not give a true geometric perspective. That is, perpendicular lines will not appear to be perpendicular and circles will not appear to be circular. To overcome this, you can use a *square setting*, as demonstrated in Example 5.

EXAMPLE 5 Sketching a Circle with a Graphing Utility

Use a graphing utility to graph

$$x^2 + y^2 = 9.$$

Solution

The graph of $x^2 + y^2 = 9$ is a circle whose center is the origin and whose radius is 3. (See Section P.5.) To graph the equation, begin by solving the equation for y.

$$x^2 + y^2 = 9 \qquad \text{Write original equation.}$$

$$y^2 = 9 - x^2 \qquad \text{Subtract } x^2 \text{ from each side.}$$

$$y = \pm\sqrt{9 - x^2} \qquad \text{Take square root of each side.}$$

The graph of

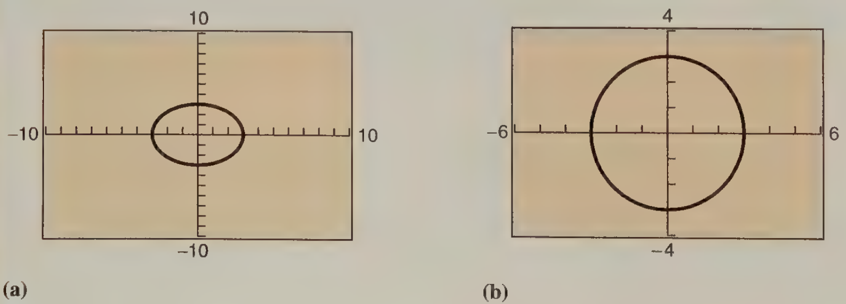
$$y = \sqrt{9 - x^2} \qquad \text{Upper semicircle}$$

is the upper semicircle. The graph of

$$y = -\sqrt{9 - x^2} \qquad \text{Lower semicircle}$$

is the lower semicircle. Enter *both* equations in your graphing utility and generate the resulting graphs. In Figure 1.6(a), note that if you use a standard viewing window, the two graphs do not appear to form a circle. You can overcome this problem by using a *square setting*, in which the horizontal and vertical tick marks have equal spacing, as shown in Figure 1.6(b). On many graphing utilities, a square setting can be obtained by using a y to x ratio of 2 to 3. For instance, in Figure 1.6(b), the y to x ratio is

$$\frac{\text{Ymax} - \text{Ymin}}{\text{Xmax} - \text{Xmin}} = \frac{4 - (-4)}{6 - (-6)} = \frac{8}{12} = \frac{2}{3}.$$



(a) (b)

Figure 1.6

The *Interactive* CD-ROM and *Internet* versions of this text show every example with its solution; clicking on the *Try It!* button brings up similar problems. Guided Examples and Integrated Examples show step-by-step solutions to additional examples. Integrated Examples are related to several concepts in the section.

Throughout this course, you will learn that there are many ways to approach a problem. Two of the three common approaches are illustrated in Example 6.

A Numerical Approach: Construct and use a table.
A Graphical Approach: Draw and use a graph.
An Algebraic Approach: Use the rules of algebra.

We recommend that you habitually use at least two approaches with every problem to help build your intuition and check that your answer is reasonable.

Applications

The following two applications show how to develop mathematical models to represent real-world situations. You will see that both a graphing utility and algebra can be used to understand and solve the problems posed.

STUDY T!P

In applications, it is convenient to use variable names that suggest real-life quantities: d for distance, t for time, and so on. Most graphing utilities, however, require the variable names to be x and y.

EXAMPLE 6 Running a Marathon

A runner runs at a constant rate of 4.9 miles per hour. The verbal model and algebraic equation relating distance run and elapsed time are as follows.

Verbal Model: Distance = Rate · Time *Equation:* $d = 4.9t$

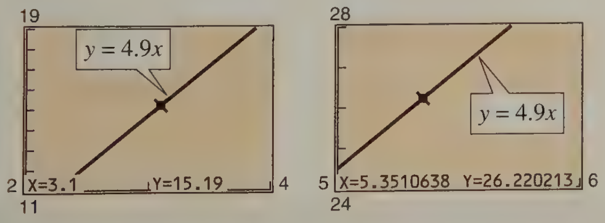
a. Determine how far the runner can run in 3.1 hours.

b. Determine how long it will take to run a 26.2-mile marathon.

Graphical Solution

a. To begin, use a graphing utility to graph the equation $d = 4.9t$. (Represent d by y and t by x.) Be sure to use a viewing window that shows the graph when $x = 3.1$. Then use the *value* feature or *zoom* and *trace* features of the graphing utility to estimate that when $x = 3.1$, the distance is $y \approx 15.2$ miles, as shown in Figure 1.7(a).

b. Adjust the viewing window so that it shows the graph when $y = 26.2$. Use the *value* feature or *zoom* and *trace* features to estimate that when $y = 26.2$, the time is $x \approx 5.4$ hours, as shown in Figure 1.7(b).



(a) (b)

Figure 1.7

Note that the viewing window on your graphing utility may differ slightly from those shown in Figure 1.7.

Algebraic Solution

a. To find how far the runner can run in 3.1 hours, substitute 3.1 for t in the equation.

$$\begin{aligned} d &= 4.9t && \text{Write original equation.} \\ &= 4.9(3.1) && \text{Substitute 3.1 for } t. \\ &\approx 15.2 && \text{Use a calculator.} \end{aligned}$$

So, a runner can run about 15.2 miles in 3.1 hours. Use estimation to check your answer. Because 4.9 is about 5 and 3.1 is about 3, the distance is about $5(3) = 15$. So 15.2 is reasonable.

b. You can find how long it will take to run a 26.2-mile marathon as follows. (For help with solving linear equations see Appendix C.)

$$\begin{aligned} d &= 4.9t && \text{Write original equation.} \\ \frac{d}{4.9} &= t && \text{Divide each side by 4.9.} \\ \frac{26.2}{4.9} &= t && \text{Substitute 26.2 for } d. \\ 5.3 &\approx t && \text{Use a calculator.} \end{aligned}$$

So, it will take about 5.3 hours to run 26.2 miles.

EXAMPLE 7 Monthly Wages

You receive a monthly salary of \$2000 plus a commission of 10% of sales. The verbal model and algebraic equations relating the wages, the salary, and the commission are as follows.

Verbal Model: Wages = Salary + Commission on Sales *Equation:* $y = 2000 + 0.1x$

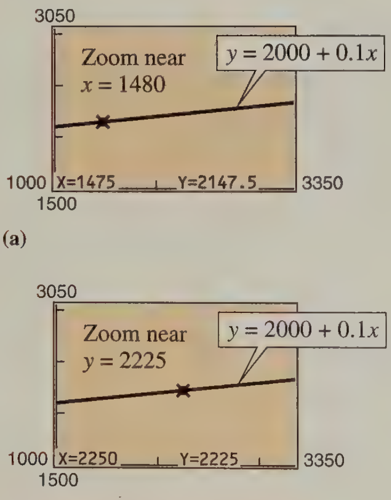
a. If sales are $x = 1480$ in August, what are your wages for that month?

b. If you receive \$2225 for September, what are your sales for that month?

Graphical Solution

a. You can use a graphing utility to graph $y = 2000 + 0.1x$ and then estimate the wages when $x = 1480$. Because $x \geq 0$ and the monthly wages are at least \$2000, a reasonable viewing window is the one shown in Figure 1.8(a). Using the *value* feature or *zoom* and *trace* features near $x = 1480$ shows that the wages are about \$2148.

b. Use the graphing utility to find the value along the x-axis (sales) that corresponds to a y-value of 2225 (wages). Beginning with Figure 1.8(b) and using the *value* feature or *zoom* and *trace* features, you can estimate the sales to be about \$2250.



(a)

(b)

Figure 1.8

Numerical Solution

a. To find the wages in August, evaluate the equation when $x = 1480$.

$$y = 2000 + 0.1x \quad \text{Write original equation.}$$
$$= 2000 + 0.1(1480) \quad \text{Substitute 1480 for } x.$$
$$= 2148 \quad \text{Simplify.}$$

So, the wages in August are \$2148.

b. The table shows the wages for different amounts of sales.

Sales x	2000	2100	2200	2300	2400
Wages y	2200	2210	2220	2230	2240

From the table, you can see that wages of \$2225 result from sales between \$2200 and \$2300. You can improve this estimate by making a table similar to the one below.

Sales x	2210	2220	2230	2240	2250
Wages y	2221	2222	2223	2224	2225

Sales x	2260	2270	2280	2290
Wages y	2226	2227	2228	2229

From the table, you can see that wages of \$2225 result from sales of \$2250.

Writing About Math *Comparison of Wages*

Your employer offers you a choice of wage scales: a monthly salary of \$3000 plus commission of 7% of sales or a salary of \$3400 plus a 5% commission. Write a short paragraph discussing how you would choose your option. At what sales level would the options yield the same salary? Be sure to take advantage of your graphing utility to make your decision.

1.1 Exercises

In Exercises 1–8, determine whether the points lie on the graph of the equation.

	Equation	*Points*	
1.	$y = \sqrt{x + 4}$	(a) $(0, 2)$	(b) $(5, 3)$
2.	$y = x^2 - 3x + 2$	(a) $(2, 0)$	(b) $(-2, 8)$
3.	$y = 4 - \lvert x - 2\rvert$	(a) $(1, 5)$	(b) $(1.2, 3.2)$
4.	$y = \dfrac{1}{x^2 + 1}$	(a) $(0, 0)$	(b) $(3, 0.1)$
5.	$2x - y - 3 = 0$	(a) $(1, 2)$	(b) $(1, -1)$
6.	$x^2 + y^2 = 20$	(a) $(3, -2)$	(b) $(-4, 2)$
7.	$x^2y - x^2 + 4y = 0$	(a) $\left(1, \frac{1}{5}\right)$	(b) $\left(2, \frac{1}{2}\right)$
8.	$y = \frac{1}{3}x^3 - 2x^2$	(a) $\left(2, -\frac{16}{3}\right)$	(b) $(-3, 9)$

In Exercises 9–14, complete the table. Use the resulting solution points to sketch the graph of the equation. Use a graphing utility to verify the graph.

9. $y = -2x + 3$

x	-1	0	1	$\frac{3}{2}$	2
y					

10. $y = \frac{3}{2}x - 1$

x	-2	0	$\frac{2}{3}$	1	2
y					

11. $y = x^2 - 2x$

x	-1	0	1	2	3
y					

12. $y = 4 - x^2$

x	-2	-1	0	1	2
y					

13. $y = 3 - \lvert x - 2\rvert$

x	0	1	2	3	4
y					

14. $y = \sqrt{x - 1}$

x	1	2	5	10	17
y					

15. ***Exploration***

(a) Complete the table for the equation $y = \frac{1}{4}x - 3$.

x	-2	-1	0	1	2
y					

(b) Use the resulting solution points to sketch its graph. Then use a graphing utility to verify the graph.

(c) Repeat parts (a) and (b) for the equation $y = -\frac{1}{4}x - 3$. Use the result to describe any differences between the graphs.

16. ***Exploration***

(a) Complete the table for the equation $y = \dfrac{6x}{x^{-2} + 1}$.

x	-2	-1	0	1	2
y					

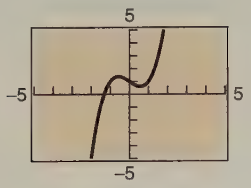
(b) Use the resulting solution points to sketch its graph. Then use a graphing utility to verify the graph.

(c) Continue the table in part (a) for x-values of 5, 10, 20, and 40. What is the value of y approaching? Can y be negative for positive values of x? Explain.

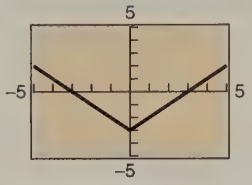
The *Interactive* CD-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered Section and Review Exercises. They also provide Tutorial Exercises, which link to Guided Examples for additional help.

In Exercises 17–22, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

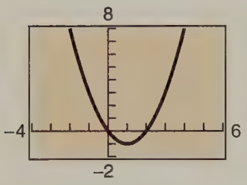
(a)



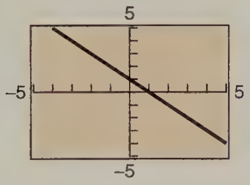
(b)



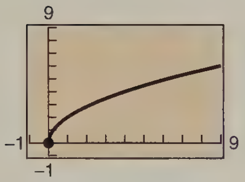
(c)



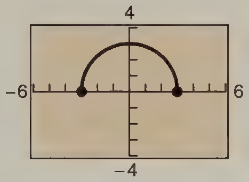
(d)



(e)



(f)



17. $y = 1 - x$

18. $y = x^2 - 2x$

19. $y = \sqrt{9 - x^2}$

20. $y = 2\sqrt{x}$

21. $y = x^3 - x + 1$

22. $y = |x| - 3$

In Exercises 23–36, sketch the graph of the equation.

23. $y = -3x + 2$

24. $y = 2x - 3$

25. $y = 1 - x^2$

26. $y = x^2 - 1$

27. $y = x^2 - 3x$

28. $y = -x^2 - 4x$

29. $y = x^3 + 2$

30. $y = x^3 - 1$

31. $y = \sqrt{x - 3}$

32. $y = \sqrt{1 - x}$

33. $y = |x - 2|$

34. $y = 4 - |x|$

35. $x = y^2 - 1$

36. $x = y^2 - 4$

In Exercises 37–50, use a graphing utility to graph the equation. Use a standard viewing window. Approximate any *x*- or *y*-intercepts of the graph.

37. $y = x - 5$

38. $y = (x + 1)(x - 3)$

39. $y = 3 - \frac{1}{2}x$

40. $y = \frac{2}{3}x - 1$

41. $y = x^2 - 4x + 3$

42. $y = \frac{1}{2}(x + 4)(x - 2)$

43. $y = x(x - 2)^2$

44. $y = \dfrac{4}{x^2 + 1}$

45. $y = \dfrac{2x}{x - 1}$

46. $y = \dfrac{4}{x}$

47. $y = x\sqrt{x + 6}$

48. $y = (6 - x)\sqrt{x}$

49. $y = \sqrt[3]{x}$

50. $y = \sqrt[3]{x + 1}$

In Exercises 51–54, use a graphing utility to sketch the graph of the equation. Begin by using a standard viewing window. Then graph the equation a second time using the specified viewing window. Which viewing window is better? Explain.

51. $y = \frac{5}{2}x + 5$

Xmin = 0
Xmax = 6
Xscl = 1
Ymin = 0
Ymax = 10
Yscl = 1

52. $y = -3x + 50$

Xmin = -1
Xmax = 4
Xscl = 1
Ymin = -5
Ymax = 60
Yscl = 5

53. $y = -x^2 + 10x - 5$

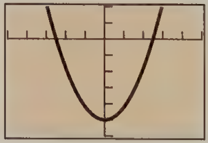
Xmin = -1
Xmax = 11
Xscl = 1
Ymin = -5
Ymax = 25
Yscl = 2

54. $y = 4(x + 5)\sqrt{4 - x}$

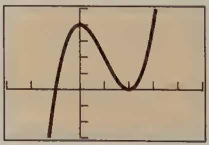
Xmin = -6
Xmax = 6
Xscl = 1
Ymin = -5
Ymax = 50
Yscl = 4

In Exercises 55–58, describe the viewing window of the graph.

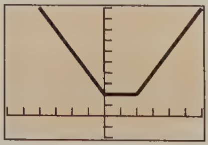
55. $y = 4x^2 - 25$



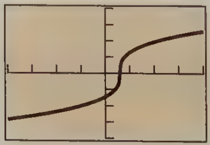
56. $y = x^3 - 3x^2 + 4$



57. $y = |x| + |x - 10|$



58. $y = 8\sqrt[3]{x - 6}$



In Exercises 59–62, solve for y and use a graphing utility to graph each of the resulting equations in the same viewing window. Adjust the viewing window so that a circle really does appear circular.

59. $x^2 + y^2 = 64$

60. $(x - 1)^2 + (y - 2)^2 = 16$

61. $x^2 + y^2 = 49$

62. $(x - 3)^2 + (y - 1)^2 = 25$

In Exercises 63–66, explain how to use a graphing utility to verify that $y_1 = y_2$. Identify the rule of algebra that is illustrated.

63. $y_1 = \frac{1}{4}(x^2 - 8)$
$y_2 = \frac{1}{4}x^2 - 2$

64. $y_1 = \frac{1}{2}x + (x + 1)$
$y_2 = \frac{3}{2}x + 1$

65. $y_1 = \frac{1}{5}[10(x^2 - 1)]$
$y_2 = 2(x^2 - 1)$

66. $y_1 = (x - 3) \cdot \frac{1}{x - 3}$
$y_2 = 1$

In Exercises 67–70, use a graphing utility to graph the equation. Move the cursor along the curve to approximate the unknown coordinate of each solution point accurate to two decimal places. (*Hint:* You may need to use the *zoom* feature of the graphing utility to obtain the required accuracy.)

67. $y = \sqrt{5 - x}$
(a) $(2, y)$
(b) $(x, 3)$

68. $y = x^3(x - 3)$
(a) $(2.25, y)$
(b) $(x, 20)$

69. $y = x^5 - 5x$
(a) $(-0.5, y)$
(b) $(x, -4)$

70. $y = |x^2 - 6x + 5|$
(a) $(2, y)$
(b) $(x, 1.5)$

71. ***Business*** A manufacturing plant purchases a new molding machine for $225,000. The depreciated value y after t years is

$$y = 225{,}000 - 20{,}000t, \qquad 0 \le t \le 8.$$

(a) Use the constraints of the model to determine an appropriate viewing window.

(b) Use a graphing utility to graph the equation.

(c) Use the *value* feature or *zoom* and *trace* features of your graphing utility to determine the value of y when $t = 5.8$. Verify your answer algebraically.

(d) Use the *value* feature or *zoom* and *trace* features of your graphing utility to determine the value of y when $t = 2.35$. Verify your answer algebraically.

72. ***Consumerism*** You purchase a personal watercraft for $8100. The depreciated value y after t years is

$$y = 8100 - 929t, \qquad 0 \le t \le 6.$$

(a) Use the constraints of the model to determine an appropriate viewing window.

(b) Use a graphing utility to graph the equation.

(c) Use the *value* feature or *zoom* and *trace* features of your graphing utility to determine the value of t when $y = 5545.25$. Verify your answer algebraically.

(d) Use the *value* feature or *zoom* and *trace* features of your graphing utility to determine the value of y when $t = 5.5$. Verify your answer algebraically.

73. ***Geometry*** A rectangle of length x and width w has a perimeter of 12 meters.

(a) Draw a diagram to represent the rectangle. Use the specified variables to label its sides.

(b) Show that $w = 6 - x$ is the width of the rectangle and that $A = x(6 - x)$ is its area.

(c) Use a graphing utility to graph the area equation.

(d) Use the *zoom* and *trace* features of your graphing utility to determine the value of A when $w = 4.9$ meters. Verify your answer algebraically.

(e) From the graph in part (c), estimate the dimensions of the rectangle that yield a maximum area.

74. ***Federal Debt*** The table shows the per capita U.S. federal debt for several years. (Sources: U.S. Treasury Department; U.S. Bureau of the Census)

Year	1950	1960	1970	1980
Per Capita Debt	\$1688	\$1572	\$1807	\$3981

Year	1990	1994	1997	1998
Per Capita Debt	\$12,848	\$15,750	\$20,063	\$20,513

A model for the per capita debt during this period is

$$y = 0.223t^3 - 0.733t^2 - 78.255t + 1837.433$$

where y represents the per capita debt and t is the time in years, with $t = 0$ corresponding to 1950.

(a) Use the *value* feature or *zoom* and *trace* features of your graphing utility to find the per capita federal debt in 1975 and 1992. Verify your answers algebraically.

(b) Use your graphing utility to determine during which year the per capita federal debt exceeded $10,200.

(c) Use the model to estimate the per capita federal debt in 2002 and 2004.

75. ***Population Statistics*** The table gives the life expectancy of a child (at birth) in the United States for selected years from 1920 to 2000. (Source: U.S. National Center for Health Statistics)

Year	1920	1930	1940	1950
Life Expectancy	54.1	59.7	62.9	68.2

Year	1960	1970	1980	1990	2000
Life Expectancy	69.7	70.8	73.7	75.4	76.4

A model for the life expectancy during this period is

$$y = \frac{66.93 + t}{1 + 0.01t}$$

where y represents the life expectancy and t is the time in years, with $t = 0$ corresponding to 1950.

(a) What does the y-intercept of the graph of the model represent?

(b) Use your graphing utility to determine the year when the life expectancy was 73.2. Verify your answer algebraically.

(c) Determine the life expectancy in 1948 both graphically and algebraically.

(d) Use the model to estimate the life expectancy of a child born in 2005.

76. ***Finance*** The dividends declared per share of Procter & Gamble Company from 1992 to 1998 can be approximated by the model

$$y = 0.464 + 0.091t, \qquad 0 \le t \le 6$$

where y is the dividend (in dollars) and t is the time (in years), with $t = 0$ corresponding to 1992. (Source: Procter & Gamble Company)

(a) Create a table showing the dividends y for the years 1992 through 1998. From your table, determine the year during which the dividend was \$0.65.

(b) Use your graphing utility to graph the model.

(c) Verify your answer to part (a) by using the *zoom* and *trace* features of your graphing utility to determine the year when $y = 0.65$.

(d) Determine the value of y in 1997 algebraically.

(e) Use the model to estimate the value of y in 2002.

77. ***Copper Wire*** The resistance y in ohms of 1000 feet of solid copper wire at 77°F can be approximated by the mathematical model

$$y = \frac{10{,}770}{x^2} - 0.37, \qquad 5 \le x \le 100$$

where x is the diameter of the wire in mils (0.001 in.). (Source: American Wire Gage)

(a) Complete the table.

x	10	20	30	40	50	60	70	80	90	100
y										

(b) Use your table to approximate the value of x when the resistance is 4.8 ohms. Then determine the answer algebraically.

(c) Use the *value* feature or *zoom* and *trace* features of your graphing utility to determine the resistance when $x = 85.5$.

(d) What can you conclude in general about the relationship between the diameter of the copper wire and the resistance?

Synthesis

True or False? **In Exercises 78 and 79, determine whether the statement is true or false. Justify your answer.**

78. A parabola can have only one x-intercept.

79. The graph of a linear equation can have either no x-intercepts or only one x-intercept.

80. ***Think About It*** Find a and b if the x-intercepts of the graph of $y = (x - a)(x - b)$ are $(-2, 0)$ and $(5, 0)$.

81. ***Writing*** Explain how to find an appropriate viewing window for the graph of an equation.

Review

In Exercises 82–85, perform the operations and simplify.

82. $7\sqrt{72} - 5\sqrt{18}$

83. $-10\sqrt{25y} - \sqrt{y}$

84. $7^{3/2} \cdot 7^{11/2}$

85. $\dfrac{10^{17/4}}{10^{5/4}}$

In Exercises 86–89, perform the operations and write the result in standard form.

86. $(9x - 4) + (2x^2 - x + 15)$

87. $4x(11 - x + 3x^2)$

88. $(2x + 9)(x - 7)$

89. $(3x^2 - 5)(-x^2 + 1)$

1.2 Lines in the Plane

The Slope of a Line

In this section, you will study lines and their equations. The **slope** of a nonvertical line represents the number of units a line rises or falls vertically for each unit of horizontal change from left to right. For instance, consider the two points (x_1, y_1) and (x_2, y_2) on the line shown in Figure 1.9. As you move from left to right along this line, a change of $(y_2 - y_1)$ units in the vertical direction corresponds to a change of $(x_2 - x_1)$ units in the horizontal direction. That is,

$$y_2 - y_1 = \text{the change in } y$$

and

$$x_2 - x_1 = \text{the change in } x.$$

The slope of the line is given by the ratio of these two changes.

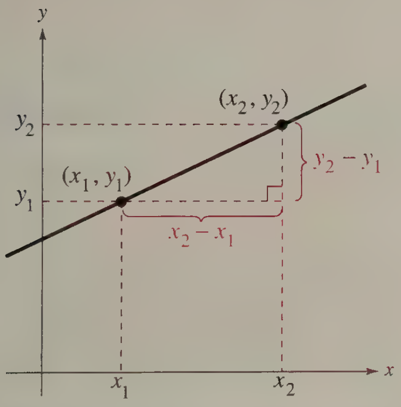


Figure 1.9

Definition of the Slope of a Line

The **slope** m of the nonvertical line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$$

where $x_1 \neq x_2$.

When this formula is used, the *order of subtraction* is important. Given two points on a line, you are free to label either one of them as (x_1, y_1) and the other as (x_2, y_2). However, once this has been done, you must form the numerator and denominator using the same order of subtraction.

$$\underbrace{m = \frac{y_2 - y_1}{x_2 - x_1}}_{\text{Correct}} \qquad \underbrace{m = \frac{y_1 - y_2}{x_1 - x_2}}_{\text{Correct}} \qquad \underbrace{\xcancel{m = \frac{y_2 - y_1}{x_1 - x_2}}}_{\text{Incorrect}}$$

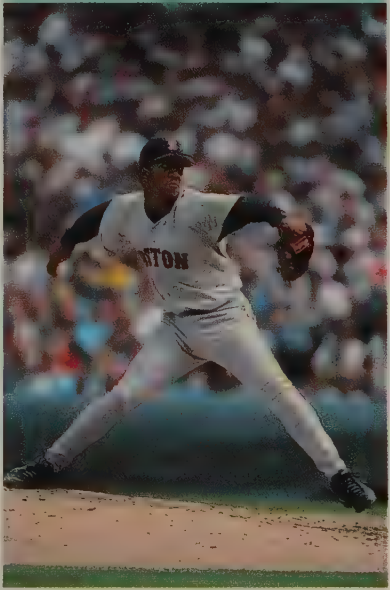
Throughout this text, the term *line* always means a *straight* line.

What You Should Learn:

- How to find the slopes of lines
- How to write linear equations given points on lines and their slopes
- How to use slope-intercept forms of linear equations to sketch graphs of lines
- How to use slope to identify parallel and perpendicular lines

Why You Should Learn It:

Linear equations can be used to model and solve real-life problems. For instance, Exercise 95 on page 91 shows how to use a linear equation to model the average annual salaries of Major League Baseball players from 1988 to 1998.



Allsport

EXAMPLE 1 Finding the Slope of a Line

Find the slope of the line passing through each pair of points.

a. $(-2, 0)$ and $(3, 1)$ **b.** $(-1, 2)$ and $(2, 2)$ **c.** $(0, 4)$ and $(1, -1)$

Solution

a. $$m = \frac{\overbrace{y_2 - y_1}^{\text{Difference in } y\text{-values}}}{\underbrace{x_2 - x_1}_{\text{Difference in } x\text{-values}}} = \frac{1 - 0}{3 - (-2)} = \frac{1}{3 + 2} = \frac{1}{5}$$

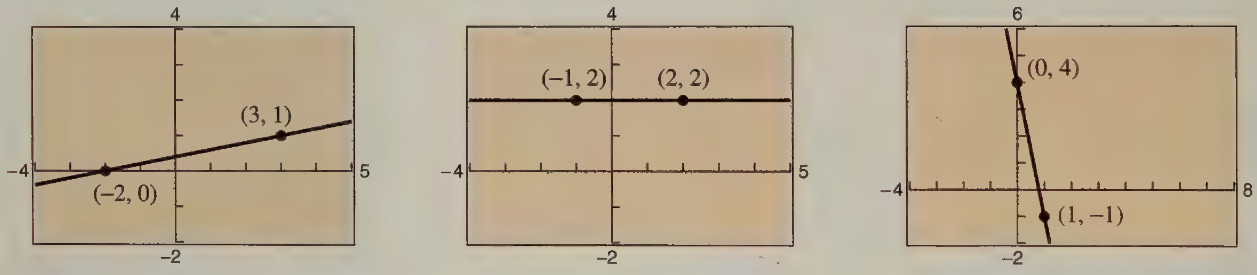
b. $$m = \frac{2 - 2}{2 - (-1)} = \frac{0}{3} = 0$$

c. $$m = \frac{-1 - 4}{1 - 0} = \frac{-5}{1} = -5$$

The graphs of the three lines are shown in Figure 1.10. Note that the square setting gives the correct "steepness" of the lines.

Exploration

Use a graphing utility to compare the slopes of the lines $y = 0.5x$, $y = x$, $y = 2x$, and $y = 4x$. What do you observe about these lines? Compare the slopes of the lines $y = -0.5x$, $y = -x$, $y = -2x$, and $y = -4x$. What do you observe about these lines? (*Hint:* Use a square setting to guarantee a true geometric perspective.)



(a) (b) (c)

Figure 1.10

The definition of slope does not apply to vertical lines. For instance, consider the points $(3, 4)$ and $(3, 1)$ on the vertical line shown in Figure 1.11. Applying the formula for slope, you obtain

$$m = \frac{4 - 1}{3 - 3} = \frac{\cancel{3}}{\cancel{0}}. \qquad \text{Undefined}$$

Because division by zero is undefined, the slope of a vertical line is undefined.

From the slopes of the lines shown in Figures 1.10 and 1.11, you can make the following generalizations about the slope of a line.

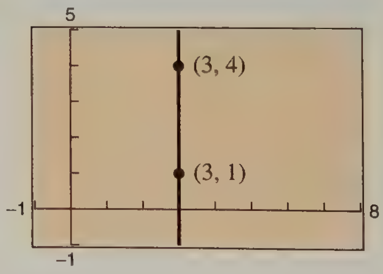


Figure 1.11

The Slope of a Line

1. A line with positive slope $(m > 0)$ *rises* from left to right.
2. A line with negative slope $(m < 0)$ *falls* from left to right.
3. A line with zero slope $(m = 0)$ is *horizontal.*
4. A line with undefined slope is *vertical.*

The Point-Slope Form of the Equation of a Line

If you know the slope of a line *and* you also know the coordinates of one point on the line, you can find an equation for the line. For instance, in Figure 1.12, let (x_1, y_1) be a given point on the line whose slope is m. If (x, y) is any *other* point on the line, it follows that

$$\frac{y - y_1}{x - x_1} = m.$$

This equation in the variables x and y can be rewritten in the **point-slope form** of the equation of a line.

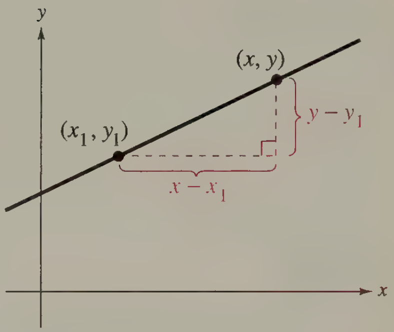


Figure 1.12

Point-Slope Form of the Equation of a Line

The **point-slope form** of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is

$$y - y_1 = m(x - x_1).$$

EXAMPLE 2 The Point-Slope Form of the Equation of a Line

Find an equation of the line that passes through the point $(1, -2)$ and has a slope of 3.

Solution

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-2) = 3(x - 1)$	Substitute for y_1, m, and x_1.
$y + 2 = 3x - 3$	
$y = 3x - 5$	

This line is shown in Figure 1.13.

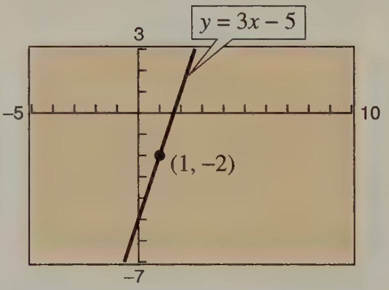


Figure 1.13

The point-slope form can be used to find an equation of a nonvertical line passing through two points (x_1, y_1) and (x_2, y_2). First, use the formula for the slope of the line passing through two points.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Then, once you know the slope, use the point-slope form to obtain the equation

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ &= \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \end{aligned}$$

This is sometimes called the **two-point form** of the equation of a line.

EXAMPLE 3 A Linear Model for Sales Prediction

During 1997, Barnes & Noble's net sales were \$2.8 billion, and in 1998 net sales were \$3.0 billion. (Source: Barnes & Noble, Inc.)

a. Write a linear equation giving the net sales y in terms of the year x.

b. Use the equation to estimate the net sales during 2000.

Solution

a. Let $x = 7$ represent 1997. In Figure 1.14, let $(7, 2.8)$ and $(8, 3.0)$ be two points on the line representing the net sales. The slope of the line passing through these two points is

$$m = \frac{3.0 - 2.8}{8 - 7} = 0.2. \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

By the point-slope form, the equation of the line is as follows.

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$

$$y - 2.8 = 0.2(x - 7) \qquad \text{Substitute for } y_1, m, \text{ and } x_1.$$

$$y = 0.2x - 1.4 + 2.8$$

$$y = 0.2x + 1.4 \qquad \text{Simplify.}$$

b. Using the equation from part (a), estimate the 2000 net sales ($x = 10$) to be

$$y = 0.2(10) + 1.4$$

$$= 2 + 1.4$$

$$= \$3.4 \text{ billion.}$$

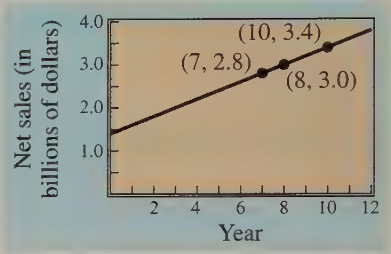
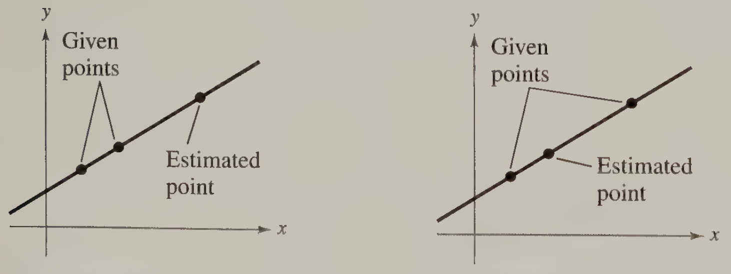


Figure 1.14

The approximation method illustrated in Example 3 is **linear extrapolation.** Note in Figure 1.15 that for linear extrapolation, the estimated point lies outside of the given points. When the estimated point lies *between* two given points, the procedure is called **linear interpolation.**



Linear Extrapolation

Linear Interpolation

Figure 1.15

Library of Functions

In the next section, you will be introduced to the precise meaning of the term *function.* The simplest type of function is a linear function and has the form

$$y = mx + b.$$

As its name implies, the graph of a linear function is a line that has a slope of m and a y-intercept at $(0, b)$.

Consult the Library of Functions Summary inside the front cover for a description of the linear function.

Sketching Graphs of Lines

Many problems in coordinate geometry can be classified as follows.

1. Given a graph (or parts of it), find its equation.
2. Given an equation, sketch its graph.

For lines, the first problem is solved easily by using the point-slope form. This formula, however, is not particularly useful for solving the second type of problem. The form that is better suited to graphing linear equations is the **slope-intercept form,** $y = mx + b$, of the equation of a line.

Slope-Intercept Form of the Equation of a Line

The graph of the equation

$$y = mx + b$$

is a line whose slope is m and whose y-intercept is $(0, b)$.

A computer simulation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 4 Using the Slope-Intercept Form

Determine the slope and y-intercept of each linear equation. Then describe its graph.

a. $x + y = 2$ **b.** $y = 2$

Algebraic Solution

a. Begin by writing the equation in slope-intercept form.

$x + y = 2$ Write original equation.

$y = 2 - x$ Subtract x from each side.

$y = -x + 2$ Slope-intercept form

From the slope-intercept form of the equation, the slope is -1 and the y-intercept is $(0, 2)$. Because the slope is negative, you know that the graph of the equation is a line that falls one unit for every unit it moves to the right.

b. By writing the equation $y = 2$ in slope-intercept form

$$y = (0)x + 2$$

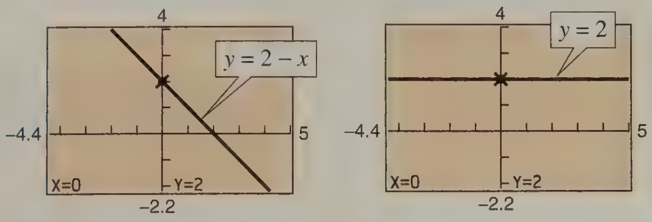
you can see that the slope is 0 and the y-intercept is $(0, 2)$. A zero slope implies that the line is horizontal.

Graphical Solution

a. Solve the equation for y to obtain $y = 2 - x$. Enter this equation in your graphing utility. Use a decimal viewing window to graph the equation as shown in Figure 1.16(a).

To find the y-intercept, use the *value* or *trace* feature. When $x = 0$, $y = 2$. So, the y-intercept is $(0, 2)$. To find the slope, continue to use the *trace* feature. Move the cursor along the line until $x = 1$. At this point, $y = 1$. So the graph falls 1 unit for every unit it moves to the right, and the slope is -1.

b. Enter the equation $y = 2$ in your graphing utility and graph the equation as shown in Figure 1.16(b). Use the *trace* feature to verify the y-intercept $(0, 2)$ and to see that the value of y is the same for all values of x. So, the slope of the horizontal line is 0.



(a) **(b)**

Figure 1.16

From the slope-intercept form of the equation of a line, you can see that a horizontal line ($m = 0$) has an equation of the form $y = b$. This is consistent with the fact that each point on a horizontal line through $(0, b)$ has a y-coordinate of b.

Similarly, each point on a vertical line through $(a, 0)$ has an x-coordinate of a. So, a vertical line has an equation of the form $x = a$. This equation cannot be written in the slope-intercept form, because the slope of a vertical line is undefined. However, *every* line has an equation that can be written in the **general form**

$$Ax + By + C = 0 \qquad \text{General form of the equation of a line}$$

where A and B are not *both* zero.

Exploration

Graph the lines $y = 2x + 1$, $y = \frac{1}{2}x + 1$, and $y = -2x + 1$ in the same viewing window. What do you observe?

Graph the lines $y = 2x + 1$, $y = 2x$, and $y = 2x - 1$ in the same viewing window. What do

Summary of Equations of Lines

1. General form:	$Ax + By + C = 0$
2. Vertical line:	$x = a$
3. Horizontal line:	$y = b$
4. Slope-intercept form:	$y = mx + b$
5. Point-slope form:	$y - y_1 = m(x - x_1)$

EXAMPLE 5 Different Viewing Windows

The graphs of the two lines

$$y = -x - 1 \qquad \text{and} \qquad y = -10x - 1$$

are shown in Figure 1.17. Even though the slopes of these lines are quite different (-1 and -10, respectively), the graphs seem misleadingly similar because the viewing windows are different.

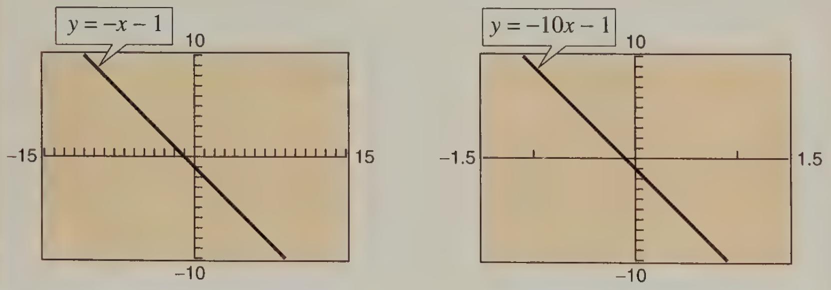


Figure 1.17

When a graphing utility is used to sketch a straight line, it is important to realize that the graph of the line may not visually appear to have the slope indicated by its equation. This occurs because of the viewing window used for the graph. For instance, Figure 1.18 shows graphs of $y = 2x + 1$ produced on a graphing utility using three different viewing windows.

Notice that the slopes in Figure 1.18(a) and (b) do not visually appear to be equal to 2. However, if you use the *square* viewing window, as in Figure 1.18(c), the slope visually appears to be 2. In general, two graphs of the same equation can appear to be quite different depending on the viewing window selected.

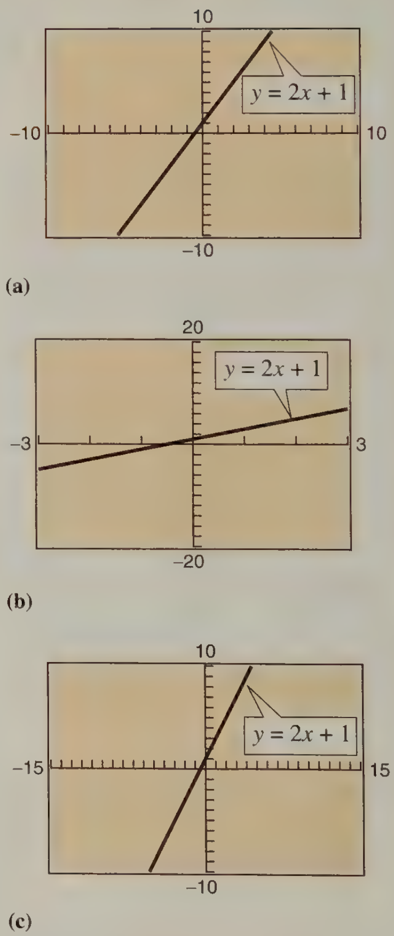


Figure 1.18

Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular.

Parallel Lines

Two distinct nonvertical lines are **parallel** if and only if their slopes are equal.

EXAMPLE 6 Equations of Parallel Lines

Find the slope-intercept form of the equation of the line that passes through the point $(2, -1)$ and is parallel to the line $2x - 3y = 5$.

Solution

Begin by finding the slope of the given line.

$$
\begin{aligned}
2x - 3y &= 5 && \text{Write original equation.}\\
-2x + 3y &= -5 && \text{Multiply by } -1.\\
3y &= 2x - 5 && \text{Add } 2x \text{ to each side.}\\
y &= \frac{2}{3}x - \frac{5}{3} && \text{Write in slope-intercept form.}
\end{aligned}
$$

Therefore, the given line has a slope of $m = \frac{2}{3}$. Because any line parallel to the given line must also have a slope of $\frac{2}{3}$, the required line through $(2, -1)$ has the following equation.

$$
\begin{aligned}
y - (-1) &= \frac{2}{3}(x - 2) && \text{Substitute for } m, x_1, \text{ and } y_1 \text{ in point-slope form.}\\
y + 1 &= \frac{2}{3}x - \frac{4}{3} && \text{Simplify.}\\
y &= \frac{2}{3}x - \frac{4}{3} - 1 && \text{Subtract 1 from each side.}\\
y &= \frac{2}{3}x - \frac{7}{3} && \text{Write in slope-intercept form.}
\end{aligned}
$$

Notice the similarity between the slope-intercept form of the original equation and the slope-intercept form of the parallel equation. The graphs of both equations are shown in Figure 1.19.

STUDY TIP

Be careful when you graph equations such as $y = \frac{2}{3}x - \frac{7}{3}$ on your graphing utility. A common mistake is to type it in as

Y1 = 2/3X − 7/3,

which may not be interpreted as the original equation by your graphing utility. You should use one of the following formulas.

Y1 = 2X/3 − 7/3

Y1 = (2/3)X − 7/3

Do you see why?

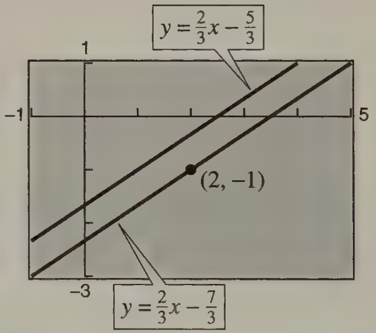


Figure 1.19

Perpendicular Lines

Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other. That is,

$$m_1 = -\frac{1}{m_2}.$$

EXAMPLE 7 Equations of Perpendicular Lines

Find an equation of the line that passes through the point $(2, -1)$ and is perpendicular to the line $2x - 3y = 5$.

Solution

By writing the given line in the form $y = \frac{2}{3}x - \frac{5}{3}$, you can see that the line has a slope of $\frac{2}{3}$. So, any line that is perpendicular to this line must have a slope of $-\frac{3}{2}$ (because $-\frac{3}{2}$ is the negative reciprocal of $\frac{2}{3}$). Therefore, the required line through the point $(2, -1)$ has the following equation.

$$y - (-1) = -\tfrac{3}{2}(x - 2) \qquad \text{Substitute for } m, x_1, \text{ and } y_1 \text{ in point-slope form.}$$

$$y + 1 = -\tfrac{3}{2}x + 3 \qquad \text{Simplify.}$$

$$y = -\tfrac{3}{2}x + 3 - 1 \qquad \text{Subtract 1 from each side.}$$

$$y = -\tfrac{3}{2}x + 2 \qquad \text{Slope-intercept form}$$

The graphs of both equations are shown in Figure 1.20.

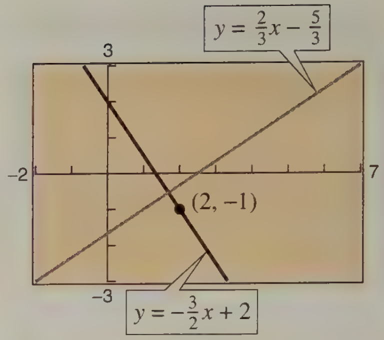


Figure 1.20

EXAMPLE 8 Graphs of Perpendicular Lines

Use a graphing utility to graph the lines

$$y = x + 1$$

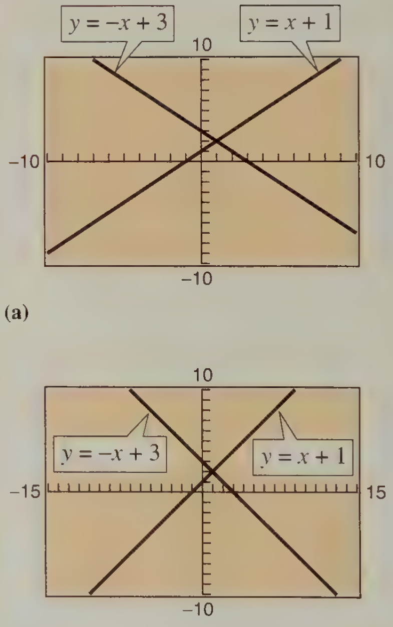
and

$$y = -x + 3.$$

Display *both* graphs in the same viewing window. The lines are supposed to be perpendicular (they have slopes of $m_1 = 1$ and $m_2 = -1$). Do they appear to be perpendicular on the display?

Solution

If the viewing window is nonsquare, as in Figure 1.21(a), the two lines will not appear perpendicular. If, however, the viewing window is square, as in Figure 1.21(b), the lines will appear perpendicular.



(a)

(b)

Figure 1.21

Writing About Math *An Application of Slope*

In 1990, a college had an enrollment of 5500 students. By 2000, the enrollment had increased to 7000 students.

a. What was the average annual change in enrollment from 1990 to 2000?

b. Use the average annual change in enrollment to estimate the enrollments in 1993, 1997, and 1999.

c. Write the equation of the line that represents the data in part (b). What is its slope? Interpret the slope in the context of the problem.

d. Write a short paragraph discussing the concepts of *slope* and *average rate of change.*

1.2 Exercises

In Exercises 1 and 2, identify the line that has the specified slope.

1. (a) $m = \frac{2}{3}$ (b) m is undefined. (c) $m = -2$

2. (a) $m = 0$ (b) $m = -\frac{3}{4}$ (c) $m = 1$

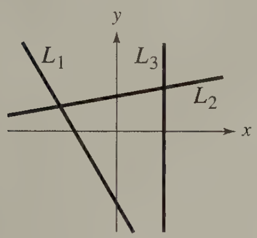


FIGURE FOR 1

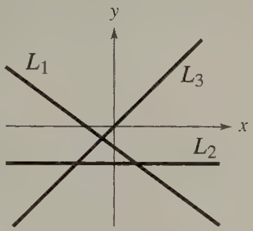


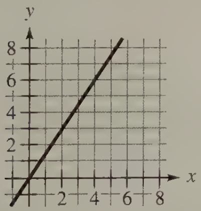
FIGURE FOR 2

In Exercises 3 and 4, sketch the line through the point with each indicated slope on the same set of coordinate axes.

	Point			*Slopes*	
3.	(2, 3)	(a) 0	(b) 1	(c) 2	(d) -3
4.	$(-4, 1)$	(a) 3	(b) -3	(c) $\frac{1}{2}$	(d) Undefined

In Exercises 5–10, estimate the slope of the line.

5.



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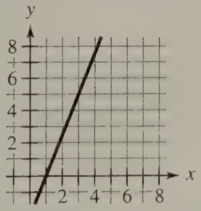

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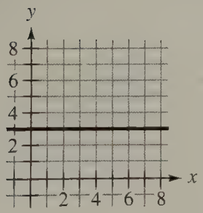

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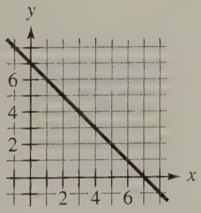

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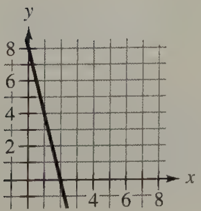
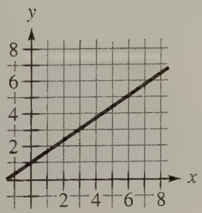


10.



In Exercises 11–14, plot the points and find the slope of the line passing through the points. Verify the slope using the *draw* feature on your graphing utility to graph the line segment connecting the two points. (Use a square setting.)

11. $(0, -10), (-4, 0)$

12. $(2, 4), (4, -4)$

13. $(-6, -1), (-6, 4)$

14. $(-3, -2), (1, 6)$

In Exercises 15–20, you are given the slope of the line and a point on the line. Find three additional points through which the line passes. (There are many correct answers.)

	Point	*Slope*
15.	(2, 1)	$m = 0$
16.	$(-4, 1)$	m is undefined.
17.	$(-5, 4)$	$m = 2$
18.	$(0, -9)$	$m = -2$
19.	$(7, -2)$	$m = \frac{1}{2}$
20.	$(-1, -6)$	$m = -\frac{1}{2}$

In Exercises 21–24, determine whether the lines L_1 and L_2 passing through the pairs of points are parallel, perpendicular, or neither. Use a graphing utility to graph the line segments connecting the pairs of points on the respective lines. (Use a square setting.)

21. L_1: $(0, -1), (5, 9)$
L_2: $(0, 3), (4, 1)$

22. L_1: $(-2, -1), (1, 5)$
L_2: $(1, 3), (5, -5)$

23. L_1: $(3, 6), (-6, 0)$
L_2: $(0, -1), \left(5, \frac{7}{3}\right)$

24. L_1: $(4, 8), (-4, 2)$
L_2: $(3, -5), \left(-1, \frac{1}{3}\right)$

In Exercises 25–32, (a) find the slope and y-intercept (if possible) of the equation of the line algebraically, (b) sketch the line by hand, and (c) use a graphing utility to verify your answers to parts (a) and (b).

25. $5x - y + 3 = 0$ y=3+5x

26. $2x + 3y - 9 = 0$

27. $5x - 2 = 0$

28. $3x + 7 = 0$

29. $3y + 5 = 0$

30. $-11 - 8y = 0$

31. $7x + 6y - 30 = 0$

32. $x - y - 10 = 0$

In Exercises 33–42, find the general form of the equation of the line that passes through the given point and has the indicated slope. Sketch the line by hand. Use a graphing utility to verify your sketch.

	Point	*Slope*
33.	$(0, -2)$	$m = 3$
34.	$(0, 10)$	$m = -1$
35.	$(-3, 6)$	$m = -2$
36.	$(0, 0)$	$m = 4$
37.	$(4, 0)$	$m = -\frac{1}{3}$
38.	$(-2, -5)$	$m = \frac{3}{4}$
39.	$(6, -1)$	m is undefined.
40.	$(-10, 4)$	$m = 0$
41.	$\left(-\frac{1}{2}, \frac{3}{2}\right)$	$m = -3$
42.	$(2.3, -8.5)$	$m = -\frac{5}{2}$

In Exercises 43–52, find the general form of the equation of the line that passes through the points. Use a graphing utility to sketch the line.

43. $(5, -1), (-5, 5)$

44. $(4, 3), (-4, -4)$

45. $(-8, 1), (-8, 7)$

46. $(-1, 4), (6, 4)$

47. $\left(2, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{5}{4}\right)$

48. $(1, 1), \left(6, -\frac{2}{3}\right)$

49. $\left(-\frac{1}{10}, -\frac{3}{5}\right), \left(\frac{9}{10}, -\frac{9}{5}\right)$

50. $\left(\frac{3}{4}, \frac{3}{2}\right), \left(-\frac{4}{3}, \frac{7}{4}\right)$

51. $(1, 0.6), (-2, -0.6)$

52. $(-8, 0.6), (2, -2.4)$

Exploration **In Exercises 53 and 54, use the values of *a* and *b* and a graphing utility to graph the equation of the line**

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad a \neq 0, b \neq 0.$$

Use the graphs to make a conjecture about what *a* and *b* represent. Verify your conjecture.

53. $a = 5;\quad b = -3$

54. $a = -6,\quad b = 2$

In Exercises 55–58, use the results of Exercises 53 and 54 to write an equation of the line that passes through the points.

55. x-intercept: $(2, 0)$
y-intercept: $(0, 3)$

56. x-intercept: $(-5, 0)$
y-intercept: $(0, -4)$

57. x-intercept: $\left(-\frac{1}{6}, 0\right)$
y-intercept: $\left(0, -\frac{2}{3}\right)$

58. x-intercept: $\left(\frac{3}{4}, 0\right)$
y-intercept: $\left(0, \frac{4}{5}\right)$

In Exercises 59 and 60, use a graphing utility to graph the equation using each of the suggested viewing windows. Describe the difference between the two graphs.

59. $y = 0.5x - 3$

Xmin = -5
Xmax = 10
Xscl = 1
Ymin = -1
Ymax = 10
Yscl = 1

Xmin = -2
Xmax = 10
Xscl = 1
Ymin = -4
Ymax = 1
Yscl = 1

60. $y = -8x + 5$

Xmin = -5
Xmax = 5
Xscl = 1
Ymin = -10
Ymax = 10
Yscl = 1

Xmin = -5
Xmax = 10
Xscl = 1
Ymin = -80
Ymax = 80
Yscl = 20

Graphical Analysis **In Exercises 61–64, use a graphing utility to graph the three equations in the same viewing window. Adjust the viewing window so that the slope appears visually correct. Identify any relationships that exist among the lines. Use the slope of the lines to verify your results.**

61. (a) $y = 2x$ (b) $y = -2x$ (c) $y = \frac{1}{2}x$

62. (a) $y = \frac{2}{3}x$ (b) $y = -\frac{3}{2}x$ (c) $y = \frac{2}{3}x + 2$

63. (a) $y = -\frac{1}{2}x$ (b) $y = -\frac{1}{2}x + 3$ (c) $y = 2x - 4$

64. (a) $y = x - 8$ (b) $y = x + 1$ (c) $y = -x + 3$

In Exercises 65–70, write equations of the lines through the given point (a) parallel to the given line and (b) perpendicular to the given line.

	Point	*Line*
65.	$(2, 1)$	$4x - 2y = 3$
66.	$(-3, 2)$	$x + y = 7$
67.	$\left(-\frac{2}{3}, \frac{7}{8}\right)$	$3x + 4y = 7$
68.	$\left(\frac{7}{8}, \frac{3}{4}\right)$	$5x + 3y = 0$
69.	$(2.5, 6.8)$	$x - y = 4$
70.	$(-3.9, -1.4)$	$6x + 2y = 9$

In Exercises 71 and 72, find a relationship between *x* and *y* such that (x, y) is equidistant from the two points.

71. $(4, -1), (-2, 3)$

72. $(3, -2), (-7, 1)$

73. ***Business*** The slopes are the slopes of lines representing annual sales y in terms of time x in years. Use each slope to interpret any change in annual sales for a 1-year increase in time.

(a) The line has a slope of $m = 135$.

(b) The line has a slope of $m = 0$.

(c) The line has a slope of $m = -40$.

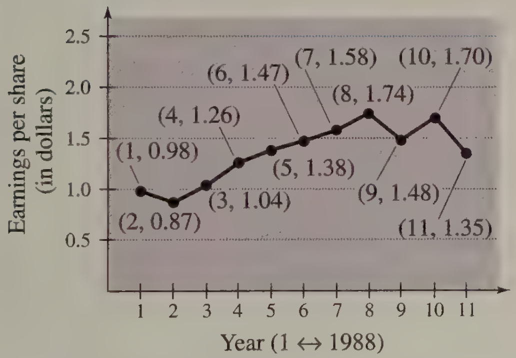
74. ***Business*** The slopes are the slopes of lines representing daily revenues y in terms of time x in days. Use each slope to interpret any change in daily revenues for a 1-day increase in time.

(a) The line has a slope of $m = 400$.

(b) The line has a slope of $m = 100$.

(c) The line has a slope of $m = 0$.

75. ***Business*** The graph shows the earnings per share of stock for the Kellogg Company for the years 1988 through 1998. (Source: Kellogg Company)



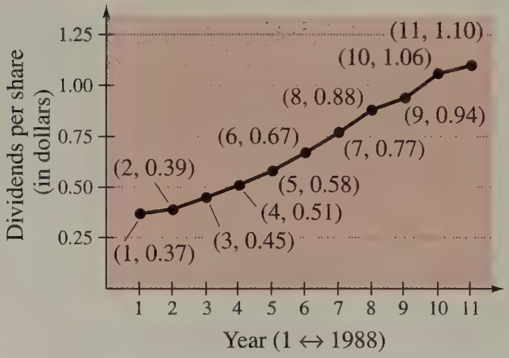
(a) Use the slopes to determine the year(s) when the earnings per share showed the greatest increase and decrease.

(b) Find the equation of the line between the years 1988 and 1998.

(c) Interpret the meaning of the slope in the equation from part (b) in the context of the problem.

(d) Use the equation from part (b) to estimate the earnings per share of stock for the year 2001. Do you think this is an accurate estimation? Explain.

76. ***Business*** The graph shows the dividends declared per share of stock for the Colgate-Palmolive Company for the years 1988 through 1998. (Source: Colgate-Palmolive Company)



(a) Use the slopes to determine the years when the dividends declared per share showed the greatest increase and the smallest increase.

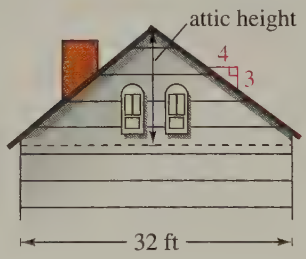
(b) Find the equation of the line between the years 1988 and 1998.

(c) Interpret the meaning of the slope in the equation from part (b) in the context of the problem.

(d) Use the equation from part (b) to estimate the dividends declared per share for the year 2001. Do you think this is an accurate estimation? Explain.

77. ***Driving*** When driving down a mountain road, you notice warning signs indicating that it is a "12% grade." This means that the slope of the road is $-\frac{12}{100}$. Approximate the amount of horizontal change in your position if you note from elevation markers that you have descended 2000 feet vertically.

78. ***Attic Height*** The "rise to run" ratio of the roof of a house determines the steepness of the roof. Suppose the rise to run ratio of a roof is 3 to 4. Determine the maximum height in the attic of the house if the house is 32 feet wide.

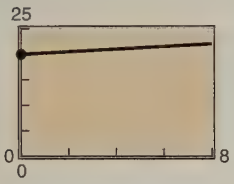


Rate of Change **In Exercises 79–82, you are given the dollar value of a product in 2001 and the rate at which the value of the product is expected to change during the next 5 years. Use this information to write a linear equation that gives the dollar value V of the item in terms of the year t. (Let $t = 1$ represent 2001.)**

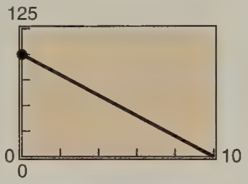
	2001 Value	*Rate*
79.	\$2540	\$125 increase per year
80.	\$156	\$4.50 increase per year
81.	\$20,400	\$2000 decrease per year
82.	\$245,000	\$5600 decrease per year

Graphical Interpretation **In Exercises 83–86, match the description with its graph. Also determine the slope and how it is interpreted in the situation. [The graphs are labeled (a), (b), (c), and (d).]**

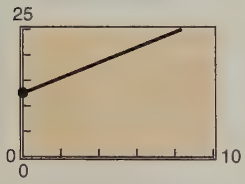
(a)



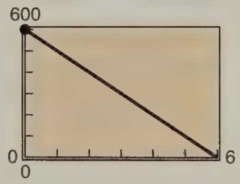
(b)



(c)



(d)



83. A person is paying \$10 per week to a friend to repay a \$100 loan.

84. An employee is paid \$12.50 per hour plus \$1.50 for each unit produced per hour.

85. A sales representative receives \$20 per day for food plus \$0.25 for each mile traveled.

86. A word processor that was purchased for \$600 depreciates \$100 per year.

87. ***Temperature*** Find the equation of the line that shows the relationship between the temperature in degrees Celsius C and degrees Fahrenheit F. Remember that water freezes at 0°C (32°F) and boils at 100°C (212°F).

88. ***Temperature*** Use the result of Exercise 87 to complete the table.

C		$-10°$	$10°$			$177°$
F	$0°$			$68°$	$90°$	

89. ***Finance*** Your salary was \$28,500 in 1998 and \$32,900 in 2000. If your salary follows a linear growth pattern, what will your salary be in 2003?

90. ***College Enrollment*** A small college had 2546 students in 1998 and 2702 students in 2000. If the enrollment follows a linear growth pattern, how many students will the college have in 2004?

91. ***Business*** A small business purchases a fax machine for \$875. After 5 years, the fax machine will be outdated and have no value.

(a) Write a linear equation giving the value V of the fax machine during the 5 years it will be used.

(b) Use a graphing utility to graph the linear equation representing the depreciation of the fax machine, and use the *value* or *trace* feature to complete the table.

t	0	1	2	3	4	5
V						

(c) Verify your answers in part (b) algebraically by using the equation you found in part (a).

92. ***Business*** A small business purchases a computer network system for \$25,000. After 10 years, the system will have to be replaced. Its value at that time is expected to be \$2000.

(a) Write a linear equation giving the value V of the system during the 10 years it will be used.

(b) Use a graphing utility to graph the linear equation representing the depreciation of the system, and use the *value* or *trace* feature to complete the table.

t	0	1	2	3	4	5	6	7	8	9	10
V											

(c) Verify your answers in part (b) algebraically by using the equation you found in part (a).

93. ***Business*** A contractor purchases a bulldozer for \$36,500. The bulldozer requires an average expenditure of \$5.25 per hour for fuel and maintenance, and the operator is paid \$11.50 per hour.

(a) Write a linear equation giving the total cost C of operating the bulldozer for t hours. (Include the purchase cost of the bulldozer.)

(b) Assuming that customers are charged \$27 per hour of bulldozer use, write an equation for the revenue R derived from t hours of use.

(c) Use the formula for profit ($P = R - C$) to write an equation for the profit derived from t hours of use.

(d) Use the result of part (c) to find the break-even point (the number of hours the bulldozer must be used to yield a profit of 0 dollars).

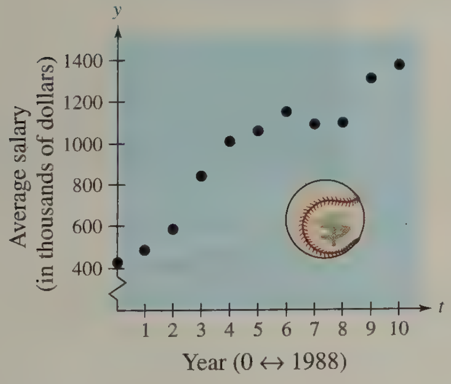
94. ***Real Estate Purchase*** A real estate office handles an apartment complex with 50 units. When the rent per unit is \$580 per month, all 50 units are occupied. However, when the rent is \$625 per month, the average number of occupied units drops to 47. Assume that the relationship between the monthly rent p and the demand x is linear.

(a) Write the equation of the line giving the demand x in terms of the rent p.

(b) Use a graphing utility to graph the demand equation and use the *trace* feature to estimate the number of units occupied if the rent is raised to \$655. Verify your answer algebraically.

(c) Use the demand equation to estimate the number of units occupied if the rent is lowered to \$595. Verify your answer graphically.

95. ***Sports*** The average annual salaries of Major League Baseball players (in thousands of dollars) from 1988 to 1998 are shown in the scatter plot. Let y represent the average salary and let t represent the year, with $t = 0$ corresponding to 1988. (Source: Major League Baseball Player Relations Committee)



(a) Use the regression capabilities of a graphing utility to find the line that best fits the data.

(b) Use the regression line to estimate the average salary in the year 2000.

(c) Interpret the meaning of the slope of the regression line.

Synthesis

True or False? **In Exercises 96 and 97, determine whether the statement is true or false. Justify your answer.**

96. The line through $(-8, 2)$ and $(-1, 4)$ and the line through $(0, -4)$ and $(-7, 7)$ are parallel.

97. If the points $(10, -3)$ and $(2, -9)$ lie on the same line, then the point $\left(-12, -\frac{37}{2}\right)$ also lies on that line.

98. ***Writing*** Explain how you could show that the points $A(2, 3)$, $B(2, 9)$, and $C(7, 3)$ are the vertices of a right triangle.

99. ***Think About It*** The slopes of two lines are -4 and $\frac{5}{2}$. Which is steeper?

100. ***Writing*** Write a brief paragraph explaining whether or not any pair of points on a line can be used to calculate the slope of the line.

101. ***Think About It*** Is it possible for two lines with positive slopes to be perpendicular? Explain.

Review

In Exercises 102–107, determine whether the expression is a polynomial. If it is, write the polynomial in standard form.

102. $x + 20$

103. $3x - 10x^2 + 1$

104. $4x^2 + x^{-1} - 3$

105. $2x^2 - 2x^4 - x^3 + 2$

106. $\dfrac{x^2 + 3x + 4}{x^2 - 9}$

107. $\sqrt{x^2 + 7x + 6}$

In Exercises 108–111, factor the trinomial.

108. $x^2 - 6x - 27$

109. $x^2 - 11x + 28$

110. $2x^2 + 11x - 40$

111. $3x^2 - 16x + 5$

In Exercises 112–115, find the standard form of the equation of the specified circle.

112. Center: $(0, 0)$; radius: 9

113. Center: $(-8, -5)$; radius: $\frac{3}{4}$

114. Center: $(10, 1)$; solution point: $(-2, -4)$

115. Endpoints of a diameter: $(6, -5)$, $(-2, 7)$

1.3 Functions

What You Should Learn:

- How to decide whether relations between two variables are functions
- How to use function notation and evaluate functions
- How to find the domains of functions
- How to use functions to model and solve real-life problems

Why You Should Learn It:

Many natural phenomena can be modeled by functions, such as the force of water against the face of a dam, in Exercise 100 on page 104.

Kunio Owaki/The Stock Market

Introduction to Functions

Many everyday phenomena involve pairs of quantities that are related to each other by some rule of correspondence. The mathematical term for such a rule of correspondence is a **relation.** Here are two examples.

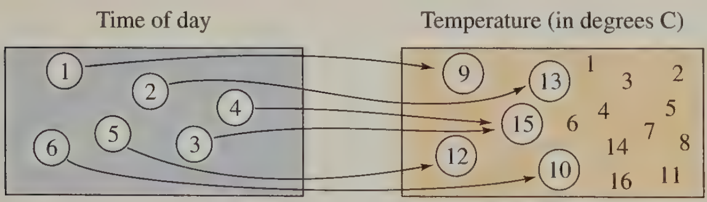
1. The simple interest I earned on an investment of \$1000 for 1 year is related to the annual interest rate r by the formula $I = 1000r$.
2. The area A of a circle is related to its radius r by the formula $A = \pi r^2$.

Not all relations have simple mathematical formulas. For instance, people commonly match up NFL starting quarterbacks with touchdown passes, and hours of the day with temperature. In each of these cases, however, there is some relation that matches each item from one set with exactly one item from a different set. Such a relation is called a **function.**

Definition of a Function

A **function** f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the **domain** (or set of inputs) of the function f, and the set B contains the **range** (or set of outputs).

To help understand this definition, look at the function in Figure 1.22.



Set A is the domain.
Inputs: 1, 2, 3, 4, 5, 6

Set B contains the range.
Outputs: 9, 10, 12, 13, 15

Figure 1.22

This function can be represented by the ordered pairs $\{(1, 9°), (2, 13°), (3, 15°), (4, 15°), (5, 12°), (6, 10°)\}$. In each ordered pair, the first coordinate is the **input** and the second coordinate is the **output.** In this example, note the following characteristics of a function.

1. Each element in A must be matched with an element of B.
2. Some elements in B may not be matched with any element in A.
3. Two or more elements of A may be matched with the same element of B.

The converse of the third statement is not true. That is, an element of A (the domain) cannot be matched with two different elements of B. In other words, each element of A must be matched with only one element of B.

Functions are commonly represented in four ways.

1. *Verbally* by a sentence that describes how the input variable is related to the output variable
2. *Numerically* by a table or a list of ordered pairs that matches input values with output values
3. *Graphically* by points on a graph in a coordinate plane in which the input values are represented by the horizontal axis and the output values are represented by the vertical axis
4. *Algebraically* by an equation in two variables

In the following example, you are asked to decide whether the given relation is a function. To do this, you must decide whether each input value is matched with exactly one output value. If any input value is matched with two or more output values, the relation is not a function.

Library of Functions

Many functions do not have simple mathematical formulas but are defined by real-life data. Such functions arise when you are using collections of data to model real-life applications. You will see that it is often convenient to approximate the data using a mathematical model or formula.

Consult the Library of Functions Summary inside the front cover for a description of functions used to model data.

EXAMPLE 1 Testing for Functions

Decide whether the description represents y as a function of x.

a. The input value x is the number of representatives from a state and the output value y is the number of senators.

b.

Input x	2	2	3	4	5
Output y	11	10	8	5	1

c.

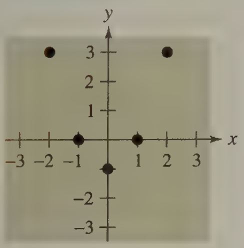


Figure 1.23

Solution

a. This verbal description does describe y as a function of x. Regardless of the value of x, the value of y is always 2. Such functions are called *constant functions.*

b. This table does not describe y as a function of x. The input value 2 is mapped to two different y-values.

c. The graph in Figure 1.23 does describe y as a function of x. No input value is mapped to two output values.

STUDY TIP

Be sure you see that the *range* of a function is not the same as the use of *range* relating to the viewing window.

Representing functions by sets of ordered pairs is common in *discrete mathematics.* For instance, a function that relates the number of units produced or sold in business to the profit could be represented by sets of ordered pairs. In algebra, however, it is more common to assume that the variables can be any real value and represent functions by equations or formulas involving two variables.

For instance, the equation

$$y = x^2 \qquad \text{y is a function of x.}$$

represents the variable y as a function of the variable x. In this equation, x is the **independent variable** and y is the **dependent variable.** The domain of the function is the set of all values taken on by the independent variable x, and the range of the function is the set of all values taken on by the dependent variable y.

Exploration

Use a graphing utility to graph $x^2 + y = 1$. Then use the graph to write a convincing argument that each x-value has at most one y-value.

Use a graphing utility to graph $-x + y^2 = 1$. (*Hint:* You will need to use two equations.) Then use the graph to find an x-value that corresponds to two y-values. Does the graph represent y as a function of x? Explain.

The *Interactive* CD-ROM and *Internet* versions of this text offer a built-in graphing calculator, which can be used with the Examples, Explorations, and Exercises.

EXAMPLE 2 Testing for Functions Represented Algebraically

Which of the equations represents y as a function of x?

a. $x^2 + y = 1$ **b.** $-x + y^2 = 1$

Solution

To determine whether y is a function of x, try to solve for y in terms of x.

a. Solving for y yields

$$x^2 + y = 1 \qquad \text{Write original equation.}$$

$$y = 1 - x^2. \qquad \text{Solve for } y.$$

To each value of x there corresponds exactly one value of y. So, y *is* a function of x.

b. Solving for y yields

$$-x + y^2 = 1 \qquad \text{Write original equation.}$$

$$y^2 = 1 + x \qquad \text{Add } x \text{ to each side.}$$

$$y = \pm\sqrt{1 + x}. \qquad \text{Solve for } y.$$

The ± indicates that for a given value of x there correspond two values of y. So, y *is not* a function of x.

Function Notation

When an equation is used to represent a function, it is convenient to name the function so that it can be referenced easily. For example, you know that the equation $y = 1 - x^2$ describes y as a function of x. Suppose you give this function the name "f." Then you can use the following **function notation.**

Input	*Output*	*Equation*
x	$f(x)$	$f(x) = 1 - x^2$

The symbol $f(x)$ is read as the *value of f at x* or simply *f of x*. The symbol $f(x)$ corresponds to the y-value for a given x. So, you can write $y = f(x)$. Keep in mind that f is the *name* of the function, whereas $f(x)$ is the *output value* of the function at the *input value* x. In function notation, the *input* is the independent variable and the *output* is the dependent variable. For instance, the function $f(x) = 3 - 2x$ has *function values* denoted by $f(-1), f(0), f(2)$, and so on. To find these values, substitute the specified input values into the given equation.

For $x = -1$, $\quad f(-1) = 3 - 2(-1) = 3 + 2 = 5.$

For $x = 0$, $\quad f(0) = 3 - 2(0) = 3 - 0 = 3.$

For $x = 2$, $\quad f(2) = 3 - 2(2) = 3 - 4 = -1.$

STUDY TIP

You can use a graphing utility to evaluate a function. On the website *college.hmco.com*, you will find the graphing utility program EVALUATE. The program will prompt you for a value of x, and then evaluate the expression in the equation editor for that value of x. Try using the program to evaluate several different functions of x.

Although f is often used as a convenient function name and x is often used as the independent variable, you can use other letters. For instance,

$$f(x) = x^2 - 4x + 7, \quad f(t) = t^2 - 4t + 7, \quad \text{and} \quad g(s) = s^2 - 4s + 7$$

all define the same function. In fact, the role of the independent variable is that of a "placeholder." Consequently, the function could be described by

$$f(\quad) = (\quad)^2 - 4(\quad) + 7.$$

Library of Functions

The function in Example 4 is a *piecewise-defined* function. This means that the function is defined by two or more equations over a specified domain. In Example 4, you use the top equation for all x-values less than 0, and the bottom equation for all x-values greater than or equal to 0.

Consult the Library of Functions Summary inside the front cover for a description of the piecewise-defined function.

EXAMPLE 3 Evaluating a Function

Let $g(x) = -x^2 + 4x + 1$ and find

a. $g(2)$ **b.** $g(t)$ **c.** $g(x + 2)$.

Solution

a. Replacing x with 2 in $g(x) = -x^2 + 4x + 1$ yields the following.

$$g(2) = -(2)^2 + 4(2) + 1 = -4 + 8 + 1 = 5$$

b. Replacing x with t yields the following.

$$g(t) = -(t)^2 + 4(t) + 1 = -t^2 + 4t + 1$$

c. Replacing x with $x + 2$ yields the following.

$$\begin{aligned} g(x + 2) &= -(x + 2)^2 + 4(x + 2) + 1 && \text{Substitute } x + 2 \text{ for } x. \\ &= -(x^2 + 4x + 4) + 4x + 8 + 1 && \text{Multiply.} \\ &= -x^2 - 4x - 4 + 4x + 8 + 1 && \text{Distributive Property} \\ &= -x^2 + 5 && \text{Simplify.} \end{aligned}$$

In Example 3, note that $g(x + 2)$ is not equal to $g(x) + g(2)$. In general, $g(u + v) \neq g(u) + g(v)$.

Sometimes a function is defined by more than one equation. An illustration of this is given in Example 4.

EXAMPLE 4 A Piecewise-Defined Function

Evaluate the function when $x = -1, 0,$ and 1.

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases}$$

Solution

Because $x = -1$ is less than 0, use $f(x) = x^2 + 1$ to obtain

$$f(-1) = (-1)^2 + 1 = 2.$$

For $x = 0$, use $f(x) = x - 1$ to obtain

$$f(0) = (0) - 1 = -1.$$

For $x = 1$, use $f(x) = x - 1$ to obtain

$$f(1) = (1) - 1 = 0.$$

STUDY T!P

Most graphing utilities can graph functions that are defined piecewise. For instructions on how to enter a piecewise-defined function into your graphing utility, consult your user's manual. You may find it helpful to set your graphing utility to *dot mode* before graphing.

The Domain of a Function

The domain of a function is the set of all values of the independent variable for which the function is defined. If x is in the domain of f, f is said to be *defined* at x. If x is not in the domain of f, f is said to be *undefined* at x.

The domain of a function can be described explicitly or it can be *implied* by the expression used to define the function. The **implied domain** is the set of all real numbers for which the expression is defined. For instance, the function

$$f(x) = \frac{1}{x^2 - 4}$$

Domain excludes x-values that result in division by zero.

has an implied domain that consists of all real x other than $x = \pm 2$. These two values are excluded from the domain because division by zero is undefined. Another common type of implied domain is that used to avoid even roots of negative numbers. For example, the function

$$f(x) = \sqrt{x}$$

Domain excludes x-values that result in even roots of negative numbers.

is defined only for $x \geq 0$. So, its implied domain is the interval $[0, \infty)$. In general, the domain of a function *excludes* values that would cause division by zero *or* result in the even root of a negative number.

EXAMPLE 5 Finding the Domain of a Function

Find the domain of each function.

a. $f: \{(-3, 0), (-1, 4), (0, 2), (2, 2), (4, -1)\}$

b. $g(x) = -3x^2 + 4x + 5$

c. $h(x) = \dfrac{1}{x + 5}$

d. Volume of a sphere: $V = \frac{4}{3}\pi r^3$

e. $k(x) = \sqrt{4 - 3x}$

Solution

a. The domain of f consists of all first coordinates in the set of ordered pairs.

$$\text{Domain} = \{-3, -1, 0, 2, 4\}.$$

b. The domain of g is the set of all real numbers.

c. Excluding x-values that yield zero in the denominator, the domain of h is the set of all real numbers $x \neq -5$.

d. Because this function represents the volume of a sphere, the values of the radius r must be positive. So, the domain is the set of all real numbers r such that $r > 0$.

e. This function is defined only for x-values for which

$$4 - 3x \geq 0.$$

The domain of k is all real numbers that are less than or equal to $\frac{4}{3}$.

In Example 5(d), note that the *domain of a function may be implied by the physical context.* For instance, from the equation $V = \frac{4}{3}\pi r^3$, you would have no reason to restrict r to positive values, but the physical context implies that a sphere cannot have a negative or zero radius.

Exploration

Use a graphing utility to graph $y = \sqrt{4 - x^2}$. What is the domain of this function? Then graph $y = \sqrt{x^2 - 4}$. What is the domain of this function? Do the domains of these two functions overlap? If so, for what values?

Library of Functions

The *square root* function $f(x) = \sqrt{x}$ is not defined for $x < 0$. This means that you must be careful when analyzing the domain of complicated functions involving the square root symbol.

Consult the Library of Functions Summary inside the front cover for a description of the square root or radical function.

STUDY T!P

In Example 5(e), $4 - 3x \geq 0$ is a *linear inequality*. For help with solving linear inequalities see Appendix C. You will study more about inequalities in Section 2.5.

Applications

EXAMPLE 6 The Dimensions of a Container

You work in the marketing department of a soft-drink company and are experimenting with a new soft-drink can that is slightly narrower and taller than a standard can. For your experimental can, the ratio of the height to the radius is 4, as shown in Figure 1.24.

a. Express the volume of the can as a function of the radius r.

b. Express the volume of the can as a function of the height h.

Solution

The volume of a right circular cylinder is $V = \pi r^2 h$.

a. To write the volume as a function of the radius, use the fact that $h = 4r$.

$$V(r) = \pi r^2 h = \pi r^2(4r) = 4\pi r^3 \qquad \text{Write } V \text{ as a function of } r.$$

b. To write the volume as a function of the height, use the fact that $r = h/4$.

$$V(h) = \pi r^2 h = \pi\left(\frac{h}{4}\right)^2 h = \frac{\pi h^3}{16} \qquad \text{Write } V \text{ as a function of } h.$$

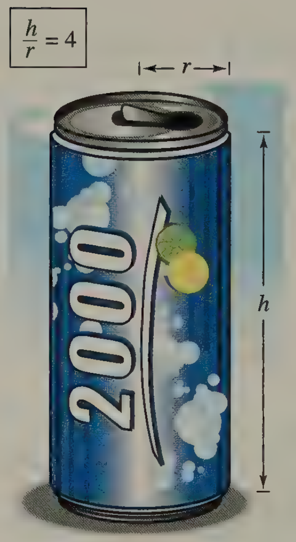


Figure 1.24

EXAMPLE 7 The Path of a Baseball

A baseball is hit at a point 3 feet above ground at a velocity of 100 feet per second and an angle of 45°. The path of the baseball is given by the function

$$f(x) = -0.0032x^2 + x + 3$$

where y and x are measured in feet. Will the baseball clear a 10-foot fence located 300 feet from home plate?

Algebraic Solution

The height of the baseball is a function of the horizontal distance from home plate. When $x = 300$, you can find the height of the baseball as follows.

$$f(x) = -0.0032x^2 + x + 3 \qquad \text{Write original equation.}$$

$$f(300) = -0.0032(300)^2 + 300 + 3 \qquad \text{Substitute 300 for } x.$$

$$= 15 \qquad \text{Simplify.}$$

When $x = 300$, the height of the baseball is 15 feet, so the baseball will clear a 10-foot fence.

Graphical Solution

Use a graphing utility to graph the function $y = -0.0032x^2 + x + 3$, as shown in Figure 1.25.

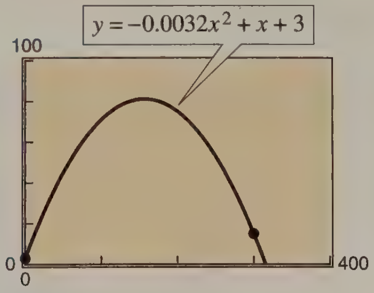


Figure 1.25

Use the *value* feature or *zoom* and *trace* features of the graphing utility to estimate that $y = 15$ when $x = 300$. So, the ball will clear a 10-foot fence.

EXAMPLE 8 Direct Mail Advertising

The money C (in billions of dollars) spent for direct mail advertising in the United States increased in a linear pattern from 1990 to 1992, as shown in Figure 1.26. Then, in 1993, the money spent took a jump and, until 1996, increased in a *different* linear pattern. These two patterns can be approximated by the function

$$C(t) = \begin{cases} 23.40 + 1.01t & 0 \le t \le 2 \\ 19.85 + 2.50t & 3 \le t \le 6 \end{cases}$$

where $t = 0$ represents 1990. Use this function to approximate the total amount spent for direct mail advertising between 1990 and 1996. (Source: McCann-Erickson)

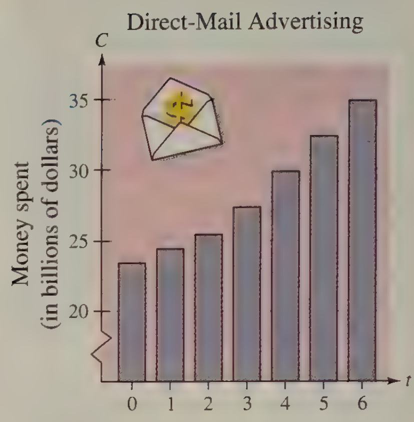


Figure 1.26

Solution

From 1990 to 1992, use the formula $C(t) = 23.40 + 1.01t$.

$$\underbrace{\$23.40}_{1990},\quad \underbrace{\$24.41}_{1991},\quad \underbrace{\$25.42}_{1992}$$

From 1993 to 1996, use the formula $C(t) = 19.85 + 2.50t$.

$$\underbrace{\$27.35}_{1993},\quad \underbrace{\$29.85}_{1994},\quad \underbrace{\$32.35}_{1995},\quad \underbrace{\$34.85}_{1996}$$

The total of these seven amounts is \$197.63, which implies that the total amount spent was approximately \$197,630,000,000.

One of the basic definitions in calculus employs the ratio

$$\frac{f(x+h) - f(x)}{h}, \qquad h \neq 0.$$

This ratio is called a **difference quotient,** as illustrated in Examples 9 and 10.

EXAMPLE 9 Evaluating a Difference Quotient

For $f(x) = -2x + 4$, find $\dfrac{f(x+h) - f(x)}{h}$.

Solution

$$\frac{f(x+h) - f(x)}{h} = \frac{[-2(x+h) + 4] - (-2x + 4)}{h} \qquad \text{Substitute } x + h \text{ for } x.$$

$$= \frac{-2x - 2h + 4 + 2x - 4}{h} \qquad \text{Distributive Property}$$

$$= \frac{-2h}{h} = -2, \quad h \neq 0 \qquad \text{Simplify and divide out common factor.}$$

STUDY TIP

Notice in Example 9 that h cannot be zero in the original expression. So, you must restrict the domain of the simplified expression by adding $h \neq 0$ so that the simplified expression is equivalent to the original expression.

The symbol indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

EXAMPLE 10 Evaluating a Difference Quotient

For $f(x) = x^2 - 4x + 7$, find $\dfrac{f(x+h) - f(x)}{h}$.

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{[(x+h)^2 - 4(x+h) + 7] - (x^2 - 4x + 7)}{h}\\ &= \frac{x^2 + 2xh + h^2 - 4x - 4h + 7 - x^2 + 4x - 7}{h}\\ &= \frac{2xh + h^2 - 4h}{h}\\ &= \frac{h(2x + h - 4)}{h} = 2x + h - 4, \quad h \neq 0\end{aligned}$$

Summary of Function Terminology

Function: A **function** is a relationship between two variables such that to each value of the independent variable there corresponds exactly one value of the dependent variable.

Function Notation: $y = f(x)$

f is the *name* of the function.

y is the **dependent variable,** or output value.

x is the **independent variable,** or input value.

$f(x)$ is the *value of the function at* x.

Domain: The **domain** of a function is the set of all values (inputs) of the independent variable for which the function is defined.

Range: The **range** of a function is the set of all values (outputs) assumed by the dependent variable (that is, the set of all function values).

Implied Domain: If f is defined by an algebraic expression and the domain is not specified, the **implied domain** consists of all real numbers for which the expression is defined.

The Granger Collection

Leonhard Euler (1707–1783), a Swiss mathematician, is considered to have been the most prolific and productive mathematician in history. One of his greatest influences on mathematics was his use of symbols, or notation. The function notation $y = f(x)$ was introduced by Euler.

Writing About Math *Modeling with Piecewise-Defined Functions*

x	y
1	5.2
2	5.6
3	6.6
4	8.3
5	11.5
6	15.8

x	y
7	12.8
8	10.1
9	8.6
10	6.9
11	4.5
12	2.7

The table at the left shows the monthly revenue y (in thousands of dollars) for one year of a landscaping business, with $x = 1$ representing January.

A mathematical model that represents this data is

$$f(x) = \begin{cases} -1.97x + 26.33 \\ 0.51x^2 - 1.47x + 6.31. \end{cases}$$

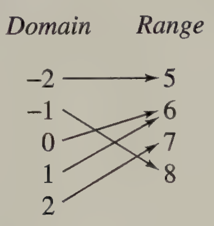
What is the domain of each part of the piecewise-defined function? How can you tell? Explain your reasoning.

Find $f(5)$ and $f(11)$, and interpret your results in the context of the problem. How do these model values compare with the actual data values?

1.3 Exercises

In Exercises 1–4, is the relationship a function?

1.



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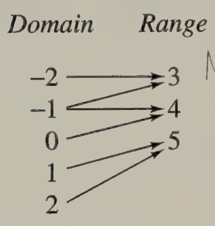
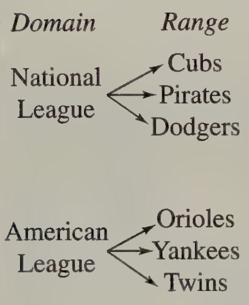


No

3.



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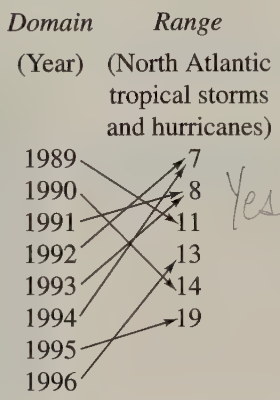


Yes

In Exercises 5–8, does the table describe a function? Explain your reasoning.

5.

Input Value	−2	−1	0	1	2
Output Value	−8	−1	0	1	8

6.

Input Value	0	1	2	1	0
Output Value	−4	−2	0	2	4

No

7.

Input Value	10	7	4	7	10
Output Value	3	6	9	12	15

8.

Input Value	0	3	9	12	15
Output Value	3	3	3	3	3

Yes

In Exercises 9 and 10, which sets of ordered pairs represent function(s) from A to B? Explain.

9. $A = \{0, 1, 2, 3\}$ and $B = \{-2, -1, 0, 1, 2\}$

(a) $\{(0, 1), (1, -2), (2, 0), (3, 2)\}$

(b) $\{(0, -1), (2, 2), (1, -2), (3, 0), (1, 1)\}$

(c) $\{(0, 0), (1, 0), (2, 0), (3, 0)\}$

(d) $\{(0, 2), (3, 0)\ (1, 1)\}$

10. $A = \{a, b, c\}$ and $B = \{0, 1, 2, 3\}$

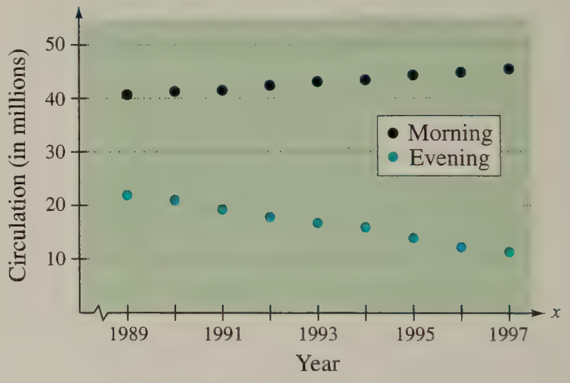
(a) $\{(a, 1), (c, 2), (c, 3), (b, 3)\}$

(b) $\{(a, 1), (b, 2), (c, 3)\}$

(c) $\{(1, a), (0, a), (2, c), (3, b)\}$

(d) $\{(c, 0), (b, 0), (a, 3)\}$

Circulation of Newspapers **In Exercises 11 and 12, use the graph, which shows the circulation (in millions) of daily newspapers in the United States.** (Source: Editor & Publisher Company)



11. Is the circulation of morning newspapers a function of the year? Is the circulation of evening newspapers a function of the year? Explain.

12. Let $f(x)$ represent the circulation of evening newspapers in year x. Find $f(1994)$.

In Exercises 13–24, determine if the equation represents y as a function of x.

13. $x^2 + y^2 = 4$

14. $x = y^2$ $y = \pm\sqrt{x}$

15. $x^2 + y = 4$

16. $x + y^2 = 4$

17. $2x + 3y = 4$

18. $x = -y + 5$

19. $y^2 = x^2 - 1$

20. $y = \sqrt{x + 5}$

21. $y = |4 - x|$

22. $|y| = 4 - x$

23. $x = 4$

24. $y = -2$

In Exercises 25 and 26, fill in the blanks using the specified function and the given values of the independent variable. Simplify the result.

25. $f(x) = \dfrac{1}{x+1}$

(a) $f(4) = \dfrac{1}{(\quad) + 1}$

(b) $f(0) = \dfrac{1}{(\quad) + 1}$

(c) $f(4t) = \dfrac{1}{(\quad) + 1}$

(d) $f(x + c) = \dfrac{1}{(\quad) + 1}$

26. $g(x) = x^2 - 2x$

(a) $g(2) = (\quad)^2 - 2(\quad)$

(b) $g(-3) = (\quad)^2 - 2(\quad)$

(c) $g(t + 1) = (\quad)^2 - 2(\quad)$

(d) $g(x + c) = (\quad)^2 - 2(\quad)$

In Exercises 27–38, evaluate the function at each specified value of the independent variable and simplify.

27. $f(x) = 2x - 3$

(a) $f(1)$ (b) $f(-3)$ (c) $f(x - 1)$

28. $g(y) = 7 - 3y$

(a) $g(0)$ (b) $g\left(\frac{7}{3}\right)$ (c) $g(s + 2)$

29. $h(t) = t^2 - 2t$

(a) $h(2)$ (b) $h(1.5)$ (c) $h(x + 2)$

30. $V(r) = \frac{4}{3}\pi r^3$

(a) $V(3)$ (b) $V\left(\frac{3}{2}\right)$ (c) $V(2r)$

31. $f(y) = 3 - \sqrt{y}$

(a) $f(4)$ (b) $f(0.25)$ (c) $f(4x^2)$

32. $f(x) = \sqrt{x + 8} + 2$

(a) $f(-8)$ (b) $f(1)$ (c) $f(x - 8)$

33. $q(x) = \dfrac{1}{x^2 - 9}$

(a) $q(0)$ (b) $q(3)$ (c) $q(y + 3)$

34. $q(t) = \dfrac{2t^2 + 3}{t^2}$

(a) $q(2)$ (b) $q(0)$ (c) $q(-x)$

35. $f(x) = \dfrac{|x|}{x}$

(a) $f(2)$ (b) $f(-2)$ (c) $f(x^2)$

36. $f(x) = |x| + 4$

(a) $f(2)$ (b) $f(-2)$ (c) $f(x^2)$

37. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$

(a) $f(-1)$ (b) $f(0)$ (c) $f(2)$

38. $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x^2 + 2, & x > 1 \end{cases}$

(a) $f(-2)$ (b) $f(1)$ (c) $f(2)$

In Exercises 39–44, complete the table.

39. $f(x) = x^2 - 3$

x	-2	-1	0	1	2
$f(x)$					

40. $g(x) = \sqrt{x - 3}$

x	3	4	5	6	7
$g(x)$					

41. $h(t) = \frac{1}{2}|t + 3|$

t	-5	-4	-3	-2	-1
$h(t)$					

42. $f(s) = \dfrac{|s - 2|}{s - 2}$

s	0	1	$\frac{3}{2}$	$\frac{5}{2}$	4
$f(s)$					

43. $f(x) = \begin{cases} -\frac{1}{2}x + 4, & x \leq 0 \\ (x - 2)^2, & x > 0 \end{cases}$

x	-2	-1	0	1	2
$f(x)$					

44. $h(x) = \begin{cases} 9 - x^2, & x < 3 \\ x - 3, & x \geq 3 \end{cases}$

x	1	2	3	4	5
$h(x)$					

In Exercises 45–52, find all real values of x such that $f(x) = 0$.

45. $f(x) = 15 - 3x$

46. $f(x) = 5x + 1$

47. $f(x) = \dfrac{3x - 4}{5}$

48. $f(x) = \dfrac{12 - x^2}{5}$

49. $f(x) = x^2 - 9$

50. $f(x) = x^3 - x$

51. $f(x) = \sqrt{x^2 - 16}$

52. $f(x) = \sqrt{4x^2 - x}$

In Exercises 53–56, find the value(s) of x for which $f(x) = g(x)$.

53. $f(x) = x^2, \quad g(x) = x + 2$

54. $f(x) = x^2 + 2x + 1, \quad g(x) = 3x + 3$

55. $f(x) = \sqrt{3x} + 1, \quad g(x) = x + 1$

56. $f(x) = x^4 - 2x^2, \quad g(x) = 2x^2$

In Exercises 57–70, determine the domain of the function.

57. $f(x) = 5x^2 + 2x - 1$

58. $g(x) = 1 - 2x^2$

59. $h(t) = \dfrac{4}{t}$

60. $s(y) = \dfrac{3y}{y + 5}$

61. $g(y) = \sqrt{y - 10}$

62. $f(t) = \sqrt[3]{t + 4}$

63. $f(x) = \sqrt[4]{1 - x^2}$

64. $f(x) = \sqrt[4]{x^2 + 3x}$

65. $g(x) = \dfrac{1}{x} - \dfrac{3}{x + 2}$

66. $h(x) = \dfrac{10}{x^2 - 2x}$

67. $f(s) = \dfrac{\sqrt{s - 1}}{s - 4}$

68. $f(x) = \dfrac{\sqrt{x + 6}}{6 + x}$

69. $f(x) = \dfrac{\sqrt[3]{x - 4}}{x}$

70. $f(x) = \dfrac{x - 5}{\sqrt[4]{x^2 - 9}}$

In Exercises 71–74, assume that the domain of f is the set $A = \{-2, -1, 0, 1, 2\}$. Determine the set of ordered pairs representing the function f.

71. $f(x) = x^2$

72. $f(x) = \dfrac{2x}{x^2 + 1}$

73. $f(x) = \sqrt{x + 2}$

74. $f(x) = |x + 1|$

Exploration **In Exercises 75–78, select a function from $f(x) = cx$, $g(x) = cx^2$, $h(x) = c\sqrt{|x|}$, or $r(x) = c/x$ and determine the value of the constant c such that the function fits the data given in the table.**

75.

x	-4	-1	0	1	4
y	-32	-2	0	-2	-32

76.

x	-4	-1	0	1	4
y	-1	$-\frac{1}{4}$	0	$\frac{1}{4}$	1

77.

x	-4	-1	0	1	4
y	-8	-32	Undef.	32	8

78.

x	-4	-1	0	1	4
y	6	3	0	3	6

In Exercises 79–86, find the difference quotient and simplify your answer.

79. $f(x) = 2x, \quad \dfrac{f(x + c) - f(x)}{c}, \; c \neq 0$

80. $g(x) = 3x - 1, \quad \dfrac{g(x + h) - g(x)}{h}, \; h \neq 0$

81. $f(x) = x^2 - x + 1, \quad \dfrac{f(2 + h) - f(2)}{h}, \; h \neq 0$

82. $f(x) = 5x - x^2, \quad \dfrac{f(5 + h) - f(5)}{h}, \; h \neq 0$

83. $f(x) = x^3, \quad \dfrac{f(x + c) - f(x)}{c}, \; c \neq 0$

84. $f(x) = x^3 + x, \quad \dfrac{f(x + h) - f(x)}{h}, \; h \neq 0$

85. $f(t) = \dfrac{1}{t}, \quad \dfrac{f(t) - f(1)}{t - 1}, \; t \neq 1$

86. $f(x) = \dfrac{4}{x + 1}, \quad \dfrac{f(x) - f(7)}{x - 7}, \; x \neq 7$

87. ***Geometry*** Express the area A of a circle as a function of its circumference C.

88. ***Geometry*** Express the area A of an equilateral triangle as a function of the length s of its sides.

89. ***Geometry*** Express the area A of an isosceles right triangle as a function of the length s of one of its two equal sides.

90. ***Geometry*** Express the area A of an equilateral triangle as a function of the height of the triangle.

The symbol indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

91. ***Exploration*** An open box of maximum volume is to be made from a square piece of material, 24 centimeters on a side, by cutting equal squares from the corners and turning up the sides.

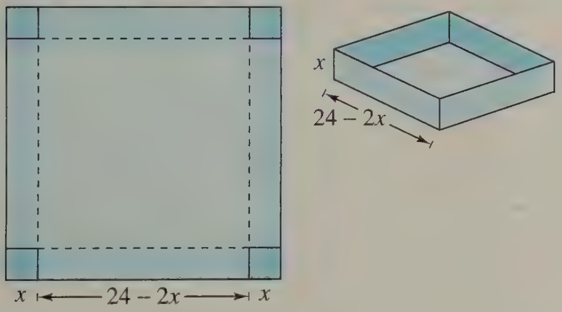
(a) Use the *table* feature of a graphing utility to complete six rows of the table. Use the result to guess the maximum volume.

Height, x	Width	Volume, V
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$

(b) Write the volume V as a function of x, and determine its domain. Use a graphing utility to graph the function.

(c) Use the *value* feature or *zoom* and *trace* features of the graphing utility to approximate V when $x = 9$ and when $x = 10$.

(d) Verify your answers to part (c) algebraically.



92. ***Exploration*** The cost per unit of a certain radio model is \$60. The manufacturer charges \$90 per unit for orders of 100 or less. To encourage large orders, the manufacturer reduces the charge by \$0.15 per radio for each unit ordered in excess of 100 (i.e., there would be a charge of \$87 per radio for an order size of 120).

(a) Use the *table* feature of a graphing utility to complete six rows of the table. Use the result to estimate the maximum profit.

Units, x	Price, p	Profit, P
110	$90 - 10(0.15)$	$xp - 110(60)$
120	$90 - 20(0.15)$	$xp - 120(60)$

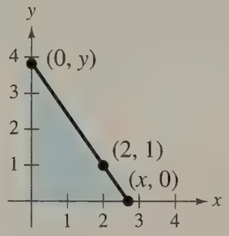
(b) Write the profit P as a function of x, and determine its domain. Use a graphing utility to graph the function.

(c) Use the *value* feature or *zoom* and *trace* features of the graphing utility to approximate P when $x = 120$, 130, and 140.

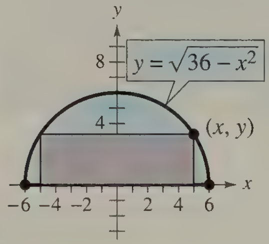
2

(d) Verify your answers to part (c) algebraically.

93. ***Geometry*** A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point (2, 1). Write the area of the triangle as a function of x, and determine the domain of the function.

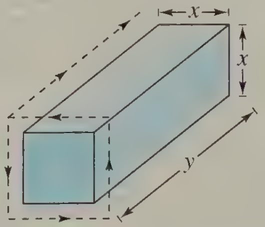


94. ***Geometry*** A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{36 - x^2}$. Write the area of the rectangle as a function of x, and determine the domain of the function.



2

95. ***Geometry*** A rectangular package to be sent by the U.S. Postal Service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches.



(a) Write the volume of the package as a function of x.

(b) What is the domain of the function?

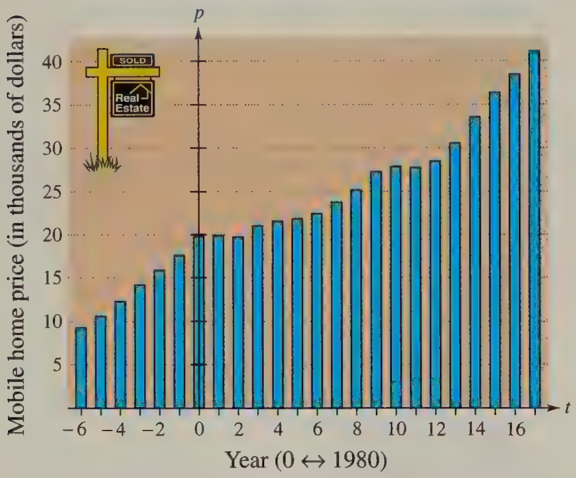
(c) Use a graphing utility to graph the function. Be sure to use the appropriate viewing window.

(d) What dimensions will maximize the volume of the package? Explain your answer.

96. ***Mobile Home Prices*** The average price p (in thousands of dollars) of a new mobile home in the United States from 1974 to 1997 can be approximated by the model

$$p(t) = \begin{cases} 17.27 + 1.036t, & -6 \le t \le 11 \\ -4.807 + 2.882t + 0.011t^2, & 12 \le t \le 17 \end{cases}$$

where $t = 0$ represents 1980. Use a graphing utility to graph this model. Then use the *value* feature or *zoom* and *trace* features to find the average price of a mobile home in 1978, 1988, 1993, and 1997. (Source: U.S. Bureau of the Census, *Construction Reports*)



97. ***Business*** A company produces a toy for which the variable cost is \$12.30 per unit and the fixed costs are \$98,000. The toy sells for \$17.98. Let x be the number of units produced and sold.

(a) Write the total cost C as a function of the number of units produced.

(b) Write the revenue R as a function of the number of units sold.

(c) Write the profit P as a function of the number of units sold. (*Note:* $P = R - C$.)

98. ***Business*** The inventor of a new game believes that the variable cost for producing the game is \$0.95 per unit and the fixed costs are \$6000. The inventor sells each game for \$1.69. Let x be the number of games sold.

(a) Write the total cost C as a function of the number of games sold.

(b) Write the average cost per unit $\overline{C} = C/x$ as a function of x.

99. ***Charter Bus Fares*** For groups of 80 or more people, a charter bus company determines the rate per person according to the formula

$$\text{Rate} = 8 - 0.05(n - 80), \quad n \ge 80$$

where the rate is given in dollars and n is the number of people.

(a) Express the revenue R for the bus company as a function of n.

(b) Use the function from part (a) to complete the table. What can you conclude?

n	90	100	110	120	130	140	150
$R(n)$							

(c) Use a graphing utility to graph R and determine the number of people that will produce a maximum revenue. Compare the result with your conclusion from part (b).

100. ***Physics*** The force F (in tons) of water against the face of a dam is a function $F(y) = 149.76\sqrt{10}y^{5/2}$, where y is the depth of the water in feet. Complete the table.

y	5	10	20	30	40
$F(y)$					

(a) What can you conclude from the table?

(b) Use a graphing utility to graph the function. Describe your viewing window.

(c) Use the table to approximate the depth at which the force against the dam is 1,000,000 tons. How could you find a better estimate?

(d) Verify your answer in part (c) graphically.

101. ***Height of a Balloon*** A balloon carrying a transmitter ascends vertically from a point 3000 feet from the receiving station.

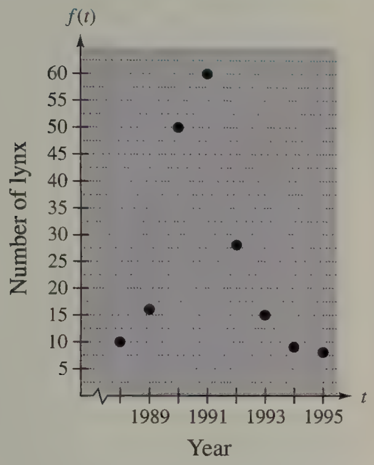
(a) Draw a diagram to represent the problem. Let h represent the height of the balloon and let d represent the distance between the balloon and the receiving station.

(b) Express the height of the balloon as a function of d. What is the domain of the function?

(c) Use a graphing utility to graph the function in part (b). Describe your viewing window.

(d) Graphically find the height of the balloon when $d = 10{,}000$ feet. Verify your answer algebraically.

102. ***Biology*** The graph below shows the lynx population from 1988 through 1995 in a 350-square-kilometer region of the Yukon territory in Canada. Let $f(t)$ represent the number of lynx in year t. (Source: Kluane Boreal Forest Ecosystem Project)



(a) Find $f(1992)$.

(b) Find $\dfrac{f(1994) - f(1991)}{1994 - 1991}$

and interpret the result in the context of the problem.

(c) An approximate model for the function is

$$N(t) = \frac{434t + 4387}{45t^2 - 55t + 100}$$

where N is the number of lynx and t is the time in years, with $t = 0$ corresponding to 1990. Complete the table and compare the result with the data. Use a graphing utility to graph the model and data in the same viewing window. Comment on the validity of the model.

t	1988	1989	1990	1991
N				

t	1992	1993	1994	1995
N				

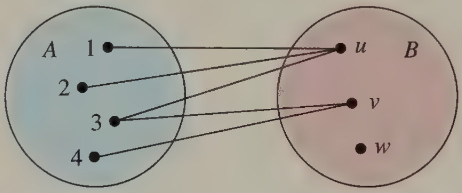
Synthesis

True or False? **In Exercises 103 and 104, determine whether the statement is true or false. Justify your answer.**

103. The domain of the function $f(x) = x^4 - 1$ is $(-\infty, \infty)$, and the range of $f(x)$ is $(0, \infty)$.

104. The set of ordered pairs $\{(-8, -2), (-6, 0), (-4, 0), (-2, 2), (0, 4), (2, -2)\}$ represents a function.

105. ***Think About It*** Does the relationship shown in the figure represent a function from set A to set B? Explain.



106. ***Writing*** In your own words, explain the meaning of domain and range.

107. ***Think About It*** Describe an advantage of function notation.

Review

In Exercises 108–111, perform the operations and simplify.

108. $12 - \dfrac{4}{x + 2}$

109. $\dfrac{3}{x^2 + x - 20} + \dfrac{x}{x^2 + 4x - 5}$

110. $\dfrac{2x^3 + 11x^2 - 6x}{5x} \cdot \dfrac{x + 10}{2x^2 + 5x - 3}$

111. $\dfrac{x + 7}{2(x - 9)} \div \dfrac{x - 7}{2(x - 9)}$

In Exercises 112–115, sketch the graph of the equation using the point-plotting method.

112. $y = -2x - 5$

113. $y = -x^2 - 3x$

114. $y = \sqrt{x - 6}$

115. $y = x^3 - 2$

1.4 Graphs of Functions

What You Should Learn:

- How to find the domains and ranges of functions
- How to use the Vertical Line Test for functions
- How to determine intervals on which functions are increasing or decreasing
- How to determine relative maximum and relative minimum values of functions
- How to identify and graph step functions and other piecewise-defined functions
- How to identify even and odd functions

Why You Should Learn It:

Graphs of functions provide a visual relationship between two variables. Exercise 87 on page 116 shows how the graph of a step function can represent the cost of a telephone call.

Jeff Greenberg/Peter Arnold, Inc.

The Graph of a Function

In Section 1.3 functions were represented graphically by points on a graph in a coordinate plane in which the input values are represented by the horizontal axis and the output values are represented by the vertical axis. The **graph of a function** f is the collection of ordered pairs $(x, f(x))$ such that x is in the domain of f. As you study this section, remember the following geometrical interpretation of x and $f(x)$.

$x =$ the directed distance from the y-axis

$f(x) =$ the directed distance from the x-axis

Example 1 shows how to use the graph of a function to find the domain and range of the function.

EXAMPLE 1 Finding the Domain and Range of a Function

Use the graph of the function f shown in Figure 1.27 to find (a) the domain of f, (b) the function values $f(-1)$ and $f(2)$, and (c) the range of f.

Solution

a. The closed dot (on the left) indicates that $x = -1$ is in the domain of f, whereas the open dot (on the right) indicates that $x = 4$ is not in the domain. So, the domain of f is all x in the interval $[-1, 4)$.

b. Because $(-1, -5)$ is a point on the graph of f, it follows that

$$f(-1) = -5.$$

Similarly, because $(2, 4)$ is a point on the graph of f, it follows that

$$f(2) = 4.$$

c. Because the graph does not extend below $f(-1) = -5$ or above $f(2) = 4$, the range of f is the interval $[-5, 4]$.

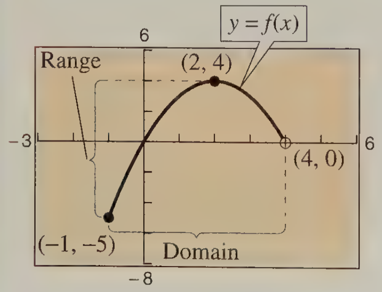


Figure 1.27

The use of dots (open or closed) at the extreme left and right points of a graph indicates that the graph does not extend beyond these points. If no such dots are shown, assume that the graph extends beyond these points.

EXAMPLE 2 Finding the Domain and Range of a Function

Find the domain and range of $f(x) = \sqrt{x-4}$.

Algebraic Solution

Because the expression under a radical cannot be negative, the domain of $f(x) = \sqrt{x-4}$ is the set of all real numbers such that $x - 4 \geq 0$. Solve this linear inequality for x as follows. (For help with solving linear inequalities, see Appendix C.)

$x - 4 \geq 0$ Write original inequality.

$x \geq 4$ Add 4 to each side.

So, the domain is the set of all real numbers greater than or equal to 4. Because the value of a radical expression is never negative, the range of $f(x) = \sqrt{x-4}$ is the set of all nonnegative real numbers.

Graphical Solution

Use a graphing utility to graph the equation $y = \sqrt{x-4}$, as shown in Figure 1.28. Use the *trace* feature to determine that the x-coordinates of points on the graph extend from 4 to the right. When x is greater than or equal to 4, the expression under the radical is nonnegative. So, you can conclude that the domain is the set of all real numbers greater than or equal to 4. From the graph, you can see that the y-coordinates of points on the graph extend from 0 upwards. So you can estimate the range to be the set of all nonnegative real numbers.

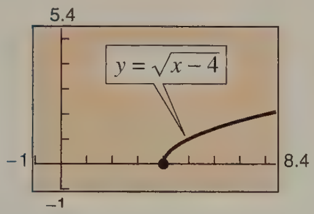


Figure 1.28

By the definition of a function, at most one y-value corresponds to a given x-value. It follows, then, that a vertical line can intersect the graph of a function at most once. This leads to the **Vertical Line Test** for functions.

Vertical Line Test for Functions

A set of points in a coordinate plane is the graph of y as a function of x if and only if no vertical line intersects the graph at more than one point.

EXAMPLE 3 Vertical Line Test for Functions

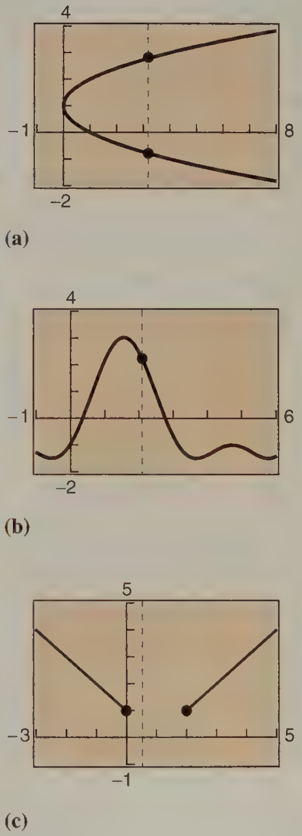
Which of the graphs in Figure 1.29 represent y as a function of x?

Solution

a. This *is not* a graph of y as a function of x because you can find a vertical line that intersects the graph twice.

b. This *is* a graph of y as a function of x because every vertical line intersects the graph at most once.

c. This *is* a graph of y as a function of x. (Note that if a vertical line does not intersect the graph, it simply means that the function is undefined for that particular value of x.)



(a)

(b)

(c)

Figure 1.29

Most graphing utilities are designed to graph functions of x more easily than other types of equations. For instance, the graph shown in Figure 1.29(a) represents the equation $x - (y-1)^2 = 0$. To use a graphing utility to duplicate this graph you must first solve the equation for y to obtain $y = 1 \pm \sqrt{x}$, and then graph the two equations $y_1 = 1 + \sqrt{x}$ and $y_2 = 1 - \sqrt{x}$ in the same viewing window.

Increasing and Decreasing Functions

The more you know about the graph of a function, the more you know about the function itself. Consider the graph shown in Figure 1.30. Moving from *left to right,* this graph falls from $x = -2$ to $x = 0$, is constant from $x = 0$ to $x = 2$, and rises from $x = 2$ to $x = 4$.

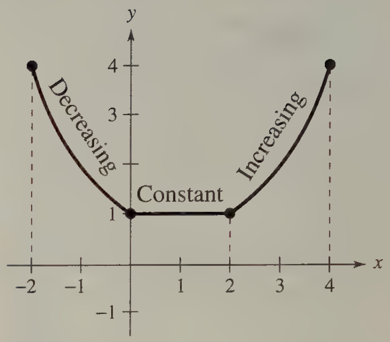


Figure 1.30

Increasing, Decreasing, and Constant Functions

A function f is **increasing** on an interval if, for any x_1 and x_2 in the interval,

$$x_1 < x_2 \text{ implies } f(x_1) < f(x_2).$$

A function f is **decreasing** on an interval if, for any x_1 and x_2 in the interval,

$$x_1 < x_2 \text{ implies } f(x_1) > f(x_2).$$

A function f is **constant** on an interval if, for any x_1 and x_2 in the interval,

$$f(x_1) = f(x_2).$$

EXAMPLE 4 Increasing and Decreasing Functions

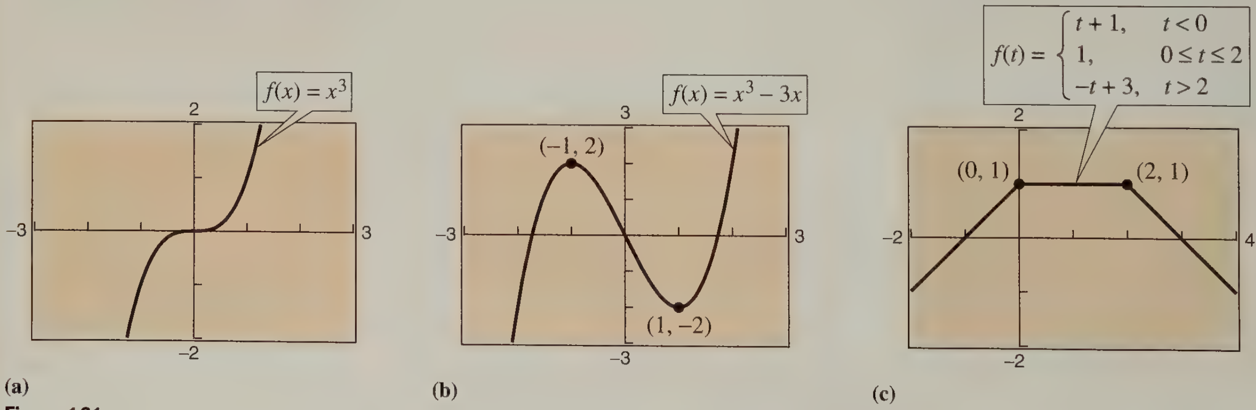
In Figure 1.31, determine the open intervals on which each function is increasing, decreasing, or constant.

Solution

a. Although it might appear that there is an interval in which this function is constant, you can see that if $x_1 < x_2$, then $(x_1)^3 < (x_2)^3$, which implies that $f(x_1) < f(x_2)$. So, the function is increasing over the entire real line.

b. This function is increasing on the interval $(-\infty, -1)$, decreasing on the interval $(-1, 1)$, and increasing on the interval $(1, \infty)$.

c. This function is increasing on the interval $(-\infty, 0)$, constant on the interval $(0, 2)$, and decreasing on the interval $(2, \infty)$.



(a) (b) (c)

Figure 1.31

Relative Minimum and Maximum Values

The points at which a function changes its increasing, decreasing, or constant behavior are helpful in determining the relative maximum or relative minimum values of the function.

Definition of Relative Minimum and Relative Maximum

A function value $f(a)$ is called a **relative minimum** of f if there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \le f(x).$$

A function value $f(a)$ is called a **relative maximum** of f if there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \ge f(x).$$

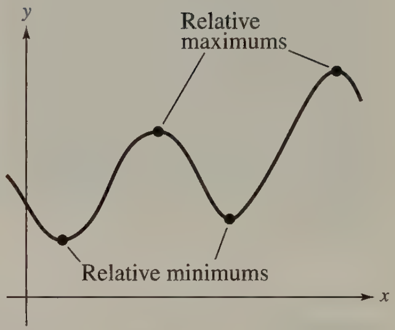


Figure 1.32

Figure 1.32 shows several different examples of relative minimums and relative maximums. In Section 3.1, you will study a technique for finding the *exact points* at which a second-degree polynomial function has a relative minimum or relative maximum. For the time being, however, you can use a graphing utility to find reasonable approximations of these points.

EXAMPLE 5 Approximating a Relative Minimum

Use a graphing utility to approximate the relative minimum of the function $f(x) = 3x^2 - 4x - 2$.

Solution

The graph of f is shown in Figure 1.33. By using the *zoom* and *trace* features of a graphing utility, you can estimate that the function has a relative minimum at the point

$(0.67, -3.33)$. Approximate relative minimum

Later, in Section 3.1, you will be able to determine that the exact point at which the relative minimum occurs is $\left(\frac{2}{3}, -\frac{10}{3}\right)$.

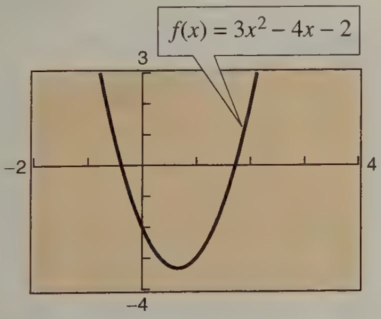


Figure 1.33

STUDY T!P

When you use a graphing utility to estimate x- and y-values of a relative minimum or relative maximum, the automatic *zoom* feature will often produce graphs that are nearly flat. To overcome this problem, you can manually change the vertical setting of the viewing window. The graph will vertically stretch if the values of Ymin and Ymax are closer together.

EXAMPLE 6 Approximating Relative Minimums and Maximums

Use a graphing utility to approximate the relative minimum and relative maximum of the function $f(x) = -x^3 + x$.

Solution

A sketch of the graph of f is shown in Figure 1.34. By using the *zoom* and *trace* features of the graphing utility, you can estimate that the function has a relative minimum at the point

$(-0.58, -0.38)$ Approximate relative minimum

and a relative maximum at the point

$(0.58, 0.38)$. Approximate relative maximum

If you go on to take a course in calculus, you will learn a technique for finding the exact points at which this function has a relative minimum and a relative maximum.

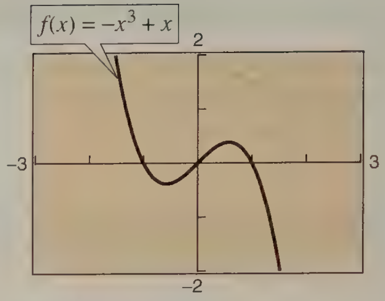


Figure 1.34

STUDY TIP

Some graphing utilities have built-in programs that will find minimum or maximum values. If your graphing utility has such features, try using them to rework Example 6.

EXAMPLE 7 Bowling Equipment Sales

During the 1990s, the sales of bowling equipment in the United States increased and then decreased according to the model

$$C = 0.165t^3 - 7.16t^2 + 100.6t - 303.1, \qquad 8 \le t \le 16$$

where C is the sales of bowling equipment (in millions of dollars) and t represents the year, with $t = 8$ corresponding to 1988. According to this model, during which years were bowling equipment sales increasing? During which years were bowling equipment sales decreasing? Approximate the maximum amount of bowling equipment sales between 1988 and 1996. (Source: National Sporting Goods Association)

Solution

To solve this problem graph the function, as shown in Figure 1.35. From the graph, you can see that the bowling equipment sales increased from 1988 until 1992. Then, from 1992 to 1996, the sales decreased. The maximum amount of bowling equipment sales during the 8-year period was approximately $158 million.

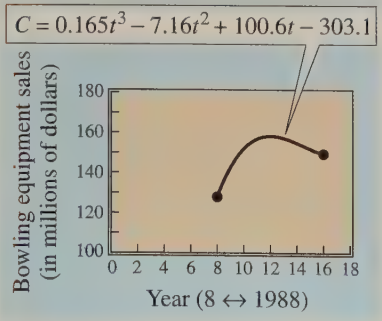


Figure 1.35

Graphing Step Functions and Piecewise-Defined Functions

The **greatest integer function** is denoted by $[\![x]\!]$ and is defined by

$$f(x) = [\![x]\!] = \text{the greatest integer less than or equal to } x.$$

The graph of this function is shown in Figure 1.36.

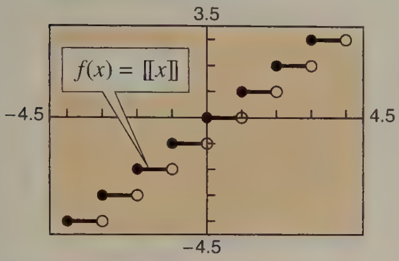


Figure 1.36

The graph of the greatest integer function jumps vertically one unit at each integer and is constant (a horizontal line segment) between each pair of consecutive integers. Because of these jumps, the greatest integer function is an example of a **step function** whose graph resembles a set of stair steps. Some values of the greatest integer function are as follows.

$[\![-2]\!] = -2 \qquad [\![-1.5]\!] = -2 \qquad [\![-1]\!] = -1$

$[\![-0.5]\!] = -1 \qquad [\![0]\!] = 0 \qquad [\![0.5]\!] = 0$

$[\![1]\!] = 1 \qquad [\![1.5]\!] = 1 \qquad [\![2]\!] = 2$

The range of the greatest integer function is the set of all integers.

In Section 1.3, you learned that a piecewise-defined function is a function that is defined by two or more equations over a specified domain. To sketch the graph of a piecewise-defined function, you need to sketch the graph of each equation on the appropriate portion of the domain.

EXAMPLE 8 Graphing a Piecewise-Defined Function

Sketch the graph of $f(x) = \begin{cases} 2x + 3, & x \le 1 \\ -x + 4, & x > 1 \end{cases}$.

Solution

This piecewise-defined function is composed of two linear functions. To the left of $x = 1$, the graph is the line given by $y = 2x + 3$. To the right of $x = 1$, the graph is the line given by $y = -x + 4$ (see Figure 1.37). Notice that the point $(1, 5)$ is a solid dot and the point $(1, 3)$ is an open dot. This is because $f(1) = 5$.

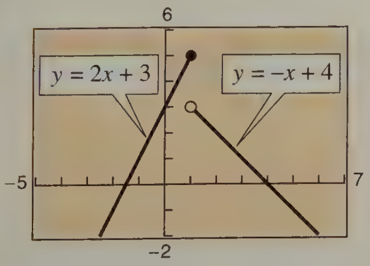


Figure 1.37

Library of Functions

The *greatest integer* function has an infinite number of breaks or steps—one at each integer value in its domain. Could you describe the greatest integer function using a piecewise-defined function? How does the graph of the greatest integer function differ from the graph of a line with zero slope?

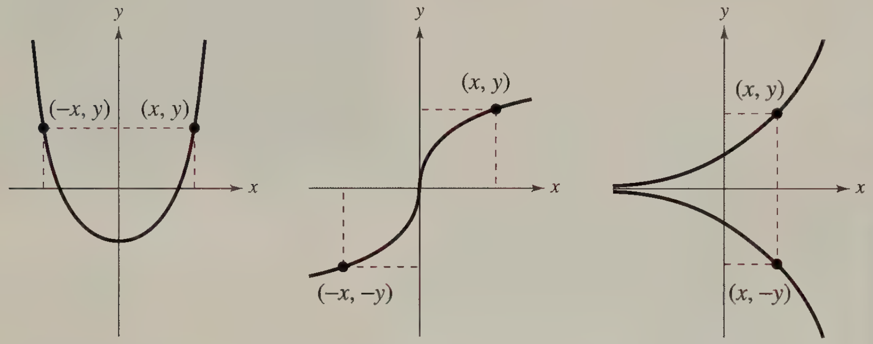
Consult the Library of Functions Summary inside the front cover for a description of the greatest integer function.

STUDY TIP

Most graphing utilities display graphs in *connected mode*, which means that the graph has no breaks. When you are sketching graphs that do have breaks, it is better to use *dot mode*. Graph the greatest integer function [often called Int (x)] in connected and dot modes, and compare the two results.

Even and Odd Functions

A graph has *symmetry with respect to the y-axis* if whenever (x, y) is on the graph, so is the point $(-x, y)$. A graph has *symmetry with respect to the origin* if whenever (x, y) is on the graph, so is the point $(-x, -y)$. A graph has *symmetry with respect to the x-axis* if whenever (x, y) is on the graph, so is the point $(x, -y)$. A function whose graph is symmetric with respect to the y-axis is an **even** function. A function whose graph is symmetric with respect to the origin is an **odd** function. The graph of a (nonzero) function cannot be symmetric with respect to the x-axis. These three types of symmetry are illustrated in Figure 1.38.



Symmetric to y-axis
Even function

Symmetric to origin
Odd function

Symmetric to x-axis
Not a function

Figure 1.38

Test for Even and Odd Functions

A function f is **even** if, for each x in the domain of f, $f(-x) = f(x)$.

A function f is **odd** if, for each x in the domain of f, $f(-x) = -f(x)$.

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

STUDY T!P

The ⋁ -shaped graph in Example 9 is a graph of the *absolute value function.* It was produced using the *abs* key on a graphing utility and the standard viewing window.

Library of Functions

The *absolute value* function can be expressed as a piecewise-defined function.

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Use this definition to show that $|-5| = 5$ and $|0| = 0$.

Consult the Library of Functions Summary inside the front cover for a description of the absolute value function.

EXAMPLE 9 Testing for Evenness and Oddness

Is the function $f(x) = |x|$ even, odd, or neither?

Algebraic Solution

This function is even because

$$\begin{aligned} f(-x) &= |-x| \\ &= |x| \\ &= f(x). \end{aligned}$$

Graphical Solution

Use a graphing utility to graph $y = |x|$, as shown in Figure 1.39. You can see that the graph appears to be symmetric about the y-axis. So, the function is even.

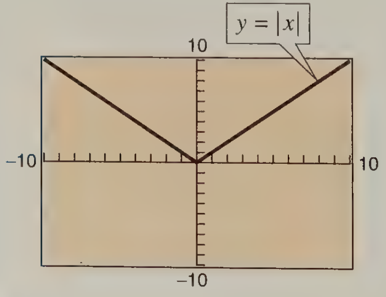


Figure 1.39

EXAMPLE 10 Even and Odd Functions

Determine whether each function is even, odd, or neither.

a. $g(x) = x^3 - x$ **b.** $h(x) = x^2 + 1$ **c.** $f(x) = x^3 - 1$

Algebraic Solution

a. This function is odd because

$$\begin{aligned} g(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -(x^3 - x) \\ &= -g(x). \end{aligned}$$

b. This function is even because

$$\begin{aligned} h(-x) &= (-x)^2 + 1 \\ &= x^2 + 1 \\ &= h(x). \end{aligned}$$

c. Substituting $-x$ for x produces

$$\begin{aligned} f(-x) &= (-x)^3 - 1 \\ &= -x^3 - 1. \end{aligned}$$

Because $f(x) = x^3 - 1$ and $-f(x) = -x^3 + 1$, you can conclude that $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. So, the function is neither even nor odd.

Graphical Solution

a. In Figure 1.40(a), the graph is symmetric with respect to the origin. So, this function is odd.

b. In Figure 1.40(b), the graph is symmetric with respect to the y-axis. So, this function is even.

c. In Figure 1.40(c), the graph is neither symmetric to the origin nor to the y-axis. So, this function is neither even nor odd.

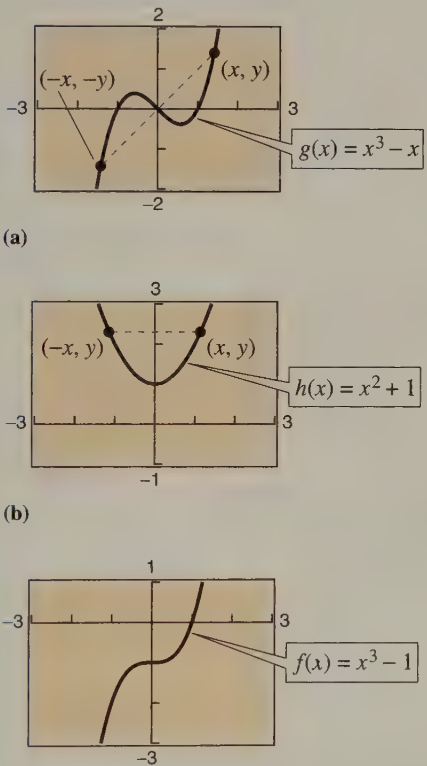


Figure 1.40

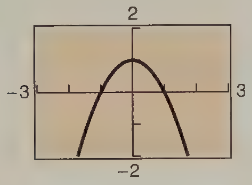
Writing About Math *Increasing and Decreasing Functions*

Write a short paragraph describing three different functions that represent quantities between 1985 and 2000. Describe one that decreased during this time, one that increased, and one that was constant. For instance, the value of the dollar decreased, the cost of first-class postage increased, and the land size of the United States remained constant. Present your results graphically.

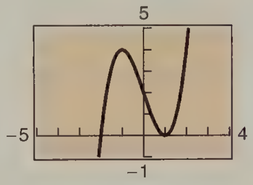
1.4 Exercises

In Exercises 1–6, find the domain and range of the function.

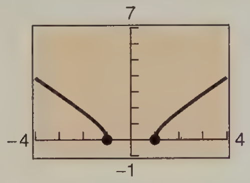
1. $f(x) = 1 - x^2$



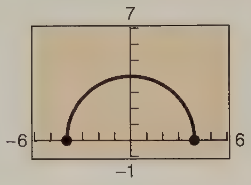
2. $f(x) = x^3 - 3x + 2$



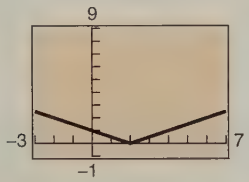
3. $f(x) = \sqrt{x^2 - 1}$



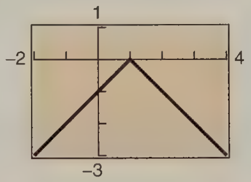
4. $h(x) = \sqrt{16 - x^2}$



5. $f(x) = \frac{1}{2}|x - 2|$



6. $g(x) = -|x - 1|$



In Exercises 7–12, use a graphing utility to graph the function and estimate its domain and range. Then find the domain and range algebraically.

7. $f(x) = 2x^2 + 3$

8. $f(x) = -x^2 - 1$

9. $f(x) = \sqrt{x - 1}$

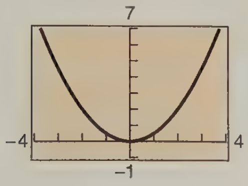
10. $h(t) = \sqrt{4 - t^2}$

11. $f(x) = |x + 3|$

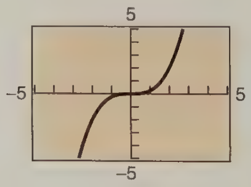
12. $f(x) = -\frac{1}{4}|x - 5|$

In Exercises 13–18, use the Vertical Line Test to determine whether y is a function of x. Describe how you can use a graphing utility to produce the given graph.

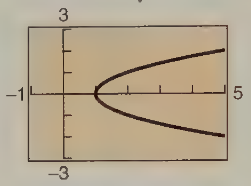
13. $y = \frac{1}{2}x^2$



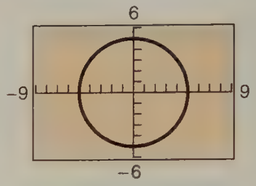
14. $y = \frac{1}{4}x^3$



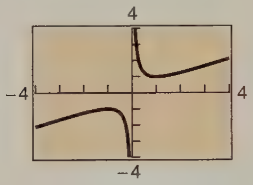
15. $x - y^2 = 1$



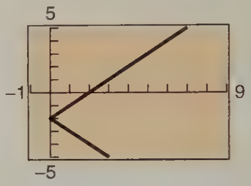
16. $x^2 + y^2 = 25$



17. $x^2 = 2xy - 1$

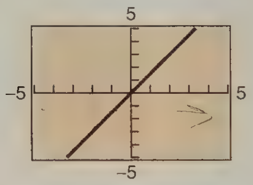


18. $x = |y + 2|$

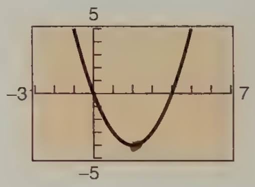


In Exercises 19–22, (a) determine the intervals over which the function is increasing, decreasing, or constant and (b) determine whether the function is even, odd, or neither.

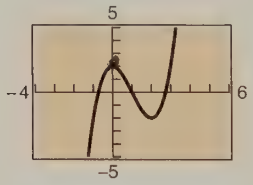
19. $f(x) = \frac{3}{2}x$



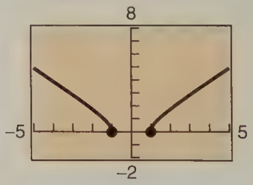
20. $f(x) = x^2 - 4x$



21. $f(x) = x^3 - 3x^2 + 2$



22. $f(x) = \sqrt{x^2 - 1}$



In Exercises 23–30, (a) use a graphing utility to graph the function; (b) determine the intervals over which the function is increasing, decreasing, or constant; and (c) determine whether the function is even, odd, or neither.

23. $f(x) = 3x^4 - 6x^2$

24. $f(x) = -x^6 - 2x^4$

25. $f(x) = x^{2/3}$

26. $f(x) = -x^{3/4}$

27. $f(x) = x\sqrt{x+3}$

28. $f(x) = x(x^2+1)^{1/2}$

29. $f(x) = |x+1| + |x-1|$

30. $f(x) = -|x+4| - |x+1|$

In Exercises 31–36, use a graphing utility to approximate (to two decimal places) any relative minimum or maximum values of the function.

31. $f(x) = x^2 - 6x$

32. $f(x) = 3x^2 - 2x - 5$

33. $y = 2x^3 + 3x^2 - 12x$

34. $y = x^3 - 6x^2 + 15$

35. $h(x) = (x-1)\sqrt{x}$

36. $g(x) = x\sqrt{4-x}$

In Exercises 37–42, (a) approximate the relative minimum or maximum values of the function by sketching its graph using the point-plotting method, (b) use a graphing utility to approximate (to two decimal places) any relative minimum or maximum values, and (c) compare your answers from parts (a) and (b).

37. $f(x) = x^2 - 4x - 5$

38. $f(x) = 3x^2 - 12$

39. $f(x) = x^3 - 8x$

40. $f(x) = -x^3 + 7x$

41. $f(x) = (x-4)^{2/3}$

42. $f(x) = \sqrt{4x^2+1}$

In Exercises 43–46, use a graphing utility to graph the piecewise-defined function.

43. $f(x) = \begin{cases} 2x+3, & x < 0 \\ 3-x, & x \ge 0 \end{cases}$

44. $f(x) = \begin{cases} x^2+5, & x \le 1 \\ -x^2+4x+3, & x > 1 \end{cases}$

45. $f(x) = \begin{cases} \sqrt{4+x}, & x < 0 \\ \sqrt{4-x}, & x \ge 0 \end{cases}$

46. $f(x) = \begin{cases} 1-(x-1)^2, & x \le 2 \\ \sqrt{x-2}, & x > 2 \end{cases}$

In Exercises 47–54, algebraically determine whether the function is even, odd, or neither. Verify your answer using a graphing utility.

47. $f(t) = t^2 + 2t - 3$

48. $f(x) = x^6 - 2x^2 + 3$

49. $g(x) = x^3 - 5x$

50. $h(x) = x^3 - 5$

51. $f(x) = x\sqrt{1-x^2}$

52. $f(x) = x\sqrt{x+5}$

53. $g(s) = 4s^{2/3}$

54. $f(s) = 4s^{3/2}$

Think About It **In Exercises 55–60, find the coordinates of a second point on the graph of a function *f* if the given point is on the graph and the function is (a) even and (b) odd.**

55. $\left(-\frac{3}{2}, 4\right)$

56. $\left(-\frac{5}{3}, -7\right)$

57. $(4, 9)$

58. $(5, -1)$

59. $(x, -y)$

60. $(2a, 2c)$

In Exercises 61–72, use a graphing utility to graph the function and determine whether it is even, odd, or neither. Verify your answer algebraically.

61. $f(x) = 5$

62. $f(x) = -9$

63. $f(x) = 3x - 2$

64. $f(x) = 5 - 3x$

65. $h(x) = x^2 - 4$

66. $f(x) = -x^2 - 8$

67. $f(x) = \sqrt{1-x}$

68. $g(t) = \sqrt[3]{t-1}$

69. $f(x) = |x+2|$

70. $f(x) = -|x-5|$

71. $f(x) = \begin{cases} x+3, & x \le 0 \\ 3, & 0 < x \le 2 \\ 2x-1, & x > 2 \end{cases}$

72. $f(x) = \begin{cases} 2x+1, & x \le -1 \\ x^2-2, & x > -1 \end{cases}$

In Exercises 73–82, graph the function and determine the interval(s) (if any) on the real axis for which $f(x) \ge 0$. Use a graphing utility to verify your results.

73. $f(x) = 4 - x$

74. $f(x) = 4x + 2$

75. $f(x) = x^2 - 9$

76. $f(x) = x^2 - 4x$

77. $f(x) = 1 - x^4$

78. $f(x) = x^2 + 1$

79. $f(x) = \sqrt{x+2}$

80. $f(x) = -2\sqrt{x-3}$

81. $f(x) = -(1 + |x|)$

82. $f(x) = \frac{1}{2}(2 + |x|)$

In Exercises 83 and 84, use a graphing utility to graph the function. State the domain and range of the function. Describe the pattern of the graph.

83. $s(x) = 2\left(\frac{1}{4}x - [\![\frac{1}{4}x]\!]\right)$

84. $g(x) = 2\left(\frac{1}{4}x - [\![\frac{1}{4}x]\!]\right)^2$

85. ***Geometry*** The perimeter of a rectangle is 100 meters.

(a) Show that the area of the rectangle is $A = x(50 - x)$, where x is its length.

(b) Use a graphing utility to graph the area function.

(c) Use a graphing utility to approximate the maximum area of the rectangle and the dimensions that yield the maximum area.

86. ***Business*** The marketing department of a company estimates that the demand for a product is

$$p = 100 - 0.0001x$$

where p is the price per unit and x is the number of units. The cost of producing x units is

$$C = 350{,}000 + 30x$$

and the profit for producing and selling x units is

$$P = R - C = xp - C.$$

Use a graphing utility to graph the profit function and estimate the number of units that would produce a maximum profit.

87. ***Communications*** The cost of using a telephone calling card is \$1.05 for the first minute and \$0.38 for each additional minute.

(a) Which of the following is the appropriate model for the cost C of a telephone call lasting t minutes? Explain.

$$C_1(t) = 1.05 + 0.38[\![t - 1]\!]$$

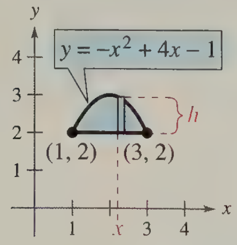
$$C_2(t) = 1.05 - 0.38[\![-(t - 1)]\!]$$

(b) Use a graphing utility to graph the appropriate model. Use the *value* feature or *zoom* and *trace* features to estimate the cost of a call lasting 18 minutes and 45 seconds.

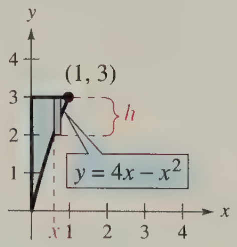
88. ***Delivery Service*** Suppose that the cost of sending an overnight package from New York to Atlanta is \$9.80 for under one pound and \$2.50 for each additional pound. Use the greatest integer function to create a model for the cost C of overnight delivery of a package weighing x pounds where $x > 0$. Sketch the graph of the function.

In Exercises 89–92, write the height h of the rectangle as a function of x.

89.



90.



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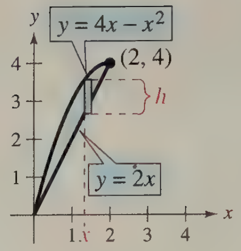
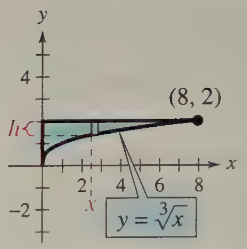
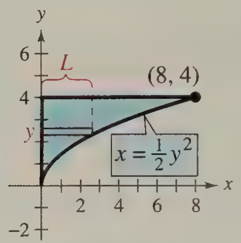


92.

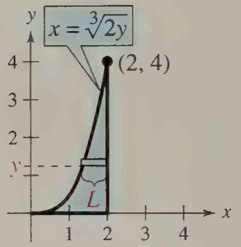


In Exercises 93 and 94, write the length L of the rectangle as a function of y.

93.



94.



95. ***Data Analysis*** The table shows the amount y (in billions of dollars) of the merchandise trade balance of the United States for the years 1990 through 1997. (Source: U.S. International Trade Administration)

Year	1990	1991	1992	1993	1994	1995	1996	1997
y	−101.7	−66.7	−84.5	−115.6	−150.6	−158.7	−170.2	−181.5

(a) Use the regression capabilities of a graphing utility to find a cubic model for the data. Let x be the time (in years), with $x = 0$ corresponding to 1990.

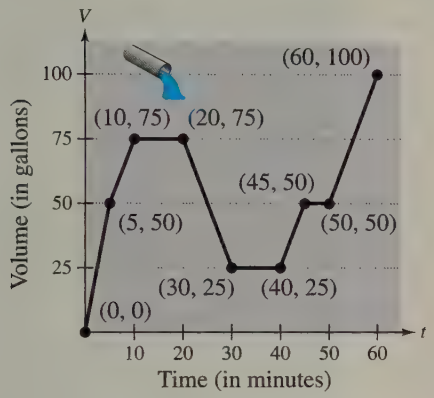
(b) What is the domain of the model?

(c) Use a graphing utility to graph the data and the model on the same viewing window.

(d) For which year does the model most accurately estimate the actual data? During which year is it least accurate?

(e) If this model remained valid in the future, would the economy show improvement? Explain your answer.

96. ***Fluid Flow*** The intake pipe of a 100-gallon tank has a flow rate of 10 gallons per minute, and two drain pipes have a flow rate of 5 gallons per minute each. The graph shows the volume V of fluid in the tank as a function of time t. Determine the pipes in which the fluid is flowing in specific subintervals of the 1 hour of time shown on the graph. (There are many correct answers.)



Synthesis

True or False? **In Exercises 97 and 98, determine whether the statement is true or false. Justify your answer.**

97. A function with a square root cannot have a domain that is the set of all real numbers.

98. It is possible for an odd function to have the interval $[0, \infty)$ as its domain.

99. Prove that a function of the following form is odd.

$$y = a_{2n+1}x^{2n+1} + a_{2n-1}x^{2n-1} + \cdot \cdot \cdot + a_3x^3 + a_1x$$

100. Prove that a function of the following form is even.

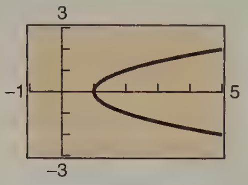
$$y = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdot \cdot \cdot + a_2x^2 + a_0$$

101. If f is an even function, determine if g is even, odd, or neither. Explain.

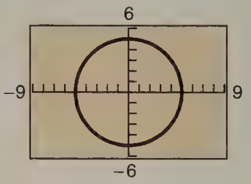
(a) $g(x) = -f(x)$ (b) $g(x) = f(-x)$

(c) $g(x) = f(x) - 2$ (d) $g(x) = -f(x - 2)$

102. ***Think About It*** Does the graph of $x - y^2 = 1$ represent x as a function of y? Explain.



103. ***Think About It*** Does the graph of $x^2 + y^2 = 25$ represent x as a function of y? Explain.



Review

In Exercises 104–107, find (a) the distance between the two points and (b) the midpoint of the segment joining the points.

104. $(-2, 7), (6, 3)$ **105.** $(-5, 0), (3, 6)$

106. $\left(\frac{5}{2}, -1\right), \left(-\frac{3}{2}, 4\right)$ **107.** $\left(-6, \frac{2}{3}\right), \left(\frac{3}{4}, \frac{1}{6}\right)$

In Exercises 108–111, evaluate the function at each specified value of the independent variable and simplify.

108. $f(x) = 5x - 1$

(a) $f(6)$ (b) $f(-1)$ (c) $f(x - 3)$

109. $f(x) = -x^2 - x + 3$

(a) $f(4)$ (b) $f(-2)$ (c) $f(x - 2)$

110. $f(x) = x\sqrt{x - 3}$

(a) $f(3)$ (b) $f(12)$ (c) $f(6)$

111. $f(x) = -\frac{1}{2}x|x + 1|$

(a) $f(-4)$ (b) $f(10)$ (c) $f\left(-\frac{2}{3}\right)$

In Exercises 112 and 113, find the difference quotient.

112. $f(x) = x^2 - 2x + 9, \quad \dfrac{f(3 + h) - f(3)}{h}, \quad h \neq 0$

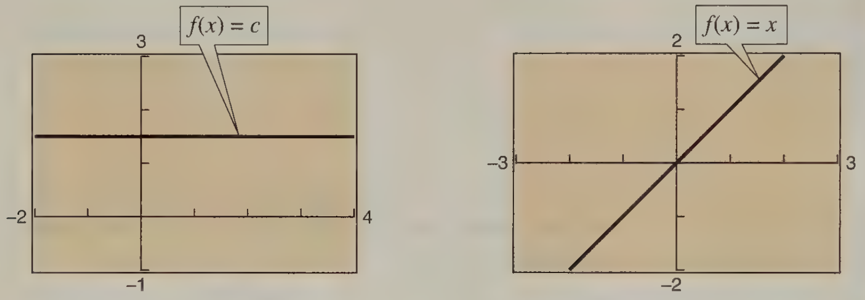
113. $f(x) = 5 + 6x - x^2, \quad \dfrac{f(6 + h) - f(6)}{h}, \quad h \neq 0$

1.5 Shifting, Reflecting, and Stretching Graphs

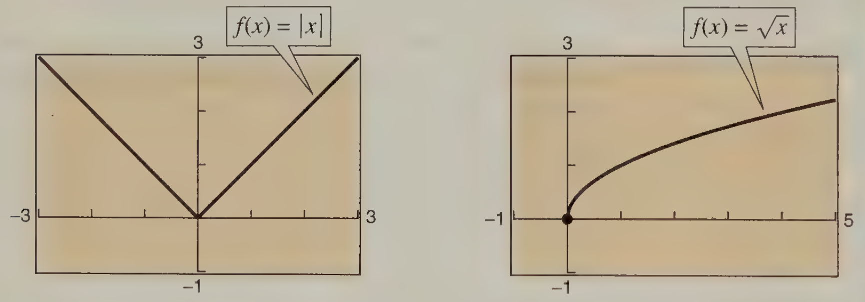
Summary of Graphs of Common Functions

One of the goals of this text is to enable you to build your intuition for the basic shapes of the graphs of different types of functions. For instance, from your study of lines in Section 1.2, you can determine the basic shape of the graph of the linear function $f(x) = mx + b$. Specifically, you know that the graph of this function is a line whose slope is m and whose y-intercept is b.

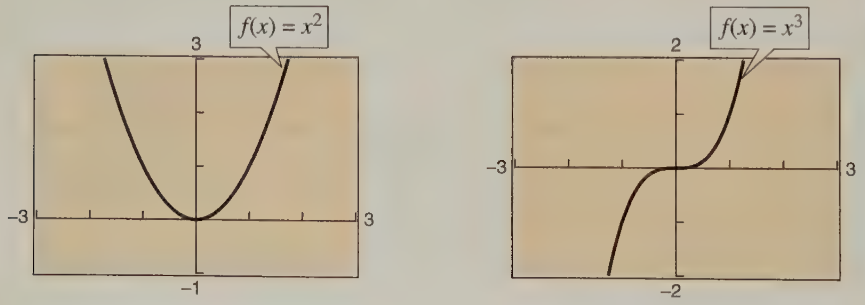
The six graphs shown in Figure 1.41 represent the most commonly used functions in algebra. Familiarity with the basic characteristics of these simple graphs will help you analyze the shapes of more complicated graphs.



(a) Constant Function **(b) Identity Function**



(c) Absolute Value Function **(d) Square Root Function**



(e) Quadratic Function **(f) Cubic Function**

Figure 1.41

What You Should Learn:

- How to recognize graphs of common functions
- How to use vertical and horizontal shifts and reflections to sketch graphs of functions
- How to use nonrigid transformations to sketch graphs of functions

Why You Should Learn It:

Knowing the graphs of common functions and knowing how to shift, reflect, and stretch graphs of functions can help you sketch a wide variety of simple functions by hand. This skill is useful in sketching graphs of functions that model real-life data, such as in Exercise 77 on page 126, where you are asked to sketch a function that models the amount of fuel used by trucks from 1980 through 1996.

Index Stock

Vertical and Horizontal Shifts

Many functions have graphs that are simple transformations of the common graphs summarized in Figure 1.41. For example, you can obtain the graph of

$$h(x) = x^2 + 2$$

by shifting the graph of $f(x) = x^2$ *up* two units, as shown in Figure 1.42. In function notation, h and f are related as follows.

$$\begin{aligned} h(x) &= x^2 + 2 \\ &= f(x) + 2 \qquad \text{Upward shift of 2} \end{aligned}$$

Similarly, you can obtain the graph of

$$g(x) = (x - 2)^2$$

by shifting the graph of $f(x) = x^2$ to the *right* two units, as shown in Figure 1.43. In this case, the functions g and f have the following relationship.

$$\begin{aligned} g(x) &= (x - 2)^2 \\ &= f(x - 2) \qquad \text{Right shift of 2} \end{aligned}$$

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Exploration

Use a graphing utility to display the graphs of $y = x^2 + c$, where $c = -4, -2, 0, 2,$ and 4. Use the result to describe the effect that c has on the graph.

Use a graphing utility to display the graphs of $y = (x + c)^2$, where $c = -2, 0, 2,$ and 4. Use the result to describe the effect that c has on the graph.

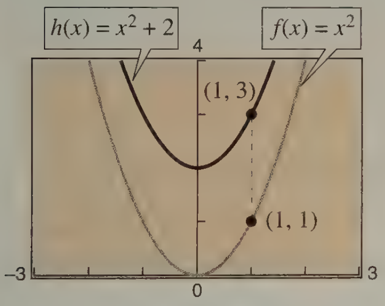


Figure 1.42 *Vertical shift upward: two units*

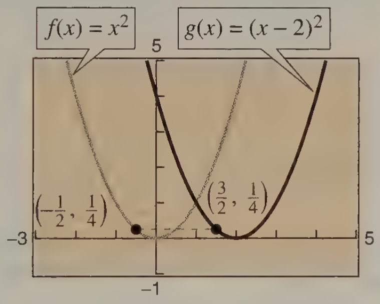


Figure 1.43 *Horizontal shift to the right: two units*

The following list summarizes this discussion about horizontal and vertical shifts.

Vertical and Horizontal Shifts

Let c be a positive real number. **Vertical and horizontal shifts** in the graph of $y = f(x)$ are represented as follows.

1. Vertical shift c units *upward:* $h(x) = f(x) + c$
2. Vertical shift c units *downward:* $h(x) = f(x) - c$
3. Horizontal shift c units to the *right:* $h(x) = f(x - c)$
4. Horizontal shift c units to the *left:* $h(x) = f(x + c)$

In items 3 and 4, be sure you see that $h(x) = f(x - c)$ corresponds to a *right* shift and $h(x) = f(x + c)$ corresponds to a *left* shift for $c > 0$.

EXAMPLE 1 Shifts in the Graph of a Function

Compare the graph of each function with the graph of $f(x) = x^3$.

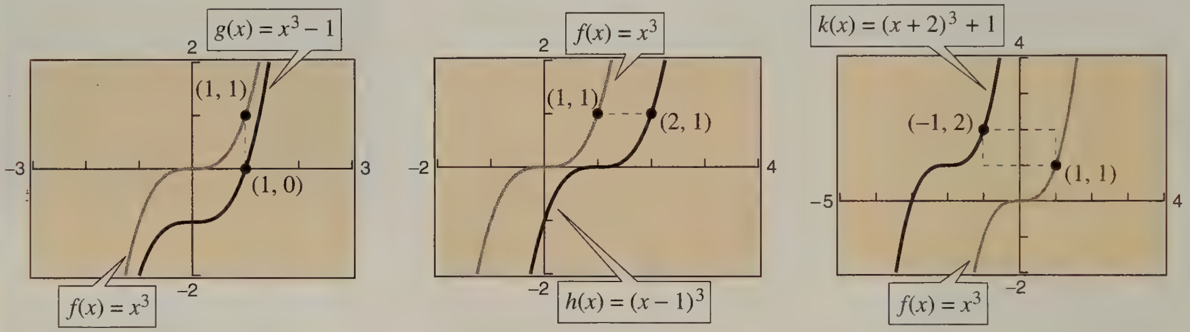
a. $g(x) = x^3 - 1$ **b.** $h(x) = (x - 1)^3$ **c.** $k(x) = (x + 2)^3 + 1$

Solution

a. Graph $f(x) = x^3$ and $g(x) = x^3 - 1$ [see Figure 1.44(a)]. You can see that you can obtain the graph of g by shifting the graph of f one unit down.

b. Graph $f(x) = x^3$ and $h(x) = (x - 1)^3$ [see Figure 1.44(b)]. You can obtain the graph of h by shifting the graph of f one unit to the right.

c. Graph $f(x) = x^3$ and $k(x) = (x + 2)^3 + 1$ [see Figure 1.44(c)]. You can see that you can obtain the graph of k by shifting the graph of f two units to the left and then one unit up.

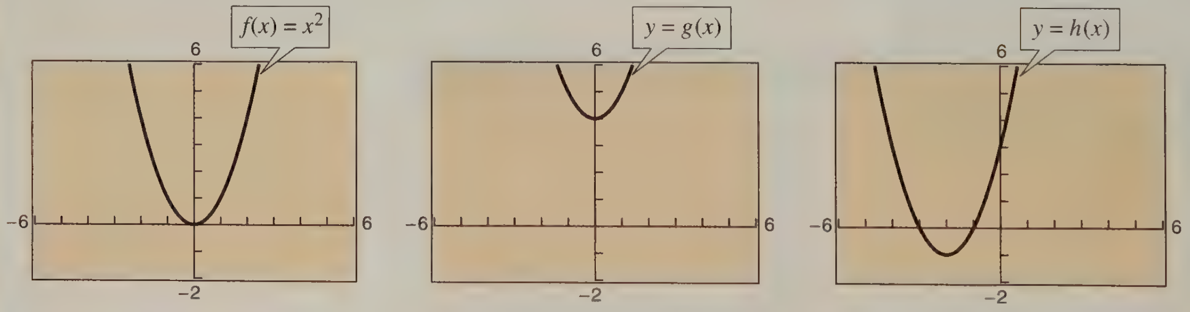


(a) Vertical shift: one unit down **(b) Horizontal shift: one unit right** **(c) Two units left and one unit up**

Figure 1.44

EXAMPLE 2 Finding Equations from Graphs

The graphs shown in Figures 1.45 (b) and (c) are shifts of the graph of $f(x) = x^2$ shown in Figure 1.45 (a). Find equations for g and h.



(a) **(b)** **(c)**

Figure 1.45

Solution

a. The graph of g is a vertical shift of four units upward of the graph of $f(x) = x^2$. So, the equation for g is $g(x) = x^2 + 4$.

b. The graph of h is a horizontal shift of two units to the left of the graph of $f(x) = x^2$ and one unit downward. So, the equation for h is $h(x) = (x + 2)^2 - 1$.

Reflecting Graphs

The second common type of transformation is called a **reflection.** For instance, if you consider the x-axis to be a mirror, the graph of $h(x) = -x^2$ is the mirror image (or reflection) of the graph of $f(x) = x^2$ (see Figure 1.46).

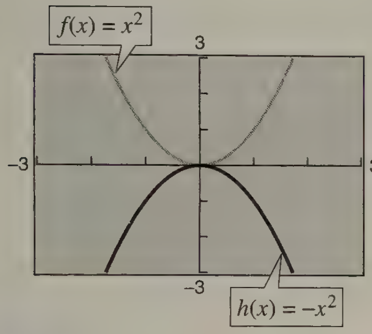


Figure 1.46

Exploration

Compare the graph of each function with the graph of $f(x) = x^2$ by using a graphing utility to graph the function and f in the same viewing window. Describe the transformation.

a. $g(x) = -x^2$

b. $h(x) = (-x)^2$

Reflections in the Coordinate Axes

Reflections in the coordinate axes of the graph of $y = f(x)$ are represented as follows.

1. Reflection in the x-axis: $h(x) = -f(x)$
2. Reflection in the y-axis: $h(x) = f(-x)$

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 3 Finding Equations from Graphs

Each of the graphs shown in Figure 1.48 is a transformation of the graph of $f(x) = x^4$ (see Figure 1.47). Find an equation of each function.

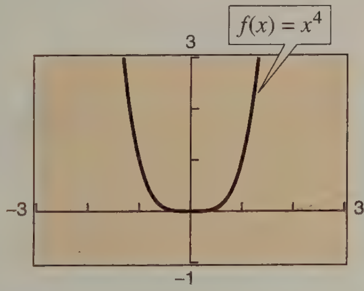
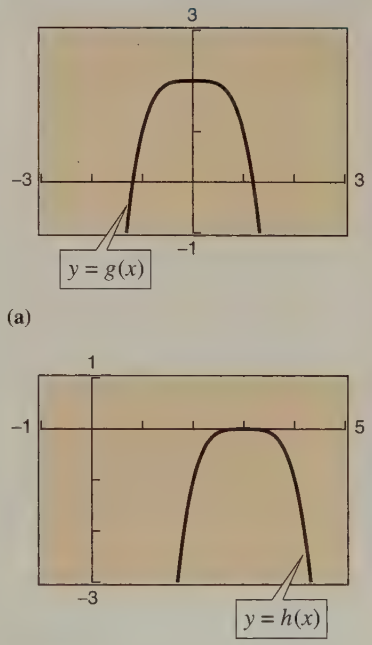


Figure 1.47



(a)

(b)

Figure 1.48

Solution

a. The graph of g is a reflection in the x-axis followed by an upward shift of two units of the graph of $f(x) = x^4$. So, the equation for g is $g(x) = -x^4 + 2$.

b. The graph of h is a horizontal shift of three units to the right followed by a reflection in the x-axis of the graph of $f(x) = x^4$. So, the equation for h is $h(x) = -(x - 3)^4$.

EXAMPLE 4 Reflections and Shifts

Compare the graph of each function with the graph of $f(x) = \sqrt{x}$.

a. $g(x) = -\sqrt{x}$ **b.** $h(x) = \sqrt{-x}$ **c.** $k(x) = -\sqrt{x+2}$

Algebraic Solution

a. Relative to the graph of $f(x) = \sqrt{x}$, the graph of g is a reflection in the x-axis because

$$\begin{aligned} g(x) &= -\sqrt{x} \\ &= -f(x). \end{aligned}$$

b. The graph of h is a reflection of the graph of $f(x) = \sqrt{x}$ in the y-axis because

$$\begin{aligned} h(x) &= \sqrt{-x} \\ &= f(-x). \end{aligned}$$

c. From the equation

$$\begin{aligned} k(x) &= -\sqrt{x+2} \\ &= -f(x+2) \end{aligned}$$

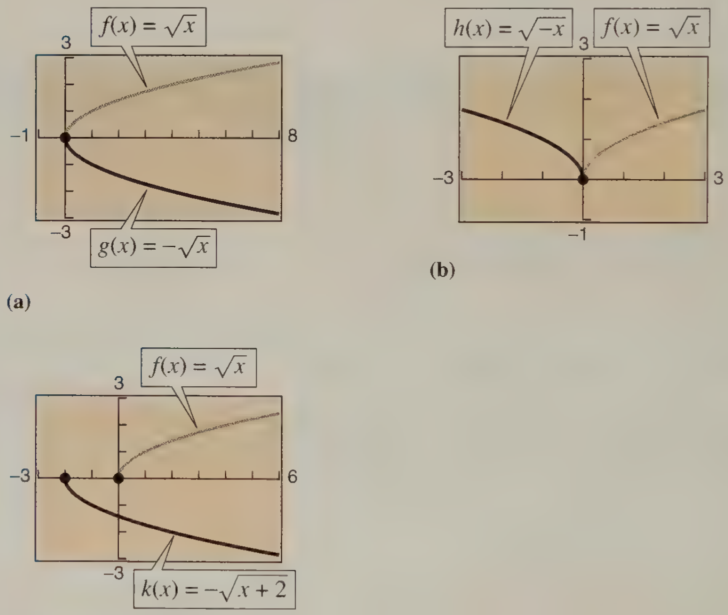
you can conclude that the graph of k is a left shift of two units, followed by a reflection in the x-axis.

Graphical Solution

a. Use a graphing utility to graph f and g in the same viewing window. From the graph in Figure 1.49(a), you can see that the graph of g is a reflection of the graph of f in the x-axis.

b. Use a graphing utility to graph f and h in the same viewing window. From the graph in Figure 1.49(b), you can see that the graph of h is a reflection of the graph of f in the y-axis.

c. Use a graphing utility to graph f and k in the same viewing window. From the graph in Figure 1.49(c), you can see that the graph of k is a left shift of f of two units, followed by a reflection in the x-axis.



(a)

(b)

(c)

Figure 1.49

When sketching the graphs of functions involving square roots, remember that the domain must be restricted to exclude negative numbers inside the radical. For instance, here are the domains of the functions in Example 4.

Domain of $g(x) = -\sqrt{x}$: $x \geq 0$

Domain of $h(x) = \sqrt{-x}$: $x \leq 0$

Domain of $k(x) = -\sqrt{x+2}$: $x \geq -2$

Nonrigid Transformations

Horizontal shifts, vertical shifts, and reflections are called **rigid transformations** because the basic shape of the graph is unchanged. These transformations change only the *position* of the graph in the *xy*-plane. **Nonrigid transformations** are those that cause a *distortion*—a change in the shape of the original graph. For instance, a nonrigid transformation of the graph of $y = f(x)$ is represented by $y = cf(x)$, where the transformation is a **vertical stretch** if $c > 1$ and a **vertical shrink** if $0 < c < 1$.

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 5 Nonrigid Transformations

Compare the graph of each function with the graph of $f(x) = |x|$.

$C = 3$ $C = 1/3$

a. $h(x) = 3|x|$ **b.** $g(x) = \frac{1}{3}|x|$

Solution

a. Relative to the graph of $f(x) = |x|$, the graph of

$$\begin{aligned} h(x) &= 3|x| \\ &= 3f(x) \end{aligned}$$

is a vertical stretch (multiply each y-value by 3) of the graph of f.

b. Similarly, the equation

$$\begin{aligned} g(x) &= \frac{1}{3}|x| \\ &= \frac{1}{3}f(x) \end{aligned}$$

indicates that the graph of g is a vertical shrink $\left(\text{multiply each } y\text{-value by } \frac{1}{3}\right)$ of the graph of f.

The graphs of all three functions are shown in Figure 1.50.

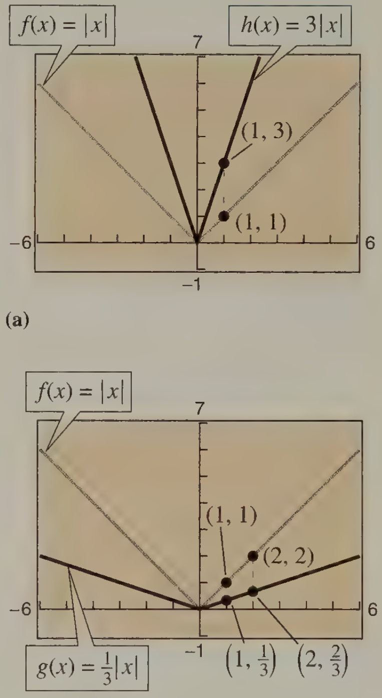


Figure 1.50

EXAMPLE 6 Sequences of Nonrigid Transformations

Use a graphing utility to graph the two functions

$$g(x) = 5(x^2 - 2) \quad \text{and} \quad h(x) = 5x^2 - 2$$

in the same viewing window. Describe how each function was obtained from $f(x) = x^2$ as a sequence of shifts and stretches.

Solution

Notice that $g(x)$ and $h(x)$ in Figure 1.51 are different. The graph of g is a downward shift of f of two units followed by a vertical stretch, whereas h is a vertical stretch of f followed by a downward shift of two units. So, the order of applying the transformations that include nonrigid transformations is important.

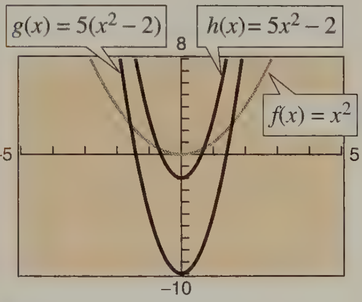


Figure 1.51

1.5 Exercises

In Exercises 1–12, sketch the graphs of the three functions by hand on the same rectangular coordinate system. Verify your result with a graphing utility.

1. $f(x) = x$
$g(x) = x - 4$
$h(x) = 3x$

2. $f(x) = \frac{1}{2}x$
$g(x) = \frac{1}{2}x + 2$
$h(x) = \frac{1}{2}(x - 2)$

3. $f(x) = x^2$
$g(x) = x^2 + 2$
$h(x) = (x - 2)^2$

4. $f(x) = x^2$
$g(x) = x^2 - 4$
$h(x) = (x + 2)^2 + 1$

5. $f(x) = -x^2$
$g(x) = -x^2 + 1$
$h(x) = -(x - 2)^2$

6. $f(x) = (x - 2)^2$
$g(x) = (x - 2)^2 + 2$
$h(x) = -(x - 2)^2 + 4$

7. $f(x) = x^2$
$g(x) = \frac{1}{2}x^2$
$h(x) = 2x^2$

8. $f(x) = x^2$
$g(x) = \frac{1}{4}x^2 + 2$
$h(x) = -\frac{1}{4}x^2$

9. $f(x) = |x|$
$g(x) = |x| - 1$
$h(x) = |x - 3|$

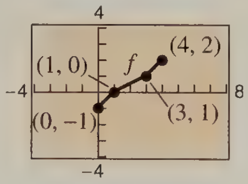
10. $f(x) = |x|$
$g(x) = 2|x|$
$h(x) = -2|x + 2| - 1$

11. $f(x) = \sqrt{x}$
$g(x) = \sqrt{x + 1}$
$h(x) = \sqrt{x - 2} + 1$

12. $f(x) = \sqrt{x}$
$g(x) = \frac{1}{2}\sqrt{x}$
$h(x) = -\frac{1}{2}\sqrt{x + 4}$

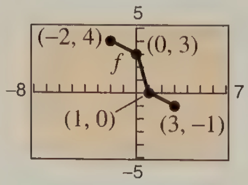
13. Use the graph of f to sketch each graph.

(a) $y = f(x) + 2$
(b) $y = -f(x)$
(c) $y = f(x - 2)$
(d) $y = f(x + 3)$
(e) $y = 2f(x)$
(f) $y = f(-x)$



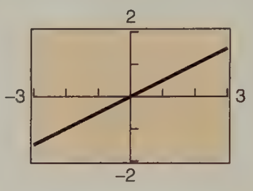
14. Use the graph of f to sketch each graph.

(a) $y = f(x) - 1$
(b) $y = f(x + 1)$
(c) $y = f(x - 1)$
(d) $y = -f(x - 2)$
(e) $y = f(-x)$
(f) $y = \frac{1}{2}f(x)$



In Exercises 15–26, identify the common function and the transformation shown in the graph. Write the formula for the graphed function.

15.



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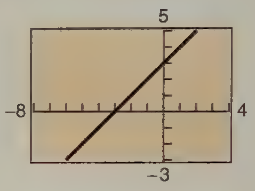

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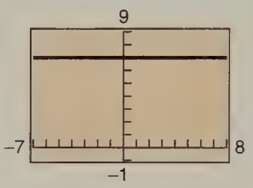

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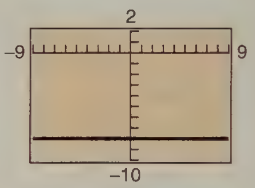

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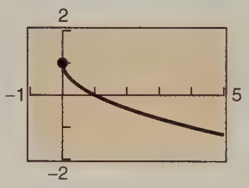

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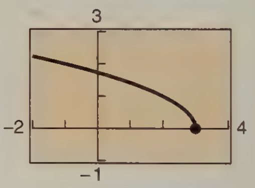

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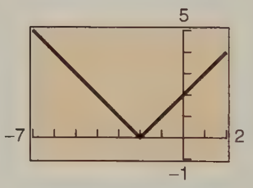

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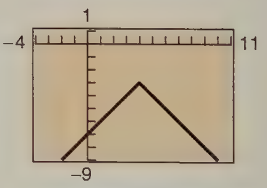

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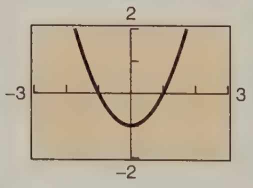

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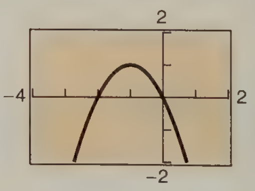

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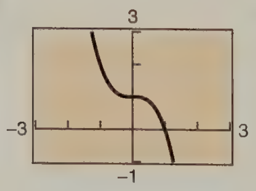

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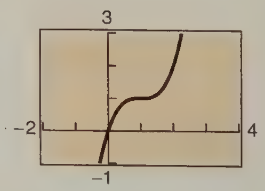


In Exercises 27–32, compare the graph of the function with the graph of $f(x) = \sqrt{x}$.

27. $y = \sqrt{x} + 2$

28. $y = -\sqrt{x} - 1$

29. $y = \sqrt{x - 2}$

30. $y = \sqrt{x + 3}$

31. $y = 2\sqrt{x}$

32. $y = \sqrt{-x + 3}$

In Exercises 33–38, compare the graph of the function with the graph of $f(x) = |x|$.

33. $y = |x + 2|$

34. $y = |x| - 3$

35. $y = -|x|$

36. $y = |-x|$

37. $y = \frac{1}{3}|x|$

38. $y = \frac{1}{2}|x|$

In Exercises 39–44, specify the sequence of transformations that will yield the graph of the given function from the graph of the function $f(x) = x^3$.

39. $g(x) = 4 - x^3$

40. $g(x) = -(x - 4)^3$

41. $h(x) = \frac{1}{4}(x + 2)^3$

42. $h(x) = -2(x - 1)^3 + 3$

43. $p(x) = \frac{1}{3}x^3 + 2$

44. $p(x) = [3(x - 2)]^3$

In Exercises 45–48, use a graphing utility to graph the three functions in the same viewing window. Describe the graphs of g and h relative to the graph of f.

45. $f(x) = x^3 - 3x^2$
$g(x) = f(x + 2)$
$h(x) = \frac{1}{2}f(x)$

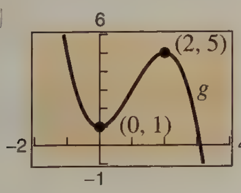
46. $f(x) = x^3 - 3x^2 + 2$
$g(x) = f(x - 1)$
$h(x) = 2f(x)$

47. $f(x) = x^3 - 3x^2$
$g(x) = -\frac{1}{3}f(x)$
$h(x) = f(-x)$

48. $f(x) = x^3 - 3x^2 + 2$
$g(x) = -f(x)$
$h(x) = f(-x)$

In Exercises 49 and 50, use the graph of $f(x) = x^3 - 3x^2$ (see Exercise 45) to write a formula for the function g shown in the graph.

49.



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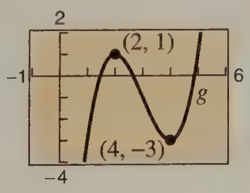


In Exercises 51–74, g is related to one of the six common functions on page 118. (a) Identify the common function f. (b) Describe the sequence of transformations from f to g. (c) Sketch the graph of g. (d) Use function notation to write g in terms of the common function f.

51. $g(x) = 12 - x^2$

52. $g(x) = (x - 8)^2$

53. $g(x) = 2 - (x + 5)^2$

54. $g(x) = -(x + 10)^2 + 5$

55. $g(x) = 3 + 2(x - 4)^2$

56. $g(x) = -\frac{1}{4}(x + 2)^2 - 2$

57. $g(x) = x^3 + 7$

58. $g(x) = -x^3 - 1$

59. $g(x) = (x - 1)^3 + 2$

60. $g(x) = -(x + 3)^3 - 10$

61. $g(x) = 3(x - 2)^3$

62. $g(x) = -\frac{1}{2}(x + 1)^3 - 5$

63. $g(x) = -|x| - 2$

64. $g(x) = 6 - |x + 5|$

65. $g(x) = -|x + 4| + 8$

66. $g(x) = |-x + 3| + 9$

67. $g(x) = -2|x - 1|$

68. $g(x) = \frac{1}{2}|x - 2| - 3$

69. $g(x) = \sqrt{x - 9}$

70. $g(x) = \sqrt{x + 4} + 8$

71. $g(x) = \sqrt{7 - x} - 2$

72. $g(x) = -\sqrt{x + 1} - 6$

73. $g(x) = 4\sqrt{x - 1}$

74. $g(x) = -\frac{1}{2}\sqrt{x + 3} - 1$

75. ***Profit*** The profit P per week on a certain product is given by the model

$$P(x) = 80 + 20x - 0.5x^2, \qquad 0 \le x \le 20$$

where x is the amount spent on advertising. In this model, x and P are both measured in hundreds of dollars.

(a) Use a graphing utility to graph the profit function.

(b) The business estimates that taxes and operating costs will increase by an average of \$2500 per week during the next year. Rewrite the profit equation to reflect this expected decrease in profits. Identify the type of transformation applied to the graph of the equation.

(c) Rewrite the profit equation so that x measures advertising expenditures in dollars. [Find $P\left(\frac{x}{100}\right)$.] Identify the type of transformation applied to the graph of the profit function.

76. ***Automobile Aerodynamics*** The number of horsepower H required to overcome wind drag on a certain automobile is approximated by

$$H(x) = 0.002x^2 + 0.005x - 0.029, \qquad 10 \le x \le 100$$

where x is the speed of the car in miles per hour.

(a) Use a graphing utility to graph the power function.

(b) Rewrite the power function so that x represents the speed in kilometers per hour. [Find $H(x/1.6)$.] Identify the type of transformation applied to the graph of the power function.

77. ***Energy*** The amount of fuel F (in billions of gallons) used by trucks from 1980 through 1996 can be approximated by the function

$$F(t) = 20.46 + 0.04t^2$$

where $t = 0$ represents 1980. (Source: U.S. Federal Highway Administration)

(a) Describe how F can be obtained from the common function $f(x) = x^2$. Then sketch the graph over the interval $0 \le t \le 16$.

(b) Rewrite the function so that $t = 0$ represents 1990. Explain how you got your answer.

78. ***Finance*** The amount of mortgage debt outstanding M (in billions of dollars) in the United States from 1985 through 1997 can be approximated by the function

$$M(t) = 1.5\sqrt{t} - 1.25$$

where $t = 5$ represents 1985. (Source: Board of Governors of the Federal Reserve System)

(a) Describe how M can be obtained from the common function $f(x) = \sqrt{x}$. Then sketch the graph over the interval $5 \le t \le 17$.

(b) Rewrite the function so that $t = 5$ represents 1995. Explain how you got your answer.

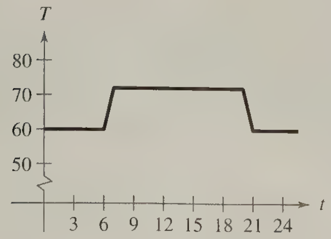
79. ***Graphical Reasoning*** An electronically controlled thermostat in a home is programmed to automatically lower the temperature at night (see figure). The temperature in the house T, in degrees Fahrenheit, is given in terms of t, the time (in hours) on a 24-hour clock.

(a) Explain why T is a function of t.

(b) Approximate $T(4)$ and $T(15)$.

(c) Suppose the thermostat were reprogrammed to produce a temperature H where $H(t) = T(t - 1)$. How would this change the temperature in the house? Explain.

(d) Suppose the thermostat were reprogrammed to produce a temperature H where $H(t) = T(t) - 1$. How would this change the temperature in the house? Explain.



Synthesis

True or False? **In Exercises 80 and 81, determine whether the statement is true or false. Justify your answer.**

80. The graphs of $f(x) = |x| - 5$ and $g(x) = |-x| - 5$ are identical.

81. If the graph of the common function $f(x) = x^2$ is moved 6 units to the right, 3 units up, and reflected in the x-axis, then the point $(-1, 28)$ will lie on the graph of the transformation.

82. ***Exploration*** Use a graphing utility to graph each function. Describe any similarities and differences you observe among the graphs.

(a) $y = x$ (b) $y = x^2$ (c) $y = x^3$

(d) $y = x^4$ (e) $y = x^5$ (f) $y = x^6$

83. ***Conjecture*** Use the results of Exercise 82 to make a conjecture about the shapes of the graphs of $y = x^7$ and $y = x^8$. Use a graphing utility to verify your conjecture.

84. Use the results of Exercise 82 to sketch the graph of $y = (x - 3)^3$ by hand. Use a graphing utility to verify your graph.

85. Use the results of Exercise 82 to sketch the graph of $y = (x + 1)^2$ by hand. Use a graphing utility to verify your graph.

Conjecture **In Exercises 86–89, use the results of Exercise 82 to make a conjecture about the shape of the graph of the function. Use a graphing utility to verify your conjecture.**

86. $f(x) = x^2(x - 6)^2$ **87.** $f(x) = x^3(x - 6)^2$

88. $f(x) = x^2(x - 6)^3$ **89.** $f(x) = x^3(x - 6)^3$

Review

In Exercises 90–93, find the domain of the function.

90. $f(x) = \dfrac{4}{9 - x}$ **91.** $f(x) = \dfrac{\sqrt{x - 5}}{x - 7}$

92. $f(x) = \sqrt{100 - x^2}$ **93.** $f(x) = \sqrt[3]{16 - x^2}$

In Exercises 94–99, sketch a graph of the equation. Use a graphing utility to verify your graph.

94. $y = -x - 7$ **95.** $y = 9 - 4x$

96. $y = x^2 - 4x$ **97.** $y = -x^3 - 3$

98. $y = \sqrt{9 - x}$ **99.** $y = 5 - 2|3x|$

1.6 Combinations of Functions

Arithmetic Combinations of Functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two *functions* can be combined to create new functions. If $f(x) = 2x - 3$ and $g(x) = x^2 - 1$ you can form the sum, difference, product, and quotient of f and g as follows.

$$f(x) + g(x) = (2x - 3) + (x^2 - 1)$$
$$= x^2 + 2x - 4 \quad \text{Sum}$$
$$f(x) - g(x) = (2x - 3) - (x^2 - 1)$$
$$= -x^2 + 2x - 2 \quad \text{Difference}$$
$$f(x) \cdot g(x) = (2x - 3)(x^2 - 1)$$
$$= 2x^3 - 3x^2 - 2x + 3 \quad \text{Product}$$
$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 - 1}, \quad x \neq \pm 1 \quad \text{Quotient}$$

The domain of an **arithmetic combination** of functions f and g consists of all real numbers that are common to the domains of f and g. In the case of the quotient $f(x)/g(x)$, there is the further restriction that $g(x) \neq 0$.

Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the sum, difference, product, and quotient of f and g are defined as follows.

1. Sum: $(f + g)(x) = f(x) + g(x)$
2. Difference: $(f - g)(x) = f(x) - g(x)$
3. Product: $(fg)(x) = f(x) \cdot g(x)$
4. Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

EXAMPLE 1 Finding the Sum of Two Functions

Find $(f + g)(x)$ for the functions $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$. Then evaluate the sum when $x = 2$.

Solution

$$(f + g)(x) = f(x) + g(x)$$
$$= (2x + 1) + (x^2 + 2x - 1)$$
$$= x^2 + 4x$$

When $x = 2$, the value of this sum is $(f + g)(2) = 2^2 + 4(2) = 12$.

What You Should Learn:

- How to add, subtract, multiply, and divide functions
- How to find compositions of one function with another function
- How to use combinations of functions to model and solve real-life problems

Why You Should Learn It:

Combining functions can sometimes help you better understand the bigger picture. For instance, Exercises 81 and 82 on page 136 show how to use combinations of functions to analyze U.S. health expenditures.

Superstock

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 2 Finding the Difference of Two Functions

Evaluate $(f-g)(x)$ for the functions

$$f(x) = 2x + 1 \quad \text{and} \quad g(x) = x^2 + 2x - 1$$

when $x = 2$.

Algebraic Solution

The difference of the functions f and g is

$$\begin{aligned}(f-g)(x) &= f(x) - g(x)\\ &= (2x+1) - (x^2+2x-1)\\ &= -x^2+2.\end{aligned}$$

When $x = 2$, the value of this difference is

$$\begin{aligned}(f-g)(2) &= -(2)^2 + 2\\ &= -2.\end{aligned}$$

Note that $(f-g)(2)$ can also be evaluated as follows.

$$\begin{aligned}(f-g)(2) &= f(2) - g(2)\\ &= [2(2)+1] - [2^2+2(2)-1]\\ &= 5-7\\ &= -2\end{aligned}$$

Graphical Solution

You can use a graphing utility to graph the difference of two functions. Enter the functions as follows.

$$y_1 = 2x + 1$$

$$y_2 = x^2 + 2x - 1$$

$$y_3 = y_1 - y_2$$

Graph y_3 as shown in Figure 1.52. Then use the *value* feature or *zoom* and *trace* features to estimate that the value of the difference when $x = 2$ is -2.

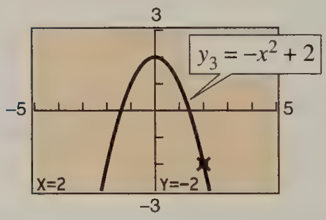


Figure 1.52

In Examples 1 and 2, both f and g have domains that consist of all real numbers. So, the domain of both $(f+g)$ and $(f-g)$ is also the set of all real numbers. Remember that any restrictions on the domains of f or g must be taken into account when forming the sum, difference, product, or quotient of f and g. For instance, the domain of $f(x) = 1/x$ is all $x \neq 0$, and the domain of $g(x) = \sqrt{x}$ is $[0, \infty)$. This implies that the domain of $f + g$ is $(0, \infty)$.

EXAMPLE 3 Finding the Product of Two Functions

Given the functions $f(x) = x^2$ and $g(x) = x - 3$, find the product of f and g. Then evaluate the product when $x = 4$.

Solution

$$\begin{aligned}(fg)(x) &= f(x)g(x)\\ &= (x^2)(x-3)\\ &= x^3 - 3x^2\end{aligned}$$

When $x = 4$, the value of this product is

$$\begin{aligned}(fg)(4) &= 4^3 - 3(4)^2\\ &= 16.\end{aligned}$$

EXAMPLE 4 Finding the Quotient of Two Functions

Find $\left(\frac{f}{g}\right)(x)$ and $\left(\frac{g}{f}\right)(x)$ for the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$. Then find the domains of f/g and g/f.

Solution

The quotient of f and g is

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4 - x^2}},$$

and the quotient of g and f is

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4 - x^2}}{\sqrt{x}}.$$

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these domains is $[0, 2]$. So, the domains for f/g and g/f are as follows.

$$\text{Domain of } \frac{f}{g}: [0, 2) \qquad \text{Domain of } \frac{g}{f}: (0, 2]$$

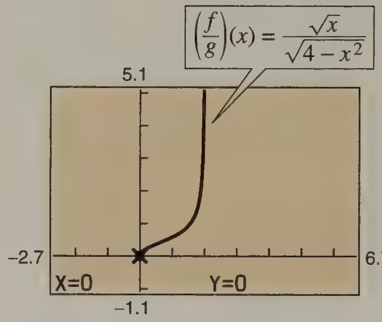


Figure 1.53

Can you see why these two domains differ slightly? You can confirm the domain of f/g in Example 4 with your graphing utility by entering the three functions

$$y_1 = \sqrt{x} \qquad y_2 = \sqrt{(4 - x^2)} \qquad y_3 = \frac{y_1}{y_2}$$

and graphing y_3 as shown in Figure 1.53.

Use the *trace* feature to determine that the x-coordinates of points on the graph extend from 0 to 2 but do not include 2. So, you can estimate the domain of f/g to be $[0, 2)$. You can confirm the domain of g/f in Example 4 by entering $y_4 = y_2/y_1$ and graphing y_4 as shown in Figure 1.54. Use the *trace* feature to determine that the x-coordinates of points on the graph extend from 0 to 2 but do not include 0. So, you can estimate the domain of g/f to be $(0, 2]$.

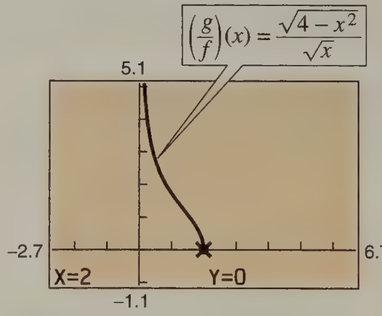


Figure 1.54

Compositions of Functions

Another way of combining two functions is to form the **composition** of one with the other. For instance, if $f(x) = x^2$ and $g(x) = x + 1$, the composition of f with g is

$$f(g(x)) = f(x + 1) = (x + 1)^2.$$

This composition is denoted as $f \circ g$.

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Definition of Composition of Two Functions

The **composition** of the function f with g is

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 1.55.)

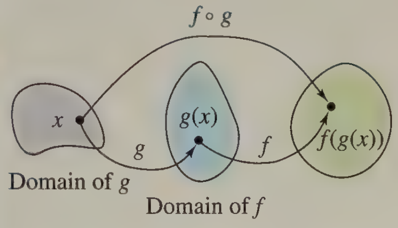


Figure 1.55

EXAMPLE 5 Forming the Composition of f with g

Find $(f \circ g)(x)$ for $f(x) = \sqrt{x}$, $x \geq 0$, and $g(x) = x - 1$, $x \geq 1$. If possible, find $(f \circ g)(2)$ and $(f \circ g)(0)$.

Solution

$$(f \circ g)(x) = f(g(x)) \qquad \text{Definition of } f \circ g$$
$$= f(x - 1) \qquad \text{Definition of } g(x)$$
$$= \sqrt{x - 1}, \quad x \geq 1 \qquad \text{Definition of } f(x)$$

The domain of $f \circ g$ is $[1, \infty)$. So, $(f \circ g)(2) = \sqrt{2 - 1} = 1$ is defined, but $(f \circ g)(0)$ is not defined because 0 is not in the domain of $f \circ g$.

The composition of f with g is generally not the same as the composition of g with f. This is illustrated in Example 6.

Exploration

Let $f(x) = x + 2$ and $g(x) = 4 - x^2$. Are the compositions $f \circ g$ and $g \circ f$ equal? You can use your graphing utility to answer this question by entering and graphing the following functions.

$$Y1 = (4 - x^2) + 2$$
$$Y2 = 4 - (x + 2)^2$$

What do you observe? Which function represents $f \circ g$ and which represents $g \circ f$?

EXAMPLE 6 Compositions of Functions

Given $f(x) = x + 2$ and $g(x) = 4 - x^2$, evaluate the following when $x = 0, 1, 2,$ and 3.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$

Algebraic Solution

a. $(f \circ g)(x) = f(g(x))$ Definition of $f \circ g$

$= f(4 - x^2)$ Definition of $g(x)$

$= (4 - x^2) + 2$ Definition of $f(x)$

$= -x^2 + 6$

$(f \circ g)(0) = -0^2 + 6 = 6$

$(f \circ g)(1) = -1^2 + 6 = 5$

$(f \circ g)(2) = -2^2 + 6 = 2$

$(f \circ g)(3) = -3^2 + 6 = -3$

b. $(g \circ f)(x) = g(f(x))$ Definition of $g \circ f$

$= g(x + 2)$ Definition of $f(x)$

$= 4 - (x + 2)^2$ Definition of $g(x)$

$= 4 - (x^2 + 4x + 4)$

$= -x^2 - 4x$

$(g \circ f)(0) = -0^2 - 4(0) = 0$

$(g \circ f)(1) = -1^2 - 4(1) = -5$

$(g \circ f)(2) = -2^2 - 4(2) = -12$

$(g \circ f)(3) = -3^2 - 4(3) = -21$

Note that $(f \circ g)(x) \neq (g \circ f)(x)$.

Numerical Solution

a. You can use a table to evaluate $f \circ g$ when $x = 0, 1, 2,$ and 3. First evaluate $g(x)$ for the values of x in the table. Then evaluate $f(g(x))$ for the values of $g(x)$ in the table.

x	0	1	2	3
$g(x)$	4	3	0	−5
$f(g(x))$	6	5	2	−3

b. To evaluate $g \circ f$ when $x = 0, 1, 2,$ and 3, first evaluate $f(x)$ for the values of x in the table. Then evaluate $g(f(x))$ for the values of $f(x)$ in the table.

x	0	1	2	3
$f(x)$	2	3	4	5
$g(f(x))$	0	−5	−12	−21

From the tables you can see that

$$(f \circ g)(x) \neq (g \circ f)(x).$$

To determine the domain of a composite function $f \circ g$, you need to restrict the outputs of g so that they are in the domain of f. For instance, to find the domain of $f \circ g$ given that $f(x) = 1/x$, and $g(x) = x + 1$, consider the outputs of g. These can be any real number. However, the domain of f is restricted to all real numbers except 0. So, the outputs of g must be restricted to all real numbers except 0. This means that $g(x) = x + 1 \neq 0$, or $x \neq -1$. So, the domain of $f \circ g$ is all real numbers except $x = -1$.

EXAMPLE 7 Finding the Domain of a Composite Function

Find the domain of the composition $(f \circ g)(x)$ for the functions

$$f(x) = x^2 - 9 \qquad \text{and} \qquad g(x) = \sqrt{9 - x^2}.$$

Algebraic Solution

The composition of the functions is as follows.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\sqrt{9 - x^2}\right) \\ &= \left(\sqrt{9 - x^2}\right)^2 - 9 \\ &= 9 - x^2 - 9 \\ &= -x^2\end{aligned}$$

From this, it might appear that the domain of the composition is the set of all real numbers. This, however, is not true because the domain of g is $-3 \le x \le 3$. So, the domain of $f \circ g$ is $-3 \le x \le 3$.

Graphical Solution

You can use a graphing utility to graph the composition of the functions $(f \circ g)(x)$ as $y = \left(\sqrt{9 - x^2}\right)^2 - 9$. Enter the functions as follows.

$$y_1 = \sqrt{(9 - x^2)}$$

$$y_2 = {y_1}^2 - 9$$

Graph y_2 as shown in Figure 1.56. Use the *trace* feature to determine that the x-coordinates of points on the graph extend from -3 to 3. So, you can graphically estimate the domain of $(f \circ g)(x)$ to be $-3 \le x \le 3$.

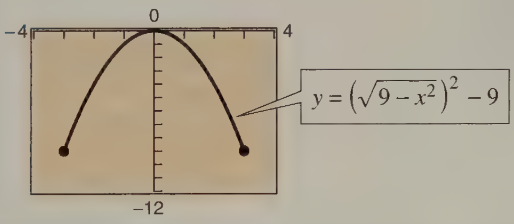


Figure 1.56

EXAMPLE 8 A Case in Which $f \circ g = g \circ f$

Given $f(x) = 2x + 3$ and $g(x) = \frac{1}{2}(x - 3)$, find the following.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$

Solution

a. $$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{1}{2}(x - 3)\right) \\ &= 2\left(\frac{1}{2}(x - 3)\right) + 3 \\ &= x - 3 + 3 \\ &= x\end{aligned}$$

b. $$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x + 3) \\ &= \frac{1}{2}((2x + 3) - 3) \\ &= \frac{1}{2}(2x) \\ &= x\end{aligned}$$

STUDY T!P

In Example 8, note that the two composite functions $f \circ g$ and $g \circ f$ are equal, and both represent the identity function. That is,

$$(f \circ g)(x) = x$$

$$(g \circ f)(x) = x.$$

You will study this special case in the next section.

In Examples 5, 6, 7, and 8 you formed the composition of two given functions. In calculus, it is also important to be able to identify two functions that make up a given composite function. For instance, the function h

$$h(x) = (3x - 5)^3$$

is the composition of f with g, where $f(x) = x^3$ and $g(x) = 3x - 5$. That is,

$$h(x) = (3x - 5)^3 = [g(x)]^3 = f(g(x)).$$

Basically, to "decompose" a composite function, look for an "inner" and an "outer" function. In the function h above, $g(x) = 3x - 5$ is the inner function and $f(x) = x^3$ is the outer function.

EXAMPLE 9 Identifying a Composite Function

Express the function

$$h(x) = \frac{1}{(x - 2)^2}$$

as a composition of two functions.

Solution

One way to write h as a composition of two functions is to take the inner function to be $g(x) = x - 2$ and the outer function to be

$$f(x) = \frac{1}{x^2} = x^{-2}.$$

Then you can write

$$h(x) = \frac{1}{(x - 2)^2} = (x - 2)^{-2} = f(x - 2) = f(g(x)).$$

Exploration

The function in Example 9 can be decomposed in other ways. For which of the following pairs of functions is $h(x)$ equal to $f(g(x))$?

a. $g(x) = \frac{1}{x - 2}$ and $f(x) = x^2$

b. $g(x) = x^2$ and $f(x) = \frac{1}{x - 2}$

c. $g(x) = \frac{1}{x}$ and $f(x) = (x - 2)^2$

Application

EXAMPLE 10 Bacteria Count

The number of bacteria in a refrigerated food is

$$N(T) = 20T^2 - 80T + 500, \qquad 2 \le T \le 14$$

where T is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature is

$$T(t) = 4t + 2, \qquad 0 \le t \le 3$$

where t is the time (in hours). Find the following.

a. The composite $N(T(t))$. What does this function represent?

b. The number of bacteria in the food when $t = 2$ hours

c. The time when the bacterial count reaches 2000

Solution

a.
$$\begin{aligned} N(T(t)) &= 20(4t + 2)^2 - 80(4t + 2) + 500 \\ &= 20(16t^2 + 16t + 4) - 320t - 160 + 500 \\ &= 320t^2 + 320t + 80 - 320t - 160 + 500 \\ &= 320t^2 + 420 \end{aligned}$$

This composite function $N(T(t))$ represents the number of bacteria as a function of the amount of time the food has been out of refrigeration.

b. When $t = 2$, the number of bacteria is

$$N = 320(2)^2 + 420 = 1280 + 420 = 1700.$$

c. The bacterial count will reach $N = 2000$ when $320t^2 + 420 = 2000$. You can solve this equation for t algebraically as follows.

$$\begin{aligned} 320t^2 + 420 &= 2000 \\ 320t^2 &= 1580 \\ t^2 &= \frac{1580}{320} = \frac{79}{16} \\ t &= \frac{\sqrt{79}}{4} \approx 2.2 \text{ hours} \end{aligned}$$

So, the count will reach 2000 when $t \approx 2.2$ hours. When you solve this equation, note that the negative value is rejected because it is not in the domain of the composite function. You can use a graphing utility to approximate the solution, as shown in Figure 1.57.

Exploration

Use a graphing utility to graph $y = 320t^2 + 420$ and $y = 2000$ in the same viewing window. (Use a viewing window in which $0 \le x \le 3$ and $420 \le y \le 4000$.) Explain how the graphs can be used to answer the question asked in Example 10(c). Compare your answer with that given in part (c). When will the bacteria count reach 3200?

Notice that the model for this bacteria count situation is valid only for a span of 3 hours. Now suppose the minimum number of bacteria in the food is reduced from 420 to 100. Will the number of bacteria still reach a level of 2000 within the 3-hour time span? Will the number of bacteria reach a level of 3200 within 3 hours?

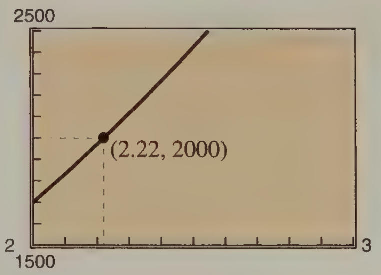


Figure 1.57

1.6 Exercises

In Exercises 1–8, find the following.

(a) $(f+g)(x)$ **(b)** $(f-g)(x)$

(c) $(fg)(x)$ **(d)** $(f/g)(x)$

(e) the domain of f/g

1. $f(x) = x + 1, \quad g(x) = x - 1$

2. $f(x) = 2x - 5, \quad g(x) = 1 - x$

3. $f(x) = x^2, \quad g(x) = 1 - x$

4. $f(x) = 2x - 5, \quad g(x) = 5$

5. $f(x) = x^2 + 5, \quad g(x) = \sqrt{1 - x}$

6. $f(x) = \sqrt{x^2 - 4}, \quad g(x) = \dfrac{x^2}{x^2 + 1}$

7. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x^2}$

8. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = x^3$

In Exercises 9–22, evaluate the indicated function for $f(x) = x^2 + 1$ and $g(x) = x - 4$ algebraically. If possible, use a graphing utility to verify your answer.

9. $(f+g)(3)$

10. $(f-g)(-2)$

11. $(f-g)(0)$

12. $(f+g)(1)$

13. $(fg)(4)$

14. $(fg)(-6)$

15. $\left(\dfrac{f}{g}\right)(5)$

16. $\left(\dfrac{f}{g}\right)(0)$

17. $(f-g)(2t)$

18. $(f+g)(t-4)$

19. $(fg)(-5t)$

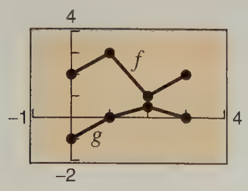
20. $(fg)(3t^2)$

21. $\left(\dfrac{f}{g}\right)(-t)$

22. $\left(\dfrac{f}{g}\right)(t+2)$

In Exercises 23–26, use the graphs to graph $h(x) = (f+g)(x)$.

23.



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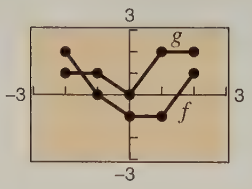

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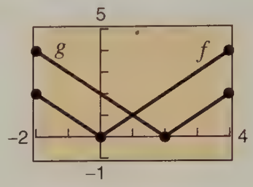

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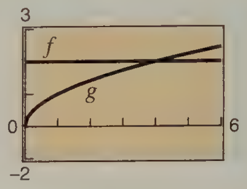


In Exercises 27–30, graph the functions f, g, and $f+g$ in the same viewing window.

27. $f(x) = \frac{1}{2}x, \quad g(x) = x - 1$

28. $f(x) = \frac{1}{3}x, \quad g(x) = -x + 4$

29. $f(x) = x^2, \quad g(x) = -2x$

30. $f(x) = 4 - x^2, \quad g(x) = x$

In Exercises 31–34, use a graphing utility to sketch the graphs of f, g, and $f+g$ in the same viewing window. Which function contributes more to the magnitude of the sum when $0 \le x \le 2$? Which function contributes more to the magnitude of the sum when $x > 6$?

31. $f(x) = 3x, \quad g(x) = -\dfrac{x^3}{10}$

32. $f(x) = \dfrac{x}{2}, \quad g(x) = \sqrt{x}$

33. $f(x) = 3x + 2, \quad g(x) = -\sqrt{x + 5}$

34. $f(x) = x^2 - \frac{1}{2}, \quad g(x) = -3x^2 - 1$

In Exercises 35–38, find (a) $f \circ g$ and (b) $g \circ f$.

35. $f(x) = x^2, \quad g(x) = x - 1$

36. $f(x) = \sqrt[3]{x - 1}, \quad g(x) = x^3 + 1$

37. $f(x) = 3x + 5, \quad g(x) = 5 - x$

38. $f(x) = x^3, \quad g(x) = \dfrac{1}{x}$

In Exercises 39–44, (a) find $f \circ g$ and $g \circ f$. (b) Use a graphing utility to graph $f \circ g$ and $g \circ f$. Determine whether $f \circ g = g \circ f$.

39. $f(x) = \sqrt{x + 4}, \quad g(x) = x^2$

40. $f(x) = \sqrt[3]{x + 1}, \quad g(x) = x^3 - 1$

41. $f(x) = \frac{1}{3}x - 3, \quad g(x) = 3x + 1$

42. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{x}$

43. $f(x) = x^{2/3}, \quad g(x) = x^6$

44. $f(x) = |x|, \quad g(x) = x + 6$

In Exercises 45–50, (a) find $(f \circ g)(x)$ and $(g \circ f)(x)$, (b) determine algebraically whether $(f \circ g)(x) = (g \circ f)(x)$, and (c) verify your answer to part (b) by comparing a table of values for each composition.

45. $f(x) = 5x + 4, \quad g(x) = 4 - x$

46. $f(x) = \frac{1}{4}(x - 1), \quad g(x) = 4x + 1$

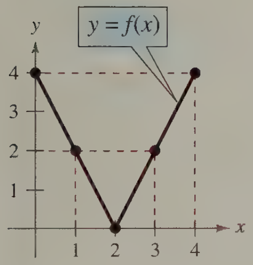
47. $f(x) = \sqrt{x + 6}, \quad g(x) = x^2 - 5$

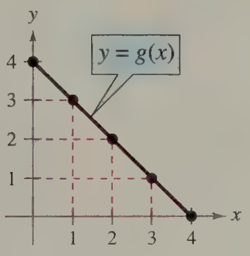
48. $f(x) = x^3 - 4, \quad g(x) = \sqrt[3]{x + 10}$

49. $f(x) = |x + 3|, \quad g(x) = 2x - 1$

50. $f(x) = \dfrac{6}{3x - 5}, \quad g(x) = -x$

In Exercises 51–56, use the graphs of f and g to evaluate the functions.





51. (a) $(f + g)(3)$ (b) $(f/g)(2)$

52. (a) $(f - g)(1)$ (b) $(fg)(4)$

53. (a) $(f \circ g)(2)$ (b) $(g \circ f)(2)$

54. (a) $(f \circ g)(1)$ (b) $(g \circ f)(3)$

55. (a) $(f \circ f)(3)$ (b) $(f \circ f)(4)$

56. (a) $(g \circ g)(1)$ (b) $(g \circ g)(0)$

In Exercises 57–64, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)

57. $h(x) = (2x + 1)^2$

58. $h(x) = (1 - x)^3$

59. $h(x) = \sqrt[3]{x^2 - 4}$

60. $h(x) = \sqrt{9 - x}$

61. $h(x) = \dfrac{1}{x + 2}$

62. $h(x) = \dfrac{4}{(5x + 2)^2}$

63. $h(x) = (x + 4)^2 + 2(x + 4)$

64. $h(x) = (x + 3)^{3/2}$

In Exercises 65–70, determine the domains of (a) f, (b) g, and (c) $f \circ g$. Use a graphing utility to verify your answer.

65. $f(x) = \sqrt{x}, \quad g(x) = x^2 + 1$

66. $f(x) = \sqrt{x + 3}, \quad g(x) = \dfrac{x}{2}$

67. $f(x) = \dfrac{1}{x}, \quad g(x) = x + 3$

68. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{2x}$

69. $f(x) = \dfrac{2}{|x|}, \quad g(x) = x - 1$

70. $f(x) = \dfrac{3}{x^2 - 1}, \quad g(x) = x + 1$

Average Rate of Change **In Exercises 71–78, find the difference quotient**

$$\frac{f(x + h) - f(x)}{h}$$

and simplify your answer.

71. $f(x) = 3x - 4$

72. $f(x) = 5x + 1$

73. $f(x) = 1 - x^2$

74. $f(x) = x^2 + 4$

75. $f(x) = \dfrac{4}{x}$

76. $f(x) = \dfrac{2}{x^2}$

77. $f(x) = \sqrt{2x + 1}$

78. $f(x) = -\sqrt{4x}$

79. ***Stopping Distance*** A car traveling x miles per hour stops quickly. The distance a car travels during the driver's reaction time is given by $R(x) = \frac{3}{4}x$. The distance traveled while braking is given by $B(x) = \frac{1}{15}x^2$.

(a) Find the stopping-distance function T.

(b) Use a graphing utility to graph the functions R, B, and T in the interval $0 \leq x \leq 60$.

(c) Which function contributes most to the magnitude of the sum at higher speeds? Explain.

80. ***Business*** You own two restaurants. From 1995 to 2000, the sales R_1 (in thousands of dollars) for one restaurant can be modeled by

$$R_1 = 480 - 8t - 0.8t^2, \qquad t = 0, 1, 2, 3, 4, 5$$

where $t = 0$ represents 1995. During the same 6-year period, the sales R_2 (in thousands of dollars) for the other restaurant can be modeled by

$$R_2 = 254 + 0.78t, \qquad t = 0, 1, 2, 3, 4, 5.$$

(a) Write a function R_3 that represents the total sales for the two restaurants.

(b) Use a graphing utility to graph R_1, R_2, and R_3 (the total sales function) in the same viewing window.

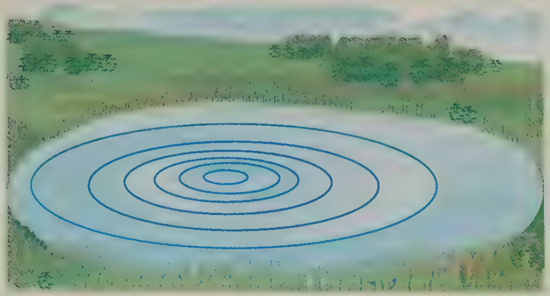
Data Analysis **In Exercises 81 and 82, use the table, which gives the total amount spent (in billions of dollars) on health services and supplies in the United States and Puerto Rico for the years 1990 through 1996. The variables y_1, y_2, and y_3 represent out-of-pocket payments, insurance premiums, and other types of payments, respectively.** (Source: U.S. Health Care Financing Administration)

Year	1990	1991	1992	1993	1994	1995	1996
y_1	144.4	151.6	159.5	163.6	164.8	166.7	171.2
y_2	238.6	259.4	282.5	303.3	315.6	326.9	337.3
y_3	21.9	24.0	25.1	27.3	29.6	31.7	32.4

81. Use a graphing utility to find a mathematical model for each of the variables. Let $t = 0$ represent 1990. Find a quadratic model $(y = ax^2 + bx + c)$ for y_1 and linear models $(y = ax + b)$ for y_2 and y_3.

82. Use a graphing utility to graph y_1, y_2, y_3, and $y_1 + y_2 + y_3$ in the same viewing window. Use the model to estimate the total amount spent on health services and supplies in 2000.

83. *Ripples* A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius (in feet) of the outer ripple is $r(t) = 0.6t$, where t is the time (in seconds) after the pebble strikes the water. The area of the circle is $A(r) = \pi r^2$. Find and interpret $(A \circ r)(t)$.



84. *Geometry* A square concrete foundation was prepared as a base for a large cylindrical gasoline tank.

(a) Express the radius r of the tank as a function of the length x of the sides of the square.

(b) Express the area A of the circular base of the tank as a function of the radius r.

(c) Find and interpret $(A \circ r)(x)$.

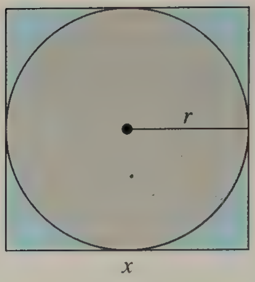


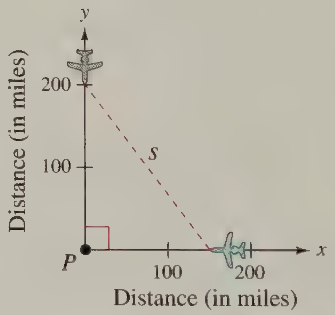
FIGURE FOR 84

85. *Business* The weekly cost of producing x units in a manufacturing process is $C(x) = 60x + 750$. The number of units produced in t hours is $x(t) = 50t$.

(a) Find and interpret $(C \circ x)(t)$.

(b) Use a graphing utility to graph the cost as a function of time. Use the *trace* feature to estimate (to two-decimal-place accuracy) the time that must elapse until the cost increases to \$15,000.

86. *Air Traffic Control* An air traffic controller spots two planes at the same altitude flying toward each other. Their flight paths form a right angle at point P. One plane is 150 miles from point P and is moving at 450 miles per hour. The other plane is 200 miles from point P and is moving at 450 miles per hour. Write the distance s between the planes as a function of time t.



87. *Business* You are a sales representative for an automobile manufacturer. You are paid an annual salary plus a bonus of 3% of your sales over \$500,000. Consider the two functions $f(x) = x - 500{,}000$ and $g(x) = 0.03x$. If x is greater than \$500,000, which of the following represents your bonus? Explain.

(a) $f(g(x))$ (b) $g(f(x))$

88. ***Exploration*** The suggested retail price of a new car is p dollars. The dealership advertised a factory rebate of \$1200 and an 8% discount.

(a) Write a function R in terms of p, giving the cost of the car after receiving the rebate from the factory.

(b) Write a function S in terms of p, giving the cost of the car after receiving the dealership discount.

(c) Form the composite functions $(R \circ S)(p)$ and $(S \circ R)(p)$, and interpret each.

(d) Find $(R \circ S)(18{,}400)$ and $(S \circ R)(18{,}400)$. Which yields the lower cost for the car? Explain.

89. ***Data Analysis*** The data in the table shows the circulations of morning and evening newspapers in the United States for the years 1988 through 1997. The variables y_1 and y_2 represent the circulations in millions of the morning and evening papers, respectively. (Source: Editor & Publisher Co.)

Year	1988	1989	1990	1991	1992
y_1	40.5	40.7	41.3	41.5	42.4
y_2	22.2	21.9	21.0	19.2	17.8

Year	1993	1994	1995	1996	1997
y_1	43.1	43.4	44.3	44.8	45.4
y_2	16.7	15.9	13.9	12.2	11.3

Use a graphing utility to create a scatter plot of each data set. What type of model would best fit the data? Use the regression capabilities of a graphing utility to find models for y_1 and y_2 (use $t = 8$ to represent 1988). Graph the models for y_1, y_2, and $y_1 - y_2$ in the same viewing window. What does the graph of the difference of the functions indicate about newspaper circulation in general?

Synthesis

True or False? **In Exercises 90 and 91, determine whether the statement is true or false. Justify your answer.**

90. If $f(x) = x + 1$ and $g(x) = 6x$, then

$(f \circ g)(x) = (g \circ f)(x)$.

91. If you are given two functions $f(x)$ and $g(x)$, you can calculate $(f \circ g)(x)$ if and only if the range of g is a subset of the domain of f.

92. ***Think About It*** Prove that the product of two odd functions is an even function and the product of two even functions is an even function.

93. ***Conjecture*** Use examples to hypothesize whether the product of an odd function and an even function is even or odd. Then prove your hypothesis.

94. Given a function f, prove that $g(x)$ is even and $h(x)$ is odd where

$$g(x) = \tfrac{1}{2}[f(x) + f(-x)] \quad \text{and}$$

$$h(x) = \tfrac{1}{2}[f(x) - f(-x)].$$

95. Use the result of Exercise 94 to prove that any function can be written as a sum of even and odd functions. (*Hint:* Add the two equations in Exercise 94.)

96. Use the result of Exercise 95 to write each function as a sum of even and odd functions.

(a) $f(x) = x^2 - 2x + 1$ (b) $f(x) = \dfrac{1}{x + 1}$

Review

In Exercises 97–100, find three points that lie on the graph of the equation.

97. $y = -x^2 + x - 5$

98. $y = \frac{1}{5}x^3 - 4x^2 + 1$

99. $x^2 + y^2 = 24$

100. $y = \dfrac{x}{x^2 - 5}$

In Exercises 101–104, find an equation of the line that passes through the two points.

101. $(-4, -2), (-3, 8)$

102. $(1, 5), (-8, 2)$

103. $\left(\frac{3}{2}, -1\right), \left(-\frac{1}{3}, 4\right)$

104. $(0, 1.1), (-4, 3.1)$

In Exercises 105–110, use the graph of f to sketch the graph of the specified function.

105. $f(x - 4)$

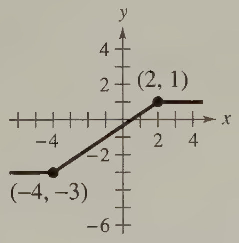
106. $f(x + 2)$

107. $f(x) + 4$

108. $f(x) - 1$

109. $2f(x)$

110. $\frac{1}{2}f(x)$



1.7 Inverse Functions

What You Should Learn:

- How to find inverse functions informally and verify that two functions are inverses of each other
- How to use graphs of functions to decide whether functions have inverses
- How to find inverse functions algebraically

Why You Should Learn It:

Inverse functions can be helpful in further exploring how two variables relate to each other. Exercise 90 on page 147 investigates the relationship between the exhaust temperature and the percent load on a diesel engine.

David J. Sams/Stock Boston

The Inverse of a Function

Recall from Section 1.3 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 4$ from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8\}$ can be written as follows.

$$f(x) = x + 4: \ \{(1, 5), (2, 6), (3, 7), (4, 8)\}$$

In this case, by interchanging the first and second coordinates of each of these ordered pairs, you can form the **inverse function** of f, which is denoted by f^{-1}. It is a function from the set B to the set A, and can be written as follows.

$$f^{-1}(x) = x - 4: \ \{(5, 1), (6, 2), (7, 3), (8, 4)\}$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 1.58. Also note that the functions f and f^{-1} have the effect of "undoing" each other. In other words, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = f(x - 4) = (x - 4) + 4 = x$$

$$f^{-1}(f(x)) = f^{-1}(x + 4) = (x + 4) - 4 = x$$

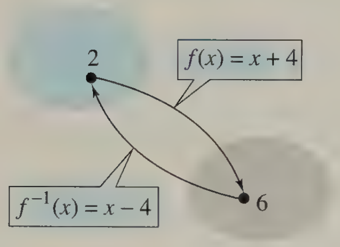


Figure 1.58

EXAMPLE 1 Finding Inverse Functions Informally

Find the inverse of $f(x) = 4x$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution

The given function *multiplies* each input by 4. To "undo" this function, you need to *divide* each input by 4. So, the inverse function of $f(x) = 4x$ is

$$f^{-1}(x) = \frac{x}{4}.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$f(f^{-1}(x)) = f\left(\frac{x}{4}\right) = 4\left(\frac{x}{4}\right) = x$$

$$f^{-1}(f(x)) = f^{-1}(4x) = \frac{4x}{4} = x$$

EXAMPLE 2 Finding Inverse Functions Informally

Find the inverse of $f(x) = x - 6$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution

The given function *subtracts* 6 from each input. To "undo" this function, you need to *add* 6 to each input. So, the inverse function of $f(x) = x - 6$ is

$$f^{-1}(x) = x + 6.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$f(f^{-1}(x)) = f(x + 6) = (x + 6) - 6 = x$$

$$f^{-1}(f(x)) = f^{-1}(x - 6) = (x - 6) + 6 = x$$

A table of values can help you understand inverse functions. For instance, the following table shows several values of the function in Example 2. Interchange the rows of this table to obtain values of the inverse function.

x	-2	-1	0	1	2
$f(x)$	-8	-7	-6	-5	-4

x	-8	-7	-6	-5	-4
$f^{-1}(x)$	-2	-1	0	1	2

In the table at the left, each output is 6 less than the input, and in the table at the right, each output is 6 more than the input.

The formal definition of the inverse of a function is as follows.

Definition of the Inverse of a Function

Let f and g be two functions such that

$f(g(x)) = x$ for every x in the domain of g

and

$g(f(x)) = x$ for every x in the domain of f.

Under these conditions, the function g is the **inverse** of the function f. The function g is denoted by f^{-1} (read "f-inverse"). So,

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.

STUDY TIP

Don't be confused by the use of -1 to denote the inverse function f^{-1}. In this text, whenever f^{-1} is written, it *always* refers to the inverse of the function f and *not* to the reciprocal of $f(x)$, which is

$$\frac{1}{f(x)}.$$

If the function g is the inverse of the function f, it must also be true that the function f is the inverse of the function g. For this reason, you can say that the functions f and g are *inverses of each other.*

EXAMPLE 3 Verifying Inverse Functions Algebraically

Show that the functions are inverses of each other.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

Solution

$$\begin{aligned} f(g(x)) &= f\left(\sqrt[3]{\frac{x+1}{2}}\right) = 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 \\ &= 2\left(\frac{x+1}{2}\right) - 1 \\ &= x + 1 - 1 \\ &= x \end{aligned}$$

$$\begin{aligned} g(f(x)) = g(2x^3 - 1) &= \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}} \\ &= \sqrt[3]{\frac{2x^3}{2}} \\ &= \sqrt[3]{x^3} \\ &= x \end{aligned}$$

STUDY TIP

Most graphing utilities can graph $y = x^{1/3}$ in two ways:

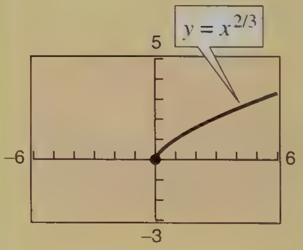
$y_1 = x \wedge (1/3)$ or

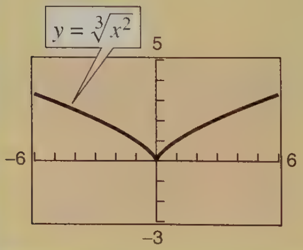
$y_1 = \sqrt[3]{x}$.

However, you may not be able to obtain the complete graph of $y = x^{2/3}$ by entering $y_1 = x \wedge (2/3)$. If not, you should use

$y_1 = (x \wedge (1/3))^2$ or

$y_1 = \sqrt[3]{x^2}$.





EXAMPLE 4 Verifying Inverse Functions Algebraically

Which of the functions is the inverse of $f(x) = \dfrac{5}{x-2}$?

$$g(x) = \frac{x-2}{5} \quad \text{or} \quad h(x) = \frac{5}{x} + 2$$

Solution

By forming the composition of f with g, you have

$$f(g(x)) = f\left(\frac{x-2}{5}\right) = \frac{5}{\dfrac{x-2}{5} - 2} = \frac{25}{x - 12} \neq x.$$

Because this composition is not equal to the identity function x, it follows that g *is not* the inverse of f. By forming the composition of f with h, you have

$$f(h(x)) = f\left(\frac{5}{x} + 2\right) = \frac{5}{\dfrac{5}{x} + 2 - 2} = \frac{5}{5/x} = x.$$

So, it appears that h is the inverse of f. You can confirm this by showing that the composition of h with f is also equal to the identity function.

The Graph of an Inverse Function

The graphs of f and f^{-1} are related to each other in the following way. If the point (a, b) lies on the graph of f, then the point (b, a) lies on the graph of f^{-1} and vice versa. This means that the graph of f^{-1} is a reflection of the graph of f in the line $y = x$, as shown in Figure 1.59.

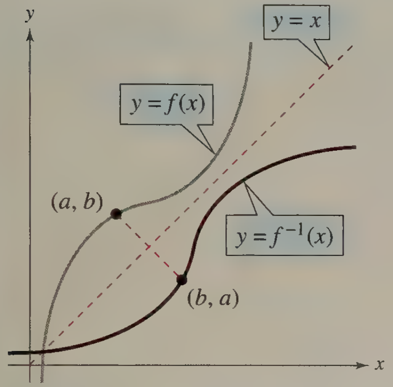


Figure 1.59

STUDY T!P

In Examples 3 and 4, inverse functions were verified algebraically. A graphing utility can also be helpful in checking to see whether one function is the inverse of another function. Use the Graph Reflection Program found on the website *college.hmco.com* to verify graphically Example 4.

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 5 Verifying Inverse Functions Graphically and Numerically

Example 3 shows how to verify *algebraically* that the functions

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

are inverses of each other. Verify that f and g are inverses of each other graphically and numerically.

Graphical Solution

You can *graphically* verify that f and g are inverses of each other by using a graphing utility to graph f and g in the same viewing window. (Be sure to use a square setting.) From the graph in Figure 1.60, you can verify that the graph of g is the reflection of the graph of f in the line $y = x$.

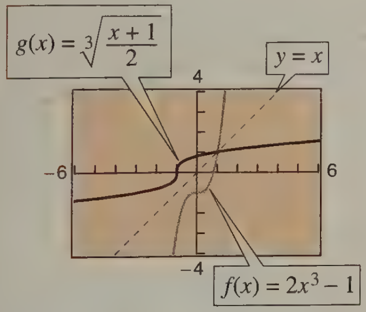


Figure 1.60

Numerical Solution

You can *numerically* verify that f and g are inverses of each other by creating two tables as shown below.

x	-2	-1	0	1	2
$f(x)$	-17	-3	-1	1	15

x	-17	-3	-1	1	15
$g(x)$	-2	-1	0	1	2

Note that the entries in the tables are the same except that their rows are interchanged. From the tables, you can verify that f and g are inverses of each other.

The Existence of an Inverse Function

A function need not have an inverse function. For instance, the function $f(x) = x^2$ has no inverse [assuming a domain of $(-\infty, \infty)$]. You can observe this numerically in the table at the right.

When you interchange the rows of the tables, you can find two different outputs correspond to the same input. For example, the input 4 corresponds to two different outputs, -2 and 2. So, the bottom table does not represent a function, and therefore $f(x) = x^2$ does not have an inverse.

To have an inverse, a function must be **one-to-one,** which means that no two elements in the domain of f correspond to the same element in the range of f.

x	-2	-1	0	1	2
$f(x)$	4	1	0	1	4

↓

x	4	1	0	1	4
$g(x)$	-2	-1	0	1	2

Definition of a One-to-One Function

A function f is **one-to-one** if, for a and b in its domain,

$$f(a) = f(b) \quad \text{implies that} \quad a = b.$$

Existence of an Inverse Function

A function f has an inverse function f^{-1} if and only if f is one-to-one.

From its graph, it is easy to tell whether a function of x is one-to-one. Simply check to see that every horizontal line intersects the graph of the function at most once. For instance, Figure 1.61 shows the graph of $y = x^4$. On the graph, you can find a horizontal line that intersects the graph twice.

Two special types of functions that pass the **Horizontal Line Test** are those that are increasing or decreasing on their entire domains.

1. If f is *increasing* on its entire domain, f is one-to-one.
2. If f is *decreasing* on its entire domain, f is one-to-one.

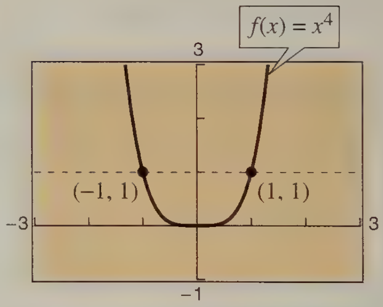


Figure 1.61 $f(x) = x^4$ *is not one-to-one.*

EXAMPLE 6 Testing for One-to-One Functions

Is the function $f(x) = \sqrt{x} + 1$ one-to-one?

Algebraic Solution

Let a and b be nonnegative real numbers with $f(a) = f(b)$.

$$\sqrt{a} + 1 = \sqrt{b} + 1 \qquad \text{Set } f(a) = f(b).$$

$$\sqrt{a} = \sqrt{b}$$

$$a = b$$

Therefore,

$$f(a) = f(b)$$

implies that

$$a = b.$$

So, f is one-to-one.

Graphical Solution

Use a graphing utility to graph the function $y = \sqrt{x} + 1$. From Figure 1.62, you can see that a horizontal line will intersect the graph at most once. So, f is one-to-one.

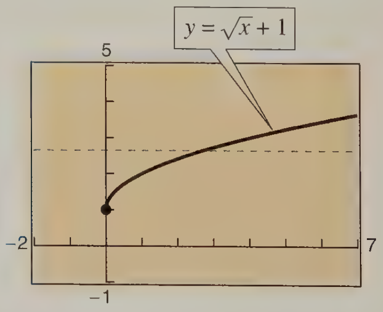


Figure 1.62

Finding Inverse Functions Algebraically

For simple functions (such as the ones in Examples 1 and 2) you can find inverse functions by inspection. For instance, the inverse of $f(x) = 8x$ is $f^{-1}(x) = x/8$. For more complicated functions, however, it is best to use the following procedure for finding the inverse of a function.

Finding the Inverse of a Function

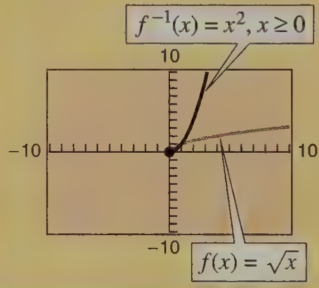
To find the inverse of f, use the following steps.

1. Use the Horizontal Line Test to decide whether f has an inverse.
2. In the equation for $f(x)$, replace $f(x)$ by y.
3. Interchange the roles of x and y, and solve for y.
4. Replace y by $f^{-1}(x)$ in the new equation.
5. Verify that f and f^{-1} are inverses of each other by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

It is important to note that in Step 1 above, the domain of f is assumed to be the entire real line. The domain of f may be restricted so that f does have an inverse. For instance, if the domain of $f(x) = x^2$ is restricted to the nonnegative real numbers, then f does have an inverse.

STUDY T!P

Many graphing utilities have a built-in feature to draw the inverse of a function. To see how this works, consider the function $f(x) = \sqrt{x}$. The inverse of f is $f^{-1}(x) = x^2$, $x \geq 0$. Enter the function $y_1 = \sqrt{x}$. Then graph it in the standard viewing window and use the *draw inverse* feature. You should obtain the figure below, which shows both f and its inverse f^{-1}.



EXAMPLE 7 Finding the Inverse of a Function

Find the inverse (if it exists) of $f(x) = \dfrac{5 - 3x}{2}$.

Solution

The graph of f in Figure 1.63 passes the Horizontal Line Test, so you can see that f is one-to-one, and therefore has an inverse.

$$f(x) = \frac{5 - 3x}{2} \qquad \text{Write original equation.}$$

$$y = \frac{5 - 3x}{2} \qquad \text{Replace } f(x) \text{ by } y.$$

$$x = \frac{5 - 3y}{2} \qquad \text{Interchange } x \text{ and } y.$$

$$2x = 5 - 3y \qquad \text{Multiply each side by 2.}$$

$$3y = 5 - 2x \qquad \text{Isolate the } y\text{-term.}$$

$$y = \frac{5 - 2x}{3} \qquad \text{Solve for } y.$$

$$f^{-1}(x) = \frac{5 - 2x}{3} \qquad \text{Replace } y \text{ by } f^{-1}(x).$$

The domain and range of both f and f^{-1} consist of all real numbers. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

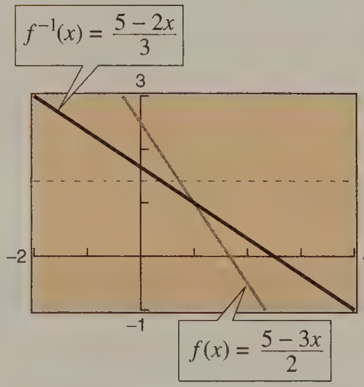


Figure 1.63

EXAMPLE 8 Finding the Inverse of a Function

Find the inverse of $f(x) = \sqrt{2x - 3}$ and sketch the graphs of f and f^{-1}.

Solution

The graph of f in Figure 1.64 passes the Horizontal Line Test, so you can see that f is one-to-one and therefore has an inverse.

$$\begin{aligned}
f(x) &= \sqrt{2x-3} && \text{Write original equation.}\\
y &= \sqrt{2x-3} && \text{Replace } f(x) \text{ by } y.\\
x &= \sqrt{2y-3} && \text{Interchange } x \text{ and } y.\\
x^2 &= 2y - 3 &&\\
2y &= x^2 + 3 &&\\
y &= \frac{x^2+3}{2} && \text{Solve for } y.\\
f^{-1}(x) &= \frac{x^2+3}{2}, \quad x \ge 0 && \text{Replace } y \text{ by } f^{-1}(x).
\end{aligned}$$

The graph of f^{-1} in Figure 1.64 is the reflection of the graph of f in the line $y = x$. Note that the range of f is the interval $[0, \infty)$, which implies that the domain of f^{-1} is the interval $[0, \infty)$. Moreover, the domain of f is the interval $\left[\frac{3}{2}, \infty\right)$, which implies that the range of f^{-1} is the interval $\left[\frac{3}{2}, \infty\right)$.

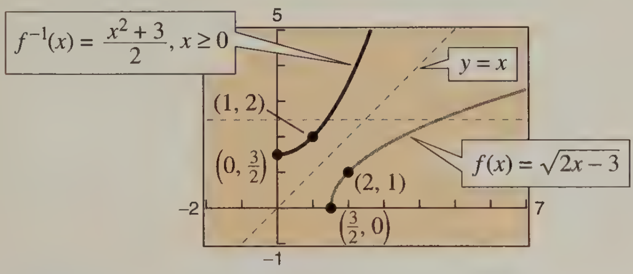


Figure 1.64

Writing About Math *The Existence of an Inverse Function*

Write a short paragraph describing why the following functions do or do not have inverse functions. Give a numerical example for each.

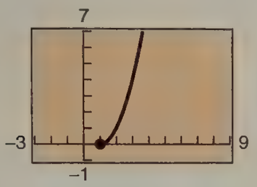
a. Your hourly wage is \$7.50 plus \$0.90 for each unit x produced per hour. Let $f(x)$ represent your weekly wage for 40 hours of work. Does this function have an inverse?

b. Let x represent the retail price of an item (in dollars), and let $f(x)$ represent the sales tax on the item. Assume that the sales tax is 7% of the retail price *and* that the sales tax is rounded to the nearest cent. Does this function have an inverse? (*Hint:* Can you undo this function? For instance, if you know that the sales tax is \$0.14, can you determine *exactly* what the retail price is?)

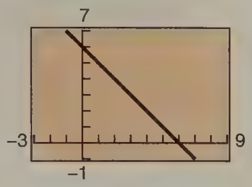
1.7 Exercises

In Exercises 1–4, match the graph of the function with the graph of its inverse. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]

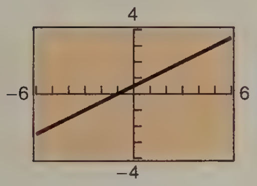
(a)



(b)



(c)



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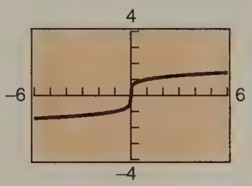
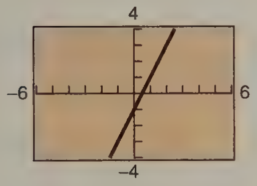


1.



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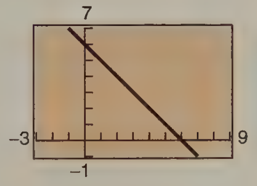

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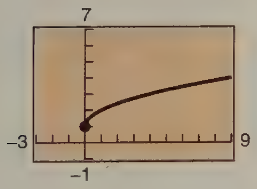

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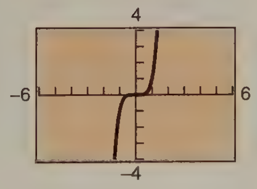


In Exercises 5–12, find the inverse of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

5. $f(x) = 8x$

6. $f(x) = \frac{1}{5}x$

7. $f(x) = x + 10$

8. $f(x) = x - 5$

9. $f(x) = 2x + 1$

10. $f(x) = \frac{x - 1}{4}$

11. $f(x) = \sqrt[3]{x}$

12. $f(x) = x^5$

In Exercises 13–16, show that f and g are inverse functions (a) algebraically and (b) graphically.

13. $f(x) = 2x, \quad g(x) = \frac{x}{2}$

14. $f(x) = x - 5, \quad g(x) = x + 5$

15. $f(x) = 5x + 1, \quad g(x) = \frac{x - 1}{5}$

16. $f(x) = 3 - 4x, \quad g(x) = \frac{3 - x}{4}$

In Exercises 17–22, show that f and g are inverse functions algebraically. Use a graphing utility to graph f and g in the same viewing window. Describe the relationship between the graphs.

17. $f(x) = x^3, \quad g(x) = \sqrt[3]{x}$

18. $f(x) = \frac{1}{x}, \quad g(x) = \frac{1}{x}$

19. $f(x) = \sqrt{x - 4}; \quad g(x) = x^2 + 4, \ x \geq 0$

20. $f(x) = 9 - x^2, \quad x \geq 0; \quad g(x) = \sqrt{9 - x}$

21. $f(x) = 1 - x^3, \quad g(x) = \sqrt[3]{1 - x}$

22. $f(x) = \frac{1}{1 + x}, \ x \geq 0; \quad g(x) = \frac{1 - x}{x}, \ 0 < x \leq 1$

In Exercises 23–28, (a) show that f and g are inverse functions algebraically and (b) verify that f and g are inverses numerically by creating a table of values for each function.

23. $f(x) = -\frac{7}{2}x - 3, \quad g(x) = -\frac{2x + 6}{7}$

24. $f(x) = \frac{x + 8}{3}, \quad g(x) = 3x - 8$

25. $f(x) = x^3 + 5, \quad g(x) = \sqrt[3]{x - 5}$

26. $f(x) = \frac{x^3}{5}, \quad g(x) = \sqrt[3]{5x}$

27. $f(x) = -\sqrt{x - 8}; \quad g(x) = 8 + x^2, \ x \leq 0$

28. $f(x) = \sqrt[3]{3x - 10}, \quad g(x) = \frac{x^3 + 10}{3}$

In Exercises 29–42, use a graphing utility to graph the function and use the Horizontal Line Test to determine whether the function is one-to-one.

29. $f(x) = 3 - \frac{1}{2}x$

30. $g(x) = \frac{4 - x}{6}$

31. $h(x) = \frac{x^2}{x^2 + 1}$

32. $f(x) = \frac{1}{8}(x + 2)^2 - 1$

33. $h(x) = \sqrt{16 - x^2}$

34. $f(x) = -2x\sqrt{16 - x^2}$

35. $f(x) = \sqrt{x - 2}$

36. $f(x) = 4 - 3x^{2/3}$

37. $f(x) = 10$

38. $f(x) = -0.65$

39. $g(x) = (x + 5)^3$

40. $f(x) = x^5 - 7$

41. $h(x) = |x + 4| - |x - 4|$

42. $f(x) = -\dfrac{|x - 6|}{|x + 6|}$

In Exercises 43–54, find the inverse of the function f. Use a graphing utility to graph both f and f^{-1} in the same viewing window. Describe the relationship between the graphs.

43. $f(x) = 2x - 3$

44. $f(x) = 3x$

45. $f(x) = x^5$

46. $f(x) = x^3 + 1$

47. $f(x) = \sqrt{x}$

48. $f(x) = x^2, \quad x \geq 0$

49. $f(x) = \sqrt{4 - x^2}, \quad 0 \leq x \leq 2$

50. $f(x) = \sqrt{16 - x^2}, \quad -4 \leq x \leq 0$

51. $f(x) = \sqrt[3]{x - 1}$

52. $f(x) = x^{3/5}$

53. $f(x) = \dfrac{4}{x}$

54. $f(x) = \dfrac{6}{\sqrt{x}}$

In Exercises 55–68, determine algebraically whether the function is one-to-one. If it is, find its inverse. Verify your answer graphically.

55. $f(x) = x^4$

56. $f(x) = \dfrac{1}{x^2}$

57. $f(x) = \dfrac{3x + 4}{5}$

58. $f(x) = 3x + 5$

59. $f(x) = (x + 3)^2, \quad x \geq -3$

60. $q(x) = (x - 5)^2, \quad x \leq 5$

61. $h(x) = \dfrac{4}{x^2}$

62. $f(x) = |x - 2|, \quad x \leq 2$

63. $f(x) = \sqrt{2x + 3}$

64. $f(x) = \sqrt{x - 2}$

65. $g(x) = x^2 - x^4$

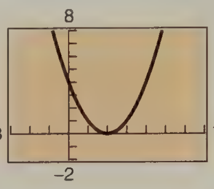
66. $f(x) = \dfrac{x^2}{x^2 + 1}$

67. $f(x) = ax + b, \quad a \neq 0$

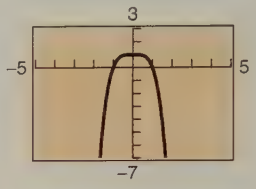
68. $f(x) = c$

Think About It **In Exercises 69–72, delete part of the graph of the function so that the part that remains is one-to-one. Find the inverse of the remaining part and give the domain of the inverse. (There are many correct answers.)**

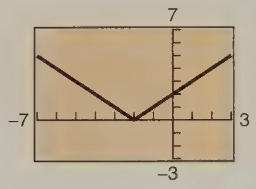
69. $f(x) = (x - 2)^2$



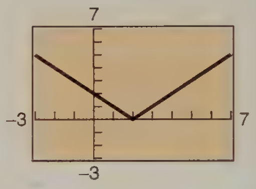
70. $f(x) = 1 - x^4$



71. $f(x) = |x + 2|$

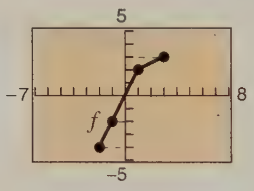


72. $f(x) = |x - 2|$



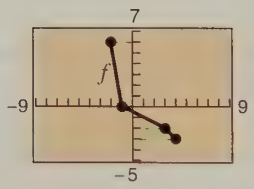
In Exercises 73 and 74, use the graph of the function f to complete the table and sketch the graph of f^{-1}.

73.



x	$f^{-1}(x)$
−4	
−2	
2	
3	

74.



x	$f^{-1}(x)$
−3	
−2	
0	
6	

Graphical Reasoning **In Exercises 75–78, (a) use a graphing utility to graph the function, (b) use the *draw inverse* feature of the graphing utility to draw the inverse of the function, and (c) determine whether the graph of the inverse relation is an inverse function, explaining your reasoning.**

75. $f(x) = x^3 + x + 1$

76. $h(x) = x\sqrt{4 - x^2}$

77. $g(x) = \dfrac{3x^2}{x^2 + 1}$

78. $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$

In Exercises 79–84, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the indicated value or function.

79. $(f^{-1} \circ g^{-1})(1)$

80. $(g^{-1} \circ f^{-1})(-3)$

81. $(f^{-1} \circ f^{-1})(6)$

82. $(g^{-1} \circ g^{-1})(-4)$

83. $(f \circ g)^{-1}$

84. $g^{-1} \circ f^{-1}$

In Exercises 85–88, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the specified functions.

85. $g^{-1} \circ f^{-1}$

86. $f^{-1} \circ g^{-1}$

87. $(f \circ g)^{-1}$

88. $(g \circ f)^{-1}$

89. ***Hourly Wage*** Your wage is \$8.00 per hour plus \$0.75 for each unit produced per hour. So, your hourly wage y in terms of the number of units produced is $y = 8 + 0.75x$.

(a) Determine the inverse of the function. What does each variable in the inverse function represent?

(b) Use a graphing utility to graph the function and its inverse.

(c) Use the *trace* feature of your graphing utility to find the hourly wage if 10 units are produced per hour.

(d) Use the *trace* feature of your graphing utility to find the number of units produced when your hourly wage is \$22.25.

90. ***Diesel Engine*** The function

$$y = 0.03x^2 + 254.50, \qquad 0 < x < 100$$

approximates the exhaust temperature y of a diesel engine in degrees Fahrenheit, where x is the percent load on the engine.

(a) Determine the inverse of the function. What does each variable in the inverse function represent?

(b) Use a graphing utility to graph the inverse function.

(c) Determine the percent load interval if the exhaust temperature of the engine must not exceed 500°F.

91. ***Transportation*** The total value of new car sales f (in billions of dollars) in the United States from 1992 through 1997 is shown in the table. The time (in years) is given by t, with $t = 2$ corresponding to 1992. (Source: National Automobile Dealers Association)

t	2	3	4	5	6	7
$f(t)$	333.8	377.3	430.6	456.2	490.0	507.5

(a) Does f^{-1} exist?

(b) If f^{-1} exists, what does it mean in the context of the problem?

(c) If f^{-1} exists, find $f^{-1}(456.2)$.

(d) If the table above was extended to 1998 and if the total value of new car sales for that year was \$430.6 billion, would f^{-1} exist? Explain.

Synthesis

True or False? **In Exercises 92 and 93, determine whether the statement is true or false. Justify your answer.**

92. If f is an even function, f^{-1} exists.

93. If the inverse of f exists, the y-intercept of f is an x-intercept of f^{-1}.

94. Prove that if f and g are one-to-one functions, $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

95. Prove that if f is a one-to-one odd function, f^{-1} is an odd function.

96. ***Think About It*** The function

$$f(x) = k(2 - x - x^3)$$

is one-to-one and $f^{-1}(3) = -2$. Find k.

Review

In Exercises 97–100, write the rational expression in simplest form.

97. $\dfrac{27x^3}{3x^2}$

98. $\dfrac{5x^2y}{xy + 5x}$

99. $\dfrac{x^2 - 36}{6 - x}$

100. $\dfrac{x^2 + 3x - 40}{x^2 - 3x - 10}$

In Exercises 101–104, evaluate the combination of functions when $f(x) = 2x^2 - 5$ and $g(x) = x - 3$ at the indicated value of x.

101. $(f + g)(-x)$ when $x = -2$

102. $(f - g)(x)$ when $x = 4$

103. $(fg)(-x)$ when $x = -3$

104. $\left(\dfrac{f}{g}\right)(x)$ when $x = \dfrac{3}{2}$

1 Chapter Summary

What did you learn?

	Review Exercises
Section 1.1	
☐ How to sketch graphs of equations by point plotting and by using a graphing utility	1–22
☐ How to use graphs of equations in real-life problems	23, 24
Section 1.2	
☐ How to find the slopes of lines	25–34
☐ How to write linear equations given points on lines and their slopes	35–46
☐ How to use slope-intercept forms of linear equations to sketch graphs of lines	47–58
☐ How to use slope to identify parallel and perpendicular lines	59–62
Section 1.3	
☐ How to decide whether relations between two variables are functions	63–68
☐ How to use function notation and evaluate functions	69, 70
☐ How to find the domains of functions	71–76
☐ How to use functions to model and solve real-life problems	77, 78
Section 1.4	
☐ How to find the domains and ranges of functions and use the Vertical Line Test for functions	79–86
☐ How to determine intervals on which functions are increasing or decreasing	87–90
☐ How to determine relative maximum and relative minimum values of functions	91–94
☐ How to graph step functions and other piecewise-defined functions	95, 96
☐ How to identify even and odd functions	97, 98
Section 1.5	
☐ How to recognize graphs of common functions	99–102
☐ How to use vertical and horizontal shifts and reflections to sketch graphs of functions	103–118
☐ How to use nonrigid transformations to sketch graphs of functions	119–126
Section 1.6	
☐ How to add, subtract, multiply, and divide functions	127–132
☐ How to find compositions of one function with another function	133–136
☐ How to use combinations of functions to model and solve real-life problems	137, 138
Section 1.7	
☐ How to find inverse functions informally and verify that two functions are inverses of each other	139–142
☐ How to use graphs of functions to decide whether functions have inverses	143–150
☐ How to find inverse functions algebraically	151–156

1 Review Exercises

1.1 In Exercises 1 and 2, complete the table. Use the resulting solution points to sketch the graph of the equation. Use a graphing utility to verify the graph.

1. $y = -\frac{1}{2}x + 2$

x	-2	0	2	3	4
y					

2. $y = x^2 - 3x$

x	-1	0	1	2	3
y					

In Exercises 3–12, sketch the graph of the equation *by hand*. Use a graphing utility to verify the graph.

3. $y - 2x - 3 = 0$

4. $3x + 2y + 6 = 0$

5. $x - 5 = 0$

6. $y = 8 - |x|$

7. $y = \sqrt{5 - x}$

8. $y = \sqrt{x + 2}$

9. $y + 2x^2 = 0$

10. $y = x^2 - 4x$

11. $x + y^2 = 9$

12. $x^2 + y^2 = 10$

In Exercises 13–20, use a graphing utility to graph the equation. Approximate any intercepts.

13. $y = \frac{1}{4}(x + 1)^3$

14. $y = 4 - (x - 4)^2$

15. $y = \frac{1}{4}x^4 - 2x^2$

16. $y = \frac{1}{4}x^3 - 3x$

17. $y = x\sqrt{9 - x^2}$

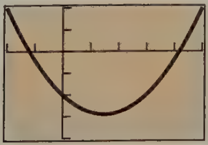
18. $y = x\sqrt{x + 3}$

19. $y = |x - 4| - 4$

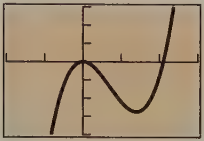
20. $y = |x + 2| + |3 - x|$

In Exercises 21 and 22, find a viewing window on a graphing utility such that the graph of the equation agrees with the graph shown.

21. $y = 0.002x^2 - 0.06x - 1$



22. $y = 10x^3 - 21x^2$



Data Analysis **In Exercises 23 and 24, (a) use a graphing utility to plot the data; (b) use the regression capabilities of a graphing utility to find the best-fitting linear model (let $t = 2$ correspond to 1992); (c) graph the model in the same viewing window with the data; and (d) use the model to estimate the values of y for the years 2000 and 2002.**

23. The average expenditures y for automobile insurance per insured vehicle from 1992 through 1996 are shown in the table. (Source: National Association of Insurance Commissioners)

x	1992	1993	1994	1995	1996
y	616	638	651	667	685

24. The total number y of Coast Guard personnel on active duty (in thousands) from 1992 through 1997 are shown in the table. (Source: U.S. Department of Transportation)

x	1992	1993	1994	1995	1996	1997
y	39.4	39.2	37.8	36.7	35.2	34.9

1.2 In Exercises 25–30, plot the two points and find the slope of the line that passes through the points.

25. $(-3, 2), (8, 2)$

26. $(7, -1), (7, 12)$

27. $\left(\frac{3}{2}, 1\right), \left(5, \frac{5}{2}\right)$

28. $\left(-\frac{3}{4}, \frac{5}{6}\right), \left(\frac{1}{2}, -\frac{5}{2}\right)$

29. $(-4.5, 6), (2.1, 3)$

30. $(-2.7, -6.3), (-1, -1.2)$

In Exercises 31–34, use the concept of slope to find t such that the three points are collinear.

31. $(-2, 5), (0, t), (1, 1)$

32. $(-6, 1), (1, t), (10, 5)$

33. $(1, -4), (t, 3), (5, 10)$

34. $(-3, 3), (t, -1), (8, 6)$

In Exercises 35–46, (a) find an equation of the line that passes through the given point and has the specified slope, and (b) find three additional points through which the line passes.

	Point	*Slope*
35.	$(2, -1)$	$m = \frac{1}{4}$
36.	$(-3, 5)$	$m = -\frac{3}{2}$
37.	$(0, -5)$	$m = \frac{3}{2}$
38.	$(3, 0)$	$m = -\frac{2}{3}$
39.	$(-6, -5)$	$m = -2$
40.	$(-7, 2)$	$m = 4$
41.	$\left(\frac{1}{5}, -5\right)$	$m = -1$
42.	$\left(0, \frac{7}{8}\right)$	$m = -\frac{4}{5}$
43.	$(-2, 6)$	$m = 0$
44.	$(-8, 8)$	$m = 0$
45.	$(10, -6)$	m is undefined.
46.	$(5, 4)$	m is undefined.

In Exercises 47–52, (a) find an equation of the line (in slope-intercept form) that passes through the points and (b) sketch the graph of the equation.

47. $(2, -1), (4, -1)$

48. $(0, 0), (0, 10)$

49. $(2, 1), (14, 6)$

50. $(-2, 2), (3, -10)$

51. $(-1, 0), (6, 2)$

52. $(1, 6), (4, 2)$

Rate of Change **In Exercises 53 and 54, you are given the dollar value of a product in 2000 *and* the rate at which the value of the item is expected to change during the 5 years following. Use this information to write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 0$ represent 2000.)**

	2000 Value	*Rate*
53.	\$12,500	\$850 increase per year
54.	\$72.95	\$5.15 increase per year

Exploration **In Exercises 55 and 56, find a relationship between x and y such that (x, y) is equidistant from the two points.**

55. $(-2, -5), (6, 3)$

56. $\left(1, \frac{7}{2}\right), (5, 0)$

57. ***Business*** During the second and third quarters of the year, a business had sales of \$160,000 and \$185,000, respectively. If the growth of sales follows a linear pattern, estimate sales during the fourth quarter.

58. ***Dollar Value*** The dollar value of a product in 2000 is \$85, and the product will increase in value at an expected rate of \$3.75 per year.

(a) Write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 0$ represent 2000.)

(b) Use a graphing utility to graph the sales equation.

(c) Use the *value* or *trace* feature of your graphing utility to estimate the dollar value of the product in 2005.

In Exercises 59–62, write equations of the lines through the point (a) parallel to the given line and (b) perpendicular to the given line. Verify your result with a graphing utility (use a square setting).

	Point	*Line*
59.	$(3, -2)$	$5x - 4y = 8$
60.	$(-8, 3)$	$2x + 3y = 5$
61.	$(-6, 2)$	$x = 4$
62.	$(3, -4)$	$y = 2$

1.3 **In Exercises 63 and 64, determine which of the sets of ordered pairs represents a function from A to B. Give reasons for your answers.**

63. $A = \{10, 20, 30, 40\}$ and $B = \{0, 2, 4, 6\}$

(a) $\{(20, 4), (40, 0), (20, 6), (30, 2)\}$

(b) $\{(10, 4), (20, 4), (30, 4), (40, 4)\}$

(c) $\{(40, 0), (30, 2), (20, 4), (10, 6)\}$

(d) $\{(20, 2), (10, 0), (40, 4)\}$

64. $A = \{u, v, w\}$ and $B = \{-2, -1, 0, 1, 2\}$

(a) $\{(v, -1), (u, 2), (w, 0), (u, -2)\}$

(b) $\{(u, -2), (v, 2), (w, 1)\}$

(c) $\{(u, 2), (v, 2), (w, 1), (w, 1)\}$

(d) $\{(w, -2), (v, 0), (w, 2)\}$

In Exercises 65–68, determine if the equation represents y as a function of x.

65. $16x - y^4 = 0$

66. $2x - y - 3 = 0$

67. $y = \sqrt{1 - x}$

68. $|y| = x + 2$

In Exercises 69 and 70, evaluate the function at each value of the specified variable. Simplify your answers.

69. $f(x) = x^2 + 1$

(a) $f(2)$ (b) $f(-4)$

(c) $f(t^2)$ (d) $-f(x)$

70. $g(x) = x^{4/3}$

(a) $g(8)$ (b) $g(t + 1)$

(c) $\dfrac{g(8) - g(1)}{8 - 1}$ (d) $g(-x)$

In Exercises 71–76, determine the domain of the function. Verify your result with a graphing utility.

71. $f(x) = (x - 1)(x + 2)$ **72.** $f(x) = x^2 - 4x - 32$

73. $f(x) = \sqrt{25 - x^2}$ **74.** $f(x) = \sqrt{x^2 + 8x}$

75. $g(s) = \dfrac{5}{3s - 9}$ **76.** $f(x) = \dfrac{2}{3x + 4}$

77. ***Business*** A company produces a product for which the variable cost is \$5.35 per unit and the fixed costs are \$16,000. The company sells the product for \$8.20 and can sell all that it produces.

(a) Find the total cost as a function of x, the number of units produced.

(b) Find the profit as a function of x.

78. ***Boating*** The retail expenditures B (in billions of dollars) on boating in the United States from 1985 to 1996 can be represented by the piecewise-defined function

$$B(t) = \begin{cases} -0.631t^2 - 2.845t + 14.160, & -5 \le t < 2 \\ 2.088t + 5.768, & 2 \le t \le 6 \end{cases}$$

where $t = 0$ represents 1990. Use a graphing utility to graph the model and find the amount spent on boating in 1985, 1990, and 1995. (Source: National Marine Manufacturers Association)

1.4 **In Exercises 79–82, find the domain and range of the function.**

79. $f(x) = 3 - 2x^2$ **80.** $f(x) = \sqrt{2x^2 - 1}$

81. $h(x) = \sqrt{36 - x^2}$ **82.** $g(x) = |x + 5|$

In Exercises 83–86, (a) use a graphing utility to graph the equation and (b) use the Vertical Line Test to determine whether y is a function of x.

83. $y = \dfrac{x^2 + 3x}{6}$ **84.** $y = -\frac{2}{3}|x + 5|$

85. $3x + y^2 = 2$ **86.** $x^2 + y^2 = 49$

In Exercises 87–90, determine the open intervals over which the function is increasing, decreasing, or constant.

87. $f(x) = x^3 - 3x$ **88.** $f(x) = \sqrt{x^2 - 9}$

89. $f(x) = x\sqrt{x - 6}$ **90.** $f(x) = \dfrac{|x + 8|}{2}$

Graphical Analysis **In Exercises 91–94, use a graphing utility to approximate (to two-decimal-place accuracy) any relative maximum or minimum values of the function.**

91. $f(x) = (x^2 - 4)^2$ **92.** $f(x) = x^2 - x - 1$

93. $h(x) = 4x^3 - x^4$ **94.** $f(x) = x^3 - 4x^2 - 1$

In Exercises 95 and 96, sketch the graph of the piecewise-defined function by hand. Verify using a graphing utility.

95. $f(x) = \begin{cases} 3x + 5, & x < 0 \\ x - 4, & x \ge 0 \end{cases}$

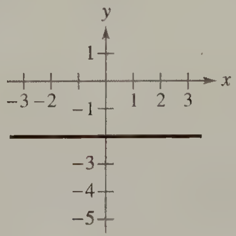
96. $f(x) = \begin{cases} x^2 + 7, & x < 1 \\ x^2 - 5x + 6, & x \ge 1 \end{cases}$

In Exercises 97 and 98, determine whether the function is even, odd, or neither.

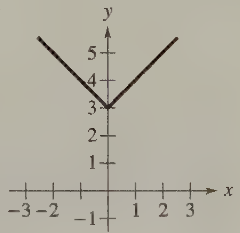
97. $f(x) = (x^2 - 8)^2$ **98.** $f(x) = 2x^3 - x^2$

1.5 **In Exercises 99–102, the graph is related to one of the common functions on page 118. Identify the common function and describe the transformation shown in the graph. Write the equation for the graphed function.**

99.



100.



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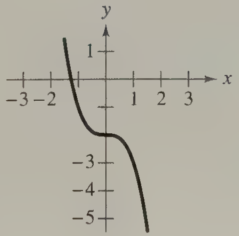
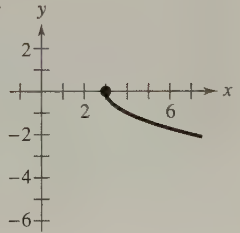


102.



In Exercises 103–126, sketch the graph of the function by hand.

103. $f(x) = x^2 - 6$ **104.** $f(x) = (x - 3)^2 - 2$

105. $f(x) = (x - 1)^3 + 7$ **106.** $f(x) = (x + 2)^3 + 5$

107. $f(x) = \sqrt{x} - 5$ **108.** $f(x) = \sqrt{x + 5} + 4$

109. $f(x) = 7 + |x|$ **110.** $f(x) = |x + 8| - 1$

111. $f(x) = -x^2 - 3$

112. $f(x) = -(x - 2)^2 - 8$

113. $f(x) = -(x + 5)^3 - 6$

114. $f(x) = 3 - x^3$ **115.** $f(x) = -\sqrt{x + 4} - 3$

116. $f(x) = \sqrt{-x - 1} + 7$

117. $f(x) = -|x + 2| - 1$

118. $f(x) = 1 - |-x|$

119. $f(x) = -2x^2 + 3$ **120.** $f(x) = \frac{1}{2}(x - 3)^2 + 6$

121. $f(x) = -\frac{1}{4}(1 - x)^3 - 2$

122. $f(x) = 5(x - 1)^3 - 5$

123. $f(x) = -\frac{1}{2}|x| + 9$ **124.** $f(x) = -2|-x - 4|$

125. $f(x) = -3\sqrt{6 - x} + 4$

126. $f(x) = 4\sqrt{x + 1} - 8$

1.6 **In Exercises 127–136, let $f(x) = 3 - 2x$, $g(x) = \sqrt{x}$, and $h(x) = 3x^2 + 2$, and find the indicated values.**

127. $(f - g)(4)$ **128.** $(f + h)(5)$

129. $(f + g)(25)$ **130.** $(g - h)(1)$

131. $(fh)(1)$ **132.** $\left(\frac{g}{h}\right)(1)$

133. $(h \circ g)(7)$ **134.** $(g \circ f)(-2)$

135. $(f \circ h)(-4)$ **136.** $(g \circ h)(6)$

Data Analysis **In Exercises 137 and 138, use the table, which shows the total value (in billions of dollars) of U.S. imports from China and Taiwan for the years 1992 through 1997. The variables y_1 and y_2 represent the total value of imports from China and Taiwan, respectively.** (Source: U.S. Bureau of the Census)

Year	1992	1993	1994	1995	1996	1997
y_1	25.7	31.5	38.8	45.5	51.5	62.6
y_2	24.6	25.1	26.7	29.0	29.9	32.6

137. Use the regression capabilities of a graphing utility to find quadratic models for each of the variables. Let $t = 2$ represent 1992.

138. Use a graphing utility to graph y_1, y_2, and $y_1 + y_2$ in the same viewing window. Use the model to estimate the total value of U.S. imports from China and Taiwan in 2002.

1.7 **In Exercises 139–142, find the inverse of f informally. Verify that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.**

139. $f(x) = 6x$ **140.** $f(x) = \frac{1}{12}x$

141. $f(x) = x - 7$ **142.** $f(x) = x + 5$

In Exercises 143–146, (a) find f^{-1}, (b) graph f and f^{-1} in the same viewing window, and (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

143. $f(x) = \frac{1}{2}x - 3$ **144.** $f(x) = 5x - 7$

145. $f(x) = \sqrt{x + 1}$ **146.** $f(x) = x^3 + 2$

In Exercises 147–150, restrict the domain of the function f to an interval over which the function is increasing, and determine f^{-1} over that interval. Use a graphing utility to graph f and f^{-1} in the same viewing window.

147. $f(x) = 2(x - 4)^2$ **148.** $f(x) = -x^2 + 4$

149. $f(x) = |x - 2|$ **150.** $f(x) = \frac{1}{2}|x + 4|$

In Exercises 151–156, find the inverse of f algebraically.

151. $f(x) = \dfrac{x}{12}$ **152.** $f(x) = \dfrac{7x + 3}{8}$

153. $f(x) = 4x^3 - 3$ **154.** $f(x) = x^3 - 2$

155. $f(x) = \sqrt{x + 10}$ **156.** $f(x) = 4\sqrt{6 - x}$

Synthesis

True or False? **In Exercises 157–159, determine whether the statement is true or false. Justify your answer.**

157. Relative to the graph of $f(x) = \sqrt{x}$, the function $h(x) = -\sqrt{x + 9} - 13$ is shifted 9 units to the left and 13 units down, then reflected in the x-axis.

158. If $f(x) = x^n$ where n is odd, f^{-1} exists.

159. There exists no function f such that $f = f^{-1}$.

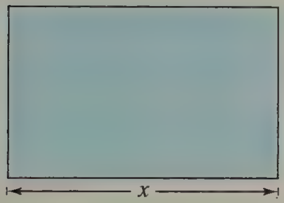
160. Explain why not all equations of lines are functions.

Chapter Project *Modeling the Area of a Plot*

Many real-life problems can be analyzed from a *graphical,* a *numerical,* and an *algebraic* perspective. In this project, you will use all three strategies to determine the maximum size of a rectangular plot that can be enclosed by a fixed amount of fencing.

You have 100 meters of fencing material to enclose a rectangular plot. Your goal is to determine the dimensions of the plot such that you enclose the maximum area possible.

a. Express the area $A(x)$ of the rectangular plot as a function of the length x of one side, as shown in the figure below.



b. Analyze the problem *numerically* by completing the table.

x	0	5	10	15	20	25	30	35	40	45	50
$A(x)$											

According to this table, what do you think the dimensions of the plot should be to enclose the maximum area? Explain.

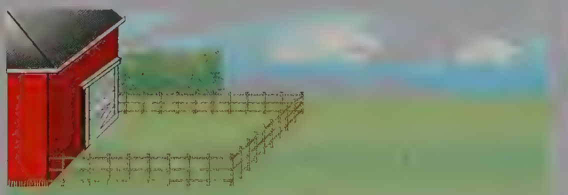
c. Use a graphing utility to graph the area function. What is the domain of the function? Solve the problem *graphically* by using the *trace* feature of your graphing utility to find the value of x that yields the maximum area.

d. Solve the problem *algebraically* by showing that the area function can be written as $A(x) = 625 - (x - 25)^2$. How does this form of the function allow you to find the dimensions that produce a maximum area?

e. Discuss the strengths and weaknesses of the three strategies used in parts (b), (c), and (d).

Questions for Further Exploration

1. Suppose you were not restricted to a rectangular plot. Would you be able to use 100 meters of fencing to enclose a greater area? Explain.
2. In the project above, you found the maximum area that can be enclosed in a rectangular plot using 100 meters of fencing. If you doubled the amount of fencing, could you enclose twice as much area? Use numerical, graphical, and algebraic approaches and explain your reasoning.
3. Suppose the rectangular plot runs along a building, so that you need to fence only three sides. What dimensions will now yield a maximum area with 100 meters of fencing?



1 Chapter Test

Take this test as you would take a test in class. After you are done, check your work against the answers in the back of the book.

The *Interactive* CD-ROM and *Internet* versions of this text provide answers to the Chapter Tests and Cumulative Tests. They also offer Chapter Pre-Tests (that test key skills and concepts covered in previous chapters) and Chapter Post-Tests, both of which have randomly generated exercises with diagnostic capabilities.

In Exercises 1–6, use the point-plotting method to graph the equation and identify any intercepts. Verify your results using a graphing utility.

1. $y = 4 - \frac{3}{4}|x|$ **2.** $y = 4 - (x - 2)^2$ **3.** $y = x - x^3$

4. $y = -x^3 + 2x - 4$ **5.** $y = \sqrt{3 - x}$ **6.** $y = \frac{1}{2}x\sqrt{x + 3}$

7. A line passes through the point $(3, -1)$ with slope $m = \frac{3}{2}$. List three additional points on the line. Then sketch the line.

8. Find the x- and y-intercepts of the graph of $3x - 2y - 9 = 0$.

9. Find an equation of the line that passes through the point $(0, 4)$ and is perpendicular to the line $5x + 2y = 3$.

10. Does the graph at the right represent y as a function of x? Explain.

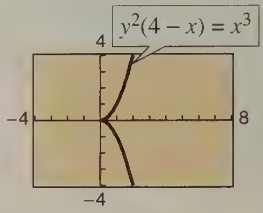


FIGURE FOR 10

In Exercises 11–14, use the function $f(x) = 10 - \sqrt{3 - x}$.

11. Evaluate: $f(-6)$ **12.** Simplify: $f(t - 3)$

13. Simplify: $\dfrac{f(x) - f(2)}{x - 2}$ **14.** Determine the domain of f.

15. A company produces a product for which the variable cost is \$5.60 and the fixed costs are \$24,000. The product sells for \$9.20. Write the total cost C as a function of x. Write the profit P as a function of x.

In Exercises 16 and 17, find the open intervals for which the function is increasing, decreasing, or constant.

16. $h(x) = \frac{1}{4}x^4 - 2x^2$ **17.** $g(t) = |t + 2| - |t - 2|$

In Exercises 18 and 19, use a graphing utility to approximate (to two decimal places) any relative minimum or maximum function values.

18. $f(x) = -x^3 - 5x^2 + 12$ **19.** $f(x) = x^5 - x^3 + 2$

In Exercises 20–22, (a) identify the common function, (b) describe the transformation, and (c) sketch the graph of the function.

20. $f(x) = -2(x - 5)^3 + 3$ **21.** $f(x) = \sqrt{-x - 7}$ **22.** $f(x) = 4|-x| - 7$

23. Use the functions $f(x) = x^2$ and $g(x) = \sqrt{2 - x}$ to find the specified function and its domain.

(a) $(f - g)(x)$ (b) $\left(\dfrac{f}{g}\right)(x)$ (c) $(f \circ g)(x)$ (d) $g^{-1}(x)$

24. A wire 100 inches long is to be cut into four pieces to form a rectangle whose shortest side has a length of x. Express the area A of the rectangle as a function of x. Determine the domain of the function and graph the function over that domain. Approximate the maximum area of the rectangle.

Intercepts, Zeros, and Solutions

2

2.1 Modeling with Linear Equations
2.2 Solving Equations Graphically
2.3 Complex Numbers
2.4 Solving Equations Algebraically
2.5 Solving Inequalities Algebraically and Graphically

Bob Krist/Tony Stone Images

The key and action combination of a piano contains more than 9000 parts. The combined tension of the more than 200 strings is greater than 18 tons.

The Big Picture

In this chapter you will learn how to

- ☐ solve and use linear equations, including those involving fractions.
- ☐ write and use mathematical models to solve real-life problems.
- ☐ find intercepts, zeros, and solutions of equations graphically.
- ☐ perform operations with complex numbers and plot complex numbers in the complex plane.
- ☐ solve quadratic equations, polynomial equations, equations involving radicals, equations involving fractions, and equations involving absolute values.
- ☐ solve linear inequalities, inequalities involving absolute values, polynomial inequalities, and rational inequalities.

Important Vocabulary

As you encounter each new vocabulary term in this chapter, add the term and its definition to your notebook glossary.

- equation (p. 156)
- solutions (p. 156)
- linear equation in one variable x (p. 156)
- extraneous (p. 157)
- mathematical modeling (p. 157)
- formulas (p. 160)
- fitting a line to data (p. 162)
- x-intercept (p. 169)
- y-intercept (p. 169)
- zero (p. 170)
- point of intersection (p. 173)
- imaginary unit i (p. 180)
- complex number (p. 180)
- complex conjugates (p. 183)
- fractal geometry (p. 184)
- complex plane (p. 184)
- imaginary axis (p. 184)
- real axis (p. 184)
- fractals (p. 184)
- Mandelbrot Set (p. 184)
- bounded (p. 184)
- unbounded (p. 184)
- quadratic equation in x (p. 188)
- second-degree polynomial equation in x (p. 188)
- position equation (p. 197)
- solutions of an inequality (p. 205)
- graph of an inequality (p. 205)
- properties of inequalities (p. 205)
- equivalent inequalities (p. 205)
- linear inequality (p. 206)
- double inequality (p. 207)
- critical numbers (p. 209)
- test intervals (p. 209)

Additional Resources Text-specific additional resources are available to help you do well in this course. See page xvi for details.

2.1 Modeling with Linear Equations

Equations and Solutions of Equations

An **equation** is a statement that two algebraic expressions are equal. For example, $3x - 5 = 7$, $x^2 - x - 6 = 0$, and $\sqrt{2x} = 4$ are equations. To **solve** an equation in x means to find all values of x for which the equation is true. Such values are **solutions.** For instance, $x = 4$ is a solution of the equation $3x - 5 = 7$, because $3(4) - 5 = 7$ is a true statement.

The solutions of an equation depend on the kinds of numbers being considered. For instance, in the set of rational numbers, $x^2 = 10$ has no solution because there is no rational number whose square is 10. However, in the set of real numbers the equation has the two solutions $\sqrt{10}$ and $-\sqrt{10}$.

An equation that is true for *every* real number in the domain of the variable is called an **identity.** For example, $x^2 - 9 = (x + 3)(x - 3)$ is an identity because it is a true statement for any real value of x, and $x/(3x^2) = 1/(3x)$, where $x \neq 0$, is an identity because it is true for any nonzero real value of x.

An equation that is true for just *some* (or even none) of the real numbers in the domain of the variable is called a **conditional equation.** For example, the equation $x^2 - 9 = 0$ is conditional because $x = 3$ and $x = -3$ are the only values in the domain that satisfy the equation. The equation $2x + 1 = 2x - 3$ is also conditional because it is not true for any value of x. Learning to solve conditional equations is the primary focus of this chapter.

A **linear equation in one variable x** is an equation that can be written in the standard form $ax + b = 0$, where a and b are real numbers, with $a \neq 0$. For a review of solving one- and two-step linear equations, see Appendix C.

To solve an equation involving fractional expressions, find the least common denominator of all terms in the equation and multiply every term by this LCD. This procedure clears the equation of fractions.

What You Should Learn:

- How to solve equations involving fractional expressions
- How to write and use mathematical models to solve real-life problems
- How to use common formulas to solve real-life problems
- How to use scatter plots and a graphing utility to find linear models for data

Why You Should Learn It:

Linear equations are useful in modeling situations in which you need to find missing information. For instance, Exercise 40 on page 165 shows how to use a linear equation to determine the score you must get on a test in order to get an A for the course you are taking.

PhotoEdit

EXAMPLE 1 Solving an Equation Involving Fractions

$$\frac{x}{3} + \frac{3x}{4} = 2 \qquad \text{Original equation}$$

$$(12)\frac{x}{3} + (12)\frac{3x}{4} = (12)2 \qquad \text{Multiply by the LCD.}$$

$$4x + 9x = 24 \qquad \text{Simplify and multiply.}$$

$$13x = 24 \qquad \text{Combine like terms.}$$

$$x = \frac{24}{13} \qquad \text{Divide each side by 13.}$$

Check

After solving an equation, check the solution in the original equation.

$$\frac{\frac{24}{13}}{3} + \frac{3\left(\frac{24}{13}\right)}{4} \overset{?}{=} 2 \qquad \text{Substitute } \tfrac{24}{13} \text{ for } x.$$

$$2 = 2 \qquad \text{Solution checks. } \checkmark$$

When multiplying or dividing an equation by a *variable* expression, it is possible to introduce an **extraneous** solution—one that does not satisfy the original equation. The next example demonstrates the importance of checking your solution when you have multiplied or divided by a variable expression.

EXAMPLE 2 An Equation with an Extraneous Solution

Solve the equation for x.

$$\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{x^2-4}$$

Algebraic Solution

In this case, the LCD is

$$x^2 - 4 = (x+2)(x-2).$$

Multiplying each term by the LCD and simplifying produces the following.

$$\frac{1}{x-2}(x+2)(x-2)$$

$$= \frac{3}{x+2}(x+2)(x-2) - \frac{6x}{x^2-4}(x+2)(x-2)$$

$$x + 2 = 3(x-2) - 6x, \quad x \neq \pm 2$$

$$x + 2 = 3x - 6 - 6x$$

$$4x = -8$$

$$x = -2$$

A check of $x = -2$ in the original equation shows that it yields a denominator of zero. So, $x = -2$ is extraneous, and the equation has *no solution.*

Graphical Solution

Use a graphing utility to graph the left and right sides of the equation in the same viewing window as the functions

$$y_1 = \frac{1}{x-2} \quad \text{and} \quad y_2 = \frac{3}{x+2} - \frac{6x}{x^2-4}$$

as shown in Figure 2.1. The graphs of the functions do not appear to intersect. This means that there is no point for which the left side of the equation $1/(x-2)$ is equal to the right side of the equation

$$\frac{3}{x+2} - \frac{6x}{x^2-4}.$$

So, the equation appears to have *no solution.*

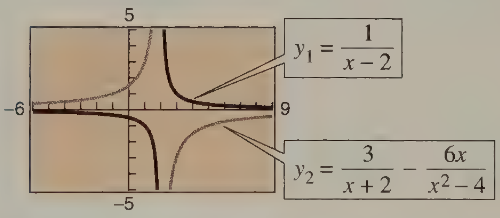


Figure 2.1

Using Mathematical Models to Solve Problems

One of the primary goals of this text is to learn how algebra can be used to solve problems that occur in real-life situations. This procedure is called **mathematical modeling.**

A good approach to mathematical modeling is to use two stages. Begin by using the verbal description of the problem to form a *verbal model.* Then, after assigning labels to the quantities in the verbal model, form a *mathematical model* or *algebraic equation.*

Verbal description ⇒ Verbal model ⇒ Algebraic equation

When you are trying to construct a verbal model, it is helpful to look for a *hidden equality*—a statement that two algebraic expressions are equal. These two expressions might be explicitly stated as being equal, or they might be known to be equal (based on prior knowledge or experience).

An ancient Egyptian papyrus, discovered in 1858, contains one of the earliest examples of mathematical writing in existence. The papyrus itself dates back to around 1650 B.C., but it is actually a copy of writings from two centuries earlier. The algebraic equations on the papyrus were written in words. Diophantus, a Greek who lived around A.D. 250, is often called the Father of Algebra. He was the first to use abbreviated word forms in equations.

EXAMPLE 3 Finding the Dimensions of a Room

A rectangular family room is twice as long as it is wide, and its perimeter is 84 feet. Find the dimensions of the family room.

Solution

For this problem, it helps to draw a diagram, as shown in Figure 2.2.

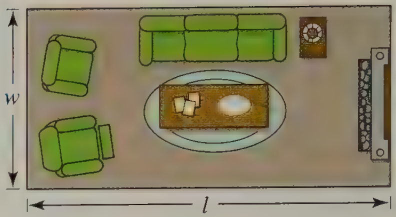


Figure 2.2

Verbal Model: 2 · Length + 2 · Width = Perimeter

Labels:
Perimeter = 84 (feet)
Width = w (feet)
Length = $l = 2w$ (feet)

Equation:

$$2(2w) + 2w = 84 \quad \text{Write original equation.}$$
$$6w = 84 \quad \text{Group like terms.}$$
$$w = 14 \quad \text{Divide each side by 6.}$$

Because the length is twice the width, you have

$$l = 2w \quad \text{Length is twice width.}$$
$$= 2(14) = 28. \quad \text{Substitute 14 for } w \text{ and simplify.}$$

So, the dimensions of the room are 14 feet by 28 feet.

You can check that the dimensions are correct as follows.

2 · Length + 2 · Width = Perimeter

$$2 \cdot 28 + 2 \cdot 14 \stackrel{?}{=} 84 \quad \text{Substitute.}$$
$$84 = 84 \quad \text{Solution checks. ✓}$$

STUDY TIP

Students sometimes say that although a solution looks easy when it is worked out in class, they don't see where to begin when solving a problem alone. Keep in mind that no one—not even great mathematicians—can expect to look at every mathematical problem and know immediately where to begin. Many problems involve some trial and error before a solution is found. To make algebra work for you, put in a lot of time, expect to try solution methods that end up not working, and learn from both your successes and your failures.

EXAMPLE 4 A Distance Problem

A plane is flying nonstop from New York to San Francisco, a distance of about 2700 miles, as shown in Figure 2.3. After $1\frac{1}{2}$ hours in the air, the plane flies over Chicago (a distance of 800 miles from New York). Estimate the time it will take the plane to fly from New York to San Francisco.

Solution

Verbal Model: Distance = Rate · Time

Labels:
Distance = 2700 (miles)
Time = t (hours)
Rate = $\dfrac{\text{Distance to Chicago}}{\text{Time to Chicago}} = \dfrac{800}{1.5}$ (miles per hour)

Equation:

$$2700 = \frac{800}{1.5}t$$
$$5.06 \approx t$$

The trip will take about 5.06 hours or about 5 hours and 4 minutes.

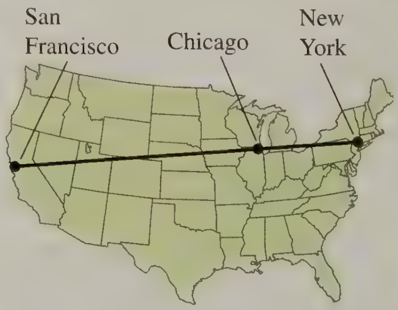


Figure 2.3

EXAMPLE 5 Height of a Building

To measure the height of the twin towers of the World Trade Center, you measure the shadow cast by one of the buildings and find it to be 170.25 feet long, as shown in Figure 2.4. Then you measure the shadow cast by a 4-foot post and find it to be 6 inches long. Estimate the building's height.

Solution

To solve this problem, you use a result from geometry that states that the ratios of corresponding sides of similar triangles are equal.

Verbal Model: $$\frac{\text{Height of building}}{\text{Length of building's shadow}} = \frac{\text{Height of post}}{\text{Length of post's shadow}}$$

Labels: Height of building $= x$ (feet)
Length of building's shadow $= 170.25$ (feet)
Height of post $= 4$ feet $= 48$ inches (inches)
Length of post's shadow $= 6$ (inches)

Equation: $$\frac{x}{170.25} = \frac{48}{6}$$

$$x = 1362$$

So, the World Trade Center is about 1362 feet high.

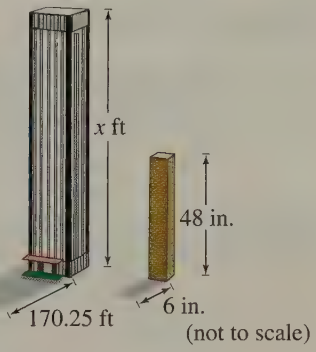


Figure 2.4

The *Interactive* CD-ROM and *Internet* versions of this text show every example with its solution; clicking on the *Try It!* button brings up similar problems. Guided Examples and Integrated Examples show step-by-step solutions to additional examples. Integrated Examples are related to several concepts in the section.

EXAMPLE 6 An Inventory Problem

A store has $30,000 of inventory in 12-inch and 19-inch color televisions. The profit on a 12-inch set is 22% and the profit on a 19-inch set is 40%. The profit for the entire stock is 35%. How much was invested in each type of television?

Solution

Verbal Model: $$\text{Profit from 12-inch sets} + \text{Profit from 19-inch sets} = \text{Total profit}$$

Labels: Inventory of 12-inch sets $= x$ (dollars)
Inventory of 19-inch sets $= 30{,}000 - x$ (dollars)
Profit from 12-inch sets $= 0.22x$ (dollars)
Profit from 19-inch sets $= 0.40(30{,}000 - x)$ (dollars)
Total profit $= 0.35(30{,}000) = 10{,}500$ (dollars)

Equation: $$0.22x + 0.40(30{,}000 - x) = 10{,}500$$

$$-0.18x = -1500$$

$$x \approx 8333.33$$

$$30{,}000 - x \approx 21{,}666.67$$

So, about $8333.33 is invested in 12-inch sets and about $21,666.67 is invested in 19-inch sets.

STUDY T!P

Notice in Example 6 that percents are expressed as decimals. For instance, 22% is written as 0.22.

Common Formulas

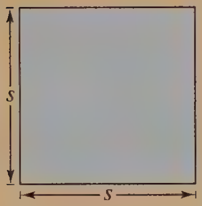
Many common types of geometric, scientific, and investment problems use ready-made equations, called **formulas.** Knowing these formulas will help you translate and solve a wide variety of real-life applications.

Common Formulas for Area A, Perimeter P, Circumference C, and Volume V

Square

$A = s^2$

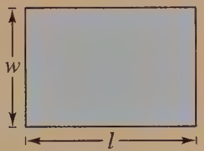
$P = 4s$



Rectangle

$A = lw$

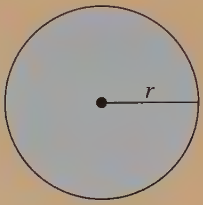
$P = 2l + 2w$



Circle

$A = \pi r^2$

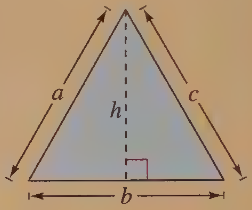
$C = 2\pi r$



Triangle

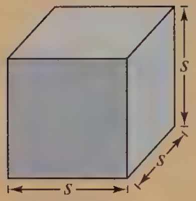
$A = \frac{1}{2}bh$

$P = a + b + c$



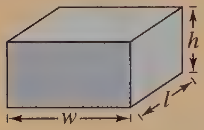
Cube

$V = s^3$



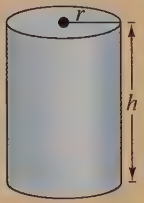
Rectangular Solid

$V = lwh$



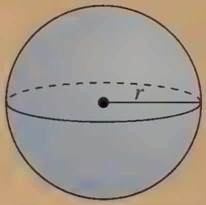
Circular Cylinder

$V = \pi r^2 h$



Sphere

$V = \frac{4}{3}\pi r^3$



Miscellaneous Common Formulas

Temperature:	$F = \frac{9}{5}C + 32$	F = degrees Fahrenheit, C = degrees Celsius
Simple Interest:	$I = Prt$	I = interest, P = principal, r = annual interest rate, t = time in years
Compound Interest:	$A = P\left(1 + \frac{r}{n}\right)^{nt}$	A = balance, P = principal, r = annual interest rate, n = compoundings per year, t = time in years
Distance:	$d = rt$	d = distance traveled, r = rate, t = time

When working with applied problems you often need to rewrite one of the common formulas. For instance, the formula for the perimeter of a rectangle, $P = 2l + 2w$, can be solved for w as $w = \frac{1}{2}(P - 2l)$.

EXAMPLE 7 Height of a Can

A cylindrical can has a volume of 300 cubic centimeters (cm^3) and a radius of 3 centimeters (cm), as shown in Figure 2.5. Find the height of the can.

Solution

The formula for the volume of a cylinder is $V = \pi r^2 h$. To find the height of the can, solve for h as $h = V/\pi r^2$.

Then, using $V = 300\ \text{cm}^3$ and $r = 3$ cm, find the height.

$$h = \frac{300}{\pi(3)^2} = \frac{300}{9\pi} \approx 10.61$$

You can use unit analysis to check your answer: $\frac{300\ \text{cm}^{\cancel{3}}}{9\pi\ \cancel{\text{cm}^2}} \approx 10.61$ cm.

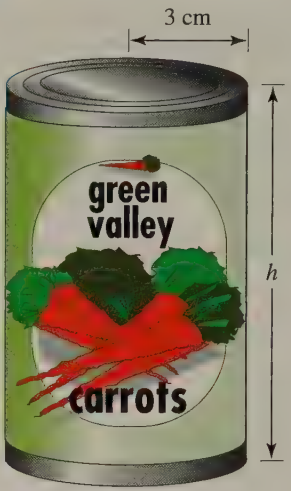


Figure 2.5

Fitting a Line to Data

Many real-life situations involve finding relationships between two variables, such as the year and the number of people in the labor force. In a typical situation, data is collected and written as a set of ordered pairs. The graph of such a set, a *scatter plot*, was discussed briefly in Section P.5.

EXAMPLE 8 Constructing a Scatter Plot

The data in the table shows the number of people P (in millions) in the United States who were part of the labor force from 1987 through 1997. In the table, t represents the year, with $t = 7$ corresponding to 1987. Sketch a scatter plot of the data. (Source: U.S. Bureau of Labor Statistics)

t	7	8	9	10	11	12	13	14	15	16	17
P	120	122	124	126	126	128	129	131	132	134	136

STUDY T!P

Most graphing utilities have built-in statistical programs that can create scatter plots. Try using your graphing utility to plot the points shown in the table at the left.

Solution

Begin by representing the data with a set of ordered pairs.

$$(7, 120), (8, 122), (9, 124), (10, 126), (11, 126), (12, 128),$$
$$(13, 129), (14, 131), (15, 132), (16, 134), (17, 136)$$

Then plot each point in a coordinate plane, as shown in Figure 2.6.

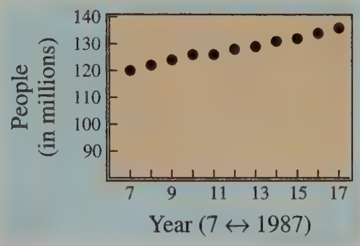


Figure 2.6

From the scatter plot in Figure 2.6, it appears that the points describe a relationship that is nearly linear. The relationship is not *exactly* linear because the labor force did not increase by precisely the same amount each year.

A mathematical equation that approximates the relationship between t and P is a *mathematical model*. When developing a mathematical model to describe a set of data, you strive for two (often conflicting) goals—accuracy and simplicity. For the data above, a linear model of the form $P = at + b$ appears to be best. It is simple and relatively accurate.

Finding a linear model to represent the relationship described by a scatter plot is called **fitting a line to data.** You can do this graphically by simply sketching the line that appears to fit the points, finding two points on the line, and then finding the equation of the line that passes through the two points.

EXAMPLE 9 Fitting a Line to Data

Find a linear model that relates the year with the number of people in the United States labor force. (See Example 8.)

t	7	8	9	10	11	12	13	14	15	16	17
P	120	122	124	126	126	128	129	131	132	134	136

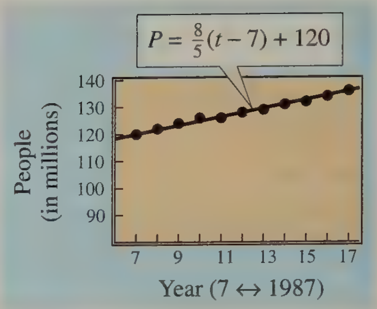


Figure 2.7

Solution

After plotting the data in the table, draw the line that you think best represents the data, as shown in Figure 2.7. Two points that lie on this line are (7, 120) and (17, 136). Using the point-slope form, you can find the equation of the line to be

$$P = \frac{8}{5}(t - 7) + 120. \qquad \text{Linear model}$$

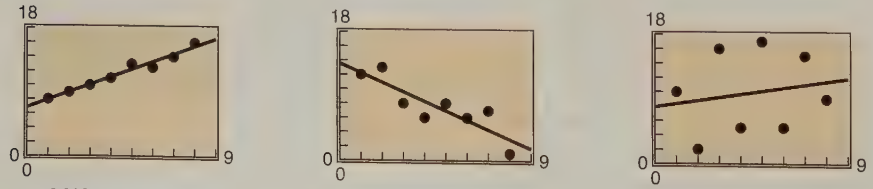
STUDY TIP

The model in Example 9 is based on the two data points chosen. If different points were chosen, the model may change somewhat. For instance, if you choose (10, 126) and (15, 132), the new model is $P = \frac{6}{5}(t - 10) + 126$.

Once you have found a model, you can measure how well the model fits the data by comparing the actual values with the values given by the model, as shown in the following table.

	t	7	8	9	10	11	12	13	14	15	16	17
Actual ⇨	P	120	122	124	126	126	128	129	131	132	134	136
Model ⇨	P	120	121.6	123.2	124.8	126.4	128	129.6	131.2	132.8	134.4	136

If you use the regression capabilities of a graphing calculator or computer program to find a linear model for the data in Example 9, you will notice that the program may also output a value of $r \approx 0.995$. This number is the **correlation coefficient** of the data. Correlation coefficients vary between -1 and 1. Basically, the closer $|r|$ is to 1, the better the points can be described by a line. Three examples are shown in Figure 2.8.



$r = 0.983$ $\qquad$ $r = -0.866$ $\qquad$ $r = 0.190$

Figure 2.8

EXAMPLE 10 A Mathematical Model

The average number of students s out of one thousand 11th- and 12th-grade students who took advanced placement exams for the years 1984 through 1995 are shown in the table. (Source: U.S. National Center for Education Statistics)

Year	1984	1985	1986	1987	1988	1989
s	24	29	33	36	39	44

Year	1990	1991	1992	1993	1994	1995
s	48	53	57	62	60	66

a. Use the regression capabilities of a graphing utility to find a linear model for the data. Let t represent the year with $t = 4$ corresponding to 1984.

b. How closely does the model represent the data?

Graphical Solution

a. Using the regression capabilities of a graphing utility, you can find that a linear model for the data is $s = 3.8t + 9.5$.

b. You can use a graphing utility to graph the actual data and the model in the same viewing window. From Figure 2.9, it appears that the model is a "good fit" for the actual data. This is verified by the fact that the graphing utility outputs a value of $r \approx 0.994$.

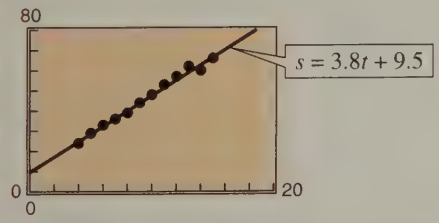


Figure 2.9

Numerical Solution

a. Using the regression capabilities of a graphing utility, you can find that a linear model for the data is $s = 3.8t + 9.5$.

b. You can see how well the model fits the data by comparing the actual values of s with the values of s given by the model, which are labeled s^* in the table below.

Year	1984	1985	1986	1987	1988	1989
s	24	29	33	36	39	44
s^*	24.7	28.5	32.3	36.1	39.9	43.7

Year	1990	1991	1992	1993	1994	1995
s	48	53	57	62	60	66
s^*	47.5	51.3	55.1	58.9	62.7	66.5

From the table, you can see that the model appears to be a good fit for the actual data.

Writing About Math *Research Project*

Use your school's library or some other reference source to locate data that you think describes a linear relationship. Create a scatter plot of the data, and use the regression capabilities of a graphing utility to find a linear model for the data. Interpret the slope and y-intercept in the context of the data. Write a summary of your findings.

2.1 Exercises

In Exercises 1–6, determine whether the given values of x are solutions of the equation.

Equation	*Values*	
1. $\frac{5}{2x} - \frac{4}{x} = 3$	(a) $x = -\frac{1}{2}$	(b) $x = 4$
	(c) $x = 0$	(d) $x = \frac{1}{4}$
2. $\frac{x}{2} + \frac{6x}{7} = \frac{19}{14}$	(a) $x = -2$	(b) $x = 1$
	(c) $x = \frac{1}{2}$	(d) $x = 7$
3. $3 + \frac{1}{x+2} = 4$	(a) $x = -1$	(b) $x = -2$
	(c) $x = 0$	(d) $x = 5$
4. $\frac{(x+5)(x-3)}{2} = 24$	(a) $x = -3$	(b) $x = -2$
	(c) $x = 7$	(d) $x = 9$
5. $\frac{\sqrt{x+4}}{6} + 3 = 4$	(a) $x = -3$	(b) $x = 0$
	(c) $x = 21$	(d) $x = 32$
6. $\frac{\sqrt[3]{x-8}}{3} = -\frac{2}{3}$	(a) $x = -16$	(b) $x = 0$
	(c) $x = 9$	(d) $x = 16$

In Exercises 7–12, determine whether the equation is an identity or a conditional equation.

7. $2(x-1) = 2x - 2$

8. $-7(x-3) + 4x = 3(7-x)$

9. $x^2 - 8x + 5 = (x-4)^2 - 11$

10. $x^2 + 2(3x-2) = x^2 + 6x - 4$

11. $3 + \frac{1}{x+1} = \frac{4x}{x+1}$

12. $\frac{5}{x} + \frac{3}{x} = 24$

In Exercises 13 and 14, solve the equation in two ways. Then explain which way was easier for you.

13. $\frac{3x}{8} - \frac{4x}{3} = 4$

14. $\frac{3z}{8} - \frac{z}{10} = 6$

The *Interactive* CD-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered Section and Review Exercises. They also provide Tutorial Exercises, which link to Guided Examples for additional help.

In Exercises 15–30, solve the equation (if possible). Then use a graphing utility to verify your solution.

15. $\frac{5x}{4} + \frac{1}{2} = x - \frac{1}{2}$

16. $\frac{x}{5} - \frac{x}{2} = 3$

17. $\frac{3}{2}(z+5) - \frac{1}{4}(z+24) = 0$

18. $\frac{3x}{2} + \frac{1}{4}(x-2) = 10$

19. $\frac{100-4u}{3} = \frac{5u+6}{4} + 6$

20. $\frac{17+y}{y} + \frac{32+y}{y} = 100$

21. $\frac{5x-4}{5x+4} = \frac{2}{3}$

22. $\frac{15}{x} - 4 = \frac{6}{x} + 3$

23. $\frac{1}{x-3} + \frac{1}{x+3} = \frac{10}{x^2-9}$

24. $\frac{1}{x-2} + \frac{3}{x+3} = \frac{4}{x^2+x-6}$

25. $\frac{7}{2x+1} - \frac{8x}{2x-1} = -4$

26. $\frac{4}{u-1} + \frac{6}{3u+1} = \frac{15}{3u+1}$

27. $\frac{1}{x} + \frac{2}{x-5} = 0$

28. $\frac{6}{x} - \frac{2}{x+3} = \frac{3(x+5)}{x(x+3)}$

29. $\frac{3}{x(x-3)} + \frac{4}{x} = \frac{1}{x-3}$

30. $3 = 2 + \frac{2}{z+2}$

In Exercises 31–34, solve for the indicated variable.

31. ***Area of a Triangle***

Solve for h: $A = \frac{1}{2}bh$

32. ***Investment at Compound Interest***

Solve for P: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

33. ***Area of a Trapezoid***

Solve for b: $A = \frac{1}{2}(a+b)h$

34. ***Geometric Progression***

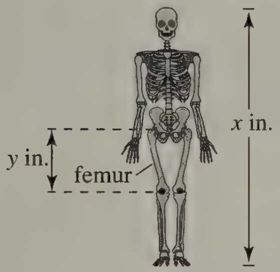
Solve for r: $S = \frac{rL-a}{r-1}$

Human Height **In Exercises 35 and 36, use the following information. The relationship between the length of an adult's thigh bone and the height of the adult can be approximated by the linear equations**

$y = 0.432x - 10.44$ **Female**

$y = 0.449x - 12.15$ **Male**

where y is the length of the femur (thigh bone) in inches and x is the height in inches (see figure).



35. An anthropologist discovers a thigh bone belonging to an adult human female. The bone is 16 inches long. Estimate the height of the female.

36. From the foot bones of an adult human male, an anthropologist estimates that the height of the male was 69 inches. A few feet away from the site where the foot bones were discovered, the anthropologist discovered a male adult thigh bone that was 19 inches long. Is it possible that both the foot bones and the thigh bone came from the same person?

37. ***Operating Cost*** A delivery company has a fleet of vans. The annual operating cost per van is

$$C = 0.32m + 2500$$

where m is the number of miles traveled by a van in a year. What number of miles will yield an annual operating cost of $10,000?

38. ***Dimensions of a Room*** A room is 1.5 times as long as it is wide, and its perimeter is 25 meters.

(a) Draw a diagram that gives a visual representation of the problem. Identify the length as l and the width as w.

(b) Write l in terms of w and write an equation for the perimeter in terms of w.

(c) Find the dimensions of the room.

39. ***Dimensions of a Picture Frame*** A picture frame has a total perimeter of 3 meters. The height of the frame is $\frac{2}{3}$ times its width.

(a) Draw a diagram that gives a visual representation of the problem. Identify the width as w and the height as h.

(b) Write h in terms of w and write an equation for the perimeter in terms of w.

(c) Find the dimensions of the picture frame.

40. ***Course Grade*** To get an A in a course, you must have an average of at least 90 on four tests of 100 points each. The scores on your first three tests were 87, 92, and 84.

(a) Write a verbal model for the test average for the course.

(b) What must you score on the fourth test to get an A for the course?

41. ***Course Grade*** Suppose you are taking a course that has four tests. The first three tests are 100 points each and the fourth test is 200 points. To get an A in the course, you must have an average of at least 90% on the four tests. Your scores on the first three tests were 87, 92, and 84. What must you score on the fourth test to get an A for the course?

42. ***Travel Time*** Suppose you are driving on a Canadian freeway to a town that is 300 kilometers from your home. After 30 minutes you pass a freeway exit that you know is 50 kilometers from your home. Assuming that you continue at the same constant speed, how long will it take for the entire trip?

43. ***Travel Time*** On the first part of a 317-mile trip, a salesman averaged 58 miles per hour. He averaged only 52 miles per hour on the last part of the trip because of an increased volume of traffic. Find the amount of time at each of the speeds if the total time was 5 hours and 45 minutes.

44. ***Travel Time*** Two families meet at a park for a picnic. At the end of the day one family travels east at an average speed of 42 miles per hour and the other travels west at an average speed of 50 miles per hour. Both families have approximately 160 miles to travel.

(a) Find the time it takes each family to get home.

(b) Find the time that will have elapsed when they are 100 miles apart.

(c) Find the distance the eastbound family has to travel after the westbound family has arrived home.

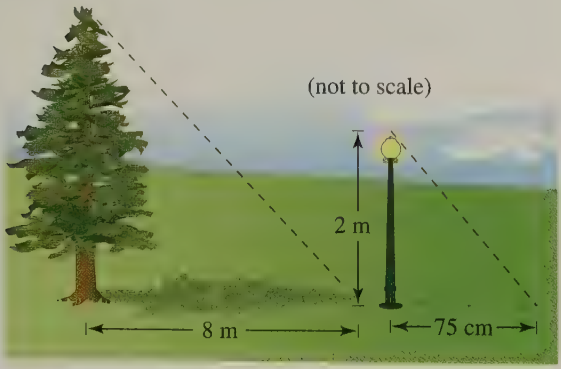
45. ***Average Speed*** A truck driver traveled at an average speed of 55 miles per hour on a 200-mile trip to pick up a load of freight. On the return trip (with the truck fully loaded), the average speed was 40 miles per hour. Find the average speed for the round trip.

46. ***Wind Speed*** An executive flew in the corporate jet to a meeting in a city 1500 kilometers away. After traveling the same amount of time on the return flight, the pilot mentioned that they still had 300 kilometers to go. If the air speed of the plane was 600 kilometers per hour, how fast was the wind blowing? (Assume that the wind direction was parallel to the flight path and constant all day.)

47. ***Speed of Light*** Light travels at the speed of 3.0×10^8 meters per second. Find the time in minutes required for light to travel from the sun to the earth (a distance of 1.5×10^{11} meters).

48. ***Radio Waves*** Radio waves travel at the same speed as light, 3.0×10^8 meters per second. Find the time required for a radio wave to travel from mission control in Houston to NASA astronauts on the surface of the moon 3.86×10^8 meters away.

49. ***Height of a Tree*** To obtain the height of a tree, you measure the tree's shadow and find that it is 8 meters long. You also measure the shadow of a 2-meter lamppost and find that it is 75 centimeters long. How tall is the tree?



50. ***Height of a Building*** To obtain the height of a building, you measure the building's shadow and find that it is 80 feet long. You also measure the shadow of a 4-foot stake and find that it is $3\frac{1}{2}$ feet long.

(a) Draw a diagram to represent the problem. Let h represent the height of the building.

(b) Find the height of the building.

51. ***Height of a Flagpole*** A person who is 6 feet tall walks away from a flagpole toward the tip of the shadow of the pole. When the person is 30 feet from the pole, the tips of the person's shadow and the shadow cast by the pole coincide at a point 5 feet in front of the person.

(a) Draw a diagram to represent the problem. Let h represent the height of the pole.

(b) Find the height of the pole.

52. ***Investment*** You plan to invest \$12,000 in two funds paying $4\frac{1}{2}\%$ and 5% simple interest. (There is more risk in the 5% fund.) Your goal is to obtain a total annual interest income of \$560 from the investments. What is the smallest amount you can invest in the 5% fund in order to meet your objective?

53. ***Investment*** You plan to invest \$25,000 in two funds paying 3% and $4\frac{1}{2}\%$ simple interest. (There is more risk in the $4\frac{1}{2}\%$ fund.) Your goal is to obtain a total annual interest income of \$1000 from the investments. What is the smallest amount you can invest in the $4\frac{1}{2}\%$ fund in order to meet your objective?

54. ***Investment*** Suppose you invested \$12,000 in a fund paying $3\frac{1}{2}\%$ simple interest and \$10,000 in a fund with a variable interest rate. At the end of the year you were notified that the total interest for both funds was \$870. Find the equivalent simple interest rate on the variable-rate fund.

55. ***Investment*** Suppose you have \$10,000 on deposit earning simple interest with the interest rate linked to the prime rate. Because of a drop in the prime rate, the rate on your investment dropped by $1\frac{1}{2}\%$ for the last quarter of the year. Your annual earnings on the fund were \$387.50. Find the interest rate for the first three quarters of the year and the interest rate for the last quarter.

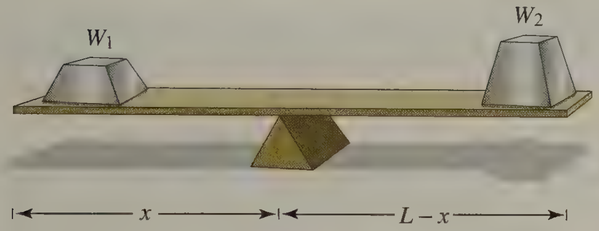
56. ***Mixture Problem*** A grocer mixes two kinds of nuts that cost \$2.49 per pound and \$3.89 per pound, respectively, to make 100 pounds of a mixture that costs \$3.19 per pound. How much of each kind of nut is put into the mixture?

57. ***Production Limit*** A company has fixed costs of \$10,000 per month and variable costs of \$8.50 per unit manufactured. The company has \$85,000 available to cover the monthly costs. How many units can the company manufacture? (Fixed costs are those that occur regardless of the level of production. Variable costs depend on the level of production.)

Statics Problems **In Exercises 58 and 59, suppose you have a uniform beam of length L with a fulcrum x feet from one end. If objects with weights W_1 and W_2 are placed at opposite ends of the beam, the beam will balance if**

$$W_1x = W_2(L - x).$$

Find x such that the beam will balance.



58. Two children weighing 50 pounds and 75 pounds are going to play on a seesaw that is 10 feet long.

59. A person weighing 200 pounds is attempting to move a 550-pound rock with a bar that is 5 feet long.

In Exercises 60–63, use the formulas on page 160.

60. ***Dimensions of a Sail*** A triangular sail has an area of 182.25 square feet. The sail has a base of 13.5 feet. Find the height of the sail.

61. ***Dimensions of a Package*** The volume of a rectangular package is 2304 cubic inches. The length of the package is 3 times its width, and the height is one and a half times its width.

(a) Draw a diagram to represent the problem. Label the height, width, and length accordingly.

(b) Find the dimensions of the package.

62. ***Geometry*** The volume of a globe is about 47,712.94 cubic centimeters. Use a graphing utility to find the radius of the globe. Round your result to two decimal places.

63. ***Temperature*** The line graph shows the temperatures (in degrees Fahrenheit) on a particular day from 10:00 A.M. to 6:00 P.M. Create a new line graph showing the temperatures throughout the day in degrees Celsius.

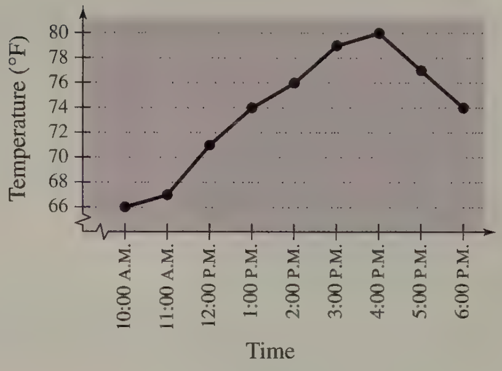


FIGURE FOR 63

64. ***Quiz Scores*** The following ordered pairs give the scores of two consecutive 15-point quizzes for a class of 18 students.

(7, 13), (9, 7), (14, 14), (15, 15), (10, 15),
(9, 7), (14, 11), (14, 15), (8, 10), (9, 10),
(15, 9), (10, 11), (11, 14), (7, 14),
(11, 10), (14, 11), (10, 15), (9, 6)

(a) Create a scatter plot for the data.

(b) Does the relationship between consecutive quiz scores appear to be approximately linear? If not, give some possible explanations.

65. ***Dentists*** The data in the table shows the number of dentists D (in thousands) in the United States from 1985 through 1996. In the table, t represents the year, with $t = 5$ representing 1985.

t	5	6	7	8	9	10
D	156	158	161	164	168	173

t	11	12	13	14	15	16
D	179	183	187	191	194	196

(a) Sketch a scatter plot of the data.

(b) Find the equation of the line that seems to best fit the data.

(c) Use the regression capabilities of a graphing utility to find a linear model for the data. Compare with your answer to part (b).

(d) Use the model from part (c) to estimate the number of dentists in the United States in 2000, 2002, and 2005.

66. ***Consumerism*** The data in the table shows the per capita consumption W (in gallons) of bottled water in the United States from 1988 through 1996. In the table, t represents the year, with $t = 0$ representing 1990. (Source: U.S. Department of Agriculture)

t	-2	-1	0	1	2
W	6.5	7.4	8	8	8.2

t	3	4	5	6
W	9.4	10.7	11.6	12.4

(a) Sketch a scatter plot of the data.

(b) Find the equation of a line that best fits the data.

(c) Use the regression capabilities of a graphing utility to find a linear model for the data. Compare with your answer to part (b).

(d) Use the model from part (c) to estimate the per capita consumption in 2000, 2004, and 2007.

67. ***Advertising and Sales*** The table shows the advertising expenditures x and sales volume y for a company for seven randomly selected months. Both are measured in thousands of dollars.

Month	1	2	3	4	5	6	7
x	2.4	1.6	2.0	2.6	1.4	1.6	2.0
y	202	184	220	240	180	164	186

(a) Use the regression capabilities of a graphing utility to find a linear model for y as a function of x.

(b) Use a graphing utility to plot the data and graph the model.

(c) Interpret the slope in the context of the problem.

(d) Use the model to estimate sales for advertising expenditures of $1500.

68. ***World Food Production*** The data in the table shows the world production P (in millions of metric tons) of vegetables (including melons) for 1990 through 1997. (Source: U.S. Department of Agriculture)

Year	1990	1991	1992	1993
P	461.4	462.4	478.6	509.5

Year	1994	1995	1996	1997
P	532.9	559.9	589.1	595.6

(a) Use the regression capabilities of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to graph the actual data and the model in the same viewing window. How closely does the model represent the data?

(c) Create a table showing the actual values of P and the values of P given by the model.

Synthesis

True or False? **In Exercises 69 and 70, determine whether the statement is true or false. Justify your answer.**

69. The volume of a cube with a side length of 9.5 inches is greater than the volume of a sphere with a radius of 5.9 inches.

70. If the correlation coefficient for a set of data with respect to a linear least squares regression model is -0.9824, you can conclude that the data points cannot be described by a linear model.

In Exercises 71 and 72, write a linear equation that has the given solution. (There are many correct answers.)

71. $x = -3$

72. $x = \frac{1}{4}$

73. ***Think About It*** What is meant by equivalent equations? Give an example of two equivalent equations.

74. ***Writing*** In your own words, describe how to clear an equation of fractions.

Review

In Exercises 75–80, sketch the graph of the equation by hand. Verify using a graphing utility.

75. $y = \frac{5}{8}x - 2$

76. $y = \dfrac{3x - 5}{2} + 2$

77. $y = (x - 3)^2 + 7$

78. $y = \frac{1}{3}x^2 - 4$

79. $y = -\frac{1}{2}|x + 4| - 1$

80. $y = |x - 2| + 10$

In Exercises 81–84, evaluate the combination of functions for $f(x) = -x^2 + 4$ and $g(x) = 6x - 5$.

81. $(f + g)(-3)$

82. $(g - f)(-1)$

83. $(f \circ g)(4)$

84. $(g \circ f)(2)$

2.2 Solving Equations Graphically

Intercepts, Zeros, and Solutions

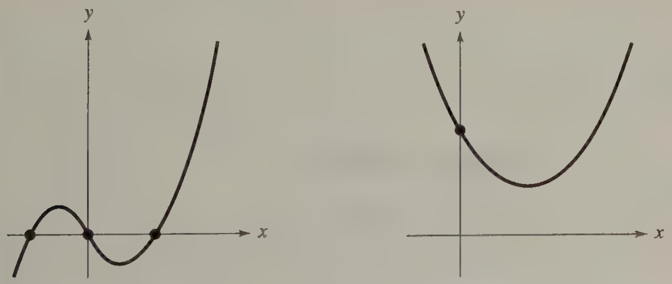
In Section 1.1, you learned that the intercepts of a graph are the points at which the graph intersects the x- or y-axis.

Definition of Intercepts

1. The point $(a, 0)$ is called an **x-intercept** of the graph of an equation if it is a solution point of the equation. To find the x-intercept(s), let $y = 0$ and solve the equation for x.
2. The point $(0, b)$ is called a **y-intercept** of the graph of an equation if it is a solution point of the equation. To find the y-intercept(s), let $x = 0$ and solve the equation for y.

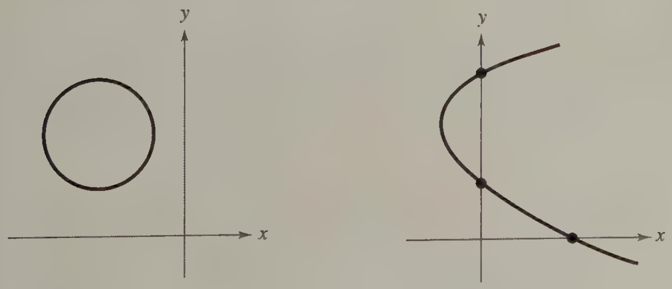
Sometimes it is convenient to denote the x-intercept as simply the x-coordinate of the point $(a, 0)$ rather than the point itself. Unless it is necessary to make a distinction, "intercept" will be used to mean either the point or the coordinate.

It is possible that a particular graph will have no intercepts or several intercepts. For instance, consider the four graphs in Figure 2.10.



Three x-Intercepts
One y-Intercept

No x-Intercepts
One y-Intercept



No Intercepts

One x-Intercept
Two y-Intercepts

Figure 2.10

What You Should Learn:

- How to find x- and y-intercepts of graphs of equations
- How to find solutions of equations graphically
- How to find the points of intersection of two graphs

Why You Should Learn It:

Because some real-life problems involve equations that are difficult to solve algebraically, it is helpful to use a graphing utility to approximate the solutions of such equations. For instance, you can use a graphing utility to find the intersection point of the equations in Exercise 75 on page 179 to find the year during which the per capita utilization of nectarines and peaches and, of cucumbers were the same.

Evan Agostini/Liaison International

EXAMPLE 1 Finding x- and y-Intercepts

Find the x- and y-intercepts of the graph of $2x + 3y = 5$.

Solution

To find the x-intercept, let $y = 0$. This produces

$$2x = 5 \quad \Longrightarrow \quad x = \frac{5}{2}$$

which implies that the graph has one x-intercept: $\left(\frac{5}{2}, 0\right)$. To find the y-intercept, let $x = 0$. This produces

$$3y = 5 \quad \Longrightarrow \quad y = \frac{5}{3}$$

which implies that the graph has one y-intercept: $\left(0, \frac{5}{3}\right)$. See Figure 2.11.

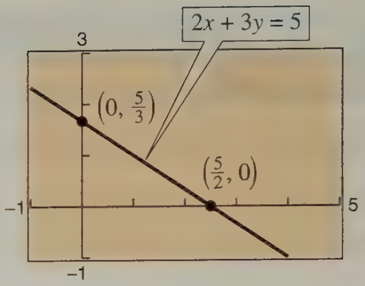


Figure 2.11

A **zero** of a function $y = f(x)$ is a number a such that $f(a) = 0$. So, to find the zeros of a function, you must solve the equation $f(x) = 0$.

The concepts of x-intercepts, zeros of functions, and solutions of equations are closely related. In fact, the following statements are equivalent.

1. The point $(a, 0)$ is an x-*intercept* of the graph of $y = f(x)$.
2. The number a is a *zero* of the function f.
3. The number a is a *solution* of the equation $f(x) = 0$.

The *Interactive* CD-ROM and *Internet* versions of this text offer a built-in graphing calculator, which can be used with the Examples, Explorations, and Exercises.

EXAMPLE 2 Verifying Zeros of Functions

Verify that the real numbers -2 and 3 are zeros of the function $f(x) = x^2 - x - 6$.

Algebraic Solution

To verify that -2 is a zero of f, check that $f(-2) = 0$.

$$\begin{aligned} f(x) &= x^2 - x - 6 && \text{Write original equation.} \\ f(-2) &= (-2)^2 - (-2) - 6 && \text{Substitute } -2 \text{ for } x. \\ &= 4 + 2 - 6 && \text{Simplify.} \\ &= 0 && -2 \text{ is a solution. } \checkmark \end{aligned}$$

To verify that 3 is a zero of $f(x) = x^2 - x - 6$, check that $f(3) = 0$.

$$\begin{aligned} f(x) &= x^2 - x - 6 && \text{Write original equation.} \\ f(3) &= (3)^2 - (3) - 6 && \text{Substitute } 3 \text{ for } x. \\ &= 9 - 3 - 6 && \text{Simplify.} \\ &= 0 && 3 \text{ is a solution. } \checkmark \end{aligned}$$

Graphical Solution

Use a graphing utility to graph $y = x^2 - x - 6$. From the graph in Figure 2.12, it appears that the function has x-intercepts (where y is zero) when $x = -2$ and when $x = 3$. So, you can approximate the zeros of the function to be -2 and 3.

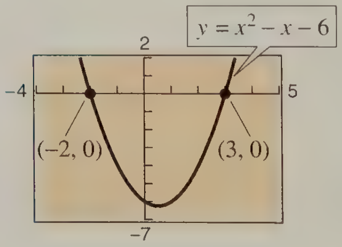


Figure 2.12

Note that the graph of $y = x^2 - x - 6$ in Figure 2.12 has x-*intercepts* of $(3, 0)$ and $(-2, 0)$ because the equation $0 = x^2 - x - 6$ has the *solutions* $x = 3$ and $x = -2$.

This close connection among x-intercepts, zeros, and solutions described on page 170 is crucial to our study of algebra, and you can take advantage of this connection in two basic ways. You can use your algebraic "equation-solving skills" to find the x-intercepts of a graph and your "graphing skills" to approximate the solutions of an equation.

Finding Solutions Graphically

Polynomial equations of degree 1 or 2 can be solved in relatively straightforward ways. Polynomial equations of higher degrees can, however, be quite difficult, especially if you rely only on algebraic techniques. For such equations, a graphing utility can be very helpful.

Graphical Approximations of Solutions of an Equation

1. Write the equation in *general form*, $f(x) = 0$, with the nonzero terms on one side of the equation and zero on the other side.
2. Use a graphing utility to graph the function $y = f(x)$. Be sure the viewing window shows all the relevant features of the graph.
3. Use the *zero* or *root* feature or the *zoom* and *trace* features of the graphing utility to approximate each of the x-intercepts of the graph of f. Remember that a graph can have more than one x-intercept, so you may need to change the viewing window a few times.

In Chapter 3 you will learn techniques for determining the number of solutions of a polynomial equation. For now, you should know that a polynomial equation of degree n cannot have more than n different solutions.

EXAMPLE 3 Finding Solutions of an Equation Graphically

Use a graphing utility to approximate the solutions of $2x^3 - 3x + 2 = 0$.

Solution

Begin by graphing the function $y = 2x^3 - 3x + 2$, as shown in Figure 2.13. You can see from the graph that there is only one x-intercept. It lies between -1 and -2 and is approximately -1.5. By using the *zero* or *root* feature of a graphing utility you can improve the approximation. To three-decimal-place accuracy, the solution is $x \approx -1.476$. Check this approximation on your calculator. You will find that the value of y is $y = 2(-1.476)^3 - 3(-1.476) + 2 \approx -0.003$.

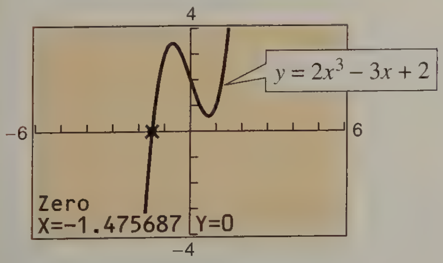


Figure 2.13

Exploration

In Chapter 3 you will learn that a cubic equation such as

$$24x^3 - 36x + 17 = 0$$

can have up to 3 real solutions. Use a graphing utility to graph

$$y = 24x^3 - 36x + 17.$$

Describe a viewing window that allows you to determine the number of real solutions of the equation

$$24x^3 - 36x + 17 = 0.$$

Use the same technique to determine the number of real solutions of

$$97x^3 - 102x^2 - 200x - 63 = 0.$$

You can also use a graphing calculator's *zoom* and *trace* features to approximate the solution of an equation. Here are some suggestions for using the *zoom-in* feature of a graphing utility.

1. With each successive zoom-in, adjust the x-scale (if necessary) so that the resulting viewing window shows at least the two scale marks between which the solution lies.
2. The accuracy of the approximation will always be such that the error is less than the distance between two scale marks.
3. If you have a *trace* feature on your graphing utility, you can generally add one more decimal place of accuracy without changing the viewing window.

Unless stated otherwise, this book will approximate all real solutions with an error of *at most* 0.01.

STUDY TIP

Remember that the more decimal places in the solution, the more accurate it is. You can reach the desired accuracy when zooming in as follows.

- To approximate the zero to the nearest hundredth, set the x-scale to 0.01.
- To approximate the zero to the nearest thousandth, set the x-scale to 0.001.

EXAMPLE 4 Approximating Solutions of an Equation Graphically

Use a graphing utility to approximate the solutions of $x^2 + 3 = 5x$.

Solution

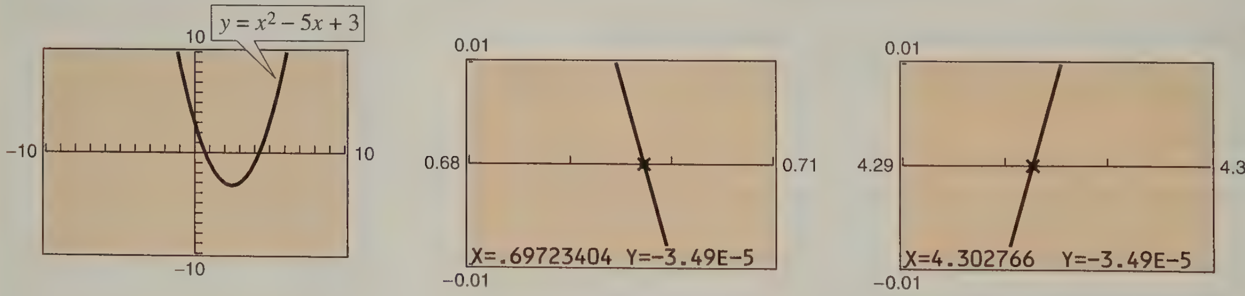
In general form, this equation is

$$x^2 - 5x + 3 = 0. \qquad \text{Equation in general form}$$

So, you can begin by graphing

$$y = x^2 - 5x + 3 \qquad \text{Function to be graphed}$$

as shown in Figure 2.14(a). This graph has two x-intercepts, and by using the *zoom* and *trace* features you can approximate the corresponding solutions to be $x \approx 0.70$ and $x \approx 4.30$, as shown in Figures 2.14(b) and 2.14(c).



(a) (b) (c)

Figure 2.14

The built-in *zero* or *root* programs of a graphing utility will approximate solutions of equations or approximate x-intercepts of graphs. If your graphing utility has such features, try using them to approximate the solutions in Example 4.

Points of Intersection of Two Graphs

An ordered pair that is a solution of two different equations is called a **point of intersection** of the graphs of the two equations. For instance, in Figure 2.15 you can see that the graphs of the following equations have two points of intersection.

$y = x + 2$ Equation 1

$y = x^2 - 2x - 2$ Equation 2

The point $(-1, 1)$ is a solution of both equations, and the point $(4, 6)$ is a solution of both equations. To check this algebraically, substitute -1 and 4 into each equation.

Check that $(-1, 1)$ *is a solution.*

Equation 1: $y = -1 + 2 = 1$ Solution checks. ✓

Equation 2: $y = (-1)^2 - 2(-1) - 2$

$= 1$ Solution checks. ✓

Check that $(4, 6)$ *is a solution.*

Equation 1: $y = 4 + 2$

$= 6$ Solution checks. ✓

Equation 2: $y = (4)^2 - 2(4) - 2$

$= 6$ Solution checks. ✓

To find the points of intersection of the graphs of two equations, solve each equation for y (or x) and set the two results equal to each other. The resulting equation will be an equation in one variable, which can be solved using standard procedures, as shown in Example 5.

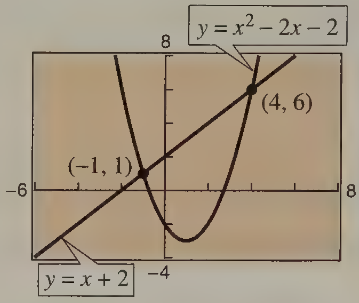
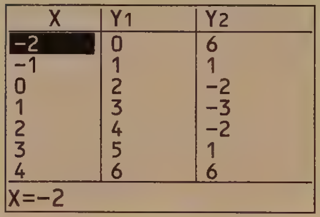


Figure 2.15

STUDY T!P

The table shows some points of the graphs of the equations at the left. Find the points of intersection of the graphs by finding the values of x for which y_1 and y_2 are equal.



X	Y1	Y2
-2	0	6
-1	1	1
0	2	-2
1	3	-3
2	4	-2
3	5	1
4	6	6

X=-2

EXAMPLE 5 Finding Points of Intersection

Find the points of intersection of the graphs of $2x - 3y = -2$ and $4x - y = 6$.

Algebraic Solution

To begin, solve each equation for y to obtain

$$y = \frac{2}{3}x + \frac{2}{3} \quad \text{and} \quad y = 4x - 6.$$

Next, set the two expressions for y equal to each other and solve the resulting equation for x, as follows.

$\frac{2}{3}x + \frac{2}{3} = 4x - 6$ Equate expressions for y.

$2x + 2 = 12x - 18$ Multiply each side by 3.

$-10x = -20$ Subtract $12x$ and 2 from each side.

$x = 2$ Divide each side by -10.

When $x = 2$, the y-value of each of the given equations is 2. So, the point of intersection is $(2, 2)$.

Graphical Solution

To begin, solve each equation for y to obtain $y_1 = \frac{2}{3}x + \frac{2}{3}$ and $y_2 = 4x - 6$. Then use a graphing utility to graph both equations in the same viewing window. In Figure 2.16, the graphs appear to have one point of intersection. Use the *intersect* feature of the graphing utility to approximate the point of intersection to be $(2, 2)$.

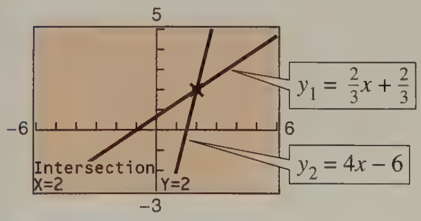


Figure 2.16

Another way to approximate points of intersection of two graphs is to graph both equations and use the *zoom* and *trace* features to find the point or points at which the two graphs intersect.

EXAMPLE 6 Approximating Points of Intersection Graphically

Approximate the point(s) of intersection of the graphs of the following equations.

$y = x^2 - 3x - 4$ Equation 1 (quadratic function)

$y = x^3 + 3x^2 - 2x - 1$ Equation 2 (cubic function)

Solution

Begin by using a graphing utility to graph both functions, as shown in Figure 2.17. From this display, you can see that the two graphs have only one point of intersection. Then, using the *zoom* and *trace* features, approximate the point of intersection to be $(-2.17, 7.25)$. To test the reasonableness of this approximation, you can evaluate both functions when $x = -2.17$.

Quadratic Function:

$$y = (-2.17)^2 - 3(\;\;2.17) - 4$$

$$\approx 7.22$$

Cubic Function:

$$y = (-2.17)^3 + 3(-2.17)^2 - 2(-2.17) - 1$$

$$\approx 7.25$$

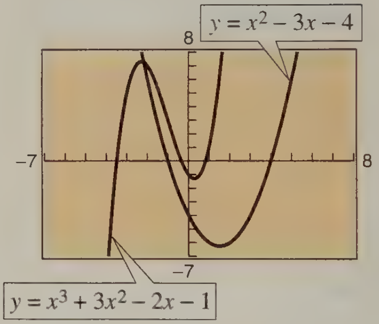


Figure 2.17

Because both functions yield approximately the same y-value, you can conclude that the approximate coordinates of the point of intersection are $x \approx -2.17$ and $y \approx 7.25$.

The method shown in Example 6 gives a nice graphical picture of the points of intersection of two graphs. However, for actual approximation purposes, it is better to use the algebraic procedure described in Example 5. That is, the point of intersection of $y = x^2 - 3x - 4$ and $y = x^3 + 3x^2 - 2x - 1$ coincides with the solution of the equation

$x^3 + 3x^2 - 2x - 1 = x^2 - 3x - 4$ Equate y-values.

$x^3 + 2x^2 + x + 3 = 0.$ Write in general form.

By graphing $y = x^3 + 2x^2 + x + 3$ on a graphing utility and using the *zoom* and *trace* features (or the *zero* or *root* feature), you can approximate the solution of this equation to be $x \approx -2.17$. The corresponding y-value for *both* of the functions given in Example 6 is $y \approx 7.25$.

EXAMPLE 7 A Historical Look at Recorded Music

Between 1990 and 1997, the number of compact discs sold each year in the United States was *increasing* and the number of cassettes was *decreasing*. Two models that approximate the sales S are

$$S = 279.67 + 78.66t, \quad 0 \le t \le 7$$

Compact discs

$$S = 442.14 - 35.46t, \quad 0 \le t \le 7$$

Cassettes

where $t = 0$ represents 1990. According to these two models, when would you expect the sales of compact discs to have exceeded the sales of cassettes? (Source: Recording Industry Association of America)

Algebraic Solution

Both equations are solved for S. So, set the two expressions for S equal to each other and solve the resulting equation for t, as follows.

$$279.67 + 78.66t = 442.14 - 35.46t$$

Equate expressions for S.

$$114.12t = 162.47$$

Subtract 279.67 from and add $35.46t$ to each side.

$$t \approx 1.42$$

Use a calculator.

So, from the given models, you would expect that compact disc sales exceeded cassette sales sometime during 1991.

Graphical Solution

Use a graphing utility to graph both equations in the same viewing window. From Figure 2.18, the graphs appear to have one point of intersection. Use the *intersect* feature of the graphing utility to approximate the point of intersection to be (1.42, 391.66). So, you would expect that compact disc sales exceeded cassette sales sometime during 1991.

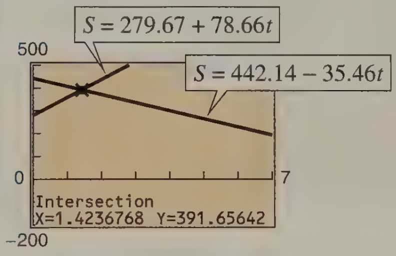


Figure 2.18

Writing About Math *Judging the Accuracy of an Approximate Solution*

Suppose you are solving the equation

$$\frac{x}{x-1} - \frac{99}{100} = 0$$

for x, and you obtain $x = -99.1$ as your solution. Substituting this value back into the equation produces

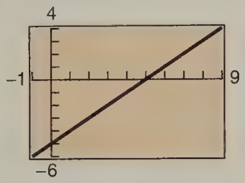
$$\frac{-99.1}{-99.1 - 1} - \frac{99}{100} = 0.00000999 = 9.99 \times 10^{-6} \approx 0.$$

Does this mean that -99.1 is a good approximation to the solution? Write a short paragraph explaining why or why not.

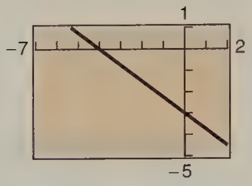
2.2 Exercises

In Exercises 1–12, find the x- and y-intercepts of the graph of the equation.

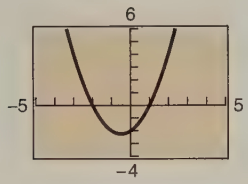
1. $y = x - 5$



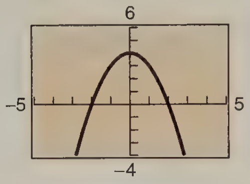
2. $y = -\frac{3}{4}x - 3$



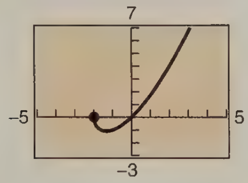
3. $y = x^2 + x - 2$



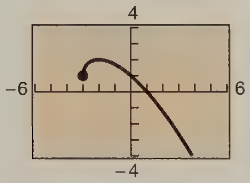
4. $y = 4 - x^2$



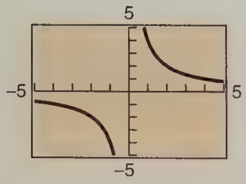
5. $y = x\sqrt{x + 2}$



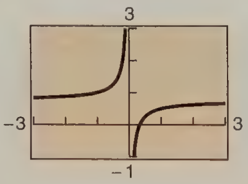
6. $y = -\frac{1}{2}x\sqrt{x + 3} + 1$



7. $xy = 4$



8. $4xy = 3x - 1$



9. $y = |x - 2| - 4$

10. $y = 3 - \frac{1}{2}|x + 1|$

11. $xy - 2y - x + 1 = 0$

12. $x^2y - x^2 + 4y = 0$

In Exercises 13–18, the zero(s) of the function are given. Verify the zero(s) both algebraically and graphically.

	Function	*Zero(s)*
13.	$f(x) = 12 - 4x$	$x = 3$
14.	$f(x) = 3(x - 5) + 9$	$x = 2$
15.	$f(x) = x^2 - 2.5x - 6$	$x = -1.5, 4$
16.	$f(x) = x^3 - 9x^2 + 18x$	$x = 0, 3, 6$
17.	$f(x) = \frac{x + 2}{3} - \frac{x - 1}{5} - 1$	$x = 1$
18.	$f(x) = x - 3 - \frac{10}{x}$	$x = -2, 5$

Graphical Analysis **In Exercises 19–22, use a graphing utility to graph the equation and approximate any x-intercepts. Set $y = 0$ and solve the resulting equation. Compare the results with the x-intercepts of the graph.**

19. $y = 2(x - 1) - 4$

20. $y = \frac{4}{3}x + 2$

21. $y = 20 - (3x - 10)$

22. $y = 10 + 2(x - 2)$

In Exercises 23–34, solve the equation algebraically. Then write the equation in the form $f(x) = 0$ and use a graphing utility to verify the algebraic solution.

23. $27 - 4x = 12$

24. $3.5x - 8 = 0.5x$

25. $25(x - 3) = 12(x + 2) - 10$

26. $1200 = 300 + 2(x - 500)$

27. $\frac{3x}{2} + \frac{1}{4}(x - 2) = 10$

28. $0.60x + 0.40(100 - x) = 50$

29. $\frac{2x}{3} = 10 - \frac{24}{x}$

30. $\frac{x - 3}{25} = \frac{x - 5}{12}$

31. $\frac{3}{x + 2} - \frac{4}{x - 2} = 5$

32. $\frac{6}{x} + \frac{8}{x + 5} = 3$

33. $3(x + 3) = 5(1 - x) - 1$

34. $(x + 1)^2 + 2(x - 2) = (x + 1)(x - 2)$

In Exercises 35–46, use a graphing utility to approximate any solutions (accurate to three decimal places) of the equation. [Remember to write the equation in the form $f(x) = 0$.]

35. $\frac{1}{4}(x^2 - 10x + 17) = 0$

36. $-2(x^2 - 6x + 6) = 0$

37. $x^3 + x + 4 = 0$

38. $\frac{1}{9}x^3 + x + 4 = 0$

39. $2x^3 - x^2 - 18x + 9 = 0$

40. $4x^3 + 12x^2 - 26x - 24 = 0$

41. $x^4 = 2x^3 + 1$

42. $x^5 = 3 + 2x^3$

43. $\dfrac{2}{x+2} = 3$

44. $\dfrac{5}{x} = 1 + \dfrac{3}{x+2}$

45. $|x - 3| = 4$

46. $\sqrt{x-2} = 3$

47. ***Exploration***

(a) Use a graphing utility to complete the table.

x	-1	0	1	2	3	4
$3.2x - 5.8$						

(b) Use the table in part (a) to determine the interval in which the solution to the equation $3.2x - 5.8 = 0$ is located. Explain your reasoning.

(c) Use a graphing utility to complete the table.

x	1.5	1.6	1.7	1.8	1.9	2
$3.2x - 5.8$						

(d) Use the table in part (c) to determine the interval in which the solution to the equation $3.2x - 5.8 = 0$ is located. Explain how this process can be used to approximate the solution to any desired degree of accuracy.

(e) Use a graphing utility to verify graphically the solution to $3.2x - 5.8 = 0$ found in part (d).

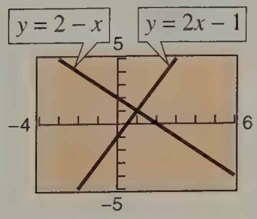
48. ***Exploration*** Use the procedure of Exercise 47 to approximate the solution of the equation

$$0.3(x - 1.5) - 2 = 0$$

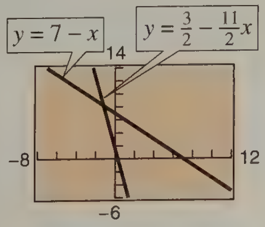
accurate to two decimal places.

In Exercises 49–54, determine any point(s) of intersection algebraically. Then verify your result numerically by creating a table of values for each function.

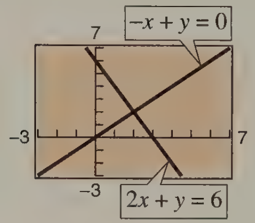
49. $y = 2 - x$
$y = 2x - 1$



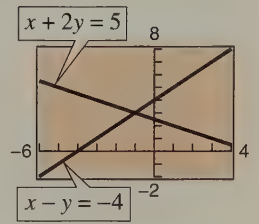
50. $y = 7 - x$
$y = \frac{3}{2} - \frac{11}{2}x$



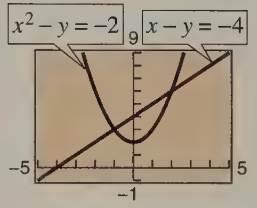
51. $2x + y = 6$
$-x + y = 0$



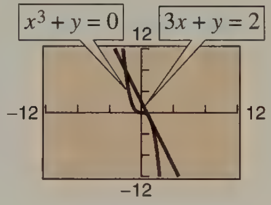
52. $x - y = -4$
$x + 2y = 5$



53. $x - y = -4$
$x^2 - y = -2$



54. $3x + y = 2$
$x^3 + y = 0$



In Exercises 55–62, use a graphing utility to approximate any points of intersection (accurate to three decimal places) of the graphs of equations. Verify your results algebraically.

55. $y = 9 - 2x$
$y = x - 3$

56. $y = \frac{1}{3}x + 2$
$y = \frac{5}{2}x - 11$

57. $y = 4 - x^2$
$y = 2x - 1$

58. $y = x^3 - 3$
$y = 5 - 2x$

59. $y = 8$
$y = 3x^2 + 2x$

60. $y = 32$
$y = x^5 - x^2$

61. $y = 2x^2$
$y = x^4 - 2x^2$

62. $y = -x$
$y = 2x - x^2$

In Exercises 63 and 64, evaluate the expression in two ways. (a) Calculate entirely on your calculator by storing intermediate results and then rounding the final answer to two decimal places. (b) Round both the numerator and denominator to two decimal places before dividing, and then round the final answer to two decimal places. Does the method in part (b) decrease the accuracy? Explain.

63. $\dfrac{1 + 0.73205}{1 - 0.73205}$

64. $\dfrac{1 + 0.86603}{1 - 0.86603}$

65. ***Travel Time*** On the first part of a 280-mile trip, a sales person averaged 63 miles per hour. The sales person averaged only 54 miles per hour on the last part of the trip because of an increased volume of traffic.

(a) Express the total time for the trip as a function of the distance x traveled at an average speed of 63 miles per hour.

(b) Use a graphing utility to graph the time function. What is the domain of the function?

(c) Approximate the number of miles traveled at 63 miles per hour if the total time was 4 hours and 45 minutes.

66. ***Production Limit*** A company has fixed costs of \$25,000 per month and a variable cost of \$18.65 per unit manufactured. (*Fixed costs* are those that occur regardless of the level of production.)

(a) Write the total monthly costs C as a function of the number of units x produced.

(b) Use a graphing utility to graph the cost function. Approximate the number of units that can be produced per month if total costs cannot exceed \$200,000. Verify algebraically. Is this better solved algebraically or graphically? Explain.

67. ***Mixture Problem*** A 55-gallon barrel contains a mixture with a concentration of 33%. You remove x gallons of this mixture and replace it with 100% concentrate.

(a) Write the amount of concentrate in the final mixture as a function of x.

(b) Use a graphing utility to graph the concentration function. What is the domain of the function?

(c) Approximate (accurate to one decimal place) the value of x if the final mixture is 60% concentrate.

68. ***Geometry*** A rectangular region with a perimeter of 230 meters has a length of x.

(a) Draw a diagram to represent the problem.

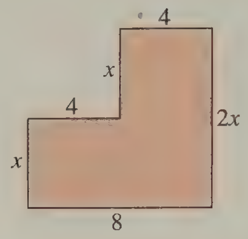
(b) Express the rectangle's area as a function of x.

(c) Use a graphing utility to graph the area function. Because area is nonnegative, what is the domain of the function?

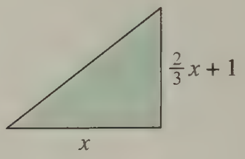
(d) Approximate (accurate to one decimal place) the dimensions of the region if its area is 2000 square feet.

Geometry **In Exercises 69 and 70, (a) write a function for the area of the region, (b) use a graphing utility to graph the function, and (c) approximate the value of x if the area of the region is 200 square units.**

69.



70.



71. ***Volume*** Consider the swimming pool in the figure below. (When finding its volume, use the fact that the volume is the area of the region on the vertical sidewall times the width of the pool.)

(a) Find the volume of the pool.

(b) Find an equation of the line representing the base of the pool.

(c) If the depth of the water at the deep end of the pool is d, show that the volume of water is

$$V(d) = \begin{cases} 80d^2, & 0 \le d \le 5 \\ 800d - 2000, & 5 < d \le 9 \end{cases}.$$

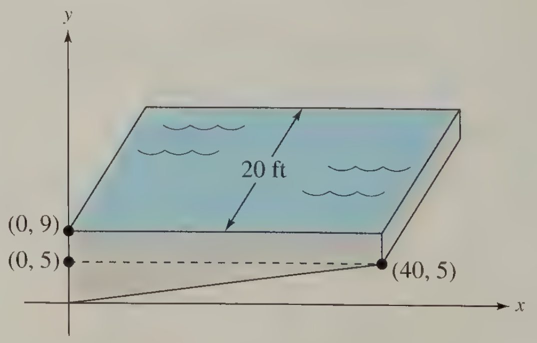
(d) Graph the volume function.

(e) Use a graphing utility to complete the table.

d	3	5	7	9
V				

(f) Approximate the depth of the water at the deep end if the volume is 4800 cubic feet.

(g) How many gallons of water are in the pool? (There are 7.48 gallons of water in 1 cubic foot.)



72. ***Negative Income Tax*** The following information describes a possible negative income tax for a family consisting of two adults and two children. The plan would guarantee the poor a minimum income while encouraging a family to increase its private income ($0 \leq x \leq 20{,}000$).

Family's earned income: $I = x$

Subsidy: $S = 10{,}000 - \frac{1}{2}x$

Total income: $T = I + S$

(a) Express the total income T in terms of x.

(b) Use a graphing utility to find the earned income x if the subsidy is \$6600. Verify your answer algebraically.

(c) Use a graphing utility to find the earned income x if the total income is \$13,800. Verify your answer algebraically.

(d) Find the subsidy S graphically if the total income is \$12,500.

73. ***Labor Force*** The number of married women y in the civilian work force (in millions) in the United States from 1990 to 1997 can be approximated by the model

$$y = 0.46x + 30.81$$

where $t = 0$ represents 1990. According to this model, during which year did this number reach 33 million? Explain how to answer the question graphically and algebraically. (Source: U.S. Bureau of Labor Statistics)

74. ***Recommended Weight*** The median recommended weight of women of small frame who are 25 to 59 years old can be approximated by the mathematical model

$$y = 0.041x^2 - 2.525x + 113.639, \quad 58 \leq x \leq 72$$

where y is the median recommended weight in pounds and x is the height in inches. Suppose you are an insurance agent. After taking information from a female client, you compute her recommended weight to be 123 pounds. Later, when you are filling out her paperwork, you are unable to remember her height. Use the model and a graphing utility to estimate your client's height based on her recommended weight. (Source: Metropolitan Life Insurance Company)

75. ***Per Capita Utilization*** The per capita utilization (in pounds) of nectarines and peaches N, and cucumbers C, from 1991 through 1996 can be modeled by

$$N = -0.37t + 6.88$$

$$C = 0.27t + 4.42$$

where $t = 1$ represents 1991. (Source: U.S. Department of Agriculture)

(a) What does the intersection of the graphs of these equations represent?

(b) Find the point of intersection of the graphs algebraically.

(c) Verify your answer to part (b) using the *zoom* and *trace* features of a graphing utility.

Synthesis

True or False? **In Exercises 76–78, determine whether the statement is true or false. Justify your answer.**

76. To find the y-intercept of a graph, let $x = 0$ and solve the equation for y.

77. Every linear equation has at least one y-intercept or x-intercept.

78. Two linear equations can have either one point of intersection or no points of intersection.

Review

In Exercises 79–82, rationalize the denominator.

79. $\dfrac{12}{5\sqrt{3}}$

80. $\dfrac{4}{\sqrt{10} - 2}$

81. $\dfrac{3}{8 + \sqrt{11}}$

82. $\dfrac{14}{3\sqrt{10} - 1}$

In Exercises 83–88, find the product.

83. $(x + 6)(3x - 5)$

84. $(3x + 13)(4x - 7)$

85. $(2x - 9)(2x + 9)$

86. $(4x + 1)^2$

87. $(2x^2 - y)(3x^2 + 4y)$

88. $(4x^3 + 7y^2)(3x^3 - y^2)$

In Exercises 89 and 90, determine the open intervals over which the function is increasing, decreasing, or constant.

89. $f(x) = -x^2 - 8$

90. $f(x) = 2x^4 - 3x^2$

2.3 Complex Numbers

The Imaginary Unit *i*

Some quadratic equations have no real solutions. For instance, the quadratic equation $x^2 + 1 = 0$ has no real solution because there is no real number x that can be squared to produce -1. To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit *i*,** defined as

$$i = \sqrt{-1} \qquad \text{Imaginary unit}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, you obtain the set of **complex numbers.** Each complex number can be written in the **standard form** $a + bi$. For instance, the standard form of the complex number $\sqrt{-9} - 5$ is $-5 + 3i$ because

$$\sqrt{-9} - 5 = \sqrt{3^2(-1)} - 5 = 3\sqrt{-1} - 5 = -5 + 3i.$$

In the standard form $a + bi$, the real number a is called the **real part** of the complex number and the number bi (where b is a real number) is called the **imaginary part** of the complex number.

Definition of a Complex Number

If a and b are real numbers, the number $a + bi$ is a **complex number,** and it is said to be written in **standard form.** If $b = 0$, the number $a + bi = a$ is a real number. If $b \neq 0$, the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

The set of real numbers is a subset of the set of complex numbers, as shown in Figure 2.19. This is true because every real number a can be written as a complex number using $b = 0$. That is, for every real number a, you can write $a = a + 0i$.

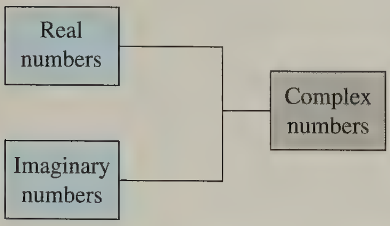


Figure 2.19

Equality of Complex Numbers

Two complex numbers $a + bi$ and $c + di$, written in standard form, are equal to each other

$$a + bi = c + di \qquad \text{Equality of two complex numbers}$$

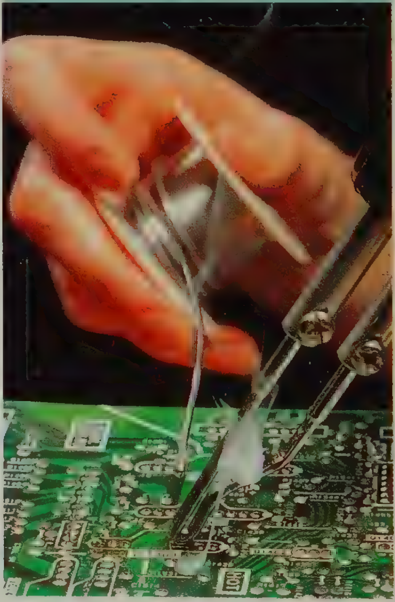
if and only if $a = c$ and $b = d$.

What You Should Learn:

- How to use the imaginary unit *i* to write complex numbers
- How to add, subtract, and multiply complex numbers
- How to use complex conjugates to divide complex numbers
- How to plot complex numbers in the complex plane

Why You Should Learn It:

Complex numbers are used to model numerous aspects of the natural world, such as the impedance of an electrical circuit, as shown in Exercise 83 on page 187.



Phil Degginger/Tony Stone Images

Operations with Complex Numbers

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers written in standard form, their sum and difference are defined as follows.

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number $a + bi$ is

$$-(a + bi) = -a - bi. \qquad \text{Additive inverse}$$

So, you have

$$(a + bi) + (-a - bi) = 0 + 0i = 0.$$

EXAMPLE 1 Adding and Subtracting Complex Numbers

a.
$$\begin{aligned} (3 - i) + (2 + 3i) &= 3 - i + 2 + 3i && \text{Remove parentheses.} \\ &= 3 + 2 - i + 3i && \text{Group like terms.} \\ &= (3 + 2) + (-1 + 3)i \\ &= 5 + 2i && \text{Write in standard form.} \end{aligned}$$

b.
$$\begin{aligned} 2i + (-4 - 2i) &= 2i - 4 - 2i && \text{Remove parentheses.} \\ &= -4 + 2i - 2i && \text{Group like terms.} \\ &= -4 && \text{Write in standard form.} \end{aligned}$$

c.
$$\begin{aligned} 3 - (-2 + 3i) + (-5 + i) &= 3 + 2 - 3i - 5 + i \\ &= 3 + 2 - 5 - 3i + i \\ &= 0 - 2i \\ &= -2i \end{aligned}$$

d.
$$\begin{aligned} (3 + 2i) + (4 - i) - (7 + i) &= 3 + 2i + 4 - i - 7 - i \\ &= 3 + 4 - 7 + 2i - i - i \\ &= 0 + 0i \\ &= 0 \end{aligned}$$

In Examples 1(b) and 1(d) note that the sum of complex numbers can be a real number.

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Carl Friedrich Gauss (1777–1855) proved that all the roots of any algebraic equation are "numbers" of the form $a + bi$, where a and b are real numbers and i is the square root of -1. These "numbers" are called complex.

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

Associative Property of Addition and Multiplication
Commutative Property of Addition and Multiplication
Distributive Property of Multiplication over Addition

Notice how these properties are used when two complex numbers are multiplied.

$$\begin{aligned}(a+bi)(c+di) &= a(c+di)+bi(c+di) && \text{Distributive Property}\\ &= ac+(ad)i+(bc)i+(bd)i^2 && \text{Distributive Property}\\ &= ac+(ad)i+(bc)i+(bd)(-1) && i^2=-1\\ &= ac-bd+(ad)i+(bc)i && \text{Commutative Property}\\ &= (ac-bd)+(ad+bc)i && \text{Distributive Property}\end{aligned}$$

Rather than trying to memorize this multiplication rule, you should simply remember how the distributive property is used to multiply two complex numbers. The procedure is similar to multiplying two polynomials and combining like terms (as in the FOIL Method) shown in Section P.3.

Exploration

Complete the table:

$i^1 = i$	$i^7 =$
$i^2 = -1$	$i^8 =$
$i^3 = -i$	$i^9 =$
$i^4 = 1$	$i^{10} =$
$i^5 =$	$i^{11} =$
$i^6 =$	$i^{12} =$

What pattern do you see? Write a brief description of how you would find i raised to any positive integer power.

EXAMPLE 2 Multiplying Complex Numbers

a. $$\begin{aligned}(i)(-3i) &= -3i^2 && \text{Multiply.}\\ &= -3(-1) && i^2=-1\\ &= 3 && \text{Simplify.}\end{aligned}$$

b. $$\begin{aligned}\sqrt{-4}\cdot\sqrt{-16} &= (2i)(4i) && \text{Write each factor in standard form.}\\ &= 8i^2 && \text{Multiply.}\\ &= 8(-1) && i^2=-1\\ &= -8 && \text{Simplify.}\end{aligned}$$

c. $$\begin{aligned}(2-i)(4+3i) &= 8+6i-4i-3i^2 && \text{Product of binomials}\\ &= 8+6i-4i-3(-1) && i^2=-1\\ &= 8+3+6i-4i && \text{Group like terms.}\\ &= 11+2i && \text{Write in standard form.}\end{aligned}$$

d. $$\begin{aligned}(3+2i)(3-2i) &= 9-6i+6i-4i^2 && \text{Product of binomials}\\ &= 9-4(-1) && i^2=-1\\ &= 9+4 && \text{Simplify.}\\ &= 13 && \text{Write in standard form.}\end{aligned}$$

e. $$\begin{aligned}(3+2i)^2 &= 9+6i+6i+4i^2 && \text{Product of binomials}\\ &= 9+4(-1)+12i && i^2=-1\\ &= 9-4+12i && \text{Simplify.}\\ &= 5+12i && \text{Write in standard form.}\end{aligned}$$

STUDY TIP

Note in Example 2(b) that

$$\begin{aligned}\sqrt{-4}\cdot\sqrt{-16} &\neq \sqrt{(-4)(-16)}\\ &= \sqrt{64}\\ &= 8.\end{aligned}$$

Complex Conjugates and Division

Notice in Example 2(d) that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the form $a + bi$ and $a - bi$, called **complex conjugates.**

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2\\ &= a^2 - b^2(-1)\\ &= a^2 + b^2\end{aligned}$$

To find the quotient of $a + bi$ and $c + di$ where c and d are not both zero, multiply the numerator and denominator by the conjugate of the *denominator* to obtain

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right)\\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.\end{aligned}$$

EXAMPLE 3 Dividing Complex Numbers

$$\frac{1}{1 + i} = \frac{1}{1 + i}\left(\frac{1 - i}{1 - i}\right)$$

Multiply numerator and denominator by conjugate of denominator.

$$= \frac{1 - i}{1^2 - i^2}$$

Expand.

$$= \frac{1 - i}{1 - (-1)}$$

$i^2 = -1$

$$= \frac{1 - i}{2}$$

Simplify.

$$= \frac{1}{2} - \frac{1}{2}i$$

Write in standard form.

EXAMPLE 4 Dividing Complex Numbers

$$\frac{2 + 3i}{4 - 2i} = \frac{2 + 3i}{4 - 2i}\left(\frac{4 + 2i}{4 + 2i}\right)$$

Multiply numerator and denominator by conjugate of denominator.

$$= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2}$$

Expand.

$$= \frac{8 - 6 + 16i}{16 + 4}$$

$i^2 = -1$

$$= \frac{(2 + 16i)}{20}$$

Simplify.

$$= \frac{1}{10} + \frac{4}{5}i$$

Write in standard form.

STUDY TIP

Some graphing utilities can perform operations with complex numbers. For instance, on some graphing utilities, to divide $2 + 3i$ by $4 - 2i$, enter

(2 + 3 *i*) ÷

(4 − 2 *i*) ENTER.

The display will be as follows.

$$.1 + .8i \quad \text{or} \quad \frac{1}{10} + \frac{4}{5}i$$

Consult your user's manual for specific instructions on performing operations with complex numbers.

Applications

Most applications involving complex numbers are either theoretical or very technical, and are therefore not appropriate for inclusion in this text. However, to give you some idea of how complex numbers can be used in applications, we give a general description of their use in **fractal geometry.**

To begin, consider a coordinate system called the **complex plane.** Just as every real number corresponds to a point on the real number line, every complex number corresponds to a point in the complex plane, as shown in Figure 2.20. In this figure, note that the vertical axis is the **imaginary axis** and the horizontal axis is the **real axis.** The point that corresponds to the complex number $a + bi$ is (a, b).

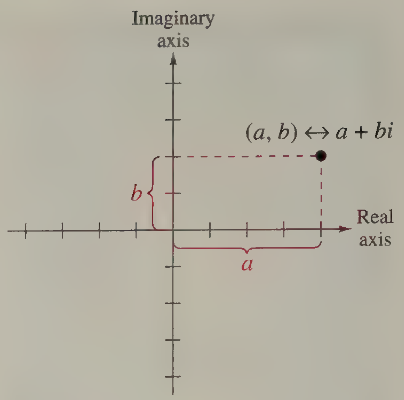


Figure 2.20

EXAMPLE 5 Plotting Complex Numbers

Plot each complex number in the complex plane.

a. $2 + 3i$ **b.** $-1 + 2i$

c. 4 **d.** $-3i$

Solution

a. To plot the complex number $2 + 3i$, move (from the origin) two units to the right on the real axis and then three units up, as shown in Figure 2.21. In other words, plotting the complex number $2 + 3i$ in the complex plane is comparable to plotting the point $(2, 3)$ in the Cartesian plane.

b. The complex number $-1 + 2i$ corresponds to the point $(-1, 2)$, as shown in Figure 2.21.

c. The complex number 4 corresponds to the point $(4, 0)$, as shown in Figure 2.21.

d. The complex number $-3i$ corresponds to the point $(0, 3)$, as shown in Figure 2.21.

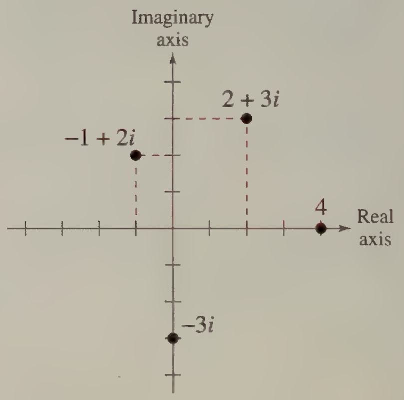


Figure 2.21

In the hands of a person who understands fractal geometry, the complex plane can become an easel on which stunning pictures, called **fractals,** can be drawn. The most famous such picture is called the **Mandelbrot Set,** named after the Polish-born mathematician Benoit Mandelbrot. To draw the Mandelbrot Set, consider the following sequence of numbers.

$$c, c^2 + c, (c^2 + c)^2 + c, [(c^2 + c)^2 + c]^2 + c, \ldots$$

The behavior of this sequence depends on the value of the complex number c. For some values of c this sequence is **bounded,** which means that all elements in the sequence are less than some fixed number N. For other values it is **unbounded,** which means that the elements in the sequence become infinitely large. If the sequence is bounded, the complex number c is in the Mandelbrot Set; if the sequence is unbounded, the complex number c is not in the Mandelbrot Set.

EXAMPLE 6 Members of the Mandelbrot Set

a. The complex number -2 is in the Mandelbrot Set because for $c = -2$, the corresponding Mandelbrot sequence is $-2, 2, 2, 2, 2, 2, \ldots$, which is bounded.

b. The complex number i is also in the Mandelbrot Set because for $c = i$, the corresponding Mandelbrot sequence is

$$i, \quad -1 + i, \quad -i, \quad -1 + i, \quad -i, \quad -1 + i, \quad \ldots$$

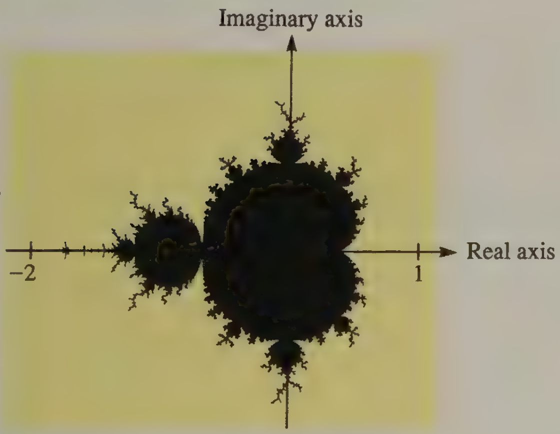
which is bounded.

c. The complex number $1 + i$ is not in the Mandelbrot Set because for $c = 1 + i$, the corresponding Mandelbrot sequence is

$$1 + i, \quad 1 + 3i, \quad -7 + 7i, \quad 1 - 97i, \quad -9407 - 193i,$$
$$88454401 + 3631103i, \quad \ldots$$

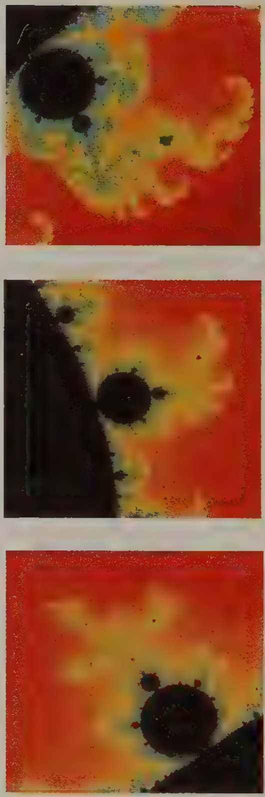
which is unbounded.

With this definition, a picture of the Mandelbrot Set would have only two colors: one color for points that are in the set (the sequence is bounded), and one for points that are outside the set (the sequence is unbounded). Figure 2.22 shows a black and yellow picture of the Mandelbrot Set. The points that are black are in the Mandelbrot Set and the points that are yellow are not.



American Mathematical Society

Figure 2.22



American Mathematical Society

Figure 2.23

To add more interest to the picture, computer scientists discovered that the points that are not in the Mandelbrot Set can be assigned a variety of colors, depending on "how quickly" their sequences diverge. Figure 2.23 shows three different appendages of the Mandelbrot Set. (The black portions of the picture represent points that are in the Mandelbrot Set.)

Figures 2.24 shows another type of fractal. From this picture, you can see why fractals have fascinated people since their discovery (around 1980). The fractal shown was produced on a graphing calculator.

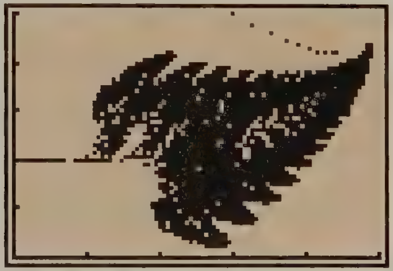


Figure 2.24 *A Fractal Fern*

2.3 Exercises

In Exercises 1–4, solve for *a* and *b*.

1. $a + bi = -9 + 4i$

2. $a + bi = 12 + 5i$

3. $(a - 1) + (b + 3)i = 5 + 8i$

4. $(a + 6) + 2bi = 6 - 5i$

In Exercises 5–14, write in standard form.

5. $4 + \sqrt{-25}$

6. $3 + \sqrt{-9}$

7. 12

8. 42

9. $-5i + i^2$

10. $-3i^2 + i$

11. $\left(\sqrt{-75}\right)^2$

12. $\left(\sqrt{-4}\right)^2 - 7$

13. $\sqrt{-0.09}$

14. $\sqrt{-0.0004}$

In Exercises 15–24, perform the addition or subtraction and write the result in standard form.

15. $(4 + i) + (7 - 2i)$

16. $(11 - 2i) + (-3 + 6i)$

17. $\left(-1 + \sqrt{-8}\right) + \left(8 - \sqrt{-50}\right)$

18. $\left(7 + \sqrt{-18}\right) - \left(3 + 3\sqrt{2}i\right)$

19. $13i - (14 - 7i)$

20. $22 + (-5 + 8i) + 10i$

21. $-\left(\frac{3}{2} + \frac{5}{2}i\right) + \left(\frac{5}{3} + \frac{11}{3}i\right)$

22. $-\left(\frac{3}{4} + \frac{7}{5}i\right) - \left(\frac{5}{6} - \frac{1}{6}i\right)$

23. $(1.6 + 3.2i) + (-5.8 + 4.3i)$

24. $-(-3.7 - 12.8i) - \left(6.1 - \sqrt{-24.5}\right)$

In Exercises 25–36, perform the operation and write the result in standard form.

25. $\sqrt{-6} \cdot \sqrt{-2}$

26. $\sqrt{-5} \cdot \sqrt{-10}$

27. $\left(\sqrt{-10}\right)^2$

28. $\left(\sqrt{-75}\right)^2$

29. $(1 + i)(3 - 2i)$

30. $(6 - 2i)(2 - 3i)$

31. $6i(5 - 2i)$

32. $-8i(9 + 4i)$

33. $\left(\sqrt{14} + \sqrt{10}i\right)\left(\sqrt{14} - \sqrt{10}i\right)$

34. $\left(3 + \sqrt{-5}\right)\left(7 - \sqrt{-10}\right)$

35. $(4 + 5i)^2$

36. $(1 - 2i)^2 - (1 + 2i)^2$

37. ***Error Analysis*** Describe the error.

$$\sqrt{-6}\sqrt{-6} = \sqrt{(-6)(-6)} = \sqrt{36} = 6 \quad \times$$

38. ***Error Analysis*** Describe the error.

$$\begin{aligned} -i\left(\sqrt{-4} - 1\right) &= -i(4i - 1) \quad \times \\ &= -4i^2 - i \\ &= 4 - i \end{aligned}$$

In Exercises 39–46, find the product of the number and its conjugate.

39. $4 + 3i$

40. $8 - 12i$

41. $-6 - \sqrt{5}i$

42. $-3 + \sqrt{2}i$

43. $22i$

44. $\sqrt{-13}$

45. $3 - \sqrt{-2}$

46. $1 + \sqrt{-8}$

In Exercises 47–58, perform the operation and write the result in standard form.

47. $\dfrac{6}{i}$

48. $-\dfrac{5}{i}$

49. $\dfrac{4}{4 - 5i}$

50. $\dfrac{3}{1 - i}$

51. $\dfrac{2 + i}{2 - i}$

52. $\dfrac{8 - 7i}{1 - 2i}$

53. $\dfrac{6 - 7i}{i}$

54. $\dfrac{8 + 20i}{2i}$

55. $\dfrac{1}{(4 - 5i)^2}$

56. $\dfrac{(2 - 3i)(5i)}{2 + 3i}$

57. $\dfrac{2}{1 + i} - \dfrac{3}{1 - i}$

58. $\dfrac{2i}{2 + i} + \dfrac{5}{2 - i}$

In Exercises 59–66, simplify the complex number and write it in standard form.

59. $-6i^3 + i^2$

60. $4i^2 - 2i^3$

61. $-5i^5$

62. $(-i)^3$

63. $\left(\sqrt{-75}\right)^3$

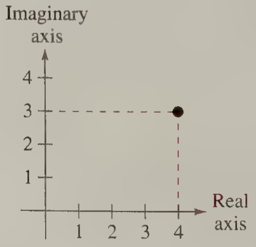
64. $\left(\sqrt{-2}\right)^6$

65. $\dfrac{1}{i^3}$

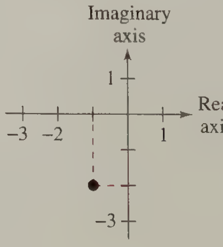
66. $\dfrac{1}{(2i)^3}$

In Exercises 67–70, determine the complex number shown in the complex plane.

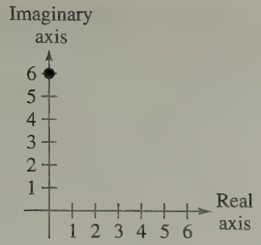
67.



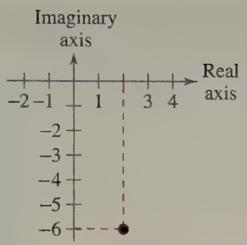
68.



69.



70.



In Exercises 71–74, plot the complex number in the complex plane.

71. $4 - 5i$

72. $3i$

73. -6

74. $-7 + 2i$

Fractals **In Exercises 75–80, find the first six terms in the following sequence.**

$c,\ c^2 + c,\ (c^2 + c)^2 + c,\ [(c^2 + c)^2 + c]^2 + c, \ldots$

From these terms, do you think the given complex number is in the Mandelbrot Set? Explain your reasoning.

75. $c = 0$

76. $c = 2$

77. $c = \frac{1}{2}i$

78. $c = -i$

79. $c = 1$

80. $c = -1$

81. Cube each complex number. What do you notice?

(a) 2 (b) $-1 + \sqrt{3}i$ (c) $-1 - \sqrt{3}i$

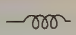
82. Raise each number to the fourth power.

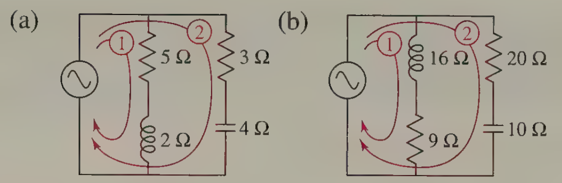
(a) 2 (b) -2 (c) $2i$ (d) $-2i$

83. ***Impedance*** The opposition to current in an electrical circuit is called its impedance. The impedance in a parallel circuit with two pathways satisfies the equation

$$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}$$

when z_1 is the impedance (in ohms) of pathway 1 and z_2 is the impedance (in ohms) of pathway 2. Use the table to determine the impedance of each parallel circuit. (*Hint:* You can find the impedance of each pathway by adding the impedance of each component in the pathway.)

	Resistor	Inductor	Capacitor
Symbol	 $a\ \Omega$	 $b\ \Omega$	 $c\ \Omega$
Impedance	a	bi	$-ci$



Synthesis

True or False? **In Exercises 84–86, determine whether the statement is true or false. Justify your answer.**

84. There is no complex number that is equal to its conjugate.

85. $-i\sqrt{6}$ is a solution of $x^4 - x^2 + 14 = 56$.

86. $i^{44} + i^{150} - i^{74} - i^{109} + i^{61} = -1$

87. Express each of the following powers of i as i, $-i$, 1, or -1.

(a) i^{40} (b) i^{25} (c) i^{50} (d) i^{67}

88. Prove that the sum of a complex number $a + bi$ and its conjugate is a real number, and that the difference of a complex number $a + bi$ and its conjugate is an imaginary number.

89. Prove that the product of a complex number $a + bi$ and its conjugate is a real number.

Review

In Exercises 90–93 perform the operations and write the result in standard form.

90. $(4 + 3x) + (8 - 6x - x^2)$

91. $(x^3 - 3x^2) - (6 - 2x - 4x^2)$

92. $\left(3x - \frac{1}{2}\right)(x + 4)$

93. $(2x - 5)^2$

In Exercises 94–97, find the x- and y-intercepts of the graph of the equation.

94. $y = -x^2 + 6$

95. $y = x^2 + 2x - 8$

96. $y = |x - 4| + 1$

97. $y = |x| - 1$

98. ***Mixture Problem*** A 5-liter container contains a mixture with a concentration of 50%. How much of this mixture must be withdrawn and replaced by 100% concentrate to bring the mixture up to 60% concentration?

2.4 Solving Equations Algebraically

Quadratic Equations

A **quadratic equation in x** is an equation that can be written in the general form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$. Another name for a quadratic equation in x is a **second-degree polynomial equation in x.** You should be familiar with the following four methods for solving quadratic equations.

Solving a Quadratic Equation

Factoring: If $ab = 0$, then $a = 0$ or $b = 0$.

Example:

$$\begin{aligned} x^2 - x - 6 &= 0 \\ (x-3)(x+2) &= 0 \\ x - 3 &= 0 \quad \Rightarrow \quad x = 3 \\ x + 2 &= 0 \quad \Rightarrow \quad x = -2 \end{aligned}$$

Extracting Square Roots: If $u^2 = c$, where $c > 0$, then $u = \pm\sqrt{c}$.

Example:

$$\begin{aligned} (x+3)^2 &= 16 \\ x + 3 &= \pm 4 \\ x &= -3 \pm 4 \\ x = 1 \quad &\text{or} \quad x = -7 \end{aligned}$$

Completing the Square: If $x^2 + bx = c$, then

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

$$\left(x + \frac{b}{2}\right)^2 = c + \frac{b^2}{4}.$$

Example:

$$\begin{aligned} x^2 + 6x &= 5 \\ x^2 + 6x + 3^2 &= 5 + 3^2 \\ (x+3)^2 &= 14 \\ x + 3 &= \pm\sqrt{14} \\ x &= -3 \pm \sqrt{14} \end{aligned}$$

Quadratic Formula: If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example: $2x^2 + 3x - 1 = 0$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-1)}}{2(2)} = \frac{-3 \pm \sqrt{17}}{4}$$

What You Should Learn:

- How to solve quadratic equations by factoring, extracting square roots, completing the square, and using the Quadratic Formula
- How to solve polynomial equations of degree three or greater
- How to solve equations involving radicals
- How to solve equations involving fractions or absolute values
- How to use quadratic equations to model and solve real-life problems

Why You Should Learn It:

Knowing how to solve quadratic and other types of equations algebraically can help you solve real-life problems, such as Exercise 132 on page 203, where you determine the greatest distance a car can travel on a tank of fuel.

Paul Souders/Tony Stone Images

EXAMPLE 1 Solving a Quadratic Equation by Factoring

Solve each quadratic equation by factoring.

a. $6x^2 = 3x$ **b.** $9x^2 - 6x + 1 = 0$

Solution

a. $6x^2 = 3x$ Write original equation.

$6x^2 - 3x = 0$ Write in general form.

$3x(2x - 1) = 0$ Factor.

$3x = 0 \quad \Rightarrow \quad x = 0$ Set 1st factor equal to 0.

$2x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{2}$ Set 2nd factor equal to 0.

b. $9x^2 - 6x + 1 = 0$ Write original equation.

$(3x - 1)^2 = 0$ Factor.

$3x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{3}$ Set repeated factor equal to 0.

Throughout the text, when solving equations, be sure to check your solutions either *algebraically* by substituting in the original equation or *graphically*.

Check

a. $6x^2 = 3x$ Write original equation.

$6(0)^2 \overset{?}{=} 3(0)$ Substitute 0 for x.

$0 = 0$ Solution checks. ✓

$6\left(\frac{1}{2}\right)^2 \overset{?}{=} 3\left(\frac{1}{2}\right)$ Substitute $\frac{1}{2}$ for x.

$\frac{6}{4} = \frac{3}{2}$ Solution checks. ✓

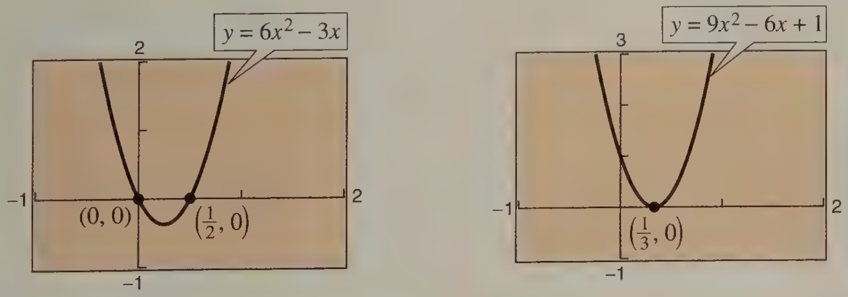
b. $9x^2 - 6x + 1 = 0$ Write original equation.

$9\left(\frac{1}{3}\right)^2 - 6\left(\frac{1}{3}\right) + 1 \overset{?}{=} 0$ Substitute $\frac{1}{3}$ for x.

$1 - 2 + 1 \overset{?}{=} 0$ Simplify.

$0 = 0$ Solution checks. ✓

Similarly, you can approximate the solutions using the graphs in Figure 2.25 to graphically check your solutions.



(a) (b)

Figure 2.25

Exploration

Try programming the Quadratic Formula into a computer or graphing calculator. Programs for several graphing calculator models may be found on the website *college.hmco.com*.

To use one of the programs, you must first write the equation in general form. Then enter the values of a, b, and c. After the final value has been entered, the program will display either two real solutions *or* the words "NO REAL SOLUTION," *or* the program will give both real and complex solutions.

Solving a quadratic equation by extracting square roots is an efficient method to use when the quadratic equation can be written in the form $ax^2 + c = 0$, as shown in Example 2.

EXAMPLE 2 Extracting Square Roots

Solve each quadratic equation.

a. $4x^2 = 12$ **b.** $(x - 3)^2 = 7$

Solution

a. $4x^2 = 12$ Write original equation.

$x^2 = 3$ Divide each side by 4.

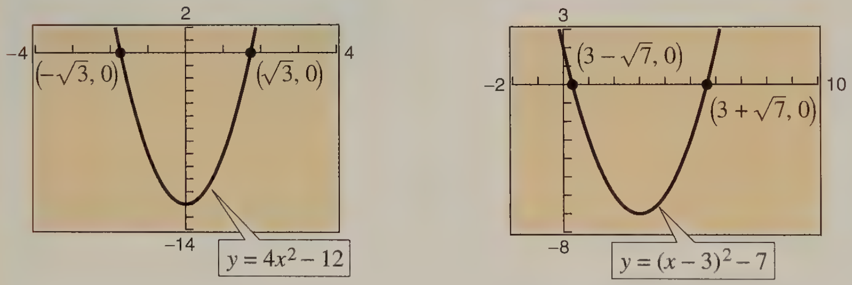
$x = \pm\sqrt{3}$ Take square root of each side.

b. $(x - 3)^2 = 7$ Write original equation.

$x - 3 = \pm\sqrt{7}$ Take square root of each side.

$x = 3 \pm \sqrt{7}$ Add 3 to each side.

The graphs of $y = 4x^2 - 12$ and $y = (x - 3)^2 - 7$, as shown in Figure 2.26, verify the solutions graphically.



(a) (b)

Figure 2.26

Note that the solutions shown in Example 2 are listed in *exact* form. Most graphing utilities produce decimal approximations of solutions rather than exact forms. For instance, if you solve the equations in Example 2 using a graphing utility, you will obtain $x \approx \pm 1.732$ in part (a) and $x \approx 5.646$ and $x \approx 0.354$ in part (b).

Some graphing utilities have symbolic algebra programs that *can* list the exact form of a solution.

Completing the square can be used to solve any quadratic equation, but it is best suited for quadratic equations in general form $ax^2 + bx + c = 0$ with $a = 1$ and b an even number. If the leading coefficient of the quadratic is not 1, divide each side of the equation by this coefficient *before* completing the square, as shown in Example 3.

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 3 Completing the Square: Leading Coefficient Is Not 1

Solve $3x^2 - 4x - 5 = 0$ by completing the square.

Solution

$$3x^2 - 4x - 5 = 0 \qquad \text{Write original equation.}$$

$$3x^2 - 4x = 5 \qquad \text{Add 5 to each side.}$$

$$x^2 - \frac{4}{3}x = \frac{5}{3} \qquad \text{Divide each side by 3.}$$

$$x^2 - \frac{4}{3}x + \underbrace{\left(\frac{2}{3}\right)^2}_{\left(\text{Half of } \frac{4}{3}\right)^2} = \frac{5}{3} + \left(\frac{2}{3}\right)^2 \qquad \text{Add } \left(\tfrac{2}{3}\right)^2 \text{ to each side.}$$

$$\left(x - \frac{2}{3}\right)^2 = \frac{19}{9}$$

$$x - \frac{2}{3} = \pm\frac{\sqrt{19}}{3} \qquad \text{Take square root of each side.}$$

$$x = \frac{2}{3} \pm \frac{\sqrt{19}}{3} \qquad \text{Solutions}$$

Using a calculator, the two solutions are approximately 2.11963 and −0.78630, which agree with the graphical solution shown in Figure 2.27.

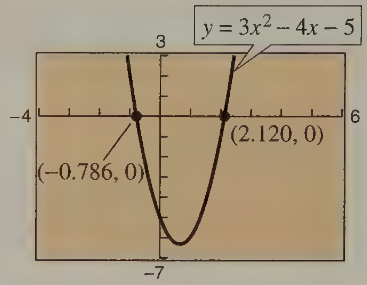


Figure 2.27

Exploration

Use a graphing utility to graph the three quadratic equations

$$y_1 = x^2 - 2x$$

$$y_2 = x^2 - 2x + 1$$

$$y_3 = x^2 - 2x + 2$$

in the same viewing window. Compute the *discriminant* $\sqrt{b^2 - 4ac}$ for each and discuss the relationship between the discriminant and the number of zeros of the quadratic function.

EXAMPLE 4 Quadratic Formula: Two Distinct Solutions

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Solve $x^3 + 3x = 9$ using the Quadratic Formula.

Algebraic Solution

$$x^2 + 3x = 9 \qquad \text{Write original equation.}$$

$$x^2 + 3x - 9 = 0 \qquad \text{Write in general form.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-9)}}{2(1)} \qquad \text{Substitute 3 for } b, 1 \text{ for } a, \text{ and } -9 \text{ for } c.$$

$$x = \frac{-3 \pm \sqrt{45}}{2} \qquad \text{Simplify.}$$

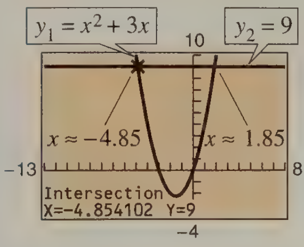
$$x = \frac{-3 \pm 3\sqrt{5}}{2} \qquad \text{Simplify radical.}$$

$$x \approx 1.85 \text{ or } -4.85 \qquad \text{Solutions}$$

The equation has two solutions: $x \approx 1.85$ and $x \approx -4.85$. Check these solutions in the original equation.

Graphical Solution

Use a graphing utility to graph $y_1 = x^2 + 3x$ and $y_2 = 9$ in the same viewing window. Use the *intersect* feature of the graphing utility to approximate the points where the graphs intersect. From Figure 2.28, it appears that the graphs intersect when $x \approx 1.85$ and $x \approx -4.85$. These x-coordinates of the intersection points are solutions of the equation $x^2 + 3x = 9$.



```
Intersection
X=-4.854102  Y=9
```

Figure 2.28

EXAMPLE 5 Quadratic Formula: One Repeated Solution

Solve $8x^2 - 24x + 18 = 0$.

Algebraic Solution

This equation has a common factor of 2. To simplify things, first divide each side of the equation by 2.

$$8x^2 - 24x + 18 = 0 \qquad \text{Write original equation.}$$

$$4x^2 - 12x + 9 = 0 \qquad \text{Divide each side by 2.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)}$$

$$x = \frac{12 \pm \sqrt{0}}{8} = \frac{3}{2} \qquad \text{Repeated solution}$$

This quadratic equation has only one solution: $\frac{3}{2}$. Check this solution in the original equation.

Graphical Solution

Use a graphing utility to graph

$$y = 8x^2 - 24x + 18.$$

Use the *zoom* and *trace* features of the graphing utility to approximate the values of x for which the function is equal to zero. From the graph in Figure 2.29, it appears that the function is equal to zero when $x = \frac{3}{2}$. This is the only solution of the equation $8x^2 - 24x + 18 = 0$.

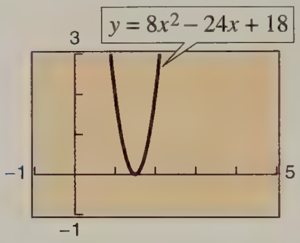


Figure 2.29

EXAMPLE 6 Complex Solutions of a Quadratic Equation

Solve $3x^2 - 2x + 5 = 0$.

Algebraic Solution

By the Quadratic Formula, you can write the solutions as follows.

$$3x^2 - 2x + 5 = 0 \qquad \text{Write original equation.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)} \qquad \text{Substitute } -2 \text{ for } b\text{, } 3 \text{ for } a\text{, and } 5 \text{ for } c.$$

$$= \frac{2 \pm \sqrt{-56}}{6} \qquad \text{Simplify.}$$

$$= \frac{2 \pm 2\sqrt{14}i}{6} \qquad \text{Simplify radical.}$$

$$= \frac{1}{3} \pm \frac{\sqrt{14}}{3}i \qquad \text{Solutions}$$

The equation has no real solution, but it has two complex solutions: $\frac{1}{3}(1 + \sqrt{14}i)$ and $\frac{1}{3}(1 - \sqrt{14}i)$.

Graphical Solution

Use a graphing utility to graph

$$y = 3x^2 - 2x + 5.$$

Note in Figure 2.30 that the graph of the function appears to have no x-intercepts. From this you can conclude that the equation $3x^2 - 2x + 5 = 0$ has no real solution. You can solve the equation algebraically to find the complex solutions.

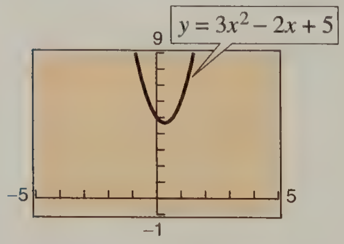


Figure 2.30

Polynomial Equations of Higher Degree

The methods used to solve quadratic equations can sometimes be extended to polynomial equations of higher degree, as shown in the next two examples.

EXAMPLE 7 Solving an Equation of Quadratic Type

Solve the equation $x^4 - 3x^2 + 2 = 0$.

Solution

The expression $x^4 - 3x^2 + 2$ is said to be in *quadratic form* because it is written in the form $au^2 + bu + c$, where u is any expression in x, namely x^2. You can use factoring to solve the equation as follows.

$$\begin{aligned} x^4 - 3x^2 + 2 &= 0 && \text{Write original equation.} \\ (x^2)^2 - 3(x^2) + 2 &= 0 && \text{Write in quadratic form in } x^2. \\ (x^2 - 1)(x^2 - 2) &= 0 && \text{Partially factor.} \\ (x + 1)(x - 1)(x^2 - 2) &= 0 && \text{Factor.} \\ x + 1 &= 0 \quad \Longrightarrow \quad x = -1 \\ x - 1 &= 0 \quad \Longrightarrow \quad x = 1 \\ x^2 - 2 &= 0 \quad \Longrightarrow \quad x = \pm\sqrt{2} \end{aligned}$$

The equation has four solutions: -1, 1, $\sqrt{2}$, and $-\sqrt{2}$. Check these solutions in the original equation. The graph of $y = x^4 - 3x^2 + 2$, as shown in Figure 2.31, verifies the solutions graphically.

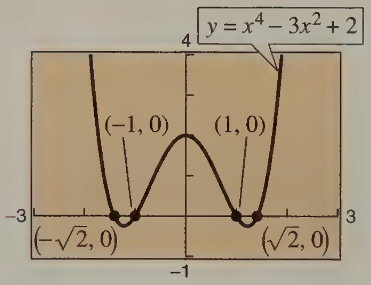


Figure 2.31

EXAMPLE 8 Solving a Polynomial Equation by Factoring

Solve the equation $2x^3 - 6x^2 - 6x + 18 = 0$.

Solution

This equation has a common factor of 2. To simplify things, first divide each side of the equation by 2.

$$\begin{aligned} 2x^3 - 6x^2 - 6x + 18 &= 0 && \text{Write original equation.} \\ x^3 - 3x^2 - 3x + 9 &= 0 && \text{Divide each side by 2.} \\ x^2(x - 3) - 3(x - 3) &= 0 && \text{Group terms.} \\ (x - 3)(x^2 - 3) &= 0 && \text{Factor by grouping.} \\ x - 3 &= 0 \quad \Longrightarrow \quad x = 3 \\ x^2 - 3 &= 0 \quad \Longrightarrow \quad x = \pm\sqrt{3} \end{aligned}$$

The equation has three solutions: 3, $\sqrt{3}$, and $-\sqrt{3}$. Check these solutions in the original equation. The graph of $y = 2x^3 - 6x^2 - 6x + 18$, as shown in Figure 2.32, verifies the solutions graphically.

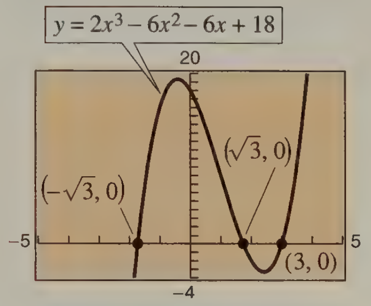


Figure 2.32

Equations Involving Radicals

An equation involving a radical expression can often be cleared of radicals by raising both sides of the equation to an appropriate power. When using this procedure, remember to check for extraneous solutions.

EXAMPLE 9 Solving an Equation Involving a Radical

Solve $\sqrt{2x+7} - x = 2$.

Algebraic Solution

$$\sqrt{2x+7} - x = 2 \qquad \text{Write original equation.}$$

$$\sqrt{2x+7} = x + 2 \qquad \text{Isolate radical.}$$

$$2x + 7 = x^2 + 4x + 4 \qquad \text{Square each side.}$$

$$x^2 + 2x - 3 = 0 \qquad \text{Write in general form.}$$

$$(x+3)(x-1) = 0 \qquad \text{Factor.}$$

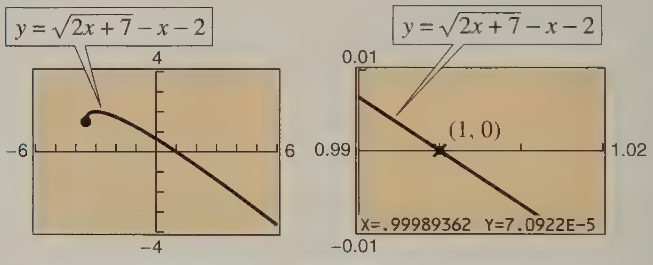
$$x + 3 = 0 \Rightarrow x = -3 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 1 = 0 \Rightarrow x = 1 \qquad \text{Set 2nd factor equal to 0.}$$

By substituting into the original equation, you can determine that -3 is extraneous, whereas 1 is valid. So, the equation has only one real solution: $x = 1$.

Graphical Solution

First rewrite the equation as $\sqrt{2x+7} - x - 2 = 0$. Then use a graphing utility to graph $y = \sqrt{2x+7} - x - 2$ as shown in Figure 2.33(a). Notice that the domain is $x \geq -\frac{7}{2}$ because the expression under the radical cannot be negative. There appears to be one solution near $x = 1$. Use the *zoom* and *trace* features, as shown in Figure 2.33(b), to approximate the only solution to be $x = 1$.



(a) **(b)**

Figure 2.33

EXAMPLE 10 Solving an Equation Involving Two Radicals

Solve the equation $\sqrt{2x+6} - \sqrt{x+4} = 1$.

Solution

$$\sqrt{2x+6} - \sqrt{x+4} = 1 \qquad \text{Write original equation.}$$

$$\sqrt{2x+6} = 1 + \sqrt{x+4} \qquad \text{Isolate radical.}$$

$$2x + 6 = 1 + 2\sqrt{x+4} + (x+4) \qquad \text{Square each side.}$$

$$x + 1 = 2\sqrt{x+4} \qquad \text{Isolate radical.}$$

$$x^2 + 2x + 1 = 4(x+4) \qquad \text{Square each side.}$$

$$x^2 - 2x - 15 = 0 \qquad \text{Write in general form.}$$

$$(x-5)(x+3) = 0 \qquad \text{Factor.}$$

$$x - 5 = 0 \Rightarrow x = 5 \qquad \text{Set 1st factor equal to 0.}$$

$$x + 3 = 0 \Rightarrow x = -3 \qquad \text{Set 2nd factor equal to 0.}$$

By substituting into the original equation, you can determine that -3 is extraneous, whereas 5 is valid. Figure 2.34 verifies that $x = 5$ is the only solution.

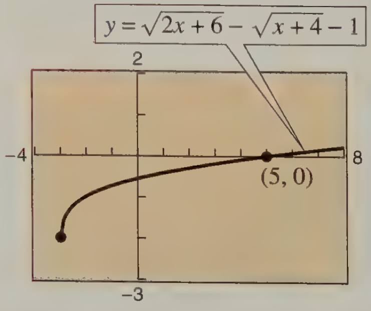


Figure 2.34

EXAMPLE 11 Solving an Equation with Rational Exponents

Solve $(x + 1)^{2/3} = 4$.

Algebraic Solution

$(x + 1)^{2/3} = 4$ Write original equation.

$\sqrt[3]{(x + 1)^2} = 4$ Rewrite with radical sign.

$(x + 1)^2 = 64$ Cube each side.

$x + 1 = \pm 8$ Take square root of each side.

$x = -9, x = 7$ Subtract 1 from each side.

Substitute $x = -9$ and $x = 7$ into the original equation to determine that both are valid solutions.

Graphical Solution

Use a graphing utility to graph $y_1 = \sqrt[3]{(x + 1)^2}$ and $y_2 = 4$ in the same viewing window, as shown in Figure 2.35. Use the *intersect* feature of the graphing utility to approximate the solutions to be $x = -9$ and $x = 7$.

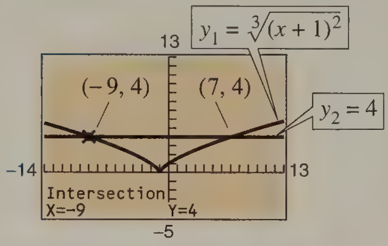


Figure 2.35

Equations Involving Fractions or Absolute Values

As demonstrated in Section 2.1, you can algebraically solve an equation involving fractions by multiplying both sides of the equation by the least common denominator of each term in the equation to "clear an equation of fractions."

EXAMPLE 12 Solving an Equation Involving Fractions

Solve $\dfrac{2}{x} = \dfrac{3}{x - 2} - 1$.

Solution

For this equation, the least common denominator of the three terms is $x(x - 2)$, so you can begin by multiplying each term in the equation by this expression.

$$\frac{2}{x} = \frac{3}{x - 2} - 1$$

$$x(x - 2)\frac{2}{x} = x(x - 2)\frac{3}{x - 2} - x(x - 2)(1)$$

$$2(x - 2) = 3x - x(x - 2), \qquad x \neq 0, 2$$

$$x^2 - 3x - 4 = 0$$

$$(x - 4)(x + 1) = 0$$

$$x - 4 = 0 \quad \Rightarrow \quad x = 4$$

$$x + 1 = 0 \quad \Rightarrow \quad x = -1$$

The equation has two solutions: 4 and −1. Check these solutions in the original equation. Use a graphing utility to verify these solutions graphically.

Exploration

Using dot mode, graph the two functions

$$y_1 = \frac{2}{x}$$

$$y_2 = \frac{3}{x - 2} - 1$$

in the same viewing window. How many times do the graphs of the functions intersect? What does this tell you about the solution to Example 12?

STUDY T!P

Graphs of functions involving variable denominators can be tricky because of the way graphing utilities skip over points where the denominator is zero. You will study graphs of such functions in Sections 3.5 and 3.6.

EXAMPLE 13 Solving an Equation Involving Absolute Value

Solve $|x^2 - 3x| = -4x + 6$.

Solution

Begin by writing the equation as $|x^2 - 3x| + 4x - 6 = 0$. From the graph of $y = |x^2 - 3x| + 4x - 6$ in Figure 2.36, you can estimate the solutions to be -3 and 1. These can be verified by substitution into the equation. To solve *algebraically* an equation involving an absolute value, you must consider the fact that the expression inside the absolute value symbols can be positive or negative. This consideration results in *two* separate equations, each of which must be solved.

First Equation:

$$x^2 - 3x = -4x + 6 \qquad \text{Use positive expression.}$$

$$x^2 + x - 6 = 0 \qquad \text{Write in general form.}$$

$$(x + 3)(x - 2) = 0 \qquad \text{Factor.}$$

$$x + 3 = 0 \Rightarrow x = -3 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 2 = 0 \Rightarrow x = 2 \qquad \text{Set 2nd factor equal to 0.}$$

Second Equation:

$$-(x^2 - 3x) = -4x + 6 \qquad \text{Use negative expression.}$$

$$x^2 - 7x + 6 = 0 \qquad \text{Write in general form.}$$

$$(x - 1)(x - 6) = 0 \qquad \text{Factor.}$$

$$x - 1 = 0 \Rightarrow x = 1 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 6 = 0 \Rightarrow x = 6 \qquad \text{Set 2nd factor equal to 0.}$$

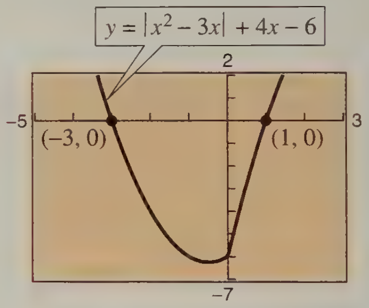


Figure 2.36

Check

$$|(-3)^2 - 3(-3)| \stackrel{?}{=} -4(-3) + 6 \qquad \text{Substitute } -3 \text{ for } x.$$

$$18 = 18 \qquad -3 \text{ checks. } \checkmark$$

$$|2^2 - 3(2)| \stackrel{?}{=} -4(2) + 6 \qquad \text{Substitute 2 for } x.$$

$$2 \neq -2 \qquad \text{2 does not check.}$$

$$|1^2 - 3(1)| \stackrel{?}{=} -4(1) + 6 \qquad \text{Substitute 1 for } x.$$

$$2 = 2 \qquad \text{1 checks. } \checkmark$$

$$|6^2 - 3(6)| \stackrel{?}{=} -4(6) + 6 \qquad \text{Substitute 6 for } x.$$

$$18 \neq -18 \qquad \text{6 does not check.}$$

The equation has only two solutions: -3 and 1, just as you obtained by graphing.

In Figure 2.36, the graph of $y = |x^2 - 3x| + 4x - 6$ appears to be a straight line to the right of the y-axis. Is it? Explain how you decided.

Applications

A common application of quadratic equations involves an object that is falling (or projected into the air). The general equation that gives the height of such an object is called a **position equation,** and on *earth's* surface it has the form

$$s = -16t^2 + v_0t + s_0.$$

In this equation, s represents the height of the object (in feet), v_0 represents the original velocity of the object (in feet per second), s_0 represents the original height of the object (in feet), and t represents the time (in seconds). Note that this position equation ignores air resistance.

STUDY T!P

Note in the position equation

$$s = -16t^2 + v_0t + s_0$$

that the initial velocity v_0 is positive when an object is rising and negative when an object is falling.

EXAMPLE 14 Falling Time

A construction worker on the 24th floor of a building project (see Figure 2.37) accidentally drops a wrench and yells, "Look out below!" Could a person at ground level hear this warning in time to get out of the way?

Solution

Assume that each floor of the building is 10 feet high, so that the wrench is dropped from a height of 240 feet. Because sound travels at about 1100 feet per second, it follows that a person at ground level hears the warning within 1 second of the time the wrench is dropped. To set up a mathematical model for the height of the wrench, use the position equation

$$s = -16t^2 + v_0t + s_0. \qquad \text{Position equation}$$

Because the object is dropped rather than thrown, the initial velocity is $v_0 = 0$. So, with an initial height of $s_0 = 240$ feet, you have the following model.

$$s = -16t^2 + 240$$

After falling for 1 second, the height of the wrench is

$$-16(1)^2 + 240 = 224.$$

After falling for 2 seconds, the height of the wrench is

$$-16(2)^2 + 240 = 176.$$

To find the number of seconds it takes the wrench to hit the ground, let the height s be zero and solve the equation for t.

$$s = -16t^2 + 240 \qquad \text{Position equation}$$

$$0 = -16t^2 + 240 \qquad \text{Substitute 0 for } s.$$

$$16t^2 = 240 \qquad \text{Add } 16t^2 \text{ to each side.}$$

$$t^2 = 15 \qquad \text{Divide each side by 16.}$$

$$t = \sqrt{15} \approx 3.87 \qquad \text{Extract positive square root.}$$

The wrench will take about 3.87 seconds to hit the ground. If the person hears the warning 1 second after the wrench is dropped, the person still has almost 3 more seconds to get out of the way.

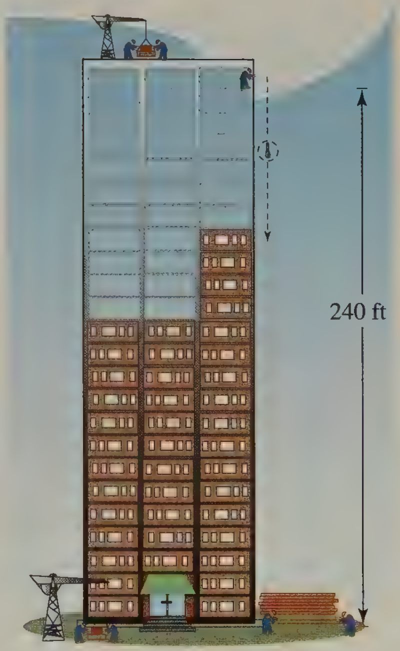


Figure 2.37

EXAMPLE 15 Quadratic Modeling: Internet Use

From 1992 to 2000, the number of hours spent annually per person using the Internet in the United States closely followed the quadratic model

$$\text{Hours} = 0.39t^2 + 0.26t - 1.12$$

where $t = 2$ represents 1992. The number of hours per year is shown graphically in Figure 2.38. According to this model, in which year will the number of hours spent per person reach or surpass 100? (Source: Veronis, Suhler, & Associates)

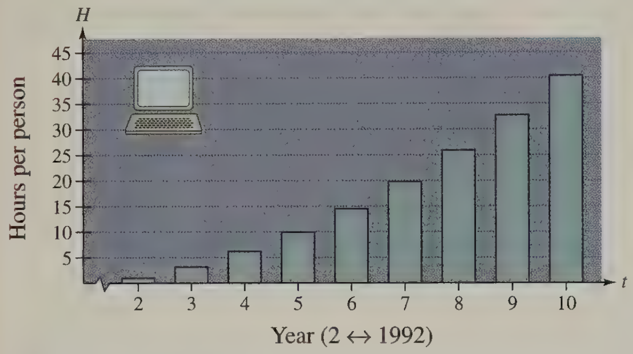


Figure 2.38

Solution

To find when the number of hours spent per person will reach 100, you need to solve the equation

$$0.39t^2 + 0.26t - 1.12 = 100$$

To begin, write the equation in general form.

$$0.39t^2 + 0.26t - 101.12 = 0$$

Then apply the Quadratic Formula.

$$t = \frac{-0.26 \pm \sqrt{(0.26)^2 - 4(0.39)(-101.12)}}{2(0.39)}$$

Because time is nonnegative, choose the positive solution

$$t = \frac{-0.26 + \sqrt{(0.26)^2 - 4(0.39)(-101.12)}}{2(0.39)} \approx 15.8.$$

Because $t = 2$ corresponds to 1992, it follows that $t = 15.8$ must correspond to some time in 2005. So, the number of hours spent annually per person using the Internet should reach 100 during 2005.

You can solve Example 15 with your graphing utility by graphing the two functions

$$y_1 = 0.39t^2 + 0.26t - 1.12$$

$$y_2 = 100$$

in the same viewing window and finding their point of intersection. You should obtain $x \approx 15.8$, which verifies the answer obtained analytically.

Another type of application that often involves a quadratic equation is one dealing with the hypotenuse of a right triangle. These types of applications often use the Pythagorean Theorem, which states that

$$a^2 + b^2 = c^2 \qquad \text{Pythagorean Theorem}$$

where a and b are the legs of a right triangle and c is the hypotenuse, as indicated in Figure 2.39.

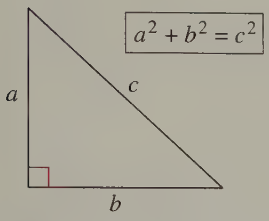


Figure 2.39

EXAMPLE 16 Cutting Across the Lawn

Your house is on a large corner lot. Several of the children in the neighborhood cut across your lawn, as shown in Figure 2.40. The distance across the lawn is 32 feet. How many feet does a person save by walking across the lawn instead of walking on the sidewalk?

Solution

In Figure 2.40, let x represent the length of the shorter part of the sidewalk. Using a ruler, you can find that the length of the longer part of the sidewalk is twice the shorter, so you can represent its length by $2x$. Now, using the Pythagorean Theorem, you have

$$a^2 + b^2 = c^2 \qquad \text{Pythagorean Theorem}$$

$$x^2 + (2x)^2 = 32^2 \qquad \text{Substitute for } a, b, \text{ and } c.$$

$$5x^2 = 1024 \qquad \text{Combine like terms.}$$

$$x^2 = 204.8 \qquad \text{Divide each side by 5.}$$

$$x = \pm\sqrt{204.8} \qquad \text{Take square root of each side.}$$

Choose the positive square root. The total distance on the sidewalk is

$$x + 2x = 3x = 3\sqrt{204.8} \approx 42.9 \text{ feet.}$$

Cutting across the lawn saves a person about 42.9 − 32 or 10.9 feet.

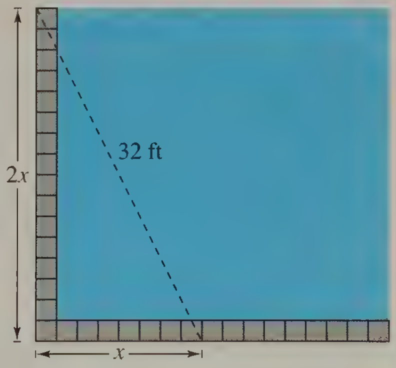


Figure 2.40

Writing About Math *Investigating Intercepts*

On a graphing utility, store the value 5 in A, -2 in B, and 1 in C. Use a graphing utility to sketch the graph of $y = C(x - A)(x - B)$. Explain how the values of A and B can be determined from the graph. Now store any other nonzero value in C. Does the value of C affect the x-intercepts of the graph? Explain. Find values of A, B, and C such that the graph opens downward and has x-intercepts at $(-5, 0)$ and $(0, 0)$. Summarize your findings.

2.4 Exercises

In Exercises 1–4, write the quadratic equation in general form.

1. $2x^2 = 3 - 5x$

2. $x^2 = 25x$

3. $\frac{1}{5}(3x^2 - 10) = 12x$

4. $x(x + 2) = 3x^2 + 1$

In Exercises 5–14, solve the quadratic equation by factoring. Check your solutions in the original equation.

5. $6x^2 + 3x = 0$

6. $9x^2 - 1 = 0$

7. $x^2 - 2x - 8 = 0$

8. $x^2 - 10x + 9 = 0$

9. $3 + 5x - 2x^2 = 0$

10. $2x^2 = 19x + 33$

11. $x^2 + 4x = 12$

12. $-x^2 + 8x = 12$

13. $(x + a)^2 - b^2 = 0$

14. $x^2 + 2ax + a^2 = 0$

In Exercises 15–22, solve the equation by extracting square roots. List both the exact solution and the decimal solution rounded to two decimal places.

15. $x^2 = 7$

16. $9x^2 = 25$

17. $(x - 12)^2 = 18$

18. $(x - 5)^2 = 20$

19. $(2x - 1)^2 = 18$

20. $(4x + 7)^2 = 44$

21. $(x - 7)^2 = (x + 3)^2$

22. $(x + 5)^2 = (x + 4)^2$

In Exercises 23–30, solve the quadratic equation by completing the square. Verify your answer graphically.

23. $x^2 + 4x - 32 = 0$

24. $x^2 - 2x - 3 = 0$

25. $x^2 + 6x + 2 = 0$

26. $x^2 + 8x + 14 = 0$

27. $9x^2 - 18x + 3 = 0$

28. $9x^2 - 12x - 14 = 0$

29. $8 + 4x - x^2 = 0$

30. $4x^2 - 4x - 99 = 0$

Graphical Reasoning **In Exercises 31–38, (a) use a graphing utility to graph the equation, (b) use the graph to approximate any x-intercepts, (c) set $y = 0$ and solve the resulting equation, and (d) compare the result with the x-intercepts of the graph.**

31. $y = (x + 3)^2 - 4$

32. $y = 1 - (x - 2)^2$

33. $y = -4x^2 + 4x + 3$

34. $y = x^2 + 3x - 4$

35. $y = \frac{1}{4}(4x^2 - 20x + 25)$

36. $y = -(x^2 - 4x + 3)$

37. $y = -(x^2 - 4x + 5)$

38. $y = \frac{1}{4}(x^2 - 2x + 9)$

In Exercises 39–44, use a graphing utility to determine the number of real solutions of the quadratic equation.

39. $2x^2 - 5x + 5 = 0$

40. $2x^2 - x - 1 = 0$

41. $\frac{1}{5}x^2 + \frac{6}{5}x - 8 = 0$

42. $\frac{1}{3}x^2 - 5x + 25 = 0$

43. $-4.1x^2 + 2.9x - 1.5 = 0$

44. $9 + 2.4x - 8.3x^2$

In Exercises 45–52, use the Quadratic Formula to solve the equation. Use a graphing utility to verify your solutions graphically.

45. $2 + 2x - x^2 = 0$

46. $x^2 - 10x + 22 = 0$

47. $x^2 + 8x - 4 = 0$

48. $4x^2 - 4x - 4 = 0$

49. $28x - 49x^2 = 4$

50. $9x^2 + 24x + 16 = 0$

51. $4x^2 + 16x + 15 = 0$

52. $9x^2 - 6x - 35 = 0$

In Exercises 53–60, solve the equation by any convenient method.

53. $x^2 - 2x - 1 = 0$

54. $11x^2 + 33x = 0$

55 $(x + 3)^2 = 81$

56. $x^2 - 14x + 49 = 0$

57. $x^2 - x - \frac{11}{4} = 0$

58. $x^2 + 3x - \frac{3}{4} = 0$

59. $(x + 1)^2 = x^2$

60. $a^2x^2 - b^2 = 0, a \neq 0$

In Exercises 61–76, find all solutions of the equation. Use a graphing utility to verify the solutions graphically.

61. $4x^4 - 18x^2 = 0$

62. $20x^3 - 125x = 0$

63. $x^4 - 81 = 0$

64. $x^6 - 64 = 0$

65. $5x^3 + 30x^2 + 45x = 0$

66. $9x^4 - 24x^3 + 16x^2 = 0$

67. $x^3 - 3x^2 - x + 3 = 0$

68. $x^4 + 2x^3 - 8x - 16 = 0$

69. $x^4 - 4x^2 + 3 = 0$

70. $x^4 + 5x^2 - 36 = 0$

71. $4x^4 - 65x^2 + 16 = 0$

72. $36t^4 + 29t^2 - 7 = 0$

73. $\frac{1}{t^2} + \frac{8}{t} + 15 = 0$

74. $6\left(\frac{s}{s + 1}\right)^2 + 5\left(\frac{s}{s + 1}\right) - 6 = 0$

75. $2x + 9\sqrt{x} - 5 = 0$

76. $3x^{1/3} + 2x^{2/3} = 5$

Graphical Analysis **In Exercises 77–80, (a) use a graphing utility to graph the equation, (b) use the graph to approximate any x-intercepts of the graph, (c) set $y = 0$ and solve the resulting equation, and (d) compare the result with the x-intercepts of the graph.**

77. $y = x^3 - 2x^2 - 3x$

78. $y = 2x^4 - 15x^3 + 18x^2$

79. $y = x^4 - 10x^2 + 9$

80. $y = x^4 - 29x^2 + 100$

In Exercises 81–90, find all solutions of the equation algebraically. Check your solutions both algebraically and graphically.

81. $\sqrt{x - 10} - 4 = 0$

82. $\sqrt[3]{2x + 5} + 3 = 0$

83. $\sqrt{x + 1} - 3x = 1$

84. $\sqrt{x + 5} = \sqrt{x - 5}$

85. $\sqrt{x} - \sqrt{x - 5} = 1$

86. $\sqrt{x} + \sqrt{x - 20} = 10$

87. $(x - 5)^{2/3} = 16$

88. $(x^2 - x - 22)^{4/3} = 16$

89. $3x(x - 1)^{1/2} + 2(x - 1)^{3/2} = 0$

90. $4x^2(x - 1)^{1/3} + 6x(x - 1)^{4/3} = 0$

Graphical Analysis **In Exercises 91–94, (a) use a graphing utility to graph the equation, (b) use the graph to approximate any x-intercepts of the graph, (c) set $y = 0$ and solve the resulting equation, and (d) compare the result with the x-intercepts of the graph.**

91. $y = \sqrt{11x - 30} - x$

92. $y = 2x - \sqrt{15 - 4x}$

93. $y = \sqrt{7x + 36} - \sqrt{5x + 16} - 2$

94. $y = 3\sqrt{x} - \dfrac{4}{\sqrt{x}} - 4$

In Exercises 95–104, find all solutions of the equation. Use a graphing utility to verify your solutions graphically.

95. $\dfrac{20 - x}{x} = x$

96. $\dfrac{4}{x} - \dfrac{5}{3} = \dfrac{x}{6}$

97. $\dfrac{1}{x} - \dfrac{1}{x + 1} = 3$

98. $\dfrac{x}{x^2 - 4} + \dfrac{1}{x + 2} = 3$

99. $x = \dfrac{3}{x} + \dfrac{1}{2}$

100. $4x + 1 = \dfrac{3}{x}$

101. $|2x - 1| = 5$

102. $|3x + 2| = 7$

103. $|x| = x^2 + x - 3$

104. $|x - 10| = x^2 - 10x$

Graphical Analysis **In Exercises 105–108, (a) use a graphing utility to graph the equation, (b) use the graph to approximate any x-intercepts of the graph, (c) set $y = 0$ and solve the resulting equation, and (d) compare the result with the x-intercepts of the graph.**

105. $y = \dfrac{1}{x} - \dfrac{4}{x - 1} - 1$

106. $y = x + \dfrac{9}{x + 1} - 5$

107. $y = |x + 1| - 2$

108. $y = |x - 2| - 3$

In Exercises 109–116, find an equation having the given solutions. (There are many correct answers.)

109. $-6, 5$

110. $0, 4, 7$

111. $-\frac{7}{3}, \frac{6}{7}$

112. $-\frac{1}{8}, -\frac{4}{5}$

113. $\sqrt{2}, -\sqrt{2}, 4$

114. $2, \sqrt{5}, -\sqrt{5}$

115. $-2, 2, i, -i$

116. $4i, -4i, 6, -6$

Think About It **In Exercises 117 and 118, find x such that the distance between the points is 13.**

117. $(1, 2), (x, -10)$

118. $(-8, 0), (x, 5)$

In Exercises 119 and 120, solve for the variable.

119. ***Surface Area of a Cone***

Solve for h: $S = \pi r\sqrt{r^2 + h^2}$

120. ***Inductance***

Solve for Q: $i = \pm\sqrt{\dfrac{1}{LC}}\sqrt{Q^2 - q}$

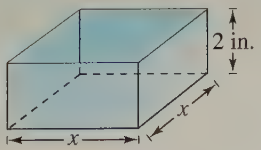
121. ***Floor Space*** The floor of a one-story building is 14 feet longer than it is wide. The building has 1632 square feet of floor space.

(a) Draw a diagram to represent the floor space. Represent the width as w and show the length in terms of w.

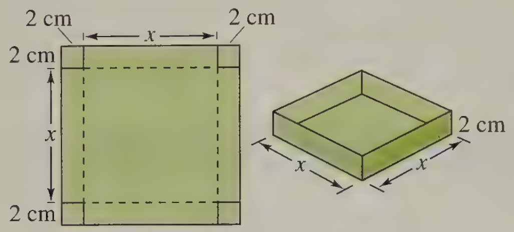
(b) Write a quadratic equation in terms of w.

(c) Find the length and width of the building floor.

122. ***Packaging*** An open box with a square base is to be constructed from 84 square inches of material. What should the dimensions of the base be if the height of the box is to be 2 inches? (*Hint:* The surface area is $S = x^2 + 4xh$.)



123. ***Packaging*** An open box is to be made from a square piece of material by cutting 2-centimeter squares from each corner and turning up the sides. The volume of the finished box is to be 200 cubic centimeters. Find the size of the original piece of material.



124. ***Exploration*** A rancher has 100 meters of fencing to enclose two adjacent rectangular corrals as shown in the figure.

(a) Write the area of the enclosed region as a function of x.

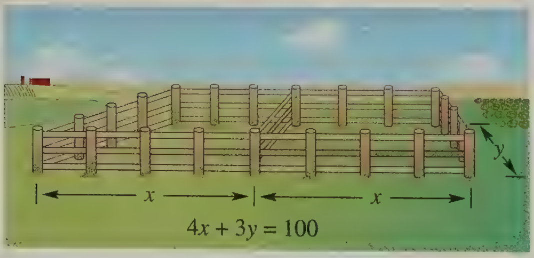
(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the dimensions that will produce a maximum area.

x	y	Area
2	$\frac{92}{3}$	$\frac{368}{3} \approx 123$
4	28	224

(c) Use a graphing utility to graph the area function, and use the graph to estimate the dimensions that will produce a maximum area.

(d) Use the graph to approximate the dimensions such that the enclosed area will be 350 square meters.

(e) Find the required dimensions of part (d) algebraically.



In Exercises 125–128, use the position equation on page 197 as the model for the problem.

125. ***CN Tower*** At 1821 feet tall, the CN Tower in Toronto, Ontario, is the world's tallest self-supporting structure. Suppose an object were dropped from the top of the tower.

(a) Find the position equation

$$s = -16t^2 + v_0 t + s_0.$$

(b) Complete the table.

t	0	2	4	6	8	10
s						

(c) From the table of part (b) you know the time until the object reaches ground level is greater than how many seconds? Find the time algebraically.

126. ***Warfare*** A cargo plane flying at 8000 feet over level terrain drops a 500-pound supply package.

(a) How long will it take until the package strikes the ground?

(b) If the plane is flying at 600 miles per hour, how far will the package travel horizontally during its descent?

127. ***World Trade Center*** Suppose you drop a penny from the top of One World Trade Center in New York City. The building has a height of 1368 feet.

(a) Use the position equation to write a mathematical model for the height of the penny.

(b) Find the height of the penny after 4 seconds.

(c) How long will it take before the penny strikes the ground?

(d) Use a graphing utility with the appropriate viewing window to verify your answer in part (c).

128. ***Baseball*** You throw a baseball straight up into the air at a velocity of 45 feet per second. You release the baseball at a height of 5.5 feet and are going to catch it when it falls back to a height of 6 feet.

(a) Use the position equation to write a mathematical model for the height of the baseball.

(b) Find the height of the baseball after 0.5 seconds.

(c) How many seconds will the baseball be in the air?

(d) Use a graphing utility to verify your answer in part (c).

129. ***Airline Traffic*** The total number y of annual aircraft departures in the United States from 1990 through 1996 can be approximated by the mathematical model

$$y = 0.036t^2 + 0.032t + 6.839, \quad 0 \le t \le 6$$

where y is in millions and t is the year, with $t = 0$ corresponding to 1990. (Source: Air Transport Association of America)

(a) Determine algebraically when the number of departures reached 8 million.

(b) Verify your answer to part (a) by creating a table of values for the model.

(c) Use a graphing utility to graph the model.

(d) Use the *zoom* and *trace* features of a graphing utility to find the year in which the total number of departures reached 7.0 million.

(e) Verify your answer to part (d) algebraically.

130. ***Boating*** The total number of dollars spent on boating in the United States from 1991 through 1996 can be approximated by the model

$$S = 0.23t^2 + 0.088t + 9.75, \quad 1 \le t \le 6$$

where S is the spending (in billions of dollars) and t is the time, with $t = 0$ corresponding to 1990. (Source: National Marine Manufacturers Association)

(a) Use a graphing utility to graph the model.

(b) Extend the model past 1996. Does the model predict that sales ever exceed 25 billion dollars? If so, estimate the year.

131. ***Biology*** The metabolic rate of ectothermic organisms increases with increasing temperature within a certain range. Experimental data for oxygen consumption (microliters per gram per hour) of a beetle for certain temperatures yielded the model

$$C = 0.45x^2 - 1.65x + 50.75, \quad 10 \le x \le 25$$

where x is the air temperature in degrees Celsius.

(a) Use a graphing utility to graph the consumption function over the specified domain.

(b) Use the graph to approximate the air temperature resulting in oxygen consumption of 150 microliters per gram per hour.

(c) If the temperature is increased from 10 to 20 degrees, the oxygen consumption is increased by approximately what factor?

132. ***Fuel Efficiency*** The distance d (in miles) a car can travel on one tank of fuel is approximated by $d = -0.024s^2 + 1.455s + 431.5$ for $0 < s \le 75$, where s is the average speed of the car in mph.

(a) Use a graphing utility to graph the distance function over the specified domain.

(b) Use the graph to determine the greatest distance that can be traveled on a tank of fuel. How long will the trip take?

(c) Determine the greatest distance that can be traveled in this car in 8 hours with no refueling. How fast should the car be driven? [*Hint:* The distance traveled in 8 hours is $8s$. Graph this expression in the same viewing window as the graph in part (a) and approximate the point of intersection.]

133. ***Saturated Steam*** The temperature T (in degrees Fahrenheit) of saturated steam increases as pressure increases. This relationship is approximated by

$$T = 75.82 - 2.11x + 43.51\sqrt{x}, \quad 5 \le x \le 40$$

where x is the absolute pressure in pounds per square inch.

(a) Use a graphing utility to graph the temperature function over the specified domain.

(b) The temperature of steam at sea level ($x = 14.696$) is 212°F. Evaluate the model at this pressure and verify the result graphically.

(c) Use the model to approximate the pressure for a steam temperature of 240°F.

134. ***Exploration*** A meteorologist is positioned 100 feet from the point where a weather balloon is launched. When the balloon is at height h, the distance d between the meteorologist and the balloon is $d = \sqrt{100^2 + h^2}$.

(a) Use a graphing utility to graph the equation. Use the trace feature to approximate the value of h when $d = 200$.

(b) Complete the table. Use the table to approximate the value of h when $d = 200$.

h	160	165	170	175	180	185
d						

(c) Find h algebraically when $d = 200$.

(d) Compare the results of each method. In each case, what information did you gain that wasn't revealed by another solution method?

135. ***Economics*** The demand equation for a certain product is $p = 20 - 0.0002x$, where p is the price per unit and x is the number of units sold. The total revenue for selling x units is

$$\text{Revenue} = xp = x(20 - 0.0002x).$$

How many units must be sold to produce a revenue of $500,000?

136. ***Economics*** The demand equation for a certain product is $p = 60 - 0.0004x$, where p is the price per unit and x is the number of units sold. The total revenue for selling x units is

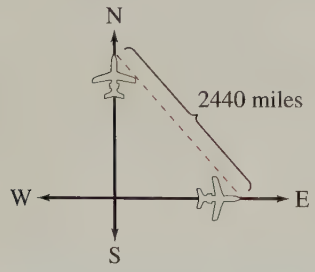
$$\text{Revenue} = xp = x(60 - 0.0004x).$$

How many units must be sold to produce a revenue of $220,000?

137. ***Geometry*** The hypotenuse of an isosceles right triangle is 5 centimeters long. How long are its right sides?

138. ***Geometry*** An equilateral triangle has a height of 10 inches. How long are each of its sides? (*Hint:* Use the height of the triangle to partition the triangle into two congruent right triangles.)

139. ***Flying Speed*** Two planes leave simultaneously from the same airport, one flying due north and the other due east. The northbound plane is flying 50 miles per hour faster than the eastbound plane. After 3 hours the planes are 2440 miles apart. Find the speed of each plane.



140. ***Flying Distance*** A small commuter airline flies to three cities whose locations form the vertices of a right triangle. The total flight distance (from City A to City B to City C and back to City A) is 1400 kilometers. It is 600 kilometers between the two cities that are farthest apart. Approximate the other two distances between cities.

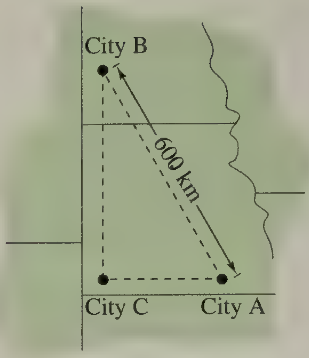


FIGURE FOR 140

Synthesis

True or False? **In Exercises 141–143, determine whether the statement is true or false. Justify your answer.**

141. The quadratic equation $-3x^2 - x = 10$ has two real solutions.

142. If $(2x - 3)(x + 5) = 8$, then $2x - 3 = 8$ or $x + 5 = 8$.

143. An equation can never have more than one extraneous solution.

144. ***Exploration*** Solve the equation $3(x + 4)^2 + (x + 4) - 2 = 0$ in two ways.

(a) Let $u = x + 4$, and solve the resulting equation for u. Then find the corresponding values of x that are solutions of the original equation.

(b) Expand and collect like terms in the original equation, and solve the resulting equation for x.

(c) Which method is easier? Explain.

145. ***Exploration*** Given that a and b are nonzero real numbers, determine the solutions of the equations.

(a) $ax^2 + bx = 0$ (b) $ax^2 - ax = 0$

Review

In Exercises 146–149, completely factor the expression over the real numbers.

146. $x^5 - 27x^2$

147. $x^3 - 5x^2 - 14x$

148. $x^3 + 5x^2 - 2x - 10$

149. $5(x + 5)x^{1/3} + 4x^{4/3}$

In Exercises 150–155, determine whether y is a function of x.

150. $5x + 8y = -1$

151. $-x^2 + y^2 = 2$

152. $x + y^2 = 10$

153. $-2y = \sqrt{x + 6}$

154. $y = |x - 3|$

155. $|y| = 1 - x$

2.5 Solving Inequalities Algebraically and Graphically

Properties of Inequalities

Simple inequalities were reviewed in Section P.1. There, the inequality symbols $<$, $\leq$, $>$, and $\geq$ were used to compare two numbers and to denote subsets of real numbers. For instance, the simple inequality $x \geq 3$ denotes all real numbers x that are greater than or equal to 3. In this section you will study inequalities that contain more involved statements such as

$$5x - 7 > 3x + 9 \quad \text{and} \quad -3 \leq 6x - 1 < 3.$$

As with an equation, you **solve an inequality** in the variable x by finding all values of x for which the inequality is true. These values are **solutions** of the inequality and are said to **satisfy** the inequality. For instance, the number 9 is a solution of the first inequality listed above because

$$5(9) - 7 > 3(9) + 9$$
$$38 > 36.$$

On the other hand, the number 7 is not a solution because

$$5(7) - 7 \ngtr 3(7) + 9$$
$$28 \ngtr 30.$$

The set of all real numbers that are solutions of an inequality is the **solution set** of the inequality.

The set of all points on the real number line that represent the solution set is the **graph of the inequality.** Graphs of many types of inequalities consist of intervals on the real number line.

The procedures for solving linear inequalities in one variable are much like those for solving linear equations. To isolate the variable you can make use of the **properties of inequalities.** These properties are similar to the properties of equality, but there are two important exceptions. When both sides of an inequality are multiplied or divided by a negative number, *the direction of the inequality symbol must be reversed.* Here is an example.

$$-2 < 5 \qquad \text{Write original inequality.}$$
$$(-3)(-2) > (-3)(5) \qquad \text{Multiply each side by } -3 \text{ and reverse inequality.}$$
$$6 > -15 \qquad \text{New inequality}$$

Two inequalities that have the same solution set are **equivalent inequalities.** The properties listed at the top of the next page describe operations that can be used to create equivalent inequalities.

What You Should Learn:

- How to use properties of inequalities to solve linear inequalities
- How to solve inequalities involving absolute values
- How to solve polynomial inequalities
- How to solve rational inequalities
- How to use inequalities to model and solve real-life problems

Why You Should Learn It:

An inequality can be used to describe the time when a level of real-life quantity is exceeded. For instance, Exercise 70 on page 215 shows how to use a quadratic inequality to determine when the total number of education degrees conferred in the United States exceeds 2.5 million.

Cliff Hollis/Liaison International

Properties of Inequalities

Let a, b, c, and d be real numbers.

1. *Transitive Property*

   $a < b$ and $b < c \quad \Rightarrow \quad a < c$

2. *Addition of Inequalities*

   $a < b$ and $c < d \quad \Rightarrow \quad a + c < b + d$

3. *Addition of a Constant*

   $a < b \quad \Rightarrow \quad a + c < b + c$

4. *Multiplying by a Constant*

   For $c > 0$, $a < b \quad \Rightarrow \quad ac < bc$

   For $c < 0$, $a < b \quad \Rightarrow \quad ac > bc$

Each of the properties above is true if the symbol $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$. For instance, another form of Property 3 would be as follows.

$$a \leq b \quad \Rightarrow \quad a + c \leq b + c$$

Solving a Linear Inequality

The simplest type of inequality to solve is a **linear inequality** in a single variable, such as $2x + 3 > 4$. (See Appendix C for help with solving one-step linear inequalities.)

EXAMPLE 1 Solving a Linear Inequality

Solve the inequality

$$5x - 7 > 3x + 9.$$

Solution

$5x - 7 > 3x + 9$	Write original inequality.
$5x > 3x + 16$	Add 7 to each side.
$5x - 3x > 16$	Subtract $3x$ from each side.
$2x > 16$	Combine like terms.
$x > 8$	Divide each side by 2.

So, the solution set consists of all real numbers that are greater than 8. The interval notation for this solution set is $(8, \infty)$. The number line graph of this solution set is shown in Figure 2.41.

Note that the five inequalities forming the solution steps of Example 1 are all *equivalent* in the sense that each has the same solution set.

Exploration

Use a graphing utility to graph $f(x) = 5x - 7$ and $g(x) = 3x + 9$ in the same viewing window. (Use $-1 \leq x \leq 15$ and $-5 \leq y \leq 50$.) For which values of x does the graph of f lie above the graph of g? Explain how the answer to this question can be used to solve the inequality in Example 1.

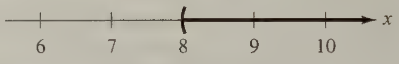


Figure 2.41 *Solution interval:* $(8, \infty)$

STUDY T!P

Checking the solution set of an inequality is not as simple as checking the solution of an equation because there are simply too many x-values to substitute into the original inequality. However, you can get an indication of the validity of the solution set by substituting a few convenient values of x. For instance, in Example 1 try substituting $x = 5$ and $x = 10$ into the original inequality.

EXAMPLE 2 Solving an Inequality

Solve $1 - \frac{3x}{2} \geq x - 4$.

Algebraic Solution

$$\begin{aligned} 1 - \frac{3x}{2} &\geq x - 4 &&\text{Write original inequality.} \\ 2 - 3x &\geq 2x - 8 &&\text{Multiply each side by the LCD.} \\ -3x &\geq 2x - 10 &&\text{Subtract 2 from each side.} \\ -5x &\geq -10 &&\text{Subtract } 2x \text{ from each side.} \\ x &\leq 2 &&\text{Divide each side by } -5 \text{ and reverse inequality.} \end{aligned}$$

The solution set consists of all real numbers that are less than or equal to 2. The interval notation for this solution set is $(-\infty, 2]$. The number line graph of this solution set is shown in Figure 2.42.

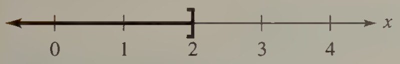


Figure 2.42 *Solution interval:* $(-\infty, 2]$

Graphical Solution

Use a graphing utility to graph $y_1 = 1 - (3x)/2$ and $y_2 = x - 4$ in the same viewing window. In Figure 2.43, you can see that the graphs appear to intersect at the point $(2, -2)$. Use the *intersect* feature of the graphing utility to confirm this. The graph of y_1 lies above the graph of y_2 to the left of their point of intersection, which implies that $y_1 \geq y_2$ for all $x \leq 2$.

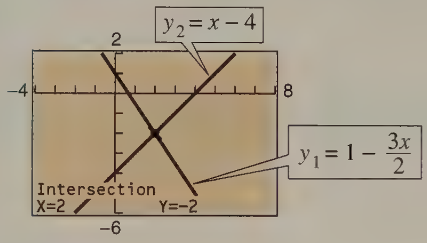


Figure 2.43

Sometimes it is possible to write two inequalities as a **double inequality.** For instance, you can write the two inequalities $-4 \leq 5x - 2$ and $5x - 2 < 7$ more simply as $-4 \leq 5x - 2 < 7$, so you can solve the two inequalities together.

EXAMPLE 3 Solving a Double Inequality

Solve $-3 \leq 6x - 1 < 3$.

Algebraic Solution

$$\begin{aligned} -3 \leq 6x - 1 &< 3 &&\text{Write original inequality.} \\ -2 \leq 6x &< 4 &&\text{Add 1 to all three parts.} \\ -\frac{1}{3} \leq x &< \frac{2}{3} &&\text{Divide by 6 and simplify.} \end{aligned}$$

The solution set consists of all real numbers that are greater than or equal to $-\frac{1}{3}$ *and* less than $\frac{2}{3}$. The interval notation for this solution set is $\left[-\frac{1}{3}, \frac{2}{3}\right)$. The number line graph of this solution set is shown in Figure 2.44.

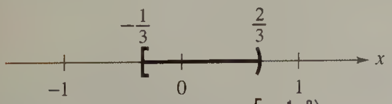


Figure 2.44 *Solution interval:* $\left[-\frac{1}{3}, \frac{2}{3}\right)$

Graphical Solution

Use a graphing utility to graph $y_1 = 6x - 1$, $y_2 = -3$, and $y_3 = 3$ in the same viewing window. In Figure 2.45, you can see that the graphs appear to intersect at the points $\left(-\frac{1}{3}, -3\right)$ and $\left(\frac{2}{3}, 3\right)$. Use the *intersect* feature of the graphing utility to confirm this. The graph of y_1 lies above the graph of y_2 to the right of $\left(-\frac{1}{3}, -3\right)$ *and* the graph of y_1 lies below the graph of y_3 to the left of $\left(\frac{2}{3}, 3\right)$. This implies that $y_2 \leq y_1 < y_3$ when $-\frac{1}{3} \leq x < \frac{2}{3}$.

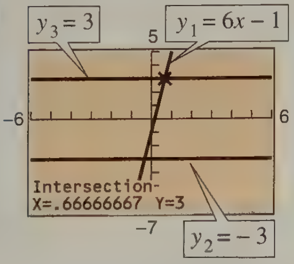


Figure 2.45

Inequalities Involving Absolute Value

Solving an Absolute Value Inequality

Let x be a variable or an algebraic expression and let a be a real number such that $a \geq 0$.

1. The solutions of $|x| < a$ are all values of x that lie between $-a$ and a.

$$|x| < a \quad \text{if and only if} \quad -a < x < a.$$

2. The solutions of $|x| > a$ are all values of x that are less than $-a$ or greater than a.

$$|x| > a \quad \text{if and only if} \quad x < -a \quad \text{or} \quad x > a.$$

These rules are also valid if $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$.

EXAMPLE 4 Solving Absolute Value Inequalities

Solve each inequality.

a. $|x - 5| < 2$ **b.** $|x - 5| > 2$

Algebraic Solution

a. $|x - 5| < 2$ Write original inequality.

$-2 < x - 5 < 2$ Equivalent inequalities

$-2 + 5 < x - 5 + 5 < 2 + 5$ Add 5 to all three parts.

$3 < x < 7$ Simplify.

The solution set is all real numbers that are greater than 3 *and* less than 7. The interval notation for this solution set is $(3, 7)$. The number line graph of this solution set is shown in Figure 2.46.

b. The absolute value inequality $|x - 5| > 2$ is equivalent to the following compound inequality.

$$x - 5 < -2 \quad or \quad x - 5 > 2$$

Solve first inequality: $x - 5 < -2$ Write first inequality.

$x < 3$ Add 5 to each side.

Solve second inequality: $x - 5 > 2$ Write second inequality.

$x > 7$ Add 5 to each side.

The solution is all real numbers that are less than -3 *or* greater than 7. The interval notation for this solution set is $(-\infty, 3) \cup (7, \infty)$. The symbol $\cup$ is called a *union* symbol and is used to denote the combining of two sets. The number line graph of this solution set is shown in Figure 2.47.

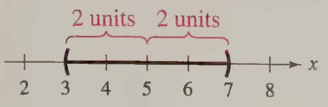


Figure 2.46 $|x - 5| < 2$

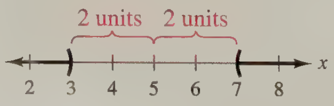


Figure 2.47 $|x - 5| > 2$

Graphical Solution

a. Use a graphing utility to graph $y_1 = |x - 5|$ and $y_2 = 2$ in the same viewing window. In Figure 2.48, you can see that the graphs appear to intersect at the points $(3, 2)$ and $(7, 2)$. Use the *intersect* feature of the graphing utility to confirm this. The graph of y_1 lies below the graph of y_2 when $3 < x < 7$. So, you can approximate the solution set to be all real numbers greater than 3 *and* less than 7.

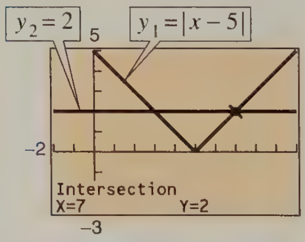


Figure 2.48

b. In Figure 2.48, you can see that the graph of y_1 lies above the graph of y_2 when $x < 3$ *or* when $x > 7$. So, you can approximate the solution set to be all real numbers that are less than 3 *or* greater than 7.

Polynomial Inequalities

To solve a polynomial inequality such as $x^2 - 2x - 3 < 0$, use the fact that a polynomial can change signs only at its zeros (the x-values that make the polynomial equal to zero). Between two consecutive zeros a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real number line into intervals in which the polynomial has no sign changes. These zeros are the **critical numbers** of the inequality, and the resulting open intervals are the **test intervals** for the inequality. For instance, the polynomial

$$x^2 - 2x - 3 = (x + 1)(x - 3)$$

has two zeros, $x = -1$ and $x = 3$, which divide the real number line into three test intervals: $(-\infty, -1)$, $(-1, 3)$, and $(3, \infty)$. To solve the inequality $x^2 - 2x - 3 < 0$, you only need to test one value from each test interval.

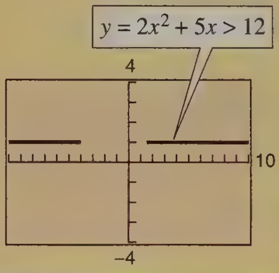
Finding Test Intervals for a Polynomial

To determine the intervals on which the values of a polynomial are entirely negative or entirely positive, use the following steps.

1. Find all real zeros of the polynomial, and arrange the zeros in increasing order. The zeros of a polynomial are its **critical numbers.**
2. Use the critical numbers of the polynomial to determine its **test intervals.**
3. Choose one representative x-value in each test interval and evaluate the polynomial at that value. If the value of the polynomial is negative, the polynomial will have negative values for *every* x-value in the interval. If the value of the polynomial is positive, the polynomial will have positive values for *every* x-value in the interval.

STUDY TIP

Some graphing utilities will produce graphs of inequalities. For instance, you can graph $2x^2 + 5x > 12$ by setting the graphing utility to *dot* mode and entering $y = 2x^2 + 5x > 12$. Using $-10 \le x \le 10$ and $-4 \le y \le 4$, your graph should look like the graph shown below. Solve the problem algebraically to verify that the solution is $(-\infty, -4) \cup \left(\frac{3}{2}, \infty\right)$.



EXAMPLE 5 Investigating Polynomial Behavior

To determine the intervals on which $x^2 - x - 6$ is entirely negative and those on which it is entirely positive, factor the quadratic as

$$x^2 - x - 6 = (x + 2)(x - 3).$$

The critical numbers occur at $x = -2$ and $x = 3$. So, the test intervals for the quadratic are $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$. In each test interval, choose a representative x-value and evaluate the polynomial, as shown in the table.

Interval	x-Value	Value of Polynomial	Sign of Polynomial
$(-\infty, -2)$	$x = -3$	$(-3)^2 - (-3) - 6 = 6$	Positive
$(-2, 3)$	$x = 0$	$(0)^2 - (0) - 6 = -6$	Negative
$(3, \infty)$	$x = 5$	$(5)^2 - (5) - 6 = 14$	Positive

The polynomial has negative values for every x in the interval $(-2, 3)$ and positive values for every x in the intervals $(-\infty, -2)$ and $(3, \infty)$. This result is shown graphically in Figure 2.49.

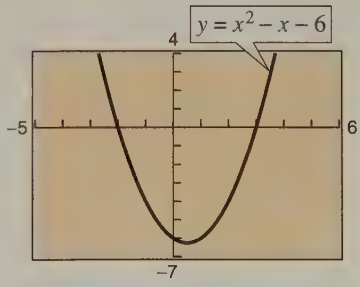


Figure 2.49

To determine the test intervals for a polynomial inequality, the inequality must first be written in general form with the polynomial on one side.

EXAMPLE 6 Solving a Polynomial Inequality

Solve $2x^2 + 5x > 12$.

Algebraic Solution

$2x^2 + 5x > 12$ Write original inequality.

$2x^2 + 5x - 12 > 0$ Write in general form.

$(x + 4)(2x - 3) > 0$ Factor.

Critical Numbers: $x = -4, x = \frac{3}{2}$

Test Intervals: $(-\infty, -4), \left(-4, \frac{3}{2}\right), \left(\frac{3}{2}, \infty\right)$

Test: Is $(x + 4)(2x - 3) > 0$?

After testing these intervals, you can see that the polynomial $2x^2 + 5x - 12$ is positive in the open intervals $(-\infty, -4)$ and $\left(\frac{3}{2}, \infty\right)$. Therefore, the solution set of the inequality is

$$(-\infty, -4) \cup \left(\tfrac{3}{2}, \infty\right).$$

Graphical Solution

First write the polynomial inequality $2x^2 + 5x > 12$ as $2x^2 + 5x - 12 > 0$. Then use a graphing utility to graph $y = 2x^2 + 5x - 12$. In Figure 2.50, you can see that the graph is *above* the x-axis when x is less than -4 *or* when x is greater than $\frac{3}{2}$. So, you can graphically approximate the solution set to be $(-\infty, -4) \cup \left(\frac{3}{2}, \infty\right)$.

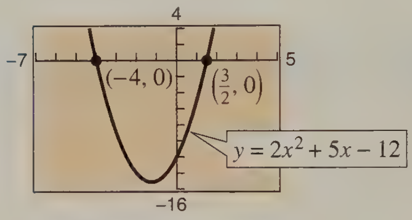


Figure 2.50

EXAMPLE 7 Solving a Polynomial Inequality

Solve $2x^3 - 3x^2 - 32x > -48$

Solution

Begin by writing the inequality in general form.

$2x^3 - 3x^2 - 32x > -48$ Write original inequality.

$2x^3 - 3x^2 - 32x + 48 > 0$ Write in general form.

$x^2(2x - 3) - 16(2x - 3) > 0$ Factor by grouping.

$(x^2 - 16)(2x - 3) > 0$ Distributive Property

$(x - 4)(x + 4)(2x - 3) > 0$ Factor difference of two squares.

The critical numbers are $x = -4, x = \frac{3}{2}$, and $x = 4$; and the test intervals are $(-\infty, -4), \left(-4, \frac{3}{2}\right), \left(\frac{3}{2}, 4\right)$, and $(4, \infty)$.

Interval	*x-Value*	*Polynomial Value*	*Conclusion*
$(-\infty, -4)$	$x = -5$	$2(-5)^3 - 3(-5)^2 - 32(-5) + 48 = -117$	Negative
$\left(-4, \frac{3}{2}\right)$	$x = 0$	$2(0)^3 - 3(0)^2 - 32(0) + 48 = 48$	Positive
$\left(\frac{3}{2}, 4\right)$	$x = 2$	$2(2)^3 - 3(2)^2 - 32(2) + 48 = -12$	Negative
$(4, \infty)$	$x = 5$	$2(5)^3 - 3(5)^2 - 32(5) + 48 = 63$	Positive

From this you can conclude that the polynomial $2x^3 - 3x^2 - 32x + 48$ is positive on the open intervals $\left(-4, \frac{3}{2}\right)$ and $(4, \infty)$. Therefore, the solution set consists of all real numbers in the intervals $\left(-4, \frac{3}{2}\right)$ and $(4, \infty)$.

STUDY TIP

When solving a quadratic inequality, be sure you have accounted for the particular type of inequality symbol given in the inequality. For instance, in Example 7, note that the original inequality contained a "greater than" symbol and the solution consisted of two open intervals. If the original inequality had been

$$2x^3 - 3x^2 + 32x \geq -48,$$

the solution would have consisted of the closed interval $\left[-4, \frac{3}{2}\right]$ and the interval $[4, \infty)$.

EXAMPLE 8 Unusual Solution Sets

a. The solution set of

$$x^2 + 2x + 4 > 0$$

consists of the entire set of real numbers, $(-\infty, \infty)$. In other words, the quadratic $x^2 + 2x + 4$ is positive for every real value of x, as indicated in Figure 2.51(a). (Note that this quadratic inequality has *no* critical numbers. In such a case, there is only one test interval—the entire real number line.)

b. The solution set of

$$x^2 + 2x + 1 \le 0$$

consists of the single real number -1, because the graph touches the x-axis only at -1, as shown in Figure 2.51(b).

c. The solution set of

$$x^2 + 3x + 5 < 0$$

is empty. In other words, the quadratic $x^2 + 3x + 5$ is not less than zero for any value of x, as indicated in Figure 2.51(c).

d. The solution set of

$$x^2 - 4x + 4 > 0$$

consists of all real numbers *except* the number 2. In interval notation, this solution set can be written as $(-\infty, 2) \cup (2, \infty)$. The graph of $x^2 - 4x + 4$ lies above the x-axis except at $x = 2$, where it touches it, as indicated in Figure 2.51(d).

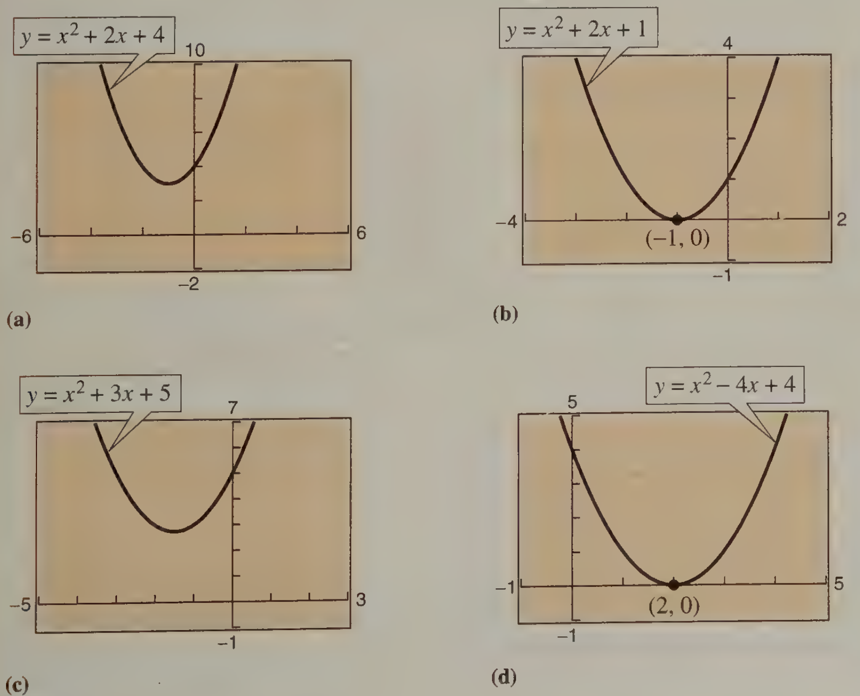


Figure 2.51

STUDY TIP

One of the advantages of technology is that you can solve complicated polynomial inequalities that might be difficult, or even impossible, to factor. For instance, you could use a graphing utility to approximate the solution to the inequality

$$x^3 - 0.26x^2 - 3.1416x + 1.414 < 0.$$

Rational Inequalities

The concepts of critical numbers and test intervals can be extended to inequalities involving rational expressions. To do this, use the fact that the value of a rational expression can change sign only at its *zeros* (the x-values for which its numerator is zero) and its *undefined values* (the x-values for which its denominator is zero). These two types of numbers make up the *critical numbers* of a rational inequality.

EXAMPLE 9 Solving a Rational Inequality

Solve $\frac{2x-7}{x-5} \le 3$.

Algebraic Solution

$$\frac{2x-7}{x-5} \le 3$$ Write original inequality.

$$\frac{2x-7}{x-5} - 3 \le 0$$ Write in standard form.

$$\frac{2x-7-3x+15}{x-5} \le 0$$ Write as single fraction.

$$\frac{-x+8}{x-5} \le 0$$ Simplify.

Now, in standard form you can see that the critical numbers are 5 and 8, and you can proceed as follows.

Critical Numbers: $x = 5, x = 8$

Test Intervals: $(-\infty, 5), (5, 8), (8, \infty)$

Test: Is $\frac{-x+8}{x-5} \le 0$?

Interval	*x-Value*	*Polynomial Value*	*Conclusion*
$(-\infty, 5)$	$x = 0$	$\frac{-0+8}{0-5} = -\frac{8}{5}$	Negative
$(5, 8)$	$x = 6$	$\frac{-6+8}{6-5} = 2$	Positive
$(8, \infty)$	$x = 9$	$\frac{-9+8}{9-5} = -\frac{1}{4}$	Negative

By testing these intervals, you can determine that the rational expression $(-x+8)/(x-5)$ is negative in the open intervals $(-\infty, 5)$ and $(8, \infty)$. Moreover, because $(-x+8)/(x-5) = 0$ when $x = 8$, you can conclude that the solution set of the inequality is $(-\infty, 5) \cup [8, \infty)$.

Graphical Solution

Use a graphing utility to graph

$$y_1 = \frac{2x-7}{x-5} \quad \text{and} \quad y_2 = 3$$

in the same viewing window. In Figure 2.52, you can see that the graphs appear to intersect at the point $(8, 3)$. Use the *intersect* feature of the graphing utility to confirm this. The graph of y_1 lies below the graph of y_2 in the intervals $(-\infty, 5)$ and $[8, \infty)$. So, you can graphically approximate the solution set to be all real numbers less than 5 *or* all real numbers greater than or equal to 8.

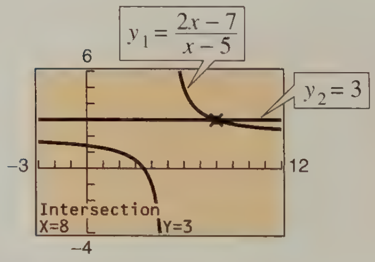


Figure 2.52

Note in Example 9 that $x = 5$ is not included in the solution set because the inequality is undefined when $x = 5$.

Applications

In Section 1.3 you studied the *implied domain* of a function, the set of all x-values for which a function is defined. A common type of implied domain is that used to avoid even roots of negative numbers, as shown in Example 10.

EXAMPLE 10 Finding the Domain of an Expression

Find the domain of

$$\sqrt{64 - 4x^2}.$$

Solution

Because $\sqrt{64 - 4x^2}$ is defined only if $64 - 4x^2$ is nonnegative, the domain is given by $64 - 4x^2 \ge 0$.

$$\begin{aligned} 64 - 4x^2 &\ge 0 && \text{Write in general form.} \\ 16 - x^2 &\ge 0 && \text{Divide each side by 4.} \\ (4 - x)(4 + x) &\ge 0 && \text{Factor.} \end{aligned}$$

The inequality has two critical numbers: -4 and 4. A test shows that $64 - 4x^2 \ge 0$ in the *closed interval* $[-4, 4]$. The graph of $y = \sqrt{64 - 4x^2}$, shown in Figure 2.53, confirms that the domain is $[-4, 4]$.

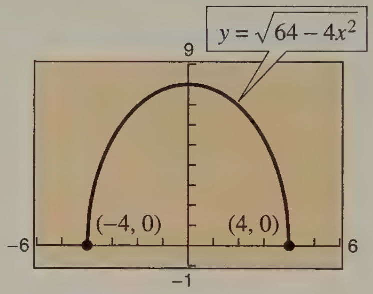


Figure 2.53

EXAMPLE 11 The Height of a Projectile

A projectile is fired straight upward from ground level with an initial velocity of 384 feet per second. During what time period will its height exceed 2000 feet?

Solution

In Section 2.4 you saw that the position of an object moving vertically can be modeled by the *position equation*

$$s = -16t^2 + v_0t + s_0,$$

where s is the height in feet and t is the time in seconds. In this case, $s_0 = 0$ and $v_0 = 384$. So, you need to solve the inequality $-16t^2 + 384t > 2000$. Using a graphing utility, graph $s = -16t^2 + 384t$ and $s = 2000$, as shown in Figure 2.54. From the graph, you can determine that $-16t^2 + 384t > 2000$ for t between approximately 7.6 and 16.4. You can verify this result algebraically as follows.

$$\begin{aligned} -16t^2 + 384t &> 2000 && \text{Write original inequality.} \\ t^2 - 24t &< -125 && \text{Divide by } -16 \text{ and reverse inequality.} \\ t^2 - 24t + 125 &< 0 && \text{Write in general form.} \end{aligned}$$

By the Quadratic Formula the critical numbers are $12 - \sqrt{19}$ and $12 + \sqrt{19}$, or approximately 7.64 and 16.36. A test will verify that the height of the projectile will exceed 2000 feet when $7.64 < t < 16.36$, that is, during the time interval $(7.64, 16.36)$ seconds.

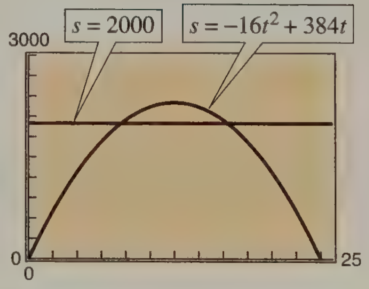
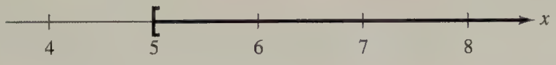
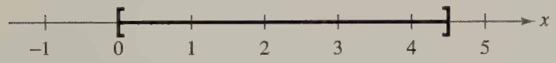


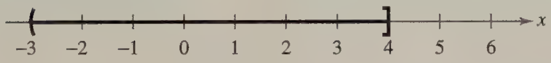
Figure 2.54

2.5 Exercises

In Exercises 1–4, match the inequality with its graph. [The graphs are labeled (a), (b), (c), and (d).]

(a)


(b)


(c)


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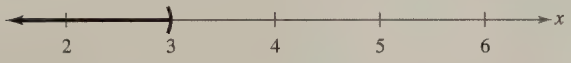


✓ **1.** $x < 3$

2. $x \geq 5$

3. $-3 < x \leq 4$

✓ **4.** $0 \leq x \leq \frac{9}{2}$

In Exercises 5–8, determine whether each given value of x is a solution of the inequality.

Inequality	*Values*	
5. $5x - 12 > 0$	(a) $x = 3$	(b) $x = -3$
	(c) $x = \frac{5}{2}$	(d) $x = \frac{3}{2}$
6. $-5 < 2x - 1 \leq 1$	(a) $x = -\frac{1}{2}$	(b) $x = -\frac{5}{2}$
	(c) $x = \frac{4}{3}$	(d) $x = 0$
✓ **7.** $-1 < \frac{3-x}{2} \leq 1$	(a) $x = 0$	(b) $x = \sqrt{5}$
	(c) $x = 1$	(d) $x = 5$
8. $\lvert x - 10\rvert \geq 3$	(a) $x = 13$	(b) $x = -1$
	(c) $x = 14$	(d) $x = 9$

In Exercises 9–18, solve the inequality and sketch the solution on the real number line. Use a graphing utility to verify your solution graphically.

9. $-10x < 40$

✓ **10.** $2x > 3$

11. $4(x + 1) < 2x + 3$

12. $2x + 7 < 3$

✓ **13.** $1 < 2x + 3 < 9$

14. $-2 < 3x + 1 < 10$

15. $-8 \leq 1 - 3(x - 2) < 13$

✓ **16.** $0 \leq 2(x + 4) < 20$

17. $-4 < \frac{2x - 3}{3} < 4$

18. $0 \leq \frac{x + 3}{2} < 5$

Graphical Analysis **In Exercises 19–24, use a graphing utility to approximate the solution.**

19. $6x > 12$

20. $3x - 1 \leq 5$

21. $5 - 2x \geq 1$

22. $3(x + 1) < x + 7$

23. $-9 < 6x - 1 < 1$

24. $-10 < 4(x - 3) \leq 8$

In Exercises 25–28, use a graphing utility to graph the equation and graphically approximate the values of x that satisfy the specified inequalities. Then solve each inequality algebraically.

Equation	*Inequalities*	
25. $y = 2x - 3$	(a) $y \geq 1$	(b) $y \leq 0$
26. $y = \frac{2}{3}x + 1$	(a) $y \leq 5$	(b) $y \geq 0$
27. $y = -\frac{1}{2}x + 2$	(a) $0 \leq y \leq 3$	(b) $y \geq 0$
28. $y = -3x + 8$	(a) $-1 \leq y \leq 3$	(b) $y \leq 0$

In Exercises 29–36, solve the inequality and sketch the solution on the real number line.

✓ **29.** $\lvert 5x\rvert > 10$

30. $\lvert x - 20\rvert \leq 4$

31. $\lvert x - 7\rvert < 6$

32. $\lvert x - 20\rvert \geq 4$

33. $\lvert x + 14\rvert + 3 > 17$

34. $\left\lvert \frac{x - 3}{2} \right\rvert \geq 5$

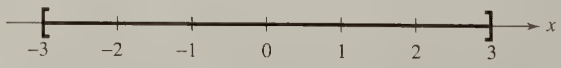
✓ **35.** $\lvert 1 - 2x\rvert < 5$

36. $3\lvert 4 - 5x\rvert \leq 9$

In Exercises 37 and 38, use a graphing utility to graph the equation and graphically approximate the values of x that satisfy the specified inequalities. Then solve each inequality algebraically.

Equation	*Inequalities*	
37. $y = \lvert x - 3\rvert$	(a) $y \leq 2$	(b) $y \geq 4$
✓ **38.** $y = \left\lvert \frac{1}{2}x + 1\right\rvert$	(a) $y \leq 4$	(b) $y \geq 1$

In Exercises 39–44, use absolute value notation to define each interval (or pair of intervals) on the real number line.

39.


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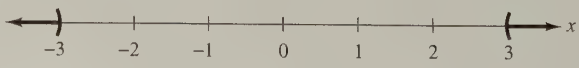

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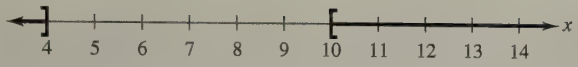
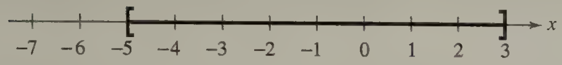


42.



43. All real numbers within 10 units of 12

44. All real numbers whose distances from −3 are more than 5

In Exercises 45–50, solve the inequality and graph the solution on the real number line. Use a graphing utility to verify your solution graphically.

45. $(x + 2)^2 < 25$

46. $(x + 6)^2 \le 8$

47. $x^2 + 4x + 4 \ge 9$

48. $x^2 - 6x + 9 < 16$

49. $x^3 - 4x \ge 0$

50. $x^4(x - 3) \le 0$

In Exercises 51–54, use a graphing utility to graph the equation and graphically approximate the values of x that satisfy the specified inequalities. Then solve each inequality algebraically.

	Equation	*Inequalities*	
51.	$y = -x^2 + 2x + 3$	(a) $y \le 0$	(b) $y \ge 3$
52.	$y = \frac{1}{2}x^2 - 2x + 1$	(a) $y \le 1$	(b) $y \ge 7$
53.	$y = \frac{1}{8}x^3 - \frac{1}{2}x$	(a) $y \ge 0$	(b) $y \le 6$
54.	$y = x^3 - x^2 - 16x + 16$	(a) $y \le 0$	(b) $y \ge 36$

In Exercises 55–58, solve the inequality and graph the solution on the real number line. Use a graphing utility to verify your solution graphically.

55. $\dfrac{1}{x} - x > 0$

56. $\dfrac{1}{x} - 4 < 0$

57. $\dfrac{x + 6}{x + 1} - 2 < 0$

58. $\dfrac{x + 12}{x + 2} - 3 \ge 0$

In Exercises 59–62, use a graphing utility to graph the equation and graphically approximate the values of x that satisfy the specified inequalities. Then solve each inequality algebraically.

	Equation	*Inequalities*	
59.	$y = \dfrac{3x}{x - 2}$	(a) $y \le 0$	(b) $y \ge 6$
60.	$y = \dfrac{2(x - 2)}{x + 1}$	(a) $y \le 0$	(b) $y \ge 8$
61.	$y = \dfrac{2x^2}{x^2 + 4}$	(a) $y \ge 1$	(b) $y \le 2$
62.	$y = \dfrac{5x}{x^2 + 4}$	(a) $y \ge 1$	(b) $y \le 0$

In Exercises 63–68, find the domain of x in the expression.

63. $\sqrt{x - 5}$

64. $\sqrt{x^2 - 4}$

65. $\sqrt[3]{6 - x}$

66. $\sqrt[3]{2x^2 - 8}$

67. $\sqrt[4]{6x + 15}$

68. $\sqrt[4]{4 - x^2}$

69. ***Data Analysis*** You want to determine whether there is a relationship between an athlete's weight x (in pounds) and the athlete's maximum bench-press weight y (in pounds). The table shows a sample of 12 athletes.

x	165	184	150	210	196	240
y	170	185	200	255	205	295

x	202	170	185	190	230	160
y	190	175	195	185	250	155

(a) Use a graphing utility to plot the data.

(b) A model for this data is $y = 1.266x - 35.766$. Use a graphing utility to graph the equation on the same display used in part (a).

(c) Use the graph to estimate the values of x that estimate a maximum bench-press weight of at least 200 pounds.

(d) Use the graph to write a statement about the accuracy of the model. If you think the graph indicates that an athlete's weight is not a particularly good indicator of the athlete's maximum bench-press weight, list other factors that may influence an individual's maximum bench-press weight.

70. ***Educational Degrees*** The number D (in thousands) of earned degrees conferred annually in the United States from 1950 to 1995 is approximated by the model

$$D = -0.0977t^2 + 47.1174t + 291.5651$$

where $t = 0$ represents 1950. (Source: U.S. National Center for Education Statistics)

(a) Use a graphing utility to graph the model.

(b) According to this model, estimate when the number of degrees will exceed 2,500,000.

71. ***Height*** The height h of two-thirds of the members of a certain population satisfies the inequality

$$\left|\frac{h - 68.5}{2.7}\right| \leq 1$$

where h is measured in inches. Determine the interval on the real number line in which these heights lie.

72. ***Meteorology*** A certain electronic device is to be operated in an environment with relative humidity h in the interval defined by

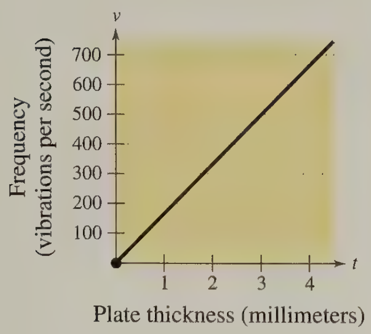
$$|h - 50| \leq 30.$$

What are the minimum and maximum relative humidities for the operation of this device?

73. ***Music*** Michael Kasha of Florida State University used physics and mathematics to design a new classical guitar. He used the model for the frequency of the vibrations on a circular plate

$$v = \frac{2.6t}{d^2}\sqrt{\frac{E}{\rho}}$$

where v is the frequency, t is the plate thickness, d is the diameter, E is the elasticity of the plate material, and ρ is the density of the plate material. For fixed values of d, E, and ρ, the graph of the equation is a line, as shown below.



(a) Estimate the frequency if the plate thickness is 2 millimeters.

(b) Estimate the plate thickness if the frequency is 600 vibrations per second.

(c) Approximate the interval for the plate thickness if the frequency is between 200 and 400 vibrations per second.

(d) Approximate the interval for the frequency if the plate thickness is less than 3 millimeters.

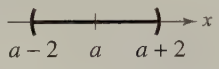
Synthesis

True or False? **In Exercises 74 and 75, determine whether the statement is true or false. Justify your answer.**

74. If a, b, and c are real numbers, and $a \leq b$, then $ac \leq bc$.

75. If $-10 \leq x \leq 8$, then $-10 \geq -x$ and $-x \geq -8$.

76. Identify the solution of the inequality $|x - a| \geq 2$.

(a)


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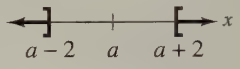
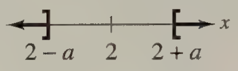


(c)


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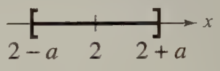
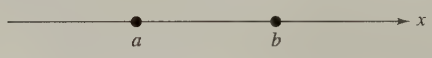


77. Consider the polynomial $(x - a)(x - b)$ and the real number line (see figure).



(a) Identify the points on the line where the polynomial is zero.

(b) In each of the three subintervals of the line, write the sign of each factor and the sign of the product.

(c) For which x-values does a polynomial possibly change signs?

Review

In Exercises 78–81, find the distance between each pair of points. Then find the midpoint of the line segment joining the points.

78. $(-4, 2), (1, 12)$

79. $(1, -2), (10, 3)$

80. $(3, 6), (-5, -8)$

81. $(0, -3), (-6, 9)$

In Exercises 82–87, sketch a graph of the function.

82. $f(x) = -x^2 + 6$

83. $f(x) = \frac{1}{3}(x - 5)^2$

84. $f(x) = -|x + 5| - 6$

85. $f(x) = \frac{1}{2}|-x| - 4$

86. $f(x) = -2\sqrt{x - 3} + 7$

87. $f(x) = \sqrt{4 - x} - 10$

In Exercises 88–91, find the inverse of the function.

88. $y = 12x$

89. $y = 5x + 8$

90. $y = x^3 + 7$

91. $y = \sqrt[3]{x - 7}$

2 Chapter Summary

What did you learn?

Section 2.1

	Review Exercises
☐ How to solve equations involving fractional expressions	1–6
☐ How to write and use mathematical models to solve real-life problems	7–13
☐ How to use common formulas to solve real-life problems	14, 15
☐ How to use scatter plots and a graphing utility to find linear models for data	16, 17

Section 2.2

☐ How to find x- and y-intercepts of graphs of equations	18–23
☐ How to find solutions of equations graphically	24–29
☐ How to find the points of intersections of two graphs	30–33

Section 2.3

☐ How to use the imaginary unit i to write complex numbers	34–37
☐ How to add, subtract, and multiply complex numbers	38–45
☐ How to use complex conjugates to divide complex numbers	46–49
☐ How to plot complex numbers in the complex plane	50–53

Section 2.4

☐ How to solve quadratic equations by factoring, extracting square roots, completing the square, and using the Quadratic Formula	54–63
☐ How to solve polynomial equations of degree three or greater	64–67
☐ How to solve equations involving radicals	68–77
☐ How to solve equations involving fractions or absolute values	78–85
☐ How to use quadratic equations to model and solve real-life problems	86, 87

Section 2.5

☐ How to use properties of inequalities to solve linear inequalities	88–91
☐ How to solve inequalities involving absolute values	92–97
☐ How to solve polynomial inequalities	98–101
☐ How to solve rational inequalities	102–105
☐ How to use inequalities to model and solve real-life problems	106, 107

2 Review Exercises

2.1 In Exercises 1 and 2, determine whether each given value of x is a solution of the equation.

	Equation	*Values*	
1.	$6+\frac{3}{x-4}=5$	(a) $x=5$	(b) $x=0$
		(c) $x=-2$	(d) $x=1$
2.	$6+\frac{2}{x+3}=\frac{6x+1}{3}$	(a) $x=-3$	(b) $x=3$
		(c) $x=0$	(d) $x=-\frac{2}{3}$

In Exercises 3–6, solve the equation (if possible) and use a graphing utility to verify your solution.

3. $14+\frac{2}{x-1}=10$

4. $6-\frac{11}{x}=3+\frac{7}{x}$

5. $\frac{9x}{3x-1}-\frac{4}{3x+1}=3$

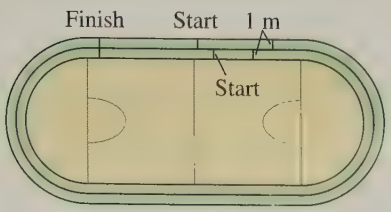
6. $\frac{5}{x-5}+\frac{1}{x+5}=\frac{2}{x^2-25}$

7. ***Monthly Profit*** In October, a company's total profit was 12% more than it was in September. The total profit for the two months was \$689,000. Find the profit for each month.

8. ***Discount Rate*** The price of a television set has been discounted \$85. The sale price is \$340. What is the percent discount?

9. ***Mixture Problem*** A car radiator contains 10 liters of a 30% antifreeze solution. How many liters will have to be replaced with pure antifreeze if the resulting solution is to be 50% antifreeze?

10. ***Starting Positions*** A fitness center has two running tracks around a rectangular playing floor. The tracks are 1 meter wide and form semicircles at the narrow ends of the rectangular floor. Determine the distance between the starting positions if two runners must run the same distance to the finish line in one lap around the track.



11. ***Cost Sharing*** A group of farmers agree to share equally in the cost of a \$48,000 piece of machinery. If they could find two more farmers to join the group, each person's share of the cost would decrease by \$4000. How many farmers are presently in the group?

12. ***Venture Capital*** An individual is planning on starting a small business that will require \$90,000 before any income can be generated. Because it is difficult to borrow for new ventures, the individual wants a group of friends to divide the cost equally for future shares of the profit. Some are willing, but three more are needed so that the price per person will be \$2500 less. How many investors are needed?

13. ***Average Speed*** You drove 56 miles one way on a service call. On the return, your average speed was 8 miles per hour greater and took 10 fewer minutes. What was your average speed on the return trip?

14. ***Geometry*** A triangle whose area is 80 square inches has a base that is two and a half times its height. Find the base and height of the triangle.

15. ***Geometry*** A basketball and a baseball have circumferences of 30 inches and $9\frac{1}{4}$ inches, respectively. Find the volume of each.

16. ***Sales*** The sales manager of a company wants to determine whether there is a relationship between sales and years of experience for sales personnel who have been with the company 4 or fewer years. The table shows the years of experience x for eight of the firm's sales personnel and monthly sales y (in thousands of dollars).

x	1.5	1.0	0.3	3.0
y	46.7	32.9	19.2	48.4

x	4.0	0.5	2.5	1.8
y	51.2	28.5	53.4	35.5

(a) Use the regression capabilities of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to plot the data and graph the model. Interpret the slope of the model in the context of the problem.

(c) Use the model to estimate monthly sales of a salesperson with 5 years of experience.

17. ***Stress Test*** A machine part was tested by bending it x centimeters 10 times per minute until it failed (y = time to failure, in hours). The results are recorded in the table.

x	3	6	9	12	15
y	61	56	53	55	48

x	18	21	24	27	30
y	35	36	33	44	23

(a) Use the regression capabilities of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the graph to determine if there may have been an error made in conducting one of the tests or in recording the results. If so, eliminate the point and find the model for the remaining data.

2.2 In Exercises 18–21, determine the x- and y-intercepts of the graph of the equation algebraically. Use a graphing utility to verify your answer.

18. $-x + y = 3$ **19.** $x - 5y = 20$

20. $y = x^2 - 9x + 8$ **21.** $y = 25 - x^2$

In Exercises 22 and 23, use a graphing utility to graph the equation and approximate the x- and y-intercepts.

22. $y = -|x + 5| - 2$ **23.** $y = 6 - 2|x - 3|$

In Exercises 24–29, use a graphing utility to approximate any solutions (accurate to three decimal places) of the equation.

24. $5(x - 2) - 1 = 0$ **25.** $12 - 5(x - 7) = 0$

26. $3x^3 - 2x + 4 = 0$ **27.** $\frac{1}{3}x^3 - x + 4 = 0$

28. $x^4 - 3x + 1 = 0$ **29.** $6 - \frac{1}{2}x^2 + \frac{5}{6}x^4 = 0$

In Exercises 30–33, determine algebraically any points of intersection of the graphs of the equations. Use a graphing utility to verify your answer(s).

30. $3x + 5y = -7$
$-x - 2y = 3$

31. $x - y = 3$
$2x + y = 12$

32. $x^2 + 2y = 14$
$3x + 4y = 1$

33. $y = -x + 7$
$y = 2x^3 - x + 9$

2.3 In Exercises 34–37, write in standard form.

34. $6 + \sqrt{-25}$ **35.** $-\sqrt{-12} + 3$

36. $-2i^2 + 7i$ **37.** $-i^2 - 4i$

In Exercises 38–49, perform the operations and write the result in standard form.

38. $(7 + 5i) + (-4 + 2i)$

39. $\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

40. $5i(13 - 8i)$ **41.** $(1 + 6i)(5 - 2i)$

42. $(10 - 8i)(2 - 3i)$ **43.** $i(6 + i)(3 - 2i)$

44. $(3 + 7i)^2 + (3 - 7i)^2$ **45.** $(4 - i)^2 - (4 + i)^2$

46. $\dfrac{6 + i}{i}$ **47.** $\dfrac{3 + 2i}{5 + i}$

48. $\dfrac{4}{-3i}$ **49.** $\dfrac{1}{(2 + i)^4}$

In Exercises 50–53, plot the complex number in the complex plane.

50. $2 - 5i$ **51.** $-1 + 4i$

52. $-6i$ **53.** $7i$

2.4 In Exercises 54–63, use any method to solve the equation. Use a graphing utility to verify your solution(s).

54. $6x = 3x^2$ **55.** $15 + x - 2x^2 = 0$

56. $(x + 4)^2 = 18$ **57.** $16x^2 = 25$

58. $x^2 - 12x + 30 = 0$ **59.** $x^2 + 6x - 3 = 0$

60. $2x^2 + 9x - 5 = 0$ **61.** $-x^2 - x + 15 = 0$

62. $x^2 - 4x - 10 = 0$ **63.** $-2x^2 - 13x = 0$

In Exercises 64–85, solve the equation (if possible) and use a graphing utility to verify your solution.

64. $3x^3 - 26x^2 + 16x = 0$ **65.** $216x^4 - x = 0$

66. $5x^4 - 12x^3 = 0$ **67.** $4x^3 - 6x^2 = 0$

68. $\sqrt{x + 4} = 3$ **69.** $\sqrt{x - 2} - 8 = 0$

70. $2\sqrt{x} - 5 = 0$ **71.** $\sqrt{3x - 2} = 4 - x$

72. $\sqrt{2x + 3} + \sqrt{x - 2} = 2$

73. $5\sqrt{x} - \sqrt{x - 1} = 6$

74. $(x - 1)^{2/3} - 25 = 0$ **75.** $(x + 2)^{3/4} = 27$

76. $(x + 4)^{1/2} + 5x(x + 4)^{3/2} = 0$

77. $8x^2(x^2 - 4)^{1/3} + (x^2 - 4)^{4/3} = 0$

78. $3\left(1 - \frac{1}{5t}\right) = 0$

79. $\frac{1}{x - 2} = 3$

80. $\frac{4}{(x - 4)^2} = 1$

81. $\frac{1}{(t + 1)^2} = 1$

82. $|x - 5| = 10$

83. $|2x + 3| = 7$

84. $|x^2 - 3| = 2x$

85. $|x^2 - 6| = x$

86. ***Population*** The population P of Wyoming (in thousands) from 1990 through 1997 can be approximated by the model

$$P = -0.435t^2 + 7.125t + 452.542$$

where $t = 0$ represents 1990. (Source: U.S. Bureau of the Census)

(a) Use a graphing utility to graph the model.

(b) Use the *zoom* and *trace* features of a graphing utility to determine when the population exceeded 465,000.

(c) According to the model, when will the population reach 490,000?

(d) Verify your answer to part (c) numerically by creating a table of values for the model.

(e) Do you think this is an accurate model for the population? Explain.

87. ***Life Insurance*** The number y of life insurance companies in the United States from 1989 through 1996 can be modeled by

$$y = 7.73t^2 - 127.7t + 2164.4$$

where $t = 0$ represents 1990. (Source: American Council of Life Insurance)

(a) Use a graphing utility to graph the model.

(b) Use the model to determine the year when there were about 2000 life insurance companies.

(c) Is this model accurate for predicting the number of life insurance companies in the future? Explain.

In Exercises 88–105, solve the inequality and graph the solution on the real number line. Use a graphing utility to verify your solution.

88. $8x - 3 < 6x + 15$

89. $\frac{1}{2}(3 - x) > \frac{1}{3}(2 - 3x)$

90. $-2 < -x + 7 \le 10$

91. $-6 \le 3 - 2(x - 5) < 14$

92. $|x - 2| < 1$

93. $|x| \le 4$

94. $\left|x - \frac{3}{2}\right| \ge \frac{3}{2}$

95. $|x - 3| > 4$

96. $4|3 - 2x| \le 16$

97. $|x + 9| + 7 > 19$

98. $x^2 - 2x \ge 3$

99. $4x^2 - 23x \le 6$

100. $x^3 - 16x \ge 0$

101. $12x^3 - 20x^2 < 0$

102. $\frac{x - 5}{3 - x} < 0$

103. $\frac{2}{x + 1} \le \frac{3}{x - 1}$

104. $\frac{3x + 8}{x - 3} \le 4$

105. $\frac{x + 8}{x + 5} - 2 < 0$

106. ***Accuracy of Measurement*** The side of a square is measured as 20.8 inches with a possible error of $\frac{1}{16}$ inch. Using these measurements, determine the interval containing the area of the square.

107. ***Fuel Consumption*** The total fuel consumption F (in billions of gallons) in the United States from 1985 through 1996 can be approximated by the model

$$F = 0.107t^2 - 0.235t + 122.446$$

where t is the time in years, with $t = 5$ corresponding to 1985. (Source: U.S. Federal Highway Administration)

(a) Use a graphing utility to graph the model over the indicated years.

(b) According to this model, estimate the year of an annual fuel consumption of more than 165 billion gallons. Solve algebraically and graphically.

Synthesis

True or False? **In Exercises 108–110, determine whether the statement is true or false. Justify your answer.**

108. The graph of a function may have two distinct y-intercepts.

109. The sum of two complex numbers cannot be a real number.

110. The sign of the slope of the regression line is always positive.

111. In your own words, explain the difference between an identity and a conditional equation.

112. Describe the relationship among the x-intercepts of a graph, the zeros of a function, and the solutions of an equation.

113. Consider the linear equation $ax + b = 0$.

(a) What is the sign of the solution if $ab > 0$?

(b) What is the sign of the solution if $ab < 0$?

Chapter Project *Mozart and the Golden Ratio*

Music and mathematics have always been closely related. In this project, you will discover a curious relationship between the structure of Mozart's (1756–1791) piano sonatas and Euclid's (ca. 300 B.C.) famous golden ratio. (Source: John F. Putz, *Mathematics Magazine,* October 1995)

Sonatas can be naturally divided into two parts: the *exposition,* which introduces the musical theme, and the *development and recapitulation,* which develop and repeat the theme. The data at the right shows all of Mozart's piano sonata movements that have these two parts. The first column identifies the sonata movement. The second column, labeled x, identifies the length of the development and recapitulation in measures. The third column, labeled y, identifies the length of the exposition in measures.

Sonata	x	y	Sonata	x	y
279, I	62	38	310, I	84	49
279, II	46	28	311, I	73	39
279, III	102	56	330, I	92	58
280, I	88	56	330, III	103	68
280, II	36	24	332, I	136	93
280, III	113	77	332, III	155	90
281, I	69	40	333, I	102	63
281, II	60	46	333, III	50	31
282, I	18	15	457, I	93	74
282, III	63	39	533, I	137	102
283, I	67	53	533, II	76	46
283, II	23	14	545, I	45	28
283, III	171	102	547a, I	118	78
284, I	76	51	570, I	130	79
309, I	97	58			

a. Enter the points (x, y) in a graphing utility and sketch a scatter plot of the points. Describe the relationship between x and y.

b. Use the statistical capabilities of a graphing utility to find the least squares regression line for the data. How well does the line fit the data? Explain. Interpret the meaning of the slope and the y-intercept.

c. The **golden section** was defined by Euclid as the point B on a line segment AC such that

$$\frac{AB}{BC} = \frac{BC}{AC}.\text{ Assume that } AC = 1 \text{ and } BC = r.$$

Then this proportion can be written as $\dfrac{1 - r}{r} = \dfrac{r}{1}$.

Use a graphing utility to graph $y_1 = \dfrac{1 - x}{x}$ and $y_2 = \dfrac{x}{1}$

and approximate their point of intersection in Quadrant I. This value is called the golden ratio. How does it compare to the line in part (b)?

Questions for Further Exploration

1. Use the data for Mozart's sonatas to complete a table that calculates ratios of the form $x/(x + y)$. How do these ratios compare with the golden ratio?

2. The Fibonacci Sequence is given by 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

   (a) Explain how to generate succeeding terms in the sequence.

   (b) Complete and expand the table, which shows decimal approximations of ratios of adjacent terms in the sequence.

$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{8}{13}$	$\frac{13}{21}$	$\frac{21}{34}$
1.000	0.500	0.667					

What can you conclude about the ratios?

3. ***Research Project*** Obtain a copy of the article "The Golden Section and the Piano Sonatas of Mozart," John F. Putz, *Mathematics Magazine*, Vol. 68, No. 4, October 1995, pp. 275–282. Read the article and write a short paper about other aspects of Mozart's sonatas and the golden ratio.

2 Chapter Test

Take this test as you would take a test in class. After you are done, check your work against the answers in the back of the book.

In Exercises 1 and 2, solve the equation and use a graphing utility to verify your solution.

1. $\frac{12}{x} - 7 = -\frac{27}{x} + 6$

2. $\frac{4}{3x - 2} - \frac{9x}{3x + 2} = -3$

In Exercises 3–8, perform the operation.

3. $(-8 - 3i) + (-1 - 15i)$

4. $\left(10 + \sqrt{-20}\right) - \left(4 - \sqrt{-14}\right)$

5. $(2 + i)(6 - i)$

6. $(4 + 3i)^2 - (5 + i)^2$

7. $\frac{8 + 5i}{6 - i}$

8. $\frac{i}{4 - 5i} + \frac{5i}{2 + i}$

The *Interactive* CD-ROM and *Internet* versions of this text provide answers to the Chapter Tests and Cumulative Tests. They also offer Chapter Pre-Tests (which test key skills and concepts covered in previous chapters) and Chapter Post-Tests, both of which have randomly generated exercises with diagnostic capabilities.

In Exercises 9–12, use a graphing utility to graph the equation. Determine the number of x-intercepts of the graph and compare these intercepts with the real zeros of the function.

9. $y = 3x^2 + 1$

10. $y = x^3 + x$

11. $y = x^3 - 4x^2 + 5x$

12. $y = 2 + 8x^{-2}$

In Exercises 13–16, use any method to solve the equation. Check your solutions algebraically and graphically.

13. $x^2 - 10x + 9 = 0$

14. $x^2 + 12x - 2 = 0$

15. $4x^2 - 81 = 0$

16. $5x^2 + 14x - 3 = 0$

In Exercises 17–20, find all real solutions of the equation. Check your solution(s) algebraically and graphically.

17. $3x^3 - 4x^2 - 12x + 16 = 0$

18. $x + \sqrt{22 - 3x} = 6$

19. $(x^2 + 6)^{2/3} = 16$

20. $|8x - 1| = 21$

In Exercises 21–23, solve the inequality and sketch the solution on the real number line. Use a graphing utility to verify your solution.

21. $-\frac{5}{6} < x - 2 < \frac{1}{8}$

22. $2|x - 8| < 10$

23. $\frac{3 - 5x}{2 + 3x} < -2$

24. The table shows the number of local telephone calls C (in billions) in the United States from 1989 through 1996, where $t = 0$ represents 1990. Use the regression capabilities of a graphing utility to find a linear model for the data. Determine algebraically and graphically when the annual number of local calls will be about 600 billion. (Source: U.S. Federal Communications Commission)

t	C
−1	389
0	402
1	416
2	434
3	447
4	465
5	484
6	504

25. You buy a bag of tomatoes for \$1.62 per pound. The weight that is listed on the bag is 4.55 pounds. The scale is accurate to within 1 ounce. How much might you have been undercharged or overcharged?

P–2 Cumulative Test

Take this test to review the material from earlier chapters. After you are done, check your work against the answers in the back of the book.

In Exercises 1–3, simplify the expression.

1. $\dfrac{14x^2y^{-3}}{32x^{-1}y^2}$ **2.** $8\sqrt{60} - 2\sqrt{135} - \sqrt{15}$ **3.** $\sqrt{28x^4y^3}$

In Exercises 4–6, perform the operations and simplify the result.

4. $4x - [2x + 5(2 - x)]$ **5.** $(x - 2)(x^2 + x - 3)$ **6.** $\dfrac{2}{x + 3} - \dfrac{1}{x + 1}$

In Exercises 7–9, factor the expression completely.

7. $25 - (x - 2)^2$ **8.** $x - 5x^2 - 6x^3$ **9.** $54 - 16x^3$

10. Find the midpoint of the line segment connecting the points $\left(-\frac{7}{2}, 4\right)$ and $(6.5, -8)$. Then find the distance between the points.

11. Find the standard form of the equation of a circle with center at $\left(-\frac{1}{2}, -8\right)$ and radius of $\frac{5}{4}$.

In Exercises 12–14, use point plotting to sketch a graph of the equation.

12. $x - 3y + 12 = 0$ **13.** $y = x^2 - 9$ **14.** $y = \sqrt{4 - x}$

In Exercises 15–17, (a) find the equation of the line that satisfies the given conditions and (b) find three additional points that lie on that line.

15. The line contains the points $(-5, 8)$ and $\left(\frac{1}{2}, -6\right)$.

16. The line contains the point $\left(-\frac{1}{2}, 1\right)$ and has a slope of -2.

17. The line has an undefined slope and contains the point $\left(-\frac{3}{7}, \frac{1}{8}\right)$.

18. Does the graph at the right represent y as a function of x? Explain.

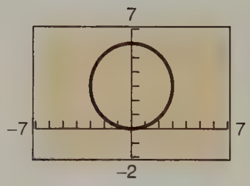


FIGURE FOR 18

In Exercises 19 and 20, evaluate the function at each specified value of the independent variable and simplify.

19. $f(x) = \dfrac{x}{x - 2}$

(a) $f(6)$ (b) $f(2)$ (c) $f(s + 2)$

20. $f(x) = \begin{cases} 3x - 8, & x \le -\frac{5}{3} \\ 3x^2 + 9x - 8, & x > -\frac{5}{3} \end{cases}$

(a) $f\left(-\frac{5}{3}\right)$ (b) $f(-1)$ (c) $f(0)$

21. Use a graphing utility to graph the function $f(x) = 2|x - 5| - |x + 5|$. Then determine the open intervals over which the function is increasing, decreasing, or constant.

22. Compare the graphs of each function with the graph of $y = \sqrt[3]{x}$.

(a) $r(x) = \dfrac{1}{2}\sqrt[3]{x}$ (b) $h(x) = \sqrt[3]{x} + 2$ (c) $g(x) = \sqrt[3]{x + 2}$

In Exercises 23–26, evaluate the indicated function for

$f(x) = -x^2 + 3x - 10$ and $g(x) = 4x + 1$.

23. $(f + g)(-4)$ **24.** $(g - f)\left(\frac{3}{4}\right)$ **25.** $(g \circ f)(-2)$ **26.** $(fg)(-1)$

27. Plot the complex number $-5 + 4i$ in the complex plane.

28. Write the quadratic equation $9x(x - 1) = -4x^2 - 6$ in general form.

In Exercises 29–32, use a graphing utility to graph the equation and approximate any x-intercepts of the graph. Find any x-intercepts algebraically and compare your results to those you approximated graphically.

29. $y = 4x^3 - 12x^2 + 8x$ **30.** $y = 12x^3 - 84x^2 + 120x$

31. $y = |2x - 3| - 5$ **32.** $y = \sqrt{x^2 + 1} + x - 9$

In Exercises 33 and 34, solve the equation for the indicated variable.

33. Solve for X: $Z = \sqrt{R^2 - X^2}$ **34.** Solve for p: $L = \dfrac{k}{3\pi r^2 p}$

In Exercises 35 and 36, find the constant C such that the ordered pair is a solution point of the equation.

35. $y = C\sqrt{x + 1}$, $(3, 8)$ **36.** $x + C(y + 2) = 0$, $(4, 3)$

In Exercises 37–39, use a graphing utility to approximate the solution to the inequality. Verify your answer algebraically.

37. $\dfrac{x}{5} - 6 \le -\dfrac{x}{2} + 6$ **38.** $2x^2 + x \ge 15$ **39.** $(x - 4)|x| > 0$

40. The total sales S (in millions of dollars) for the Cooper Tire and Rubber Company from 1989 through 1998 is shown in the table, where $t = 0$ represents 1990. Use the regression capabilities of a graphing utility to find a linear model that represents the data. Estimate algebraically the total sales for 2000, 2002, and 2004. (Source: Cooper Tire and Rubber Company)

t	-1	0	1	2	3	4	5	6	7	8
S	866.8	895.9	1001.1	1174.7	1193.6	1403.2	1493.6	1619.3	1813.0	1876.1

41. A bowling ball has a volume of about 332.38 cubic inches. Find the radius of the bowling ball (accurate to three decimal places).

42. A rectangular plot of land with a perimeter of 546 feet has a width of x.

(a) Express the area of the plot as a function of x.

(b) Use a graphing utility to graph the area function. What is the domain of the function?

(c) Approximate the dimensions of the plot if its area is 15,000 square feet.

Polynomial and Rational Functions

3

3.1 Quadratic Functions
3.2 Polynomial Functions of Higher Degree
3.3 Real Zeros of Polynomial Functions
3.4 The Fundamental Theorem of Algebra
3.5 Rational Functions and Asymptotes
3.6 Graphs of Rational Functions

Pictures Colour Library Ltd./Leo de Wys

U.S. wheat production increased from 2183 million bushels in 1995 to 2527 million bushels in 1997, while the price per bushel dropped from $4.55 to $3.45. (Source: U.S. Department of Agriculture)

The Big Picture

In this chapter you will learn how to

☐ sketch and analyze graphs of quadratic and polynomial functions.
☐ use long division and synthetic division to divide polynomials by other polynomials.
☐ determine the number of rational and real zeros of polynomial functions, and find them.
☐ determine the domain, find asymptotes, and sketch the graphs of rational functions.

Important Vocabulary

As you encounter each new vocabulary term in this chapter, add the term and its definition to your notebook glossary.

- polynomial function of x with degree n (p. 226)
- constant function (p. 226)
- linear function (p. 226)
- quadratic function (p. 226)
- parabola (p. 226)
- axis of symmetry (p. 227)
- vertex (p. 227)
- standard form of a quadratic function (p. 229)
- continuous (p. 237)
- Leading Coefficient Test (p. 239)
- extrema (p. 240)
- relative minimum (p. 240)
- relative maximum (p. 240)
- repeated zero (p. 242)
- multiplicity (p. 242)
- Intermediate Value Theorem (p. 244)
- long division of polynomials (p. 250)
- Division Algorithm (p. 251)
- synthetic division (p. 253)
- Remainder Theorem (p. 254)
- Factor Theorem (p. 254)
- Rational Zero Test (p. 256)
- upper bound (p. 258)
- lower bound (p. 258)
- Fundamental Theorem of Algebra (p. 264)
- Linear Factorization Theorem (p. 264)
- conjugates (p. 266)
- rational function (p. 271)
- vertical asymptote (p. 272)
- horizontal asymptote (p. 272)
- slant (or oblique) asymptote (p. 284)

Additional Resources Text-specific additional resources are available to help you do well in this course. See page xvi for details.

3.1 Quadratic Functions

What You Should Learn:

- How to analyze graphs of quadratic functions
- How to write quadratic functions in standard form and use the results to sketch graphs of functions
- How to use quadratic functions to model and solve real-life problems

Why You Should Learn It:

Quadratic functions can be used to model data to analyze consumer behavior. For instance, Exercise 78 on page 236 shows how a quadratic function can model VCR usage in the United States.

Mary K. Kenny/PhotoEdit

The Graph of a Quadratic Function

In this and the next section, you will study the graphs of polynomial functions.

Definition of Polynomial Function

Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function of x with degree n.**

Polynomial functions are classified by degree. For instance, the polynomial function

$$f(x) = a, \quad a \neq 0 \qquad \text{Constant function}$$

has degree 0 and is called a **constant function.** In Chapter 1, you learned that the graph of this type of function is a horizontal line. The polynomial function

$$f(x) = mx + b, \quad m \neq 0 \qquad \text{Linear function}$$

has degree 1 and is called a **linear function.** You also learned in Chapter 1 that the graph of the linear function $f(x) = mx + b$ is a line whose slope is m and whose y-intercept is $(0, b)$. In this section you will study second-degree polynomial functions, which are called **quadratic functions.**

Definition of Quadratic Function

Let a, b, and c be real numbers with $a \neq 0$. The function

$$f(x) = ax^2 + bx + c \qquad \text{Quadratic function}$$

is called a **quadratic function.**

Often real-life data can be modeled by quadratic functions. For instance, the table shows height h (in feet) of a projectile fired from a height of 6 feet with an initial velocity of 256 feet per second at any time t (in seconds). A quadratic model for the data in the table is $h(t) = -16t^2 + 256t + 6$ for $0 \leq t \leq 16$.

t	0	2	4	6	8	10	12	14	16
h	6	454	774	966	1030	966	774	454	6

The graph of a quadratic function is a special type of U-shaped curve called a **parabola.** Parabolas occur in many real-life applications—especially those involving reflective properties, such as satellite dishes or flashlight reflectors. You will study these properties in Section 8.1.

All parabolas are symmetric with respect to a line called the **axis of symmetry,** or simply the **axis** of the parabola. The point where the axis intersects the parabola is the **vertex** of the parabola, as shown in Figure 3.1. If the leading coefficient a is positive, the graph of $f(x) = ax^2 + bx + c$ is a parabola that opens upward; and if the leading coefficient a is negative, the graph of $f(x) = ax^2 + bx + c$ is a parabola that opens downward.

Exploration

Use a graphing utility to graph the parabola

$$y = ax^2$$

for $a = -3, -2, -1, 1, 2,$ and 3. What can you conclude about the parabola when $a < 0$? When $a > 0$?

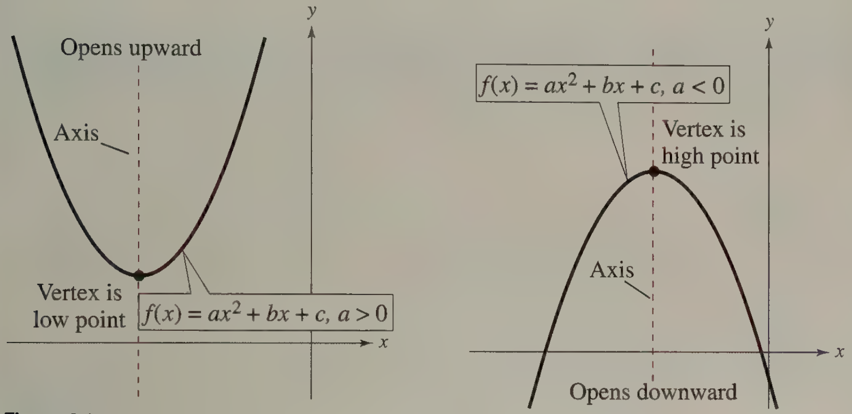


Figure 3.1

The simplest type of quadratic function is

$$f(x) = ax^2.$$

Its graph is a parabola whose vertex is (0, 0). If $a > 0$, the vertex is the *minimum* point on the graph; and if $a < 0$, the vertex is the *maximum* point on the graph, as shown in Figure 3.2.

Library of Functions

The graph of a quadratic function is called a parabola. Graph $y_1 = x^2$ and $y_2 = |x|$ in the same viewing window. Zoom in near the origin and compare the shape of the two graphs. Which graph grows faster as x gets larger and larger? Why does the definition of a quadratic function require that $a \neq 0$?

Consult the Library of Functions Summary inside the front cover for a description of the quadratic function.

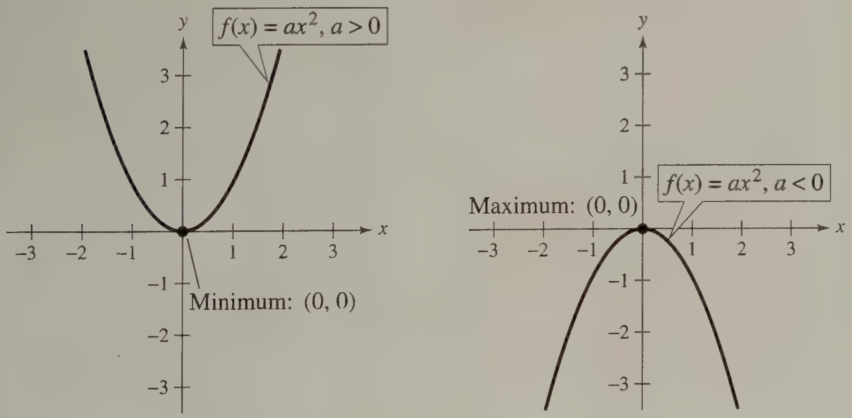


Figure 3.2

When sketching the graph of $f(x) = ax^2$, it is helpful to use the graph of $y = x^2$ as a reference, as discussed in Section 1.5. There you saw that when $a > 1$, the graph of $y = af(x)$ is a vertical stretch of the graph of $y = f(x)$. When $0 < a < 1$, the graph of $y = af(x)$ is a vertical shrink of the graph of $y = f(x)$. This is demonstrated again in Example 1.

EXAMPLE 1 Graphing Simple Quadratic Functions

Describe how the graph of each function is related to the graph of $y = x^2$.

a. $f(x) = \frac{1}{3}x^2$ **b.** $g(x) = 2x^2$

c. $h(x) = -x^2 + 1$ **d.** $k(x) = (x + 2)^2 - 3$

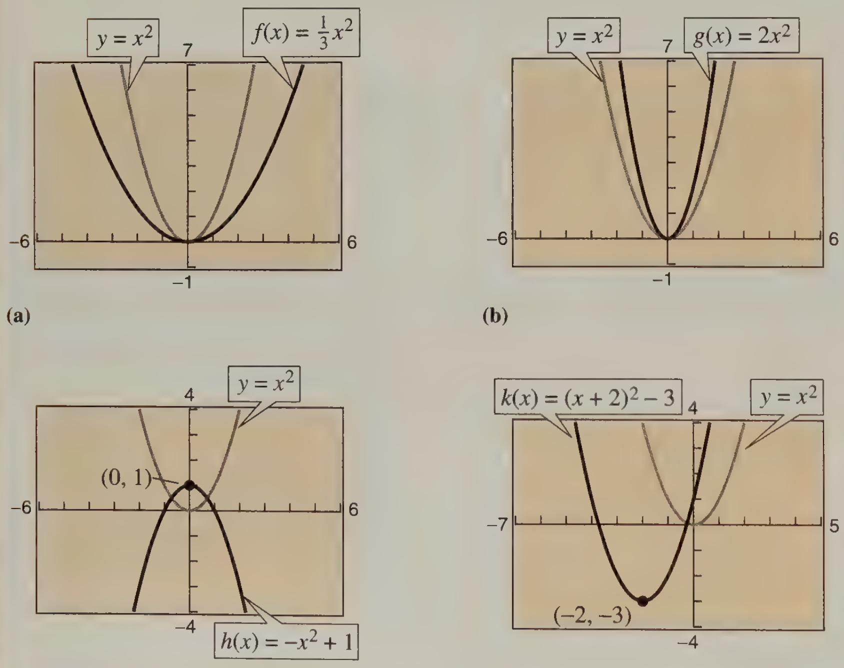
Solution

a. Compared with $y = x^2$, each output of f "shrinks" by a factor of $\frac{1}{3}$. The result is a parabola that opens upward and is broader than the parabola represented by $y = x^2$, as shown in Figure 3.3(a).

b. Compared with $y = x^2$, each output of g "stretches" by a factor of 2, creating a narrower parabola, as shown in Figure 3.3(b).

c. With respect to the graph of $y = x^2$, the negative coefficient in $h(x) = -x^2 + 1$ reflects the graph *downward* and the positive constant term shifts the vertex *up* one unit. The graph of h is shown in Figure 3.3(c).

d. With respect to the graph of $y = x^2$, the graph of $k(x) = (x + 2)^2 - 3$ is obtained by a horizontal shift two units *to the left* and a vertical shift three units *down*, as shown in Figure 3.3(d).



(a) (b)

(c) (d)

Figure 3.3

STUDY TIP

In Example 1, note that the coefficient a determines how widely the parabola given by $f(x) = ax^2$ opens. If $|a|$ is small, the parabola opens more widely than if $|a|$ is large.

The *Interactive* CD-ROM and *Internet* versions of this text show every example with its solution; clicking on the *Try It!* button brings up similar problems. Guided Examples and Integrated Examples show step-by-step solutions to additional examples. Integrated Examples are related to several concepts in the section.

Recall from Section 1.5 that the graphs of $y = f(x \pm c)$, $y = f(x) \pm c$, $y = -f(x)$, and $y = f(-x)$ are rigid transformations of the graph of $y = f(x)$.

$y = f(x \pm c)$	Horizontal shift	$y = -f(x)$	Reflection in x-axis
$y = f(x) \pm c$	Vertical shift	$y = f(-x)$	Reflection in y-axis

The Standard Form of a Quadratic Function

The equation in Example 1(d) is written in the **standard form**

$$f(x) = a(x - h)^2 + k.$$

This form is especially convenient for sketching a parabola because it identifies the vertex of the parabola as (h, k).

Standard Form of a Quadratic Function

The quadratic function

$$f(x) = a(x - h)^2 + k, \qquad a \neq 0$$

is said to be in **standard form.** The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). If $a > 0$, the parabola opens upward, and if $a < 0$, the parabola opens downward.

Exploration

Use a graphing utility to graph $y = ax^2$ with $a = -2, -1, -0.5, 0.5, 1$, and 2. How does the value of a affect the graph?

Use a graphing utility to graph $y = (x - h)^2$ with $h = -4, -2, 2$, and 4. How does the value of h affect the graph?

Use a graphing utility to graph $y = x^2 + k$ with $k = -4, -2, 2$, and 4. How does the value of k affect the graph?

EXAMPLE 2 Identifying the Vertex of a Quadratic Function

Describe the graph of

$$f(x) = 2x^2 + 8x + 7$$

and identify the vertex.

Solution

Write the quadratic function in standard form by completing the square. Recall that the first step is to factor out any coefficient of x^2 that is different from 1.

$$\begin{aligned}
f(x) &= 2x^2 + 8x + 7 && \text{Write original function.}\\
&= 2(x^2 + 4x) + 7 && \text{Factor 2 out of } x\text{-terms.}\\
&= 2(x^2 + 4x \underbrace{+ 4}_{\left(\frac{b}{2}\right)^2} - 4) + 7 && \text{Because } b = 4, \text{ add and subtract } (b/2)^2 = 4 \text{ within parentheses.}\\
&= 2(x^2 + 4x + 4) - 2(4) + 7 && \text{Regroup terms.}\\
&= 2(x + 2)^2 - 1 && \text{Write in standard form.}
\end{aligned}$$

From the standard form, you can see that the graph of f is a parabola that opens upward with vertex $(-2, -1)$, as shown in Figure 3.4.

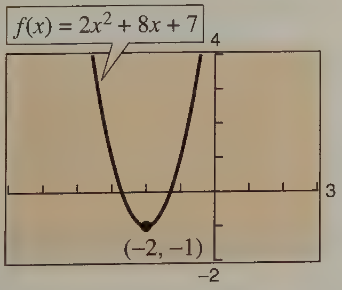


Figure 3.4

To find the x-intercepts of the graph of $f(x) = ax^2 + bx + c$, solve the equation $ax^2 + bx + c = 0$. If $ax^2 + bx + c$ does not factor, you can use the Quadratic Formula to find the x-intercepts, or a graphing utility to approximate the x-intercepts. Remember, however, that a parabola may have no x-intercept.

EXAMPLE 3 Writing a Quadratic Function in Standard Form

Describe the graph of $f(x) = -x^2 + 6x - 8$. Identify any x-intercepts.

Solution

$$
\begin{aligned}
f(x) &= -x^2 + 6x - 8 && \text{Write original function.}\\
&= -(x^2 - 6x) - 8 && \text{Factor } -1 \text{ out of } x\text{-terms.}\\
&= -(x^2 - 6x \underbrace{+\,9}_{\left(\frac{b}{2}\right)^2} - 9) - 8 && \text{Because } b = 6, \text{ add and subtract } (b/2)^2 = 9 \text{ within parentheses.}\\
&= -(x^2 - 6x + 9) - (-9) - 8 && \text{Regroup terms.}\\
&= -(x - 3)^2 + 1 && \text{Write in standard form.}
\end{aligned}
$$

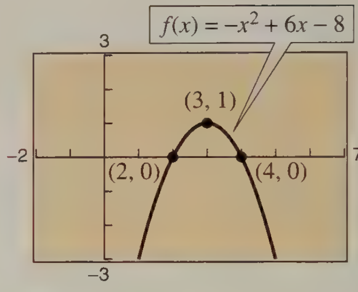


Figure 3.5

The graph of f is a parabola that opens downward with vertex at (3, 1), as shown in Figure 3.5. The x-intercepts are determined as follows.

$$
\begin{aligned}
-x^2 + 6x - 8 &= 0\\
x^2 - 6x + 8 &= 0\\
(x - 2)(x - 4) &= 0\\
x = 2,\ x &= 4
\end{aligned}
$$

So, the x-intercepts are (2, 0) and (4, 0), as shown in Figure 3.5.

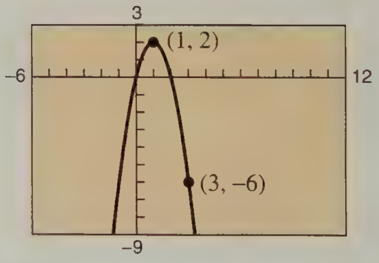


Figure 3.6

EXAMPLE 4 Finding the Equation of a Parabola in Standard Form

Find the standard form of the equation for the parabola that has its vertex at (1, 2) and passes through the point (3, −6), as shown in Figure 3.6.

Solution

Because the parabola has a vertex at $(h, k) = (1, 2)$, the equation has the form

$$f(x) = a(x - 1)^2 + 2. \qquad \text{Standard form}$$

Because the parabola passes through the point (3, −6), it follows that $f(3) = -6$. So, you obtain

$$
\begin{aligned}
-6 &= a(3 - 1)^2 + 2\\
-6 &= 4a + 2\\
-2 &= a.
\end{aligned}
$$

The equation in standard form is $f(x) = -2(x - 1)^2 + 2$. Try graphing $f(x) = -2(x - 1)^2 + 2$ with a graphing utility to confirm that its vertex is (1, 2) and that it passes through the point (3, −6).

STUDY TIP

In Example 4, there are infinitely many different parabolas that have a vertex at (1, 2). Of these, however, the only one that passes through the point (3, −6) is the one given by

$$f(x) = -2(x - 1)^2 + 2.$$

Applications

Many applications involve finding the maximum or minimum value of a quadratic function. By writing the quadratic function $f(x) = ax^2 + bx + c$ in standard form,

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

you can see that the vertex occurs at $x = -b/(2a)$, which implies the following.

1. If $a > 0$, f has a *minimum* that occurs at

$$x = -\frac{b}{2a}.$$

2. If $a < 0$, f has a *maximum* that occurs at

$$x = -\frac{b}{2a}.$$

STUDY TIP

To obtain the standard form at the left, you can complete the square of the form

$$f(x) = ax^2 + bx + c.$$

Try verifying this computation.

EXAMPLE 5 The Maximum Height of a Baseball

A baseball is hit 3 feet above ground at a velocity of 100 feet per second and at an angle of 45 degrees with respect to level ground. The path of the baseball is given by the function $f(x) = -0.0032x^2 + x + 3$, where $f(x)$ is the height of the baseball (in feet) and x is the horizontal distance from home plate (in feet). What is the maximum height reached by the baseball? (See Example 7 in Section 1.3.)

A computer simulation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Algebraic Solution

For this quadratic function, you have

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= -0.0032x^2 + x + 3 \end{aligned}$$

which implies that $a = -0.0032$ and $b = 1$. Because the function has a maximum when $x = -b/(2a)$, you can conclude that the baseball reaches its maximum height when

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{1}{2(-0.0032)} \\ &= 156.25 \text{ feet} \end{aligned}$$

from home plate. At this distance, the maximum height is

$$\begin{aligned} f(156.25) &= -0.0032(156.25)^2 + 156.25 + 3 \\ &= 81.125 \text{ feet.} \end{aligned}$$

Graphical Solution

Use a graphing utility to graph $y = -0.0032x^2 + x + 3$ so that you can see the important features of the parabola. In Figure 3.7 the graph appears to have a maximum near $x = 150$. Use the *maximum* feature of the graphing utility or the *zoom* and *trace* features to approximate the maximum height on the graph to be $y \approx 81.125$ feet at $x \approx 156.25$. Note that when using the *zoom* and *trace* features of a graphing utility, you might have to change the y-scale in order to avoid a graph that is "too flat."

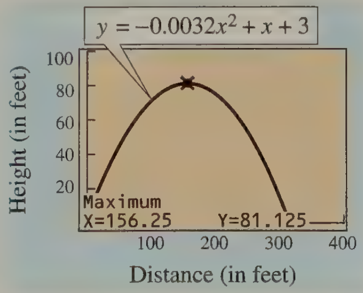


Figure 3.7

EXAMPLE 6 Charitable Contributions

According to a survey conducted by *Independent Sector*, the percent of their income that Americans give to charities is related to their household income. For families with an annual income of \$100,000 or less, the percent is approximately

$$P = 0.0014x^2 - 0.1529x + 5.855, \qquad 5 \le x \le 100$$

where P is the percent of annual income given and x is the annual income (in thousands of dollars). According to this model, what income level corresponds to the minimum percent of charitable contributions?

Algebraic Solution

Use the fact that the minimum point of the parabola occurs when $x = -b/(2a)$. For this function, you have $a = 0.0014$ and $b = -0.1529$. So,

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{-0.1529}{2(0.0014)} \\ &\approx 54.6. \end{aligned}$$

From this x-value, you can conclude that the minimum percent corresponds to an income level of about \$54,600.

Graphical Solution

Use a graphing utility to graph

$$y_1 = 0.0014x^2 - 0.1529x + 5.855$$

for $5 \le x \le 100$, as shown in Figure 3.8. The graph appears to have a minimum near $x = 55$. Use the *minimum* feature of the graphing utility or the *zoom* and *trace* features to approximate the minimum point of the parabola to be $x \approx 54.6$. So, the minimum point corresponds to an income level of about \$54,600.

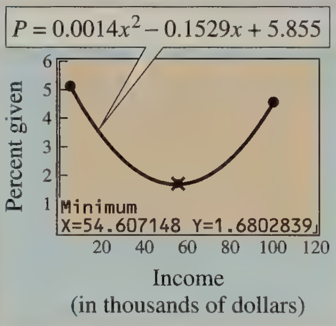


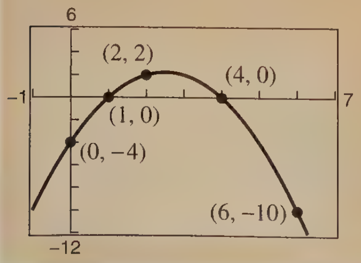
Figure 3.8

Writing About Math *Finding an Equation for a Curve*

The parabola in the figure below has an equation of the form

$$y = ax^2 + bx - 4.$$

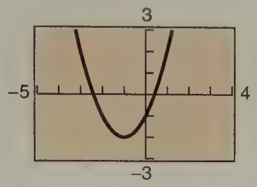
Find the equation for this parabola in two different ways, by hand and with technology (graphing utility or computer software). Write a paragraph describing the methods you used and comparing the results of the two methods.



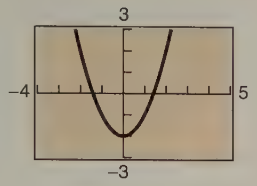
3.1 Exercises

In Exercises 1–8, match the quadratic function with the correct graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

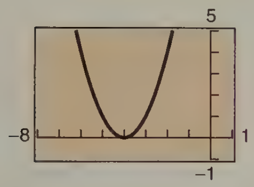
(a)



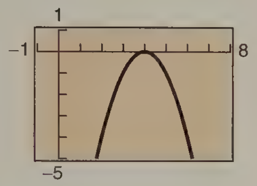
(b)



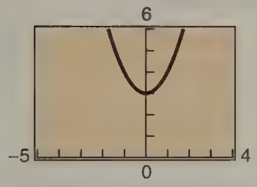
(c)



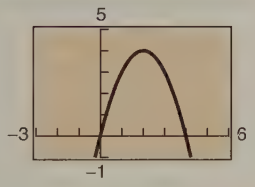
(d)



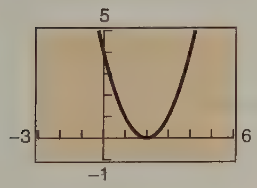
(e)



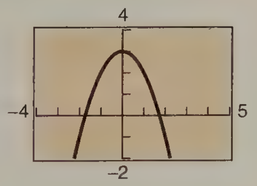
(f)



(g)



(h)



1. $f(x) = (x - 2)^2$

2. $f(x) = (x + 4)^2$

3. $f(x) = x^2 - 2$

4. $f(x) = 3 - x^2$

5. $f(x) = 4 - (x - 2)^2$

6. $f(x) = (x + 1)^2 - 2$

7. $f(x) = x^2 + 3$

8. $f(x) = -(x - 4)^2$

Exploration **In Exercises 9–12, use a graphing utility to graph each equation. Describe how the graph of each equation is related to the graph of $y = x^2$.**

9. (a) $y = \frac{1}{2}x^2$ (b) $y = -\frac{1}{8}x^2$
(c) $y = \frac{3}{2}x^2$ (d) $y = -3x^2$

10. (a) $y = x^2 + 1$ (b) $y = x^2 - 1$
(c) $y = x^2 + 3$ (d) $y = x^2 - 3$

11. (a) $y = (x - 1)^2$ (b) $y = (x + 1)^2$
(c) $y = (x - 3)^2$ (d) $y = (x + 3)^2$

12. (a) $y = -\frac{1}{2}(x - 2)^2 + 1$ (b) $y = \frac{1}{2}(x - 2)^2 + 1$
(c) $y = -\frac{1}{2}(x + 2)^2 - 1$ (d) $y = \frac{1}{2}(x + 2)^2 - 1$

In Exercises 13–26, sketch the graph of the quadratic function. Identify the vertex and intercepts. Use a graphing utility to verify your results.

13. $f(x) = 25 - x^2$

14. $f(x) = x^2 - 7$

15. $f(x) = \frac{1}{2}x^2 - 4$

16. $f(x) = 16 - \frac{1}{4}x^2$

17. $f(x) = (x + 4)^2 - 3$

18. $f(x) = (x - 6)^2 + 3$

19. $h(x) = x^2 - 8x + 16$

20. $g(x) = x^2 + 2x + 1$

21. $f(x) = x^2 - x + \frac{5}{4}$

22. $f(x) = x^2 + 3x + \frac{1}{4}$

23. $f(x) = -x^2 + 2x + 5$

24. $f(x) = -x^2 - 4x + 1$

25. $h(x) = 4x^2 - 4x + 21$

26. $f(x) = 2x^2 - x + 1$

In Exercises 27–34, use a graphing utility to graph the quadratic function. Identify the vertex and intercepts. Then check your results algebraically by completing the square.

27. $f(x) = -(x^2 + 2x - 3)$

28. $f(x) = -(x^2 + x - 30)$

29. $g(x) = x^2 + 8x + 11$

30. $f(x) = x^2 + 10x + 14$

31. $f(x) = 2x^2 - 16x + 31$

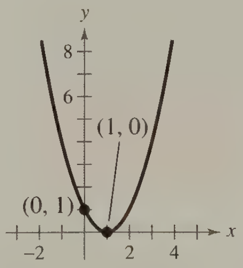
32. $f(x) = -4x^2 + 24x - 41$

33. $g(x) = \frac{1}{2}(x^2 + 4x - 2)$

34. $f(x) = \frac{3}{5}(x^2 + 6x - 5)$

In Exercises 35–38, find an equation for the parabola. Use a graphing utility to graph the equation and verify your result.

35.



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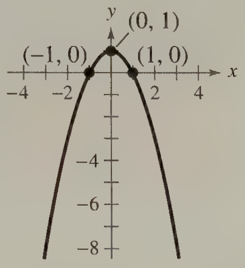
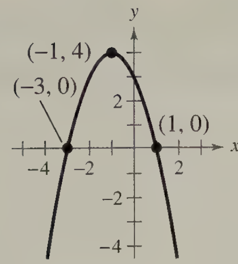


The *Interactive* CD-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered Section and Review Exercises. They also provide Tutorial Exercises, which link to Guided Examples for additional help.

37.



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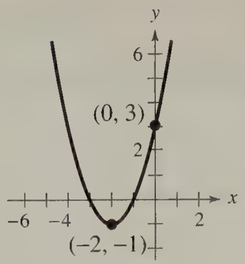


In Exercises 39–46, find the quadratic function that has the indicated vertex and whose graph passes through the given point. Confirm your result with a graphing utility.

39. Vertex: $(-2, 5)$; Point: $(0, 9)$

40. Vertex: $(4, -1)$; Point: $(2, 3)$

41. Vertex: $(3, 4)$; Point: $(1, 2)$

42. Vertex: $(2, 3)$; Point: $(0, 2)$

43. Vertex: $(-2, -2)$; Point: $(-1, 0)$

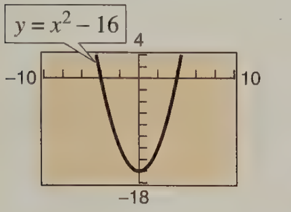
44. Vertex: $\left(-\frac{1}{4}, \frac{3}{2}\right)$; Point: $(-2, 0)$

45. Vertex: $\left(\frac{5}{2}, -\frac{3}{4}\right)$; Point: $(-2, 4)$

46. Vertex: $\left(-\frac{5}{2}, 0\right)$; Point: $\left(-\frac{7}{2}, -\frac{16}{3}\right)$

Graphical Reasoning **In Exercises 47–50, determine the *x*-intercepts of the graph visually. How do the *x*-intercepts correspond to the solutions of the quadratic equation when $y = 0$?**

47.



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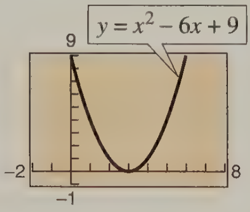

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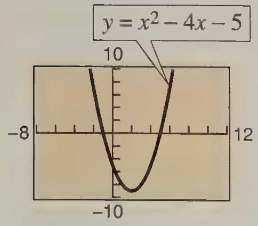

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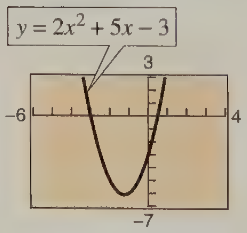


In Exercises 51–56, use a graphing utility to graph the quadratic function. Find the *x*-intercepts of the graph and compare them with the solutions of the corresponding quadratic equation when $y = 0$.

51. $y = x^2 - 4x$

52. $y = -2x^2 + 10x$

53. $y = 2x^2 - 7x - 30$

54. $y = 4x^2 + 25x - 21$

55. $y = -\frac{1}{2}(x^2 - 6x - 7)$

56. $y = \frac{7}{10}(x^2 + 12x - 45)$

In Exercises 57–60, find two quadratic functions, one that opens upward and one that opens downward, whose graphs have the given *x*-intercepts. (There are many correct answers.)

57. $(-1, 0), (3, 0)$

58. $(0, 0), (10, 0)$

59. $(-3, 0), \left(-\frac{1}{2}, 0\right)$

60. $\left(-\frac{5}{2}, 0\right), (2, 0)$

In Exercises 61–64, find two positive real numbers whose product is a maximum.

61. The sum is 110.

62. The sum is S.

63. The sum of the first and twice the second is 24.

64. The sum of the first and 3 times the second is 42.

Maximum Area **In Exercises 65 and 66, consider a rectangle of length *x* and perimeter *P*.**

(a) Express the area *A* as a function of *x* and determine the domain of the function.

(b) Use a graphing utility to graph the area function.

(c) Use the graph to approximate the length and width of the rectangle of maximum area, and verify algebraically.

65. $P = 100$ feet

66. $P = 36$ meters

67. *Geometry* An indoor physical fitness room consists of a rectangular region with a semicircle on each end. The perimeter of the room is to be a 200-meter running track.

(a) Draw a diagram to represent the problem. Let x and y represent the length and width of the rectangular region.

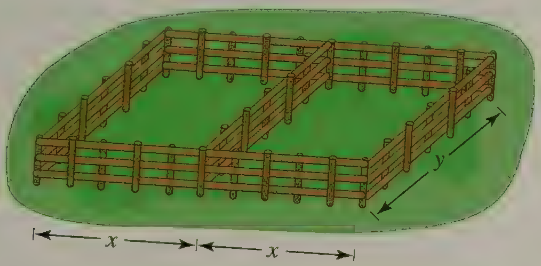
(b) Determine the radius of the semicircular ends of the track. Determine the distance, in terms of y, around the two semicircular parts of the track.

(c) Use the result of part (b) to write an equation, in terms of x and y, for the distance traveled in one lap around the track. Solve for y.

(d) Use the result of part (c) to write the area A of the rectangular region as a function of x.

(e) Use a graphing utility to graph the area function of part (d). Use the graph to approximate the dimensions that will produce a maximum area of the rectangle.

68. ***Numerical, Graphical, and Analytical Analysis*** A rancher has 200 feet of fencing to enclose two adjacent rectangular corrals. Use the following methods to determine the dimensions that will produce a maximum enclosed area.



(a) Complete six rows of a table such as the one below. (The first two rows are shown.)

x	y	Area
2	$\frac{1}{3}[200 - 4(2)]$	$2xy = 256$
4	$\frac{1}{3}[200 - 4(4)]$	$2xy \approx 491$

(b) Use a graphing utility to generate additional rows of the table in part (a). Use the table to estimate the dimensions that will produce the maximum enclosed area.

(c) Write the area A as a function of x.

(d) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions that will produce the maximum enclosed area.

(e) Write the area function in standard form to find algebraically the dimensions that will produce the maximum area.

(f) Compare your results from parts (b), (d), and (e).

69. ***Business*** A manufacturer of lighting fixtures has daily production costs of

$$C = 800 - 10x + 0.25x^2$$

where C is the total cost (in dollars) and x is the number of units produced. How many fixtures should be produced each day to yield a minimum cost?

70. ***Business*** A textile manufacturer has daily production costs of

$$C = 10{,}000 - 110x + 0.45x^2$$

where C is the total cost (in dollars) and x is the number of units produced. How many units should be produced each day to yield a minimum cost?

71. ***Business*** The profit P (in dollars) for a company is

$$P = -0.0002x^2 + 140x - 250{,}000$$

where x is the number of units sold. What sales level will yield maximum profit?

72. ***Business*** The profit P (in hundreds of dollars) that a company makes depends on the amount x (in hundreds of dollars) the company spends on advertising according to the model

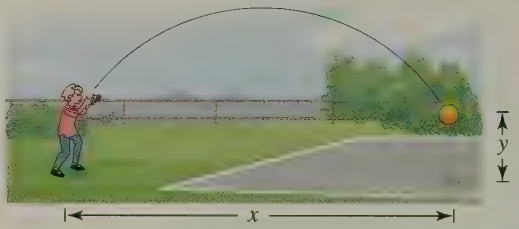
$$P = 230 + 20x - 0.5x^2.$$

What expenditure for advertising results in the maximum profit?

73. ***Trajectory of a Ball*** The height y (in feet) of a ball thrown by a child is

$$y = -\frac{1}{12}x^2 + 2x + 4$$

where x is the horizontal distance (in feet) from where the ball is thrown.



(a) Use a graphing utility to graph the path of the ball.

(b) How high is the ball when it leaves the child's hand? (*Hint:* Find y when $x = 0$.)

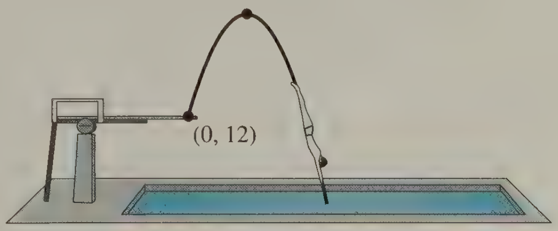
(c) How high is the ball when it is at its maximum height?

(d) How far from the child does the ball strike the ground?

74. ***Maximum Height of a Dive*** The path of a diver is

$$y = -\frac{4}{9}x^2 + \frac{24}{9}x + 12$$

where y is the height (in feet) and x is the horizontal distance (in feet) from the end of the diving board. What is the maximum height of the dive? Verify your answer using a graphing utility.



75. ***Forestry*** The number of board feet V in a 16-foot log is approximated by the model

$$V = 0.77x^2 - 1.32x - 9.31, \quad 5 \le x \le 40$$

where x is the diameter (in inches) of the log at the small end. (One board foot is a measure of volume equivalent to a board that is 12 inches wide, 12 inches long, and 1 inch thick.)

(a) Use a graphing utility to graph the function.

(b) Estimate the number of board feet in a 16-foot log with a diameter of 16 inches. Use a graphing utility to verify your answer.

(c) Estimate the diameter of a 16-foot log that scaled 500 board feet when the lumber was sold. Use a graphing utility to verify your answer.

76. ***Automobile Aerodynamics*** The number of horsepower y required to overcome wind drag on a certain automobile is approximated by

$$y = 0.002s^2 + 0.005s - 0.029, \quad 0 \le s \le 100$$

where s is the speed of the car in miles per hour.

(a) Use a graphing utility to graph the function.

(b) Graphically estimate the maximum speed of the car if the power required to overcome wind drag is not to exceed 10 horsepower. Verify your result algebraically.

77. ***Graphical Analysis*** For certain years from 1950 to 1990, the average annual per capita consumption C of cigarettes by Americans (18 and older) can be modeled by $C = 3248.89 + 108.64t - 2.97t^2$ for $0 \le t \le 40$, where t is the year, with $t = 0$ corresponding to 1950. (Source: U.S. Department of Agriculture)

(a) Use a graphing utility to graph the model.

(b) Use the graph of the model to approximate the maximum average annual consumption. Beginning in 1966, all cigarette packages were required by law to carry a health warning. Do you think the warning had any effect? Explain.

(c) In 1960, the U.S. population (18 and over) was 116,530,000. Of those, about 48,500,000 were smokers. What was the average annual cigarette consumption *per smoker* in 1960? What was the average daily cigarette consumption *per smoker*?

78. ***Data Analysis*** The number y (in millions) of VCRs in use in the United States for the years 1987 through 1996 are shown in the table. The variable t represents time (in years), with $t = 7$ corresponding to 1987.

t	7	8	9	10	11	12	13	14	15	16
y	43	51	58	63	67	69	72	74	77	79

(Source: Television Bureau of Advertising, Inc.)

(a) Use a graphing utility to sketch a scatter plot of the data.

(b) Use the regression capabilities of a graphing utility to fit a quadratic model to the data.

(c) Use a graphing utility to graph the model in the same viewing window as the scatter plot.

(d) Do you think the model can be used to estimate VCR utilization in the year 2005? Explain.

Synthesis

True or False? **In Exercises 79 and 80, determine whether the statement is true or false. Justify your answer.**

79. The function $f(x) = -12x^2 - 1$ has no x-intercepts.

80. The graphs of $f(x) = -4x^2 - 10x + 7$ and $g(x) = 12x^2 + 30x + 1$ have the same axis of symmetry.

81. ***Think About It*** The profits P (in millions of dollars) for a company are modeled by a quadratic function of the form $P = at^2 + bt + c$, where t represents the year. If you were president of the company, which of the models would you prefer? Explain your reasoning.

(a) a is positive and $t \ge -b/(2a)$.

(b) a is positive and $t \le -b/(2a)$.

(c) a is negative and $t \ge -b/(2a)$.

(d) a is negative and $t \le -b/(2a)$.

Review

In Exercises 82–85, determine algebraically any points of intersection of the graphs of the equations. Verify your results using the *intersect* feature of a graphing utility.

82. $x + y = 8$
$-\frac{2}{3}x + y = 6$

83. $y = 3x - 10$
$y = \frac{1}{4}x + 1$

84. $y = 9 - x^2$
$y = x + 3$

85. $y = x^3 + 2x - 1$
$y = -2x + 15$

In Exercises 86–89, perform the operation and write the result in standard form.

86. $(6 - i) - (2i + 11)$

87. $(2i + 5)^2 - 21$

88. $(3i + 7)(-4i + 1)$

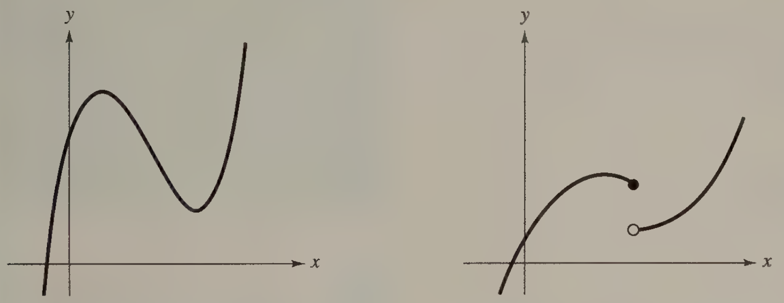
89. $(4 - i)^3$

3.2 Polynomial Functions of Higher Degree

Graphs of Polynomial Functions

You should be able to sketch accurate graphs of polynomial functions of degrees 0, 1, and 2. The graphs of polynomial functions of degree greater than 2 are more difficult to sketch by hand. However, in this section you will learn how to recognize some of the basic features of the graphs of polynomial functions.

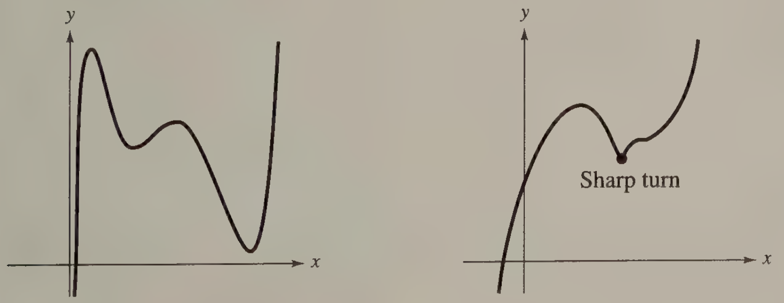
The graph of a polynomial function is **continuous.** Essentially, this means that the graph of a polynomial function has no breaks, holes, or gaps, as shown in Figure 3.9. Another feature of the graph of a polynomial function is that it has only smooth, rounded turns, as shown in Figure 3.10(a). It cannot have a sharp, pointed turn such as the one shown in Figure 3.10(b).



(a) Polynomial functions have continuous graphs.

(b) Functions with graphs that are not continuous are not polynomial functions.

Figure 3.9



(a) Polynomial functions have smooth, rounded graphs.

(b) Graphs of polynomial functions cannot have sharp turns.

Figure 3.10

Informally, you can say that a function is continuous if its graph can be drawn with a pencil without lifting the pencil from the paper.

What You Should Learn:

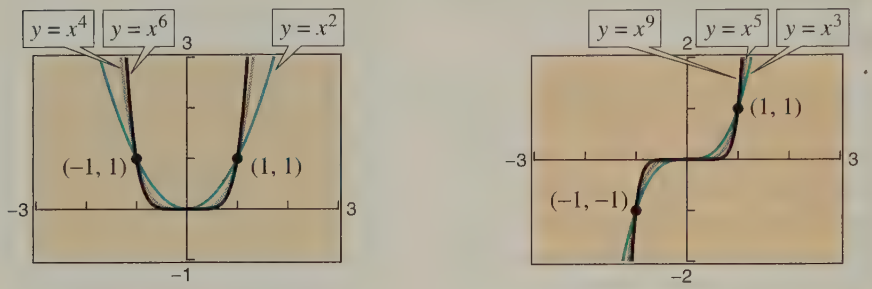
- How to use transformations to sketch graphs of polynomial functions
- How to use the Leading Coefficient Test to determine the end behavior of graphs of polynomial functions
- How to find and use zeros of polynomial functions to sketch their graphs
- How to use the Intermediate Value Theorem to help locate zeros of polynomial functions

Why You Should Learn It:

You can use polynomial functions to model various aspects of nature, such as the growth of a red oak tree, as shown in Exercise 98 on page 248.

The Granger Collection

The polynomial functions that have the simplest graphs are monomials of the form $f(x) = x^n$, where n is an integer greater than zero.



If n is even, the graph of $y = x^n$ touches the axis at the x-intercept.

If n is odd, the graph of $y = x^n$ crosses the axis at the x-intercept.

Figure 3.11

In Figure 3.11, you can see that when n is *even* the graph is similar to the graph of $f(x) = x^2$, and when n is *odd* the graph is similar to the graph of $f(x) = x^3$. Moreover, the greater the value of n, the flatter the graph is on the interval $(-1, 1)$.

Exploration

Use a graphing utility to graph $y = x^n$ with $n = 2, 4$, and 8. (Use the viewing window $-1.5 \le x \le 1.5$ and $-1 \le y \le 6$.) Compare the graphs. In the interval $(-1, 1)$, which graph is on the bottom? Outside the interval $(-1, 1)$, which graph is on the bottom?

Use a graphing utility to graph $y = x^n$ with $n = 3, 5$, and 7. (Use the viewing window $-1.5 \le x \le 1.5$ and $-4 \le y \le 4$.) Compare the graphs. In the intervals $(-\infty, -1)$ and $(0, 1)$, which graph is on the bottom? In the intervals $(-1, 0)$ and $(1, \infty)$, which graph is on the bottom?

The *Interactive* CD-ROM and *Internet* versions of this text offer a built-in graphing calculator, which can be used with the Examples, Explorations, and Exercises.

EXAMPLE 1 Sketching Transformations of Polynomial Functions

Sketch the graph of each polynomial function.

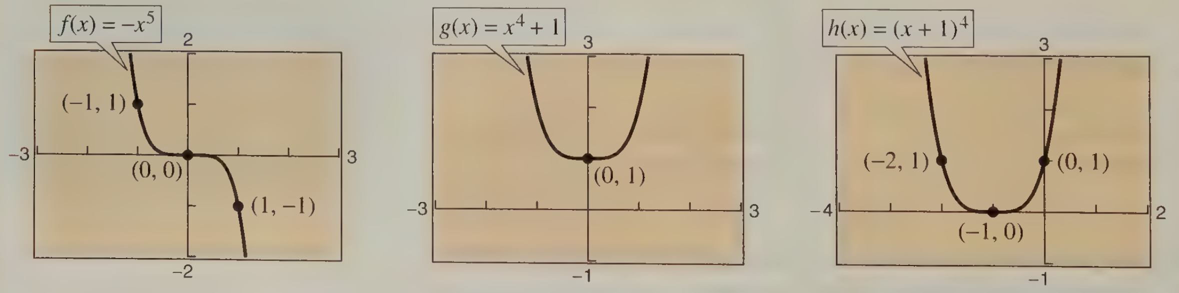
a. $f(x) = -x^5$ **b.** $g(x) = x^4 + 1$ **c.** $h(x) = (x + 1)^4$

Solution

a. Because the degree of $f(x) = -x^5$ is odd, the graph is similar to the graph of $y = x^3$. Moreover, the negative coefficient reflects the graph in the x-axis, as shown in Figure 3.12(a).

b. The graph of $g(x) = x^4 + 1$ is an upward shift, by one unit, of the graph of $y = x^4$, as shown in Figure 3.12(b).

c. The graph of $h(x) = (x + 1)^4$ is a left shift, by one unit, of the graph of $y = x^4$, as shown in Figure 3.12(c).



(a) **(b)** **(c)**

Figure 3.12

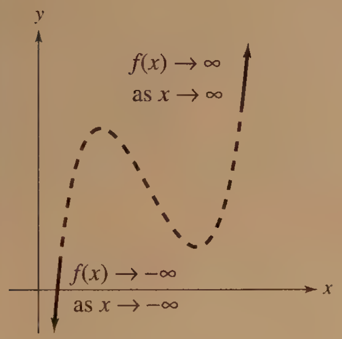
The Leading Coefficient Test

In Example 1, note that all three graphs eventually rise or fall without bound as x moves to the right. Whether the graph of a polynomial eventually rises or falls can be determined by the function's degree (even or odd) and by its leading coefficient, as indicated in the **Leading Coefficient Test.**

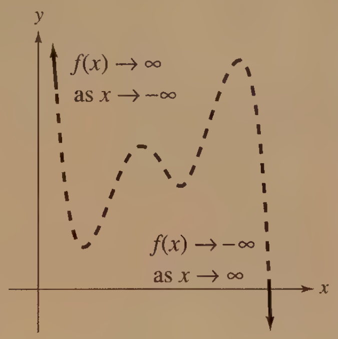
Leading Coefficient Test

As x moves without bound to the left or to the right, the graph of the polynomial function $f(x) = a_nx^n + \cdot \cdot \cdot + a_1x + a_0$ eventually rises or falls in the following manner.

1. When n is odd:

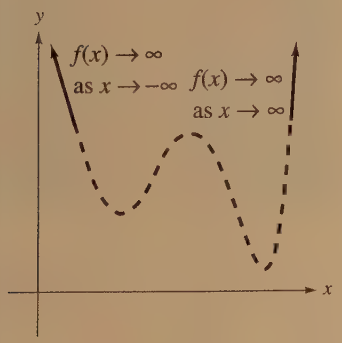


If the leading coefficient is positive ($a_n > 0$), the graph falls to the left and rises to the right.

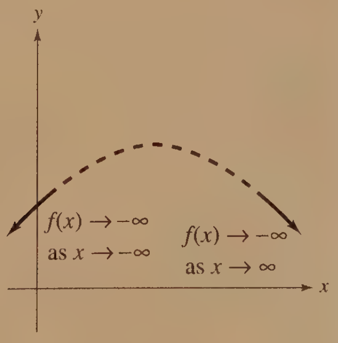


If the leading coefficient is negative ($a_n < 0$), the graph rises to the left and falls to the right.

2. When n is even:



If the leading coefficient is positive ($a_n > 0$), the graph rises to the left and right.



If the leading coefficient is negative ($a_n < 0$), the graph falls to the left and right.

Note that the dashed portions of the graphs indicate that the test determines *only* the right and left behavior of the graph.

Exploration

For each function, identify the degree of the function and whether the degree of the function is even or odd. Identify the leading coefficient and note whether the leading coefficient is positive or negative. Use a graphing utility to graph each function. Describe the relationship between the degree and leading coefficient of the function and the behavior of the ends of the graph of the function.

a. $y = x^3 - 2x^2 - x + 1$
b. $y = 2x^5 + 2x^2 - 5x + 1$
c. $y = -2x^5 - x^2 + 5x + 3$
d. $y = -x^3 + 5x - 2$
e. $y = 2x^2 + 3x - 4$
f. $y = x^4 - 3x^2 + 2x - 1$
g. $y = -x^2 + 3x + 2$
h. $y = -x^6 - x^2 - 5x + 4$

Library of Functions

The graphs of polynomials of degree 1 are lines, and those of degree 2 are parabolas. The graphs of polynomials of higher degree are smooth and continuous. The graphs eventually rise or fall without bound as x moves to the right (or left).

Consult the Library of Functions Summary inside the front cover for a description of the polynomial function.

EXAMPLE 2 Applying the Leading Coefficient Test

Use the Leading Coefficient Test to determine the left and right behavior of the graph of each polynomial function.

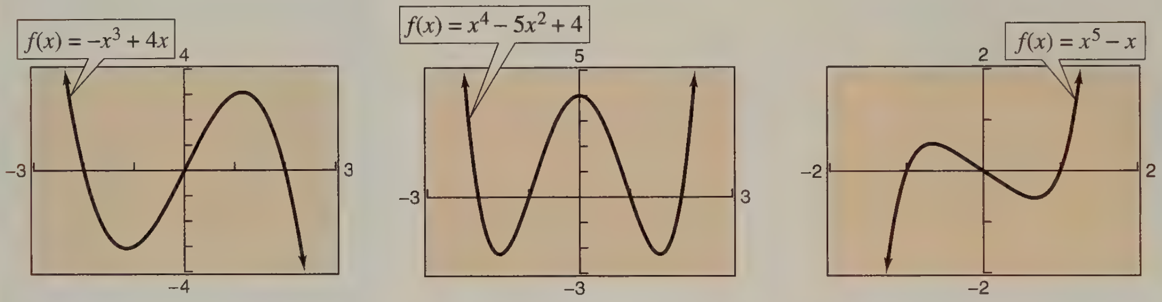
a. $f(x) = -x^3 + 4x$ **b.** $f(x) = x^4 - 5x^2 + 4$ **c.** $f(x) = x^5 - x$

Solution

a. Because the degree is odd and the leading coefficient is negative, the graph rises to the left and falls to the right, as shown in Figure 3.13(a).

b. Because the degree is even and the leading coefficient is positive, the graph rises to the left and right, as shown in Figure 3.13(b).

c. Because the degree is odd and the leading coefficient is positive, the graph falls to the left and rises to the right, as shown in Figure 3.13(c).



(a) (b) (c)

Figure 3.13

In Example 2, note that the Leading Coefficient Test only tells you whether the graph *eventually* rises or falls to the right or left. Other characteristics of the graph, such as intercepts and minimum and maximum points, must be determined by other tests.

Exploration

For each of the graphs in Figure 3.13, count the number of zeros of the polynomial function and the number of relative extrema, and compare these numbers with the degree of the polynomial. What do you observe?

Zeros of Polynomial Functions

It can be shown that for a polynomial function f of degree n, the following statements are true.

1. The graph of f has at most n real zeros. (This result is discussed in detail in Section 3.4.)
2. The function f has at most $n - 1$ relative **extrema** (relative **minimums** or **maximums**).

Recall that a **zero** of a function f is a number x for which $f(x) = 0$. Finding the zeros of polynomial functions is one of the most important problems in algebra. You have already seen that there is a strong interplay between graphical and algebraic approaches to this problem. Sometimes you can use information about the graph of a function to help find its zeros. In other cases you can use information about the zeros of a function to find a good viewing window.

Real Zeros of Polynomial Functions

If f is a polynomial function and a is a real number, the following statements are equivalent.

1. $x = a$ is a *zero* of the function f.
2. $x = a$ is a *solution* of the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a *factor* of the polynomial $f(x)$.
4. $(a, 0)$ is an *x-intercept* of the graph of f.

Finding zeros of polynomial functions is closely related to factoring and finding x-intercepts, as demonstrated in Examples 3, 4, and 5.

STUDY TIP

Some graphing utilities have two features that analyze the graph of a function: (1) the *root* feature or the *zero* feature for finding zeros of a function, and (2) the *minimum* and *maximum* features for finding the relative extrema. If your graphing utility has these features, use them to confirm the results in Examples 3 and 4.

EXAMPLE 3 Finding Zeros of a Polynomial Function

Find all real zeros of $f(x) = x^3 - x^2 - 2x$.

Algebraic Solution

$$f(x) = x^3 - x^2 - 2x \qquad \text{Write original function.}$$

$$0 = x^3 - x^2 - 2x \qquad \text{Substitute 0 for } f(x).$$

$$= x(x^2 - x - 2) \qquad \text{Remove common monomial factor.}$$

$$= x(x - 2)(x + 1) \qquad \text{Factor completely.}$$

So, the real zeros are $x = 0$, $x = 2$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(2, 0)$, and $(-1, 0)$.

Check

$$(0)^3 - (0)^2 - 2(0) = 0 \qquad x = 0 \text{ is a zero. } \checkmark$$

$$(2)^3 - (2)^2 - 2(2) = 0 \qquad x = 2 \text{ is a zero. } \checkmark$$

$$(-1)^3 - (-1)^2 - 2(-1) = 0 \qquad x = -1 \text{ is a zero. } \checkmark$$

Graphical Solution

Use a graphing utility to graph $y = x^3 - x^2 - 2x$. In Figure 3.14, the graph appears to have the x-intercepts $(0, 0)$, $(2, 0)$, and $(-1, 0)$. Use the *zero* or *root* feature, or the *zoom* and *trace* features, of the graphing utility to verify these intercepts. Note that this third-degree polynomial has two relative extrema.

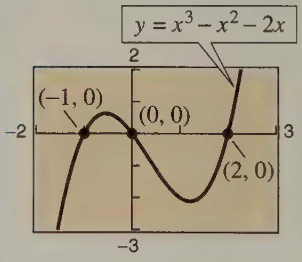


Figure 3.14

EXAMPLE 4 Analyzing a Polynomial Function

Find all real zeros and relative extrema of $f(x) = -2x^4 + 2x^2$.

Solution

$$f(x) = -2x^4 + 2x^2 \qquad \text{Write original function.}$$

$$0 = -2x^4 + 2x^2 \qquad \text{Substitute 0 for } f(x).$$

$$= -2x^2(x^2 - 1) \qquad \text{Remove common monomial factor.}$$

$$= -2x^2(x - 1)(x + 1) \qquad \text{Factor completely.}$$

So, the real zeros are $x = 0$, $x = 1$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(1, 0)$, and $(-1, 0)$, as shown in Figure 3.15. Using the *minimum* and *maximum* features of a graphing utility, you can approximate the three relative extrema to be $(-0.7071, 0.5)$, $(0, 0)$, and $(0.7071, 0.5)$.

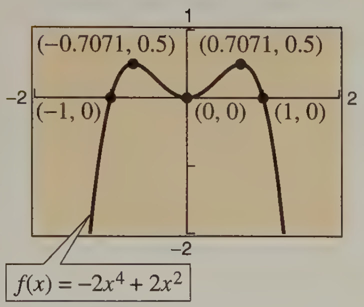


Figure 3.15

In Example 4, the real zero $x = 0$ arising from $-2x^2 = 0$ is a **repeated zero.** In general, a factor of $(x - a)^k$ yields a repeated zero $x = a$ of **multiplicity** k. If k is odd, the graph *crosses* the x-axis at $x = a$. If k is even, the graph *touches* (but does not cross) the x-axis at $x = a$, as shown in Figure 3.15.

EXAMPLE 5 Finding Zeros of a Polynomial Function

Find all real zeros of $f(x) = x^5 - 3x^3 - x^2 - 4x - 1$.

Solution

Use a graphing utility to obtain the graph shown in Figure 3.16. From the graph, you can see that there are three zeros. Using the *zero* or *root* feature, you can determine that the zeros are approximately $x \approx -1.861$, $x \approx -0.254$, and $x \approx 2.115$. It should be noted that this fifth-degree polynomial factors as

$$\begin{aligned} f(x) &= x^5 - 3x^3 - x^2 - 4x - 1 \\ &= (x^2 + 1)(x^3 - 4x - 1). \end{aligned}$$

The three zeros obtained above are the zeros of the cubic $x^3 - 4x - 1$ (the quadratic $x^2 + 1$ has two complex zeros and, so, no *real* zeros).

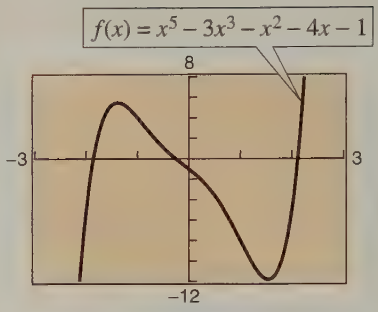


Figure 3.16

EXAMPLE 6 Finding a Polynomial Function with Given Zeros

Find polynomial functions with the following zeros. (There are many correct solutions.)

a. $-2, -1, 1, 2$ **b.** $-\frac{1}{2}, 3, 3$ **c.** $3, 2 + \sqrt{11}, 2 - \sqrt{11}$

Solution

a. For each of the given zeros, form a corresponding factor. For instance, the zero given by $x = -2$ corresponds to the factor $(x + 2)$. So, you can write

$$\begin{aligned} f(x) &= (x + 2)(x + 1)(x - 1)(x - 2) \\ &= (x^2 - 4)(x^2 - 1) = x^4 - 5x^2 + 4. \end{aligned}$$

b. Note that the zero $x = -\frac{1}{2}$ corresponds to either $\left(x + \frac{1}{2}\right)$ or $(2x + 1)$. To avoid fractions, choose the second factor and write

$$\begin{aligned} f(x) &= (2x + 1)(x - 3)^2 \\ &= (2x + 1)(x^2 - 6x + 9) = 2x^3 - 11x^2 + 12x + 9. \end{aligned}$$

c. For each of the given zeros, form a corresponding factor and write

$$\begin{aligned} f(x) &= (x - 3)\left[x - \left(2 + \sqrt{11}\right)\right]\left[x - \left(2 - \sqrt{11}\right)\right] \\ &= (x - 3)\left[(x - 2) - \sqrt{11}\right]\left[(x - 2) + \sqrt{11}\right] \\ &= (x - 3)\left[(x - 2)^2 - \left(\sqrt{11}\right)^2\right] \\ &= (x - 3)(x^2 - 4x + 4 - 11) \\ &= (x - 3)(x^2 - 4x - 7) = x^3 - 7x^2 + 5x + 21. \end{aligned}$$

Exploration

Use a graphing utility to graph

$$y_1 = x + 2$$

$$y_2 = (x + 2)(x - 1).$$

Predict the shape of the curve $y = (x + 2)(x - 1)(x - 3)$, and verify your answer with a graphing utility.

EXAMPLE 7 Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = 3x^4 - 4x^3$ by hand.

Solution

1. *Apply Leading Coefficient Test.* Because the leading coefficient is positive and the degree is even, you know that the graph eventually rises to the left and to the right [see Figure 3.17(a)].
2. *Find the Zeros of the Polynomial.* By factoring

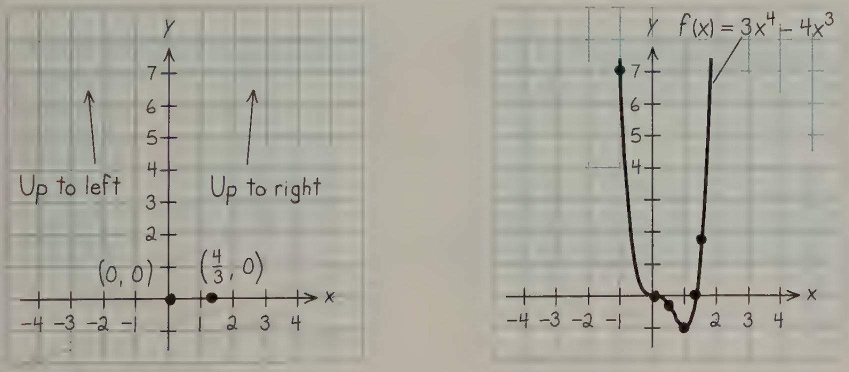
$$f(x) = 3x^4 - 4x^3$$
$$= x^3(3x - 4)$$

you can see that the zeros of f are $x = 0$ (of odd multiplicity 3) and $x = \frac{4}{3}$ (of odd multiplicity 1). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{4}{3}, 0\right)$. Add these points to your graph, as shown in Figure 3.17(a).

3. *Plot a Few Additional Points.* To sketch the graph by hand, find a few additional points, as shown in the table. Be sure to choose points between the zeros and to the left and right of the zeros. Then plot the points [see Figure 3.17(b)].

x	-1	0.5	1	1.5
$f(x)$	7	-0.3125	-1	1.6875

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 3.17(b). If you are unsure of the shape of the portion of the graph, plot some additional points.



(a) (b)

Figure 3.17

STUDY T!P

It is easy to make mistakes when entering functions into a graphing utility. So, it is important to have an understanding of the basic shapes of graphs and to be able to graph simple polynomials *by hand*. For example, suppose you had entered the function in Example 7 as $y = 3x^5 - 4x^3$. By looking at the graph, what mathematical principles would alert you to the fact that you had made a mistake?

Exploration

Partner Activity Multiply 3, 4, or 5 distinct linear factors to obtain the equation of a polynomial function that has a degree of 3, 4, or 5. Exchange equations with your partner and sketch, *by hand*, the graph of the equation that your partner wrote. When you are finished, use a graphing utility to check each other's work.

EXAMPLE 8 Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = -2x^3 + 6x^2 - \frac{9}{2}x$.

Solution

1. *Apply Leading Coefficient Test.* Because the leading coefficient is negative and the degree is odd, you know that the graph eventually rises to the left and falls to the right [see Figure 3.18(a)].
2. *Find the Zeros of the Polynomial.* By factoring

   $$f(x) = -2x^3 + 6x^2 - \tfrac{9}{2}x = -\tfrac{1}{2}x(4x^2 - 12x + 9) = -\tfrac{1}{2}x(2x - 3)^2$$

   you can see that the zeros of f are

   $$x = 0 \text{ (of odd multiplicity 1) and } x = \tfrac{3}{2} \text{ (of even multiplicity 2).}$$

   So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{3}{2}, 0\right)$. Add these points to your graph, as shown in Figure 3.18(a).
3. *Plot a Few Additional Points.* To sketch the graph by hand, find a few additional points, as shown in the table. Then plot the points [see Figure 3.18(b)].

x	-0.5	0.5	1	2
$f(x)$	4	-1	-0.5	-1

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 3.18(b). Notice that the graph crosses the x-axis at $(0, 0)$ and touches (but does not cross) the x-axis at $\left(\frac{3}{2}, 0\right)$.

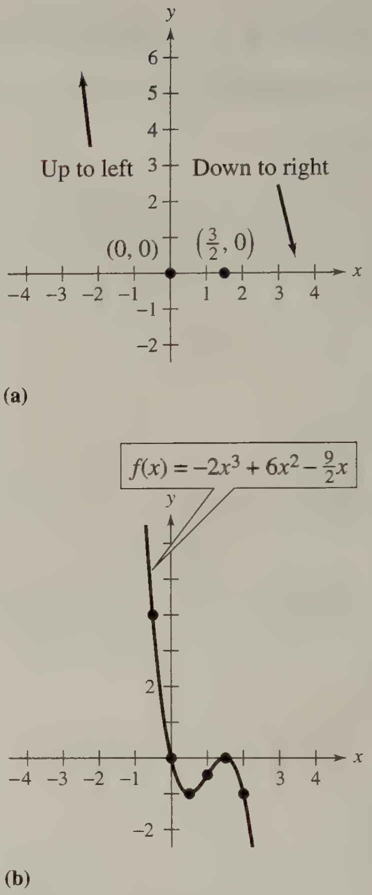


Figure 3.18

The Intermediate Value Theorem

The **Intermediate Value Theorem** concerns the existence of real zeros of polynomial functions. The theorem states that if $(a, f(a))$ and $(b, f(b))$ are two points on the graph of a polynomial function such that $f(a) \neq f(b)$, then for any number d between $f(a)$ and $f(b)$ there must be a number c between a and b such that $f(c) = d$. (See Figure 3.19.)

Intermediate Value Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

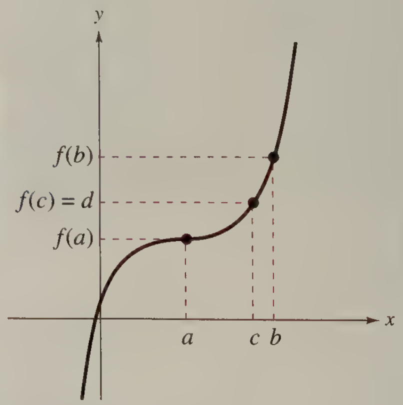


Figure 3.19

This theorem helps locate the real zeros of a polynomial function in the following way. If you can find a value $x = a$ where a polynomial function is positive, and another $x = b$ where it is negative, you can conclude that the function has at least one real zero between these two values. For example, the function $f(x) = x^3 + x^2 + 1$ is negative when $x = -2$ and positive when $x = -1$. Therefore, it follows from the Intermediate Value Theorem that f must have a real zero somewhere between -2 and -1.

EXAMPLE 9 Approximating the Zeros of a Function

Find three intervals of length 1 in which the polynomial

$$f(x) = 12x^3 - 32x^2 + 3x + 5$$

is guaranteed to have a zero.

Graphical Solution

Use a graphing utility to graph

$$y = 12x^3 - 32x^2 + 3x + 5$$

as shown in Figure 3.20.

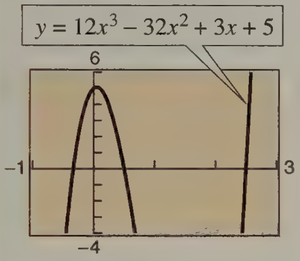
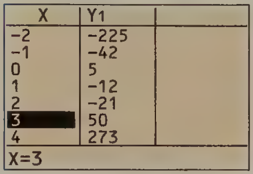


Figure 3.20

From the figure, you can see that the graph crosses the x-axis three times—between -1 and 0, between 0 and 1, and another between 2 and 3. So, you can conclude that the function has zeros in the intervals $(-1, 0)$, $(0, 1)$, and $(2, 3)$.

Numerical Solution

Use the *table* feature of a graphing utility to create a table that shows the function values for $-20 \le x \le 20$ as shown in Figure 3.21.



```
 X   | Y1
-2   | -225
-1   | -42
0    | 5
1    | -12
2    | -21
3    | 50
4    | 273
X=3
```

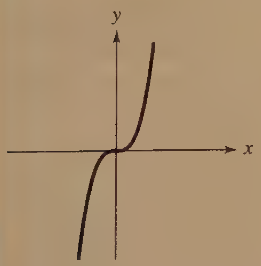
Figure 3.21

Scroll through the table looking for consecutive function values that differ in sign. For instance, from the table you can see that $f(-1)$ and $f(0)$ differ in sign. So, you can conclude from the Intermediate Value Theorem that the function has a zero between -1 and 0. Similarly, $f(0)$ and $f(1)$ differ in sign, so the function has a zero between 0 and 1. Likewise, $f(2)$ and $f(3)$ differ in sign, so the function has a zero between 2 and 3. So, you can conclude that the function has zeros in the intervals $(-1, 0)$, $(0, 1)$, and $(2, 3)$.

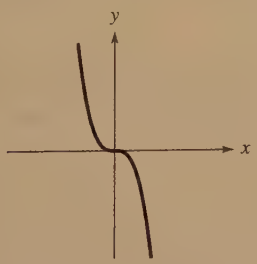
Writing About Math *The Graphs of Cubic Polynomials*

The graphs of cubic polynomials can be categorized according to the four basic shapes below. Match the graph of each function with one of the basic shapes and write a short paragraph describing how you reached your conclusion. Is it possible for a polynomial of odd degree to have no real zeros? Explain.

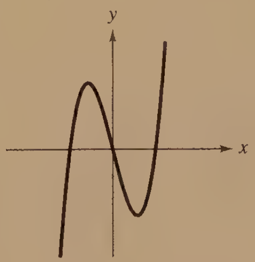
(a)



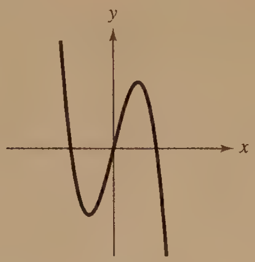
(b)



(c)



(d)

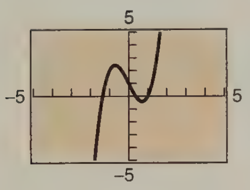


1. $f(x) = -x^3$
2. $f(x) = -x^3 + 4x$
3. $f(x) = x^3$
4. $f(x) = x^3 - 4x$

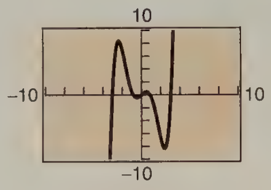
3.2 Exercises

In Exercises 1–8, match the polynomial function with its graph. [The graphs are labeled (a) through (h).]

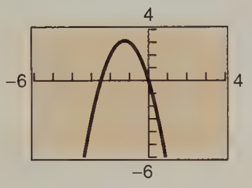
(a)



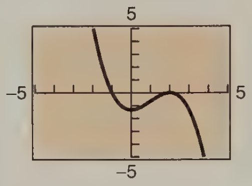
(b)



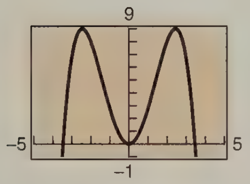
(c)



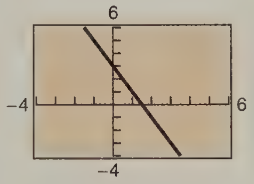
(d)



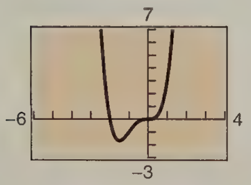
(e)



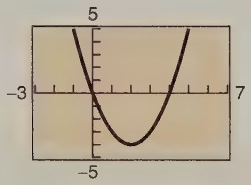
(f)



(g)



(h)



1. $f(x) = -2x + 3$ **2.** $f(x) = x^2 - 4x$
3. $f(x) = -2x^2 - 5x$ **4.** $f(x) = 2x^3 - 3x + 1$
5. $f(x) = -\frac{1}{4}x^4 + 3x^2$ **6.** $f(x) = -\frac{1}{3}x^3 + x^2 - \frac{4}{3}$
7. $f(x) = x^4 + 2x^3$ **8.** $f(x) = \frac{1}{5}x^5 - 2x^3 + \frac{9}{5}x$

In Exercises 9–12, sketch the graphs of $y = x^n$ and each specified transformation.

9. $y = x^3$
(a) $f(x) = (x - 2)^3$ (b) $f(x) = x^3 - 2$
(c) $f(x) = -\frac{1}{2}x^3$ (d) $f(x) = (x - 2)^3 - 2$

10. $y = x^5$
(a) $f(x) = (x + 3)^5$ (b) $f(x) = x^5 + 3$
(c) $f(x) = 1 - \frac{1}{2}x^5$ (d) $f(x) = -\frac{1}{2}(x + 1)^5$

11. $y = x^4$
(a) $f(x) = (x + 5)^4$ (b) $f(x) = x^4 - 5$
(c) $f(x) = 4 - x^4$ (d) $f(x) = \frac{1}{2}(x - 1)^4$

12. $y = x^6$
(a) $f(x) = -\frac{1}{8}x^6$ (b) $f(x) = x^6 - 4$
(c) $f(x) = -\frac{1}{4}x^6 + 1$ (d) $f(x) = (x + 2)^6 - 4$

Graphical Analysis **In Exercises 13–16, use a graphing utility to graph the functions f and g in the same viewing window. Zoom out sufficiently far to show that the right-hand and left-hand behavior of f and g appear identical. Show the graphs.**

13. $f(x) = 3x^3 - 9x + 1,\quad g(x) = 3x^3$
14. $f(x) = -\frac{1}{3}(x^3 - 3x + 2),\quad g(x) = -\frac{1}{3}x^3$
15. $f(x) = -(x^4 - 4x^3 + 16x),\quad g(x) = -x^4$
16. $f(x) = 3x^4 - 6x^2,\quad g(x) = 3x^4$

In Exercises 17–26, use the Leading Coefficient Test to determine the right-hand and left-hand behavior of the graph of the polynomial function. Verify your result by using a graphing utility to graph the function.

17. $f(x) = 2x^2 - 3x + 1$ **18.** $f(x) = \frac{1}{3}x^3 + 5x$
19. $g(x) = 5 - \frac{7}{2}x - 3x^2$ **20.** $h(x) = 1 - x^6$
21. $f(x) = -2.1x^5 + 4x^3 - 2$
22. $f(x) = 2x^5 - 5x + 7.5$
23. $f(x) = 6 - 2x + 4x^2 - 5x^3$
24. $f(x) = \dfrac{3x^4 - 2x + 5}{4}$
25. $h(t) = -\frac{2}{3}(t^2 - 5t + 3)$
26. $f(s) = -\frac{7}{8}(s^3 + 5s^2 - 7s + 1)$

In Exercises 27–36, find all the real zeros of the polynomial function. Verify your result by using a graphing utility to graph the function.

27. $f(x) = x^2 - 25$ **28.** $f(x) = 49 - x^2$
29. $h(t) = t^2 - 6t + 9$ **30.** $f(x) = x^2 + 10x + 25$
31. $f(x) = x^2 + x - 2$ **32.** $f(x) = 2x^2 - 14x + 24$
33. $f(t) = t^3 - 4t^2 + 4t$ **34.** $f(x) = x^4 - x^3 - 20x^2$
35. $f(x) = \frac{1}{2}x^2 + \frac{5}{2}x - \frac{3}{2}$ **36.** $f(x) = \frac{5}{3}x^2 + \frac{8}{3}x - \frac{4}{3}$

Graphical Analysis **In Exercises 37–48, (a) use a graphing utility to graph the function, (b) use the graph to approximate any zeros, and (c) find the zeros algebraically.**

37. $f(x) = 3x^2 - 12x + 3$

38. $g(x) = 5(x^2 - 2x - 1)$

39. $g(t) = \frac{1}{2}t^4 - \frac{1}{2}$

40. $y = \frac{1}{4}x^3(x^2 - 9)$

41. $f(x) = x^5 + x^3 - 6x$

42. $g(t) = t^5 - 6t^3 + 9t$

43. $f(x) = 2x^4 - 2x^2 - 40$

44. $f(x) = 5x^4 + 15x^2 + 10$

45. $f(x) = x^3 - 4x^2 - 25x + 100$

46. $y = 4x^3 + 4x^2 - 7x + 2$

47. $y = 4x^3 - 20x^2 + 25x$

48. $y = x^5 - 5x^3 + 4x$

In Exercises 49–52, use a graphing utility to graph the function and approximate (accurate to three decimal places) any relative extrema.

49. $f(x) = 2x^4 - 6x^2 + 1$

50. $f(x) = -3x^3 - 4x^2 + x - 3$

51. $f(x) = x^5 + 3x^3 - x + 6$

52. $f(x) = -\frac{3}{8}x^4 - x^3 + 2x^2 + 5$

In Exercises 53–64, find a polynomial function that has the given zeros. (There are many correct answers.)

53. $0, 12$

54. $0, -8$

55. $2, -6$

56. $-4, 5$

57. $0, -4, -3$

58. $0, 2, 7$

59. $4, -3, 3, 0$

60. $-2, -1, 0, 1, 2$

61. $1 + \sqrt{3}, 1 - \sqrt{3}$

62. $6 + \sqrt{3}, 6 - \sqrt{3}$

63. $2, 4 + \sqrt{5}, 4 - \sqrt{5}$

64. $4, 2 + \sqrt{7}, 2 - \sqrt{7}$

In Exercises 65–78, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.

65. $f(x) = x^3 - 9x$

66. $g(x) = x^4 - 4x^2$

67. $f(t) = \frac{1}{4}(t^2 - 2t + 15)$

68. $g(x) = -x^2 + 10x - 16$

69. $f(x) = x^3 - 3x^2$

70. $f(x) = 1 - x^3$

71. $f(x) = 3x^3 - 15x^2 + 18x$

72. $f(x) = -4x^3 + 4x^2 + 15x$

73. $f(x) = -x^3 - 5x^2$

74. $f(x) = 3x^4 - 48x^2$

75. $f(x) = x^2(x - 4)$

76. $h(x) = \frac{1}{3}x^3(x - 4)^2$

77. $g(t) = -\frac{1}{4}(t - 2)^2(t + 2)^2$

78. $g(x) = \frac{1}{10}(x + 1)^2(x - 3)^2$

In Exercises 79–82, (a) use the Intermediate Value Theorem and a graphing utility to find intervals of length 1 in which the polynomial function is guaranteed to have a zero, (b) use the *root* or *zero* feature of a graphing utility to approximate the zeros of the function, and (c) verify your answers in part (a) by using the *table* feature of a graphing utility.

79. $f(x) = x^3 - 3x^2 + 3$

80. $f(x) = 0.11x^3 - 2.07x^2 + 9.81x - 6.88$

81. $g(x) = 3x^4 + 4x^3 - 3$

82. $h(x) = x^4 - 10x^2 + 2$

In Exercises 83–86, use a graphing utility to graph the function. Describe a viewing window that gives a good view of the basic characteristics of the graph. (There are many correct answers.)

83. $f(x) = -\frac{3}{2}$

84. $h(x) = \frac{1}{3}x - 3$

85. $f(t) = \frac{1}{6}(t^2 - 4t + 21)$

86. $g(x) = -x^2 + 9x - 14$

In Exercises 87–94, use a graphing utility to graph the function. Identify any symmetry with respect to the *x*-axis, *y*-axis, or origin. Determine the number of *x*-intercepts of the graph.

87. $f(x) = x^2(x + 6)$

88. $h(x) = x^3(x - 4)^2$

89. $g(t) = -\frac{1}{2}(t - 4)^2(t + 4)^2$

90. $g(x) = \frac{1}{8}(x + 1)^2(x - 3)^3$

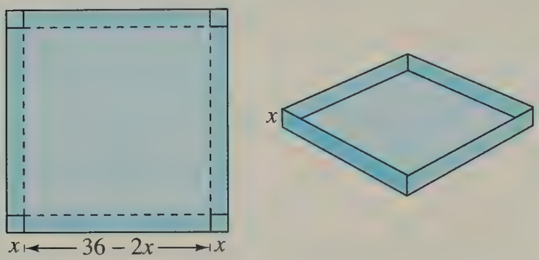
91. $f(x) = x^3 - 4x$

92. $f(x) = x^4 - 2x^2$

93. $g(x) = \frac{1}{5}(x + 1)^2(x - 3)(2x - 9)$

94. $h(x) = \frac{1}{5}(x + 2)^2(3x - 5)^2$

95. ***Numerical and Graphical Analysis*** An open box is to be made from a square piece of material 36 centimeters on a side by cutting equal squares from the corners and turning up the sides.



(a) Complete four rows of a table like the one below.

Height	Width	Volume
1	$36 - 2(1)$	$1[36 - 2(1)]^2 = 1156$
2	$36 - 2(2)$	$2[36 - 2(2)]^2 = 2048$

(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate a range of dimensions within which the maximum volume is produced.

(c) Verify that the volume of the box is $V(x) = x(36 - 2x)^2$. Determine the domain of the function.

(d) Use a graphing utility to graph V, and use the range of dimensions from part (b) to find the x-value for which $V(x)$ is maximum.

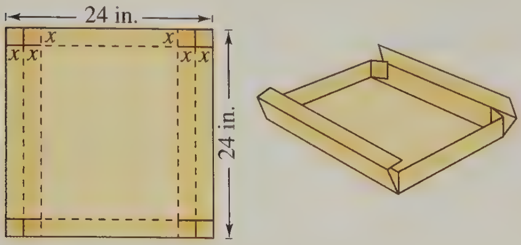
96. ***Geometry*** An open box with locking tabs is to be made from a square piece of material 24 inches on a side. This is done by cutting equal squares from the corners and folding along the dashed lines, as shown in the figure.

(a) Show that the volume of the box is

$$V(x) = 8x(6 - x)(12 - x).$$

(b) Determine the domain of the function V.

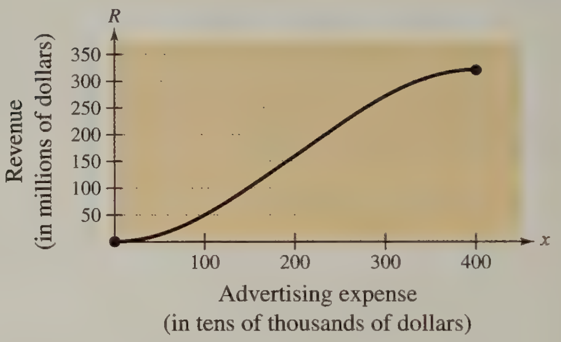
(c) Sketch the graph of the function and estimate the value of x for which $V(x)$ is maximum.



97. ***Business*** The total revenue R (in millions of dollars) for a company is related to its advertising expense by the function

$$R = \frac{1}{100{,}000}(-x^3 + 600x^2), \quad 0 \le x \le 400$$

where x is the amount spent on advertising (in tens of thousands of dollars). Use the graph of the function shown in the figure to estimate the point on the graph at which the function is increasing most rapidly. This point is called the **point of diminishing returns** because any expense above this amount will yield less return per dollar invested in advertising.



98. ***Environment*** The growth of a red oak tree is approximated by the function

$$G = -0.003t^3 + 0.137t^2 + 0.458t - 0.839$$

where G is the height of the tree (in feet) and t $(2 \le t \le 34)$ is its age (in years). Use a graphing utility to graph the function and estimate the age of the tree when it is growing most rapidly. This point is called the **point of diminishing returns** because the increase in growth will be less with each additional year. (*Hint:* Use a viewing window in which $-10 \le x \le 45$ and $-5 \le y \le 60$.)

99. ***Data Analysis*** The vertical deflection y of a 4-meter beam is measured in $\frac{1}{2}$-meter intervals x. The measurements are given by the following ordered pairs.

(0, 0), (0.5, 0.06), (1, 0.11), (1.5, 0.15), (2, 0.16), (2.5, 0.15), (3, 0.11), (3.5, 0.06), (4, 0)

(a) Use the regression capabilities of a graphing utility to fit a quartic (fourth-degree polynomial) equation to the data.

(b) Use a graphing utility to graph the data and the regression equation in the same viewing window. How do they compare?

(c) Because $y = 0$ when $x = 0$, what should the constant term in the model be? Does your answer agree with the result of part (a)? Explain.

100. ***Data Analysis*** The table gives the median values of new privately owned U.S. homes for 1983 through 1997. In the table, t is the time (in years), with $t = 3$ corresponding to 1983, and y_1 and y_2 are the median prices (in thousands of dollars) in the Northeast and the South, respectively. (Sources: U.S. Bureau of the Census: U.S. Department of Housing and Urban Development)

t	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
y_1	82.2	88.6	103.3	125	140	149	159.6	159	155.9	169	162.6	169	180	186	190
y_2	70.9	72	75	80.2	88	92	96.4	99	100	105.5	115	116.9	124.5	126.2	129.6

(a) Use the regression capabilities of a graphing utility to fit a cubic model to the median prices of homes in the Northeast.

(b) Use the regression capabilities of a graphing utility to fit a cubic model to the median prices of homes in the South.

(c) Use the graphs of the models in parts (a) and (b) to write a short paragraph about the relationship between the median prices of homes in the two regions.

Synthesis

True or False? **In Exercises 101 and 102, determine whether the statement is true or false. Justify your answer.**

101. A fourth-degree polynomial can have four turning points.

102. The graph of the function

$$f(x) = 2 + x - x^2 + x^3 - x^4 + x^5 + x^6 - x^7$$

rises to the left and falls to the right.

103. ***Graphical Reasoning*** Use a graphing utility to graph the function $f(x) = x^4$. Explain how the graph of g differs (if it does) from the graph of f and confirm your result with a graphing utility. Determine whether g is odd, even, or neither.

(a) $g(x) = f(x) + 2$ (b) $g(x) = f(x + 2)$

(c) $g(x) = f(-x)$ (d) $g(x) = -f(x)$

(e) $g(x) = f\left(\frac{1}{2}x\right)$ (f) $g(x) = \frac{1}{2}f(x)$

(g) $g(x) = f\left(x^{3/4}\right)$ (h) $g(x) = (f \circ f)(x)$

Review

In Exercises 104–109, let $f(x) = 14x - 3$, and $g(x) = 8x^2$. Find the indicated value.

104. $(f + g)(-4)$ **105.** $(g - f)(3)$

106. $(fg)\left(-\frac{4}{7}\right)$ **107.** $\left(\frac{f}{g}\right)(-1.5)$

108. $(f \circ g)(-1)$ **109.** $(g \circ f)(0)$

In Exercises 110–113, solve the inequality and sketch the solution on the real number line. Use a graphing utility to verify your solution graphically.

110. $3(x - 5) < 4x - 7$ **111.** $2x^2 - x \geq 1$

112. $\dfrac{5x - 2}{x - 7} \leq 4$ **113.** $|x + 8| - 1 \geq 15$

In Exercises 114–117, find the quadratic function that has the indicated vertex and whose graph passes through the given point.

114. Vertex: $(3, -6)$; Point: $(-1, 2)$

115. Vertex: $(0, -8)$; Point: $(5, 9)$

116. Vertex: $(4, -4)$; Point: $(1, 10)$

117. Vertex: $(-5, -2)$; Point: $(0, 3)$

3.3 Real Zeros of Polynomial Functions

Long Division of Polynomials

Consider the graph of

$$f(x) = 6x^3 - 19x^2 + 16x - 4.$$

Notice in Figure 3.22 that $x = 2$ appears to be a zero of f. Because $f(2) = 0$, you know that $x = 2$ is a zero of the polynomial function f, and that $(x - 2)$ is a factor of $f(x)$. This means that there exists a second-degree polynomial $q(x)$ such that $f(x) = (x - 2) \cdot q(x)$. To find $q(x)$, you can use **long division of polynomials.**

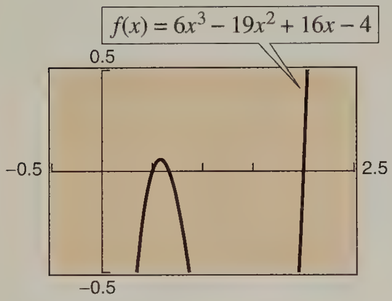


Figure 3.22

EXAMPLE 1 Long Division of Polynomials

Divide $f(x) = 6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the function f completely.

Solution

Partial quotients

$$
\begin{array}{rl}
\downarrow \quad\quad \downarrow \quad \downarrow & \\
6x^2 - 7x + 2 & \\
x - 2\,\overline{)\,6x^3 - 19x^2 + 16x - 4} & \\
\underline{6x^3 - 12x^2} \qquad\qquad\quad & \text{Multiply: } 6x^2(x - 2). \\
-7x^2 + 16x \qquad\; & \text{Subtract.} \\
\underline{-7x^2 + 14x} \qquad\; & \text{Multiply: } -7x(x - 2). \\
2x - 4 & \text{Subtract.} \\
\underline{2x - 4} & \text{Multiply: } 2(x - 2). \\
0 & \text{Subtract.}
\end{array}
$$

You can see that

$$
\begin{aligned}
6x^3 - 19x^2 + 16x - 4 &= (x - 2)(6x^2 - 7x + 2) \\
&= (x - 2)(2x - 1)(3x - 2).
\end{aligned}
$$

Note that this factorization agrees with the graph of f (Figure 3.22) in that the three x-intercepts occur at $x = 2$, $x = \frac{1}{2}$, and $x = \frac{2}{3}$.

What You Should Learn:

- How to use long division to divide polynomials by other polynomials
- How to use synthetic division to divide polynomials by binomials of the form $(x - k)$
- How to use the Remainder Theorem and the Factor Theorem
- How to use the Rational Zero Test to determine possible rational zeros of polynomial functions
- How to determine upper and lower bounds for zeros of polynomial functions

Why You Should Learn It:

Polynomial division can help you rewrite polynomials that are used to model real-life problems. For instance, Exercise 87 on page 262 shows how polynomial division can be used to model the average monthly rates for cable television in the United States from 1988 through 1997.

Stephen Ferry/Liaison International

In Example 1, $x - 2$ is a factor of the polynomial $6x^3 - 19x^2 + 16x - 4$, and the long division process produces a remainder of zero. Often, long division will produce a nonzero remainder. For instance, if you divide $x^2 + 3x + 5$ by $x + 1$, you obtain the following.

$$\begin{array}{lrl}
 & x + 2 & \Leftarrow \text{Quotient}\\
\text{Divisor} \Rightarrow \quad x + 1 & \overline{)\,x^2 + 3x + 5} & \Leftarrow \text{Dividend}\\
 & \underline{x^2 + x\phantom{{}+5}} & \\
 & 2x + 5 & \\
 & \underline{2x + 2} & \\
 & 3 & \Leftarrow \text{Remainder}
\end{array}$$

In fractional form, you can write this result as follows.

$$\frac{\overbrace{x^2 + 3x + 5}^{\text{Dividend}}}{\underbrace{x + 1}_{\text{Divisor}}} = \overbrace{x + 2}^{\text{Quotient}} + \frac{\overset{\overset{\text{Remainder}}{\downarrow}}{3}}{\underbrace{x + 1}_{\text{Divisor}}}$$

This implies that

$$x^2 + 3x + 5 = (x + 1)(x + 2) + 3$$

which illustrates the following well-known theorem called the **Division Algorithm.**

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\underset{\underset{\text{Dividend}}{\uparrow}}{f(x)} = \underset{\underset{\text{Divisor}}{\uparrow}}{d(x)}\,\underset{\underset{\text{Quotient}}{\uparrow}}{q(x)} + \underset{\underset{\text{Remainder}}{\uparrow}}{r(x)}$$

where $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $d(x)$ *divides evenly* into $f(x)$.

The Division Algorithm can also be written as

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

In the Division Algorithm, the rational expression $f(x)/d(x)$ is **improper** because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$. On the other hand, the rational expression $r(x)/d(x)$ is **proper** because the degree of $r(x)$ is less than the degree of $d(x)$.

EXAMPLE 2 Long Division of Polynomials

Divide $x^3 - 1$ by $x - 1$.

Solution

Because there is no x^2-term or x-term in the dividend, you need to line up the subtraction by using zero coefficients (or leaving spaces) for the missing terms.

$$\begin{array}{rrrrr}
 & & x^2 & +\;x & +\;1 \\ \cline{2-5}
x-1\,\big) & x^3 & +\;0x^2 & +\;0x & -\;1 \\
 & x^3 & -\;x^2 & & \\ \cline{2-3}
 & & x^2 & & \\
 & & x^2 & -\;x & \\ \cline{3-4}
 & & & x & -\;1 \\
 & & & x & -\;1 \\ \cline{4-5}
 & & & & 0
\end{array}$$

So, $x - 1$ divides evenly into $x^3 - 1$, and you can write

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1, \qquad x \neq 1.$$

You can check the result of a division problem by multiplying. For instance, in Example 2, try checking that

$$(x - 1)(x^2 + x + 1) = x^3 - 1.$$

EXAMPLE 3 Long Division of Polynomials

Divide $2x^4 + 4x^3 - 5x^2 + 3x - 2$ by $x^2 + 2x - 3$.

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Solution

$$\begin{array}{rrrrrr}
 & & & 2x^2 & & +\;1 \\ \cline{2-6}
x^2+2x-3\,\big) & 2x^4 & +\;4x^3 & -\;5x^2 & +\;3x & -\;2 \\
 & 2x^4 & +\;4x^3 & -\;6x^2 & & \\ \cline{2-4}
 & & & x^2 & +\;3x & -\;2 \\
 & & & x^2 & +\;2x & -\;3 \\ \cline{4-6}
 & & & & x & +\;1
\end{array}$$

Note that the first subtraction eliminated two terms from the dividend. When this happens, the quotient skips a term. You can write the result as

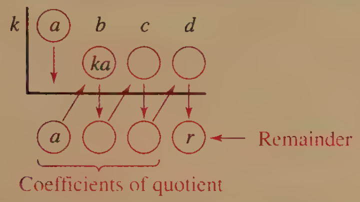
$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x + 1}{x^2 + 2x - 3}.$$

Synthetic Division

There is a nice shortcut for long division of polynomials by divisors of the form $x - k$. The shortcut is called **synthetic division.** The pattern for synthetic division of a cubic polynomial is summarized as follows. (The pattern for higher-degree polynomials is similar.)

Synthetic Division (of a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.



Vertical pattern: Add terms.
Diagonal pattern: Multiply by k.

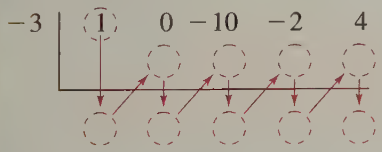
Note that synthetic division works *only* for divisors of the form $x - k$. [Remember that $x + k = x - (-k)$.] You cannot use synthetic division to divide a polynomial by a quadratic such as $x^2 - 3$.

EXAMPLE 4 Using Synthetic Division

Use synthetic division to divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$.

Solution

You should set up the array as follows. Note that a zero is included for each missing term in the dividend.



Then, use the synthetic division pattern by adding terms in columns and multiplying the results by -3.

Divisor: $x + 3$ Dividend: $x^4 - 10x^2 - 2x + 4$

$$\begin{array}{r|rrrrr} -3 & 1 & 0 & -10 & -2 & 4 \\ & & -3 & 9 & 3 & -3 \\ \hline & 1 & -3 & -1 & 1 & 1 \end{array}$$

Remainder: 1

Quotient: $x^3 - 3x^2 - x + 1$

So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}.$$

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

The Remainder and Factor Theorems

The remainder obtained in the synthetic division process has an important interpretation, as described in the **Remainder Theorem.** (A proof of this theorem is given in Appendix A.)

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, the remainder is

$$r = f(k).$$

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial function $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$, as illustrated in Example 5.

EXAMPLE 5 Using the Remainder Theorem

Use the Remainder Theorem to evaluate the following function at $x = -2$.

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

Solution

Using synthetic division, you obtain the following.

$$\begin{array}{r|rrrr} -2 & 3 & 8 & 5 & -7 \\ & & -6 & -4 & -2 \\ \hline & 3 & 2 & 1 & -9 \end{array}$$

Because the remainder is $r = -9$, you can conclude that

$$f(-2) = -9.$$

This means that $(-2, -9)$ is a point on the graph of f. You can check this by substituting $x = -2$ in the original function.

Check

$$\begin{aligned} f(-2) &= 3(-2)^3 + 8(-2)^2 + 5(-2) - 7 \\ &= 3(-8) + 8(4) - 10 - 7 \\ &= -24 + 32 - 10 - 7 \\ &= -9 \end{aligned}$$

Another important theorem is the **Factor Theorem,** which is stated below. This theorem states that you can test to see whether a polynomial has $(x - k)$ as a factor by evaluating the polynomial at $x = k$. If the result is 0, $(x - k)$ is a factor. For a proof of the Factor Theorem, see Appendix A.

The Factor Theorem

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

EXAMPLE 6 Factoring a Polynomial: Repeated Division

Show that $(x-2)$ and $(x+3)$ are factors of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

Algebraic Solution

Using synthetic division with 2 and -3 *repeatedly*, you obtain the following.

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & & 4 & 22 & 36 & 18 \\ \hline & 2 & 11 & 18 & 9 & 0 \end{array} \Longrightarrow$$

0 remainder
$(x-2)$ is
a factor.

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & & -6 & -15 & -9 \\ \hline & 2 & 5 & 3 & 0 \end{array} \Longrightarrow$$

0 remainder
$(x+3)$ is
a factor.

Because the resulting quadratic factors as

$$2x^2 + 5x + 3 = (2x+3)(x+1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x-2)(x+3)(2x+3)(x+1).$$

Graphical Solution

The graph of a polynomial with factors of $(x-2)$ and $(x+3)$ has x-intercepts at $x = 2$ and $x = -3$. Use a graphing utility to graph

$$y = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

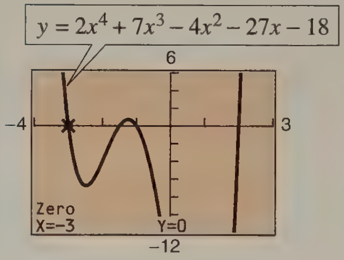


Figure 3.23

From Figure 3.23, you can see that the graph appears to cross the x-axis in two other places, near $x = -1$ and $x = -\frac{3}{2}$. Use the *zero* or *root* feature or the *zoom* and *trace* features to approximate the other two intercepts to be $x = -1$ and $x = -\frac{3}{2}$. So, the factors of f are $(x-2)$, $(x+3)$, $\left(x+\frac{3}{2}\right)$, and $(x+1)$. You can rewrite the factor $\left(x+\frac{3}{2}\right)$ as $(2x+3)$, so the complete factorization of f is $f(x) = (x-2)(x+3)(2x+3)(x+1)$.

Using the Remainder in Synthetic Division

In summary, the remainder r, obtained in the synthetic division of $f(x)$ by $x - k$, provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.
2. If $r = 0$, $(x-k)$ is a factor of $f(x)$.
3. If $r = 0$, $(k, 0)$ is an x-intercept of the graph of f.

Throughout this text, the importance of developing several problem-solving strategies has been emphasized. In the exercises for this section, try using more than one strategy to solve several of the exercises. For instance, if you find that $x - k$ divides evenly into $f(x)$, try sketching the graph of f. You should find that $(k, 0)$ is an x-intercept of the graph.

The Rational Zero Test

The **Rational Zero Test** relates the possible rational zeros of a polynomial (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.

The Rational Zero Test

If the polynomial

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

has integer coefficients, every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

To use the Rational Zero Test, first list all rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient.

$$\text{Possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$

Now that you have formed this list of *possible rational zeros,* use a trial-and-error method to determine which, if any, are actual zeros of the polynomial. Note that when the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term. This case is illustrated in Example 7.

STUDY TIP

Graph the polynomial $y = x^3 - 53x^2 + 103x - 51$ in the standard viewing window. From the graph alone, it appears that there is only one zero. From the Leading Coefficient Test, you know that because the degree of the polynomial is odd and the leading coefficient is positive, the graph falls to the left and rises to the right. So, the function must have another zero. From the Rational Zero Test, you know that ± 51 might be zeros of the function. If you zoom out several times, you will see a more complete picture of the graph. Your graph should confirm that $x = 51$ is a zero of f.

EXAMPLE 7 Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of $f(x) = x^3 + x + 1$.

Solution

Because the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term.

Possible Rational Zeros: ± 1

By testing these possible zeros, you can see that neither works.

$$f(1) = (1)^3 + 1 + 1 = 3$$

$$f(-1) = (-1)^3 + (-1) + 1 = -1$$

So, the polynomial has *no* rational zeros. Note from the graph of f in Figure 3.24 that f does have one real zero (between -1 and 0). However, by the Rational Zero Test, you know that this real zero is *not* a rational number.

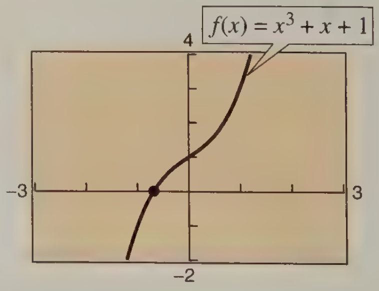


Figure 3.24

If the leading coefficient of a polynomial is not 1, the list of possible rational zeros can increase dramatically. In such cases the search can be shortened in several ways.

1. A programmable calculator can be used to speed up the calculations.
2. A graphing utility can give a good estimate of the locations of the zeros.
3. The Factor Theorem and synthetic division can be used to test the possible rational zeros.

Finding the first zero is often the most difficult part. After that, the search is simplified by working with the lower-degree polynomial obtained in synthetic division.

EXAMPLE 8 Using the Rational Zero Test

Find the rational zeros of

$$f(x) = 2x^3 + 3x^2 - 8x + 3.$$

Solution

The leading coefficient is 2 and the constant term is 3.

Possible Rational Zeros:

$$\frac{\text{Factors of } 3}{\text{Factors of } 2} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$$

By synthetic division, you can determine that $x = 1$ is a zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$\begin{aligned} f(x) &= (x - 1)(2x^2 + 5x - 3) \\ &= (x - 1)(2x - 1)(x + 3) \end{aligned}$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$, as shown in Figure 3.25.

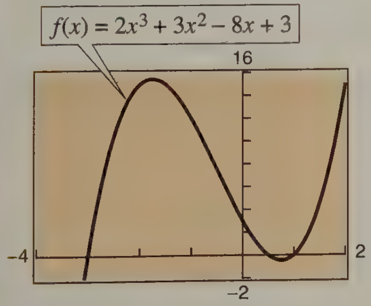


Figure 3.25

A graphing utility can help you determine which possible rational zeros to test, as demonstrated in Example 9.

EXAMPLE 9 Finding Real Zeros of a Polynomial Function

Find all the real zeros of $f(x) = 10x^3 - 15x^2 - 16x + 12$.

Solution

Because the leading coefficient is 10 and the constant term is 12, there is a long list of possible rational zeros.

Possible Rational Zeros:

$$\frac{\text{Factors of } 12}{\text{Factors of } 10} = \frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 5, \pm 10}$$

With so many possibilities (32, in fact), it is worth your time to use a graphing utility to focus on just a few. From Figure 3.26, it looks like three reasonable choices would be $x = -\frac{6}{5}$, $x = \frac{1}{2}$, and $x = 2$. Synthetic division shows that only $x = 2$ works. (You could also use the Factor Theorem to test these choices.)

$$\begin{array}{r|rrrr} 2 & 10 & -15 & -16 & 12 \\ & & 20 & 10 & -12 \\ \hline & 10 & 5 & -6 & 0 \end{array}$$

So, $x = 2$ is one zero and you have

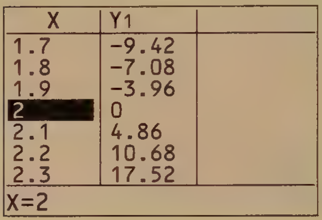
$$f(x) = (x - 2)(10x^2 + 5x - 6).$$

Using the Quadratic Formula, you find that the two additional zeros are irrational numbers.

$$x = \frac{-5 + \sqrt{265}}{20} \approx 0.5639 \quad \text{and} \quad x = \frac{-5 - \sqrt{265}}{20} \approx -1.0639$$

STUDY TIP

You can use the *table* feature of a graphing utility to test the possible rational zeros of the function in Example 9, as shown below. The table should be set to begin at -12, increase by increments of 0.1, and end at 12. Look through the table to determine the values of x for which y_1 is 0.



```
X      Y1
1.7    -9.42
1.8    -7.08
1.9    -3.96
2      0
2.1    4.86
2.2    10.68
2.3    17.52
X=2
```

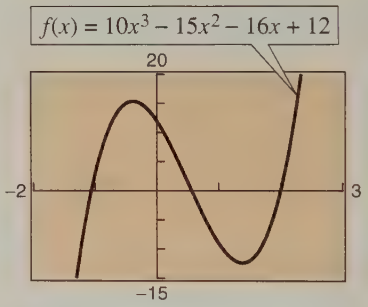


Figure 3.26

Bounds for Real Zeros of Polynomial Functions

The third test for zeros of a polynomial function is related to the sign pattern in the last row of the synthetic division tableau. This test can give you an upper or lower bound of the real zeros of f, which can help you eliminate possible real zeros.

A real number b is an **upper bound** for the real zeros of f if no zeros are greater than b. Similarly, b is a **lower bound** if no real zeros of f are less than b.

Upper and Lower Bound Rule

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$, using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, c is an **upper bound** for the real zeros of f.
2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), c is a **lower bound** for the real zeros of f.

EXAMPLE 10 Finding the Zeros of a Polynomial Function

Find the real zeros of $f(x) = 6x^3 - 4x^2 + 3x - 2$.

Solution

The possible real zeros are as follows.

$$\frac{\text{Factors of } 2}{\text{Factors of } 6} = \frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 6} = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{2}{3}, \pm 2$$

Trying $x = 1$ produces the following.

$$\begin{array}{r|rrrr} 1 & 6 & -4 & 3 & -2 \\ & & 6 & 2 & 5 \\ \hline & 6 & 2 & 5 & 3 \end{array}$$

So, $x = 1$ is not a zero, but because the last row has all positive entries, you know that $x = 1$ is an upper bound for the real zeros. Therefore, you can restrict the search to zeros less than 1. By testing the possible rational zeros less than 1, you can determine that $x = \frac{2}{3}$ is a zero. So,

$$f(x) = \left(x - \frac{2}{3}\right)(6x^2 + 3).$$

Because $6x^2 + 3$ has no real zeros, it follows that $x = \frac{2}{3}$ is the only real zero.

Exploration

Use a graphing utility to graph

$$y_1 = 6x^3 - 4x^2 + 3x - 2.$$

Notice that the graph intersects the x-axis at the point $\left(\frac{2}{3}, 0\right)$. How does this relate to the real zero found in Example 10? Use a graphing utility to graph

$$y_2 = x^4 - 5x^3 + 3x^2 + x.$$

How many times does the graph intersect the x-axis? How many real zeros does y_2 have?

Before concluding this section, two additional hints that can help you find the real zeros of a polynomial are listed below.

1. If the terms of $f(x)$ have a common monomial factor, it should be factored out before applying the tests in this section. For instance, by writing

   $$f(x) = x^4 - 5x^3 + 3x^2 + x = x(x^3 - 5x^2 + 3x + 1)$$

   you can see that $x = 0$ is a zero of f and that the remaining zeros can be obtained by analyzing the cubic factor.

2. If you are able to find all but two zeros of $f(x)$, you can always use the Quadratic Formula on the remaining quadratic factor. For instance, if you succeeded in writing

   $$f(x) = x^4 - 5x^3 + 3x^2 + x = x(x - 1)(x^2 - 4x - 1)$$

   you can apply the Quadratic Formula to $x^2 - 4x - 1$ to conclude that the two remaining zeros are $x = 2 + \sqrt{5}$ and $x = 2 - \sqrt{5}$.

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Writing About Math *Finding Patterns in Polynomial Division*

Complete the following polynomial divisions.

a. $\dfrac{x^2 - 1}{x - 1} = \square$ **b.** $\dfrac{x^3 - 1}{x - 1} = \square$ **c.** $\dfrac{x^4 - 1}{x - 1} = \square$

Write a brief description of the pattern that you obtain, and use your result to find a formula for the polynomial division $(x^n - 1)/(x - 1)$. Create a numerical example to test your formula.

Exploration

Use a graphing utility to graph

$$y = x^3 + 4.9x^2 - 126x + 382.5$$

in the standard viewing window. From the graph, what do the real zeros appear to be? Discuss how the mathematical tools of this section might help you realize that the graph does not show all the important features of the polynomial function. Now use the *zoom* feature to find all the zeros of this function.

3.3 Exercises

Graphical Analysis **In Exercises 1–6, use a graphing utility to graph the two equations in the same viewing window. Use the graphs to verify that the expressions are equivalent. Verify the results algebraically.**

1. $y_1 = \dfrac{4x}{x-1}, \quad y_2 = 4 + \dfrac{4}{x-1}$

2. $y_1 = \dfrac{3x-5}{x-3}, \quad y_2 = 3 + \dfrac{4}{x-3}$

3. $y_1 = \dfrac{x^2}{x+2}, \quad y_2 = x - 2 + \dfrac{4}{x+2}$

4. $y_1 = \dfrac{x^4 - 3x^2 - 1}{x^2 + 5}, \quad y_2 = x^2 - 8 + \dfrac{39}{x^2 + 5}$

5. $y_1 = \dfrac{x^5 - 3x^3}{x^2 + 1}, \quad y_2 = x^3 - 4x + \dfrac{4x}{x^2 + 1}$

6. $y_1 = \dfrac{x^3 - 2x^2 + 5}{x^2 + x + 1}, \quad y_2 = x - 3 + \dfrac{2(x+4)}{x^2 + x + 1}$

In Exercises 7–18, divide using long division.

7. Divide $2x^2 + 10x + 12$ by $x + 3$.

8. Divide $5x^2 - 17x - 12$ by $x - 4$.

9. Divide $4x^3 - 7x^2 - 11x + 5$ by $4x + 5$.

10. Divide $x^4 + 5x^3 + 6x^2 - x - 2$ by $x + 2$.

11. Divide $7x + 3$ by $x + 2$.

12. Divide $8x - 5$ by $2x + 1$.

13. $(6x^3 + 10x^2 + x + 8) \div (2x^2 + 1)$

14. $(x^3 - 9) \div (x^2 + 1)$

15. $\dfrac{x^4 + 3x^2 + 1}{x^2 - 2x + 3}$

16. $\dfrac{x^5 + 7}{x^3 - 1}$

17. $\dfrac{2x^3 - 4x^2 - 15x + 5}{(x-1)^2}$

18. $\dfrac{x^4}{(x-1)^3}$

In Exercises 19–28, divide using synthetic division.

19. $(3x^3 - 10x^2 + 12x - 22) \div (x - 4)$

20. $(2x^3 + 6x^2 - 14x + 9) \div (x - 1)$

21. $(6x^3 + 7x^2 - x + 26) \div (x - 3)$

22. $(2x^3 + 14x^2 - 20x + 7) \div (x + 6)$

23. $(9x^3 - 18x^2 - 16x + 32) \div (x - 2)$

24. $(5x^3 + 6x + 8) \div (x + 2)$

25. $\dfrac{x^3 + 512}{x + 8}$

26. $\dfrac{x^3 - 729}{x - 9}$

27. $\dfrac{4x^3 + 16x^2 - 23x - 15}{x + \frac{1}{2}}$

28. $\dfrac{3x^3 - 4x^2 + 5}{x - \frac{3}{2}}$

In Exercises 29–34, express the function in the form $f(x) = (x - k)q(x) + r(x)$ for the given value of k. Use a graphing utility to demonstrate that $f(k) = r$.

Function	*Value of k*
29. $f(x) = x^3 - x^2 - 14x + 11$	$k = 4$
30. $f(x) = 15x^4 + 10x^3 - 6x^2 + 14$	$k = -\frac{2}{3}$
31. $f(x) = x^3 + 3x^2 - 2x - 14$	$k = \sqrt{2}$
32. $f(x) = x^3 + 2x^2 - 5x - 4$	$k = -\sqrt{5}$
33. $f(x) = 4x^3 - 6x^2 - 12x - 4$	$k = 1 - \sqrt{3}$
34. $f(x) = -3x^3 + 8x^2 + 10x - 8$	$k = 2 + \sqrt{2}$

In Exercises 35–38, use synthetic division to find each function value. Use a graphing utility to verify your result.

35. $f(x) = 4x^3 - 13x + 10$

(a) $f(1)$ (b) $f(-2)$ (c) $f\left(\frac{1}{2}\right)$ (d) $f(8)$

36. $g(x) = x^6 - 4x^4 + 3x^2 + 2$

(a) $g(2)$ (b) $g(-4)$ (c) $g(3)$ (d) $g(-1)$

37. $h(x) = 3x^3 + 5x^2 - 10x + 1$

(a) $h(3)$ (b) $h\left(\frac{1}{3}\right)$ (c) $h(-2)$ (d) $h(-5)$

38. $f(x) = 0.4x^4 - 1.6x^3 + 0.7x^2 - 2$

(a) $f(1)$ (b) $f(-2)$ (c) $f(5)$ (d) $f(-10)$

In Exercises 39–44, use synthetic division to show that x is a solution of the third-degree polynomial equation, and use the result to factor the polynomial completely. List all the real zeros of the function.

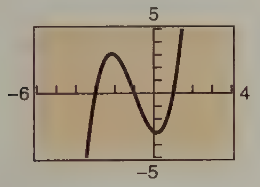
Polynomial Equation	*Value of x*
39. $x^3 - 7x + 6 = 0$	$x = 2$
40. $x^3 - 28x - 48 = 0$	$x = -4$
41. $2x^3 - 15x^2 + 27x - 10 = 0$	$x = \frac{1}{2}$
42. $x^3 + 2x^2 - 3x - 6 = 0$	$x = -2$
43. $x^3 + 2x^2 - 2x - 4 = 0$	$x = -2$
44. $x^3 - x^2 - 13x - 3 = 0$	$x = -3$

In Exercises 45–50, (a) verify the given factors of the function f, (b) find the remaining factors of f, (c) use your results to write the complete factorization of f, (d) list all real zeros of f, and (e) confirm your results by using a graphing utility to graph the function.

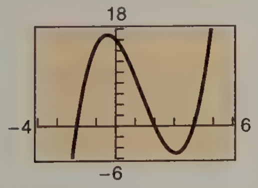
	Function	*Factors*
45.	$f(x) = 2x^3 + x^2 - 5x + 2$	$(x + 2), (x - 1)$
46.	$f(x) = 3x^3 + 2x^2 - 19x + 6$	$(x + 3), (x - 2)$
47.	$f(x) = x^4 - 4x^3 - 15x^2 + 58x - 40$	$(x - 5), (x + 4)$
48.	$f(x) = 8x^4 - 14x^3 - 71x^2 - 10x + 24$	$(x + 2), (x - 4)$
49.	$f(x) = 6x^3 + 41x^2 - 9x - 14$	$(2x + 1), (3x - 2)$
50.	$f(x) = 2x^3 - x^2 - 10x + 5$	$(2x - 1), (x + \sqrt{5})$

In Exercises 51–54, use the Rational Zero Test to list all possible rational zeros of f. Verify that the zeros of f shown on the graph are contained in the list.

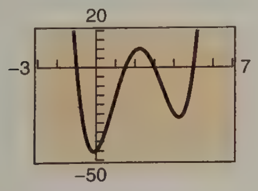
51. $f(x) = x^3 + 3x^2 - x - 3$



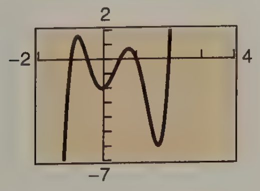
52. $f(x) = x^3 - 4x^2 - 4x + 16$



53. $f(x) = 2x^4 - 17x^3 + 35x^2 + 9x - 45$



54. $f(x) = 4x^5 - 8x^4 - 5x^3 + 10x^2 + x - 2$



In Exercises 55–62, (a) list the possible rational zeros of f, (b) use a graphing utility to graph f so that some of the possible zeros in part (a) can be disregarded, and (c) determine all the real zeros of f.

55. $f(x) = x^3 + x^2 - 4x - 4$

56. $f(x) = -3x^3 + 20x^2 - 36x + 16$

57. $f(x) = -4x^3 + 15x^2 - 8x - 3$

58. $f(x) = 4x^3 - 12x^2 - x + 15$

59. $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$

60. $f(x) = 4x^4 - 17x^2 + 4$

61. $f(x) = 6x^3 - x^2 - 13x + 8$

62. $f(x) = 4x^3 + 7x^2 - 11x - 18$

In Exercises 63–68, find all the real solutions of the polynomial equation.

63. $z^4 - z^3 - 2z - 4 = 0$

64. $x^4 - x^3 - 29x^2 - x - 30 = 0$

65. $x^4 - 13x^2 - 12x = 0$

66. $2y^4 + 7y^3 - 26y^2 + 23y - 6 = 0$

67. $2x^4 - 11x^3 - 6x^2 + 64x + 32 = 0$

68. $x^5 - x^4 - 3x^3 + 5x^2 - 2x = 0$

Graphical Analysis **In Exercises 69–74, (a) use the *zero* or *root* feature of a graphing utility to approximate (accurate to three decimal places) the zeros of the function, (b) determine one of the exact zeros and use synthetic division to verify your result, and (c) factor the polynomial completely.**

69. $f(x) = x^3 - 2x^2 - 5x + 10$

70. $g(x) = x^3 - 4x^2 - 2x + 8$

71. $h(t) = t^3 - 2t^2 - 7t + 2$

72. $f(s) = s^3 - 12s^2 + 40s - 24$

73. $h(x) = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$

74. $g(x) = 6x^4 - 11x^3 - 51x^2 + 99x - 27$

In Exercises 75–78, use synthetic division to verify the upper bound and lower bound of the zeros of f.

75. $f(x) = x^4 - 4x^3 + 15$

Upper bound: $x = 4$; Lower bound: $x = -1$

76. $f(x) = 2x^3 - 3x^2 - 12x + 8$

Upper bound: $x = 4$; Lower bound: $x = -3$

77. $f(x) = x^4 - 4x^3 + 16x - 16$

Upper bound: $x = 5$; Lower bound: $x = -3$

78. $f(x) = 2x^4 - 8x + 3$

Upper bound: $x = 3$; Lower bound: $x = -4$

In Exercises 79–82, find the rational zeros of the polynomial function.

79. $P(x) = x^4 - \frac{25}{4}x^2 + 9 = \frac{1}{4}(4x^4 - 25x^2 + 36)$

80. $f(x) = x^3 - \frac{3}{2}x^2 - \frac{23}{2}x + 6 = \frac{1}{2}(2x^3 - 3x^2 - 23x + 12)$

81. $f(x) = x^3 - \frac{1}{4}x^2 - x + \frac{1}{4} = \frac{1}{4}(4x^3 - x^2 - 4x + 1)$

82. $f(z) = z^3 + \frac{11}{6}z^2 - \frac{1}{2}z - \frac{1}{3} = \frac{1}{6}(6z^3 + 11z^2 - 3z - 2)$

In Exercises 83–86, match the cubic function with the correct number of rational and irrational zeros.

(a) Rational zeros: 0; Irrational zeros: 1
(b) Rational zeros: 3; Irrational zeros: 0
(c) Rational zeros: 1; Irrational zeros: 2
(d) Rational zeros: 1; Irrational zeros: 0

83. $f(x) = x^3 - 1$

84. $f(x) = x^3 - 2$

85. $f(x) = x^3 - x$

86. $f(x) = x^3 - 2x$

87. ***Data Analysis*** The average monthly basic rates R for cable television in the United States for the years 1988 through 1997 are given in the table, where t represents the time (in years), with $t = 0$ corresponding to 1990. (Source: Paul Kagan Associates, Inc.)

t	−2	−1	0	1	2
R	13.86	15.21	16.78	18.10	19.08

t	3	4	5	6	7
R	19.39	21.62	23.07	24.41	26.48

(a) Use a graphing utility to sketch a scatter plot of the data.

(b) Use the regression capabilities of a graphing utility to find a cubic model for the data. Then graph the model in the same viewing window as the scatter plot. Compare the model with the data.

(c) Use a graphing utility and the model to create a table of estimated values of R. Compare the estimated values with the actual data.

(d) Use the Remainder Theorem to evaluate the model for the year 2002. Even though the model is relatively accurate for estimating the given data, do you think it is accurate to predict future cable rates? Explain.

88. ***Data Analysis*** The number of United States military personnel M (in thousands) on active duty for the years 1989 through 1996 is shown in the table, where t represents the time (in years), with $t = 0$ corresponding to 1990. (Source: U.S. Department of Defense)

t	−1	0	1	2
M	2130	2044	1986	1807

t	3	4	5	6
M	1705	1611	1518	1472

(a) Use a graphing utility to sketch a scatter plot of the data.

(b) Use the regression capabilities of a graphing utility to find a cubic model for the data. Then graph the model in the same viewing window as the scatter plot. Compare the model with the data.

(c) Use a graphing utility and the model to create a table of estimated values of M. Compare the estimated values with the actual data.

(d) Use the Remainder Theorem to evaluate the model for the year 2001. Even though the model is relatively accurate for estimating the given data, would you use this model to predict the number of military personnel in the future? Explain.

89. ***Geometry*** An open box is to be made from a rectangular piece of material 15 centimeters by 9 centimeters by cutting equal squares from the corners and turning up the sides.

(a) Let x represent the length of the sides of the squares. Draw a diagram showing the squares removed from the original piece of material and the resulting dimensions of the open box.

(b) Use the diagram in part (a) to write the volume V of the box as a function of x. Determine the domain of the function.

(c) Use a graphing utility to graph the function and approximate the dimensions of the box that yield maximum volume.

(d) Find values of x such that $V = 56$. Which of these values is a physical impossibility in the construction of the box? Explain.

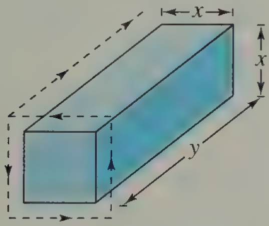
90. ***Geometry*** A rectangular package sent by a delivery service can have a maximum combined length and girth (perimeter of a cross section) of 120 inches.

(a) Show that the volume of the package is

$$V(x) = 4x^2(30 - x).$$

(b) Use a graphing utility to graph the function and approximate the dimensions of the package that yield a maximum volume.

(c) Find values of x such that $V = 13{,}500$. Which of these values is a physical impossibility in the construction of the package? Explain.



91. ***Automobile Emissions*** The number of parts per million of nitric oxide emissions y from a certain car engine is approximated by the model

$$y = -5.05x^3 + 3857x - 38{,}411.25, \quad 13 \le x \le 18$$

where x is the air-fuel ratio.

(a) Use a graphing utility to graph the emissions function.

(b) It is observed from the graph that two air-fuel ratios produce 2400 parts per million of nitric oxide, with one being 15. Use the graph to approximate the second air-fuel ratio.

(c) Algebraically approximate the second air-fuel ratio that produces 2400 parts per million of nitric oxide. (*Hint:* Because you know that an air-fuel ratio of 15 produces the specified nitric oxide emission, you can use synthetic division.)

92. ***Advertising Costs*** A company that manufactures bicycles estimates that the profit for selling a particular model is

$$P = -45x^3 + 2500x^2 - 275{,}000, \quad 0 \le x \le 50$$

where P is the profit (in dollars) and x is the advertising expense (in tens of thousands of dollars). According to this model, find the smaller of two advertising amounts that yield a profit of $800,000.

Synthesis

True or False? **In Exercises 93–95, determine whether the statement is true or false. Justify your answer.**

93. If $(7x + 4)$ is a factor of some polynomial function f, then $\frac{4}{7}$ is a zero of f.

94. The rational expression $\dfrac{x^3 + 2x^2 - 13x + 10}{x^2 - 4x - 12}$ is improper.

95. $(2x - 1)$ is a factor of the polynomial

$$6x^6 + x^5 - 92x^4 + 45x^3 + 184x^2 + 4x - 48.$$

Think About It **In Exercises 96 and 97, perform the division by assuming that *n* is a positive integer.**

96. $\dfrac{x^{3n} + 9x^{2n} + 27x^n + 27}{x^n + 3}$

97. $\dfrac{x^{3n} - 3x^{2n} + 5x^n - 6}{x^n - 2}$

98. ***Think About It*** What does it mean for a divisor to divide evenly into a dividend?

99. ***Writing*** Write a short paragraph explaining how you can check polynomial division. Give an example.

Review

In Exercises 100–103, perform the operations and write the result in standard form.

100. $12 - (-7 + 5i) + (-2 + 3i)$

101. $-8i - i(2 + 3i)$

102. $(-3 - 8i)^2$

103. $\dfrac{6 - i}{1 - 4i}$

In Exercises 104–107, find a polynomial function that has the given zeros.

104. $0, -12$

105. $1, -3, 8$

106. $0, -1, 2, 5$

107. $2 + \sqrt{3}, 2 - \sqrt{3}$

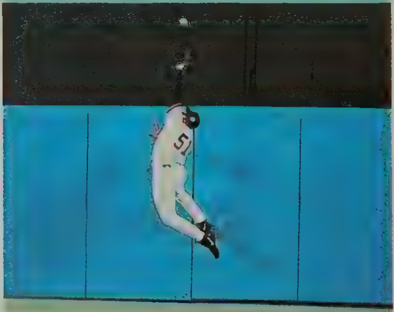
3.4 The Fundamental Theorem of Algebra

What You Should Learn:

- How to use the Fundamental Theorem of Algebra to determine the number of zeros of polynomial functions
- How to find all zeros of polynomial functions, including complex zeros
- How to find conjugate pairs of complex zeros
- How to find zeros of polynomials by factoring

Why You Should Learn It:

Being able to find zeros of polynomial functions is an important part of modeling real-life problems. For instance, Exercise 65 on page 270 shows how to determine whether a ball thrown with a given velocity can reach a certain height.



Jed Jacobsohn/Allsport

The Fundamental Theorem of Algebra

You have been using the fact that an nth-degree polynomial can have at most n real zeros. In the complex number system, this statement can be improved. That is, in the complex number system, every nth-degree polynomial function has *precisely n* zeros. This important result is derived from the **Fundamental Theorem of Algebra,** first proved by the famous German mathematician Carl Friedrich Gauss (1777–1855).

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, f has at least one zero in the complex number system.

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the **Linear Factorization Theorem**.

Linear Factorization Theorem

If $f(x)$ is a polynomial of degree n where $n > 0$, f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

(A proof of the Linear Factorization Theorem is found in Appendix A.)

Note that neither the Fundamental Theorem of Algebra nor the Linear Factorization Theorem tells you *how* to find the zeros or factors of a polynomial. Such theorems are called *existence theorems*. To find the zeros of a polynomial function, you still must rely on other techniques.

Remember that the n zeros of a polynomial function can be real or complex, and they may be repeated. Examples 1 and 2 illustrate several cases.

EXAMPLE 1 Real Zeros of Polynomial Functions

a. The first-degree polynomial $f(x) = x - 2$ has exactly *one* zero: $x = 2$.

b. Counting multiplicity, the second-degree polynomial function

$$f(x) = x^2 - 6x + 9 = (x - 3)(x - 3)$$

has exactly *two* zeros: $x = 3$ and $x = 3$.

Note in Example 1 that you can use a graphing utility to verify graphically the zeros of the polynomial functions.

EXAMPLE 2 Real and Complex Zeros of Polynomial Functions

a. The third-degree polynomial function

$$f(x) = x^3 + 4x = x(x^2 + 4) = x(x - 2i)(x + 2i)$$

has exactly *three* zeros: $x = 0$, $x = 2i$, and $x = -2i$. In the graph in Figure 3.27(a), only the *real* zero $x = 0$ appears as an intercept.

b. The fourth-degree polynomial function

$$f(x) = x^4 - 1 = (x - 1)(x + 1)(x - i)(x + i)$$

has exactly *four* zeros: $x = 1$, $x = -1$, $x = i$, and $x = -i$. In the graph in Figure 3.27(b), only the *real* zeros $x = -1$ and $x = 1$ appear as x-intercepts.

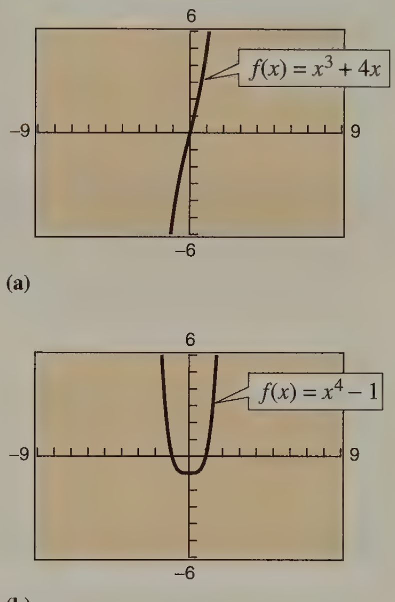


Figure 3.27

Example 3 shows how to use the methods described in Sections 3.2 and 3.3 (the Rational Zero Test, synthetic division, and factoring) to find all the zeros of a polynomial function, including complex zeros.

EXAMPLE 3 Finding the Zeros of a Polynomial Function

Write

$$f(x) = x^5 + x^3 + 2x^2 - 12x + 8$$

as the product of linear factors, and list all of its zeros.

Solution

The possible rational zeros are ±1, ±2, ±4, and ±8. The graph shown in Figure 3.28 indicates that 1 and -2 are good guesses, and that 1 is possibly a repeated zero because it touches the x-axis.

Using synthetic division, you can determine that -2 is a zero and 1 is a repeated zero of f. So, you have

$$\begin{aligned} f(x) &= x^5 + x^3 + 2x^2 - 12x + 8 \\ &= (x - 1)(x - 1)(x + 2)(x^2 + 4). \end{aligned}$$

By factoring $x^2 + 4$ as

$$x^2 - (-4) = \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right) = (x - 2i)(x + 2i)$$

you obtain

$$f(x) = (x - 1)(x - 1)(x + 2)(x - 2i)(x + 2i)$$

which gives the following five zeros of f.

$$1, \quad 1, \quad -2, \quad 2i, \quad \text{and} \quad -2i$$

Note from the graph of f shown in Figure 3.28 that the *real* zeros are the only ones that appear as x-intercepts.

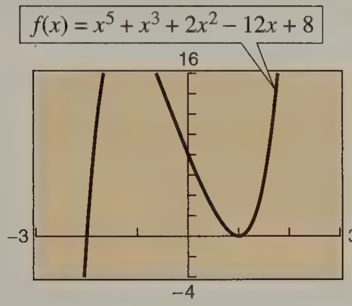


Figure 3.28

Conjugate Pairs

In Example 3, note that the two complex zeros are **conjugates.** That is, they are of the form $a + bi$ and $a - bi$.

Complex Zeros Occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has *real coefficients.* If $a + bi$, where $b \neq 0$, is a zero of the function, the conjugate $a - bi$ is also a zero of the function.

Be sure you see that this result is true only if the polynomial function has *real coefficients.* For instance, the result applies to the function $f(x) = x^2 + 1$, but not to the function $g(x) = x - i$.

EXAMPLE 4 Finding a Polynomial with Given Zeros

Find a *fourth-degree* polynomial function, with real coefficients, that has -1, -1, and $3i$ as zeros.

Solution

Because $3i$ is a zero *and* the polynomial is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero. So, from the Linear Factorization Theorem, $f(x)$ can be written as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).$$

For simplicity, let $a = 1$, to obtain

$$\begin{aligned} f(x) &= (x^2 + 2x + 1)(x^2 + 9) \\ &= x^4 + 2x^3 + 10x^2 + 18x + 9. \end{aligned}$$

Factoring a Polynomial

The Linear Factorization Theorem shows that you can write any nth-degree polynomial as the product of n linear factors.

$$f(x) = a(x - c_1)(x - c_2)(x - c_3) \cdot \cdot \cdot (x - c_n)$$

However, this result includes the possibility that some of the values of c_i are complex. The following theorem says that even if you do not want to get involved with "complex factors," you can still write $f(x)$ as the product of linear and/or quadratic factors.

Factors of a Polynomial

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

(For a proof of this theorem, see Appendix A.)

The Fogg Art Museum

Jean Le Rond d'Alembert (1717–1783) worked independently of Carl Gauss trying to prove the Fundamental Theorem of Algebra. His efforts were such that in France, the Fundamental Theorem of Algebra is frequently known as the Theorem of d'Alembert.

STUDY TIP

Some graphing utilities can find both real and complex zeros of a function. For instance, to find all the zeros of

$$f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$$

use the graphing utility's *solve* feature. If your graphing utility can find complex zeros, you will obtain the following solutions.

$$x = 1 + 3i, \quad x = 1 - 3i, \quad x = 3, \quad x = -2$$

A quadratic factor with no real zeros is said to be **irreducible over the reals.** Be sure you see that this is not the same as being *irreducible over the rationals.* For example, the quadratic

$$x^2 + 1 = (x - i)(x + i)$$

is irreducible over the reals (and therefore over the rationals). On the other hand, the quadratic

$$x^2 - 2 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$$

is irreducible over the rationals, but *reducible* over the reals.

EXAMPLE 5 Factoring a Polynomial

Write the polynomial

$$f(x) = x^4 - x^2 - 20$$

a. as the product of factors that are irreducible over the *rationals,*

b. as the product of linear factors and quadratic factors that are irreducible over the *reals,* and

c. in completely factored form.

Solution

a. Begin by factoring the polynomial into the product of two quadratic polynomials.

$$x^4 - x^2 - 20 = (x^2 - 5)(x^2 + 4)$$

Both of these factors are irreducible over the rationals.

b. By factoring over the reals, you have

$$x^4 - x^2 - 20 = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)(x^2 + 4)$$

where the quadratic factor is irreducible over the reals.

c. In completely factored form, you have

$$x^4 - x^2 - 20 = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)(x - 2i)(x + 2i).$$

In Example 5, notice from the completely factored form that the fourth-degree polynomial has four zeros.

Throughout this chapter, the results and theorems have basically been stated in terms of zeros of polynomial functions. Be sure you see that the same results could have been stated in terms of solutions of polynomial equations. This is true because the zeros of the polynomial function

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

are precisely the solutions of the polynomial equation

$$a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0 = 0.$$

EXAMPLE 6 Finding the Zeros of a Polynomial Function

Find all the zeros of

$$f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$$

given that $1 + 3i$ is a zero of f.

Algebraic Solution

Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$x - (1 + 3i) \qquad \text{and} \qquad x - (1 - 3i)$$

are factors of $f(x)$. Multiplying these two factors produces

$$\begin{aligned}[x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\ &= (x - 1)^2 - 9i^2 \\ &= x^2 - 2x + 10.\end{aligned}$$

Using long division, you can divide $x^2 - 2x + 10$ into $f(x)$ to obtain the following.

$$\begin{array}{r} x^2 - x - 6 \\ x^2 - 2x + 10\,\overline{)\,x^4 - 3x^3 + 6x^2 + 2x - 60} \\ \underline{x^4 - 2x^3 + 10x^2} \phantom{{}+2x-60} \\ -x^3 - 4x^2 + 2x \phantom{{}-60} \\ \underline{-x^3 + 2x^2 - 10x} \phantom{{}-60} \\ -6x^2 + 12x - 60 \\ \underline{-6x^2 + 12x - 60} \\ 0 \end{array}$$

Therefore, you have

$$\begin{aligned}f(x) &= (x^2 - 2x + 10)(x^2 - x - 6) \\ &= (x^2 - 2x + 10)(x - 3)(x + 2)\end{aligned}$$

and you can conclude that the zeros of f are $1 + 3i$, $1 - 3i$, 3, and -2.

Graphical Solution

Because complex zeros always occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. Because the polynomial is a fourth-degree polynomial, you know that there are at most two other zeros of the function. Use a graphing utility to graph

$$y = x^4 - 3x^3 + 6x^2 + 2x - 60$$

as shown in Figure 3.29.

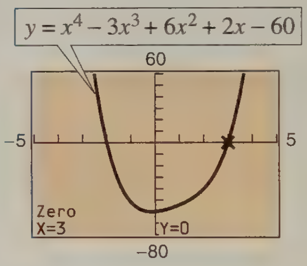


Figure 3.29

You can see that -2 and 3 appear to be x-intercepts of the graph of the function. Use the *zero* or *root* feature or the *zoom* and *trace* features of the graphing utility to confirm that $x = -2$ and $x = 3$ are x-intercepts of the graph. So, you can conclude that the zeros of f are

$$1 + 3i, \quad 1 - 3i, \quad 3, \quad \text{and} \quad -2.$$

In Example 6, if you had not been told that $1 + 3i$ is a zero of f, you could still find all zeros of the function. You could use synthetic division to find the real zeros -2 and 3. Then, you could factor the polynomial as $(x + 2)(x - 3)(x^2 - 2x + 10)$. Finally, by using the Quadratic Formula, you could determine that the zeros are $1 + 3i$, $1 - 3i$, 3, and -2.

Writing About Math *Factoring a Polynomial*

Compile a list of all the various techniques for factoring a polynomial that have been covered so far in the text. Give an example illustrating each technique, and write a paragraph discussing when the use of each technique is appropriate.

3.4 Exercises

In Exercises 1–8, find all the zeros of the function.

1. $f(x) = x(x - 6)^2$

2. $f(x) = x^2(x + 3)(x^2 - 1)$

3. $g(x) = (x - 2)(x + 4)^3$ **4.** $f(x) = (x + 5)(x - 8)^2$

5. $f(x) = (x + 6)(x + i)(x - i)$

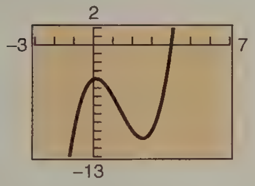
6. $h(t) = (t - 3)(t - 2)(t - 3i)(t + 3i)$

7. $f(x) = (x - 2)(x + 3 - 5i)(x + 3 + 5i)$

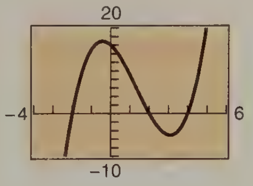
8. $h(m) = (m - 4)^2(m - 2 + 4i)(m - 2 - 4i)$

Graphical and Analytical Analysis **In Exercises 9–12, find all the zeros of the function. Is there a relationship between the number of real zeros and the number of x-intercepts of the graph? Explain.**

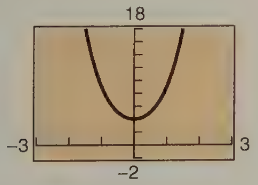
9. $f(x) = x^3 - 4x^2 + x - 4$



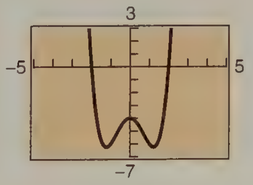
10. $f(x) = x^3 - 4x^2 - 4x + 16$



11. $f(x) = x^4 + 4x^2 + 4$



12. $f(x) = x^4 - 3x^2 - 4$



In Exercises 13–32, find all the zeros of the function and write the polynomial as a product of linear factors. Use a graphing utility to graph the function to verify your results graphically. (If your graphing utility can find complex zeros, use it to verify the complex zeros.)

13. $h(x) = x^2 - 4x + 1$ **14.** $g(x) = x^2 + 10x + 23$

15. $f(x) = x^2 - 12x + 26$ **16.** $f(x) = x^2 + 6x - 2$

17. $f(x) = x^2 + 25$ **18.** $f(x) = x^2 - x + 56$

19. $f(x) = x^4 - 81$ **20.** $f(y) = y^4 - 625$

21. $f(z) = z^2 - 2z + 2$

22. $h(x) = x^3 - 3x^2 + 4x - 2$

23. $f(t) = t^3 - 3t^2 - 15t + 125$

24. $f(x) = x^3 + 11x^2 + 39x + 29$

25. $f(x) = 16x^3 - 20x^2 - 4x + 15$

26. $f(s) = 2s^3 - 5s^2 + 12s - 5$

27. $f(x) = x^4 + 10x^2 + 9$

28. $f(x) = x^4 + 29x^2 + 100$

29. $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$

30. $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

31. $f(x) = 2x^4 + 5x^3 + 4x^2 + 5x + 2$

32. $g(x) = x^5 - 8x^4 + 28x^3 - 56x^2 + 64x - 32$

In Exercises 33–40, (a) find all zeros of the function, (b) write the polynomial as a product of linear factors, (c) use your factorization to determine the x-intercepts of the graph of the function, and (d) use a graphing utility to verify that the real zeros are the only x-intercepts.

33. $f(x) = x^2 - 14x + 46$ **34.** $f(x) = x^2 - 12x + 34$

35. $f(x) = x^2 + 14x + 44$ **36.** $f(x) = x^2 - 16x + 62$

37. $f(x) = x^3 - 11x + 150$

38. $f(x) = x^3 + 10x^2 + 33x + 34$

39. $f(x) = x^4 + 25x^2 + 144$

40. $f(x) = x^4 - 8x^3 + 17x^2 - 8x + 16$

In Exercises 41–48, find a polynomial function with integer coefficients that has the given zeros. (There are many correct answers.)

41. $1, 5i, -5i$ **42.** $4, 3i, -3i$

43. $2, 4 + i, 4 - i$ **44.** $6, -5 + 2i, -5 - 2i$

45. $i, -i, 6i, -6i$ **46.** $2, 2, 2, 4i, -4i$

47. $-5, -5, 1 + \sqrt{3}i$ **48.** $0, 0, 4, 1 + \sqrt{2}i$

In Exercises 49–52, write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.

49. $f(x) = x^4 - 6x^2 - 7$ **50.** $f(x) = x^4 + 6x^2 - 27$

51. $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$
(*Hint:* One factor is $x^2 - 6$.)

52. $f(x) = x^4 - 3x^3 - x^2 - 12x - 20$
(*Hint:* One factor is $x^2 + 4$.)

In Exercises 53–60, use the given zero to find all the zeros of the function.

	Function	*Zero*
53.	$f(x) = 2x^3 + 3x^2 + 50x + 75$	$5i$
54.	$f(x) = 2x^4 - x^3 + 7x^2 - 4x - 4$	$2i$
55.	$g(x) = x^3 - 7x^2 - x + 87$	$5 + 2i$
56.	$g(x) = 4x^3 + 23x^2 + 34x - 10$	$-3 + i$
57.	$h(x) = 3x^3 - 4x^2 + 8x + 8$	$1 - \sqrt{3}i$
58.	$f(x) = x^3 + 4x^2 + 14x + 20$	$-1 - 3i$
59.	$h(x) = 8x^3 - 14x^2 + 18x - 9$	$\frac{1}{2}(1 - \sqrt{5}i)$
60.	$f(x) = 25x^3 - 55x^2 - 54x - 18$	$\frac{1}{5}(-2 + \sqrt{2}i)$

Graphical Analysis **In Exercises 61–64, (a) use the *zero* or *root* feature of a graphing utility to approximate the zeros of the function accurate to three decimal places, (b) determine one of the exact zeros and use synthetic division to verify your result, and (c) find the exact values of the remaining zeros.**

61. $f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22$

62. $f(x) = x^3 + 4x^2 + 14x + 20$

63. $h(x) = 8x^3 - 14x^2 + 18x - 9$

64. $f(x) = 25x^3 - 55x^2 - 54x - 18$

65. ***Maximum Height*** A baseball is thrown upward from ground level with an initial velocity of 48 feet per second, and its height h (in feet) is

$$h = -16t^2 + 48t, \quad 0 \leq t \leq 3$$

where t is the time (in seconds). Suppose you are told that the ball reaches a height of 64 feet. Is this possible? Explain.

66. ***Profit*** The demand equation for a microwave is

$$p = 140 - 0.0001x$$

where p is the unit price (in dollars) of the microwave and x is the number of units produced and sold. The cost equation for the microwave is

$$C = 80x + 150{,}000$$

where C is the total cost (in dollars) and x is the number of units produced. The total profit obtained by producing and selling x units is

$$P = R - C = xp - C.$$

Suppose you are working in the marketing department that produces this microwave, and you are asked to determine a price p that would yield a profit of 9 million dollars. Is this possible? Explain.

Synthesis

True or False? **In Exercises 67 and 68, decide whether the statement is true or false. Justify your answer.**

67. It is possible for a third-degree polynomial function with integer coefficients to have no real zeros.

68. If $x = 4 + 3i$ is a zero of the function $f(x) = x^4 - 7x^3 - 13x^2 + 265x - 750$, then $x = -3i + 4$ must also be a zero of f.

69. ***Exploration*** Use a graphing utility to graph the function $f(x) = x^4 - 4x^2 + k$ for different values of k. Find values of k such that the zeros of f satisfy the specified characteristics. (Some parts have many correct answers.)

(a) Four real zeros

(b) Two real zeros each of multiplicity 2

(c) Two real zeros and two complex zeros

(d) Four complex zeros

70. ***Think About It*** Will the answers to Exercise 69 change for the function g?

(a) $g(x) = f(x - 2)$ (b) $g(x) = f(2x)$

71. Find a quadratic function f (with integer coefficients) that has $\pm\sqrt{b}i$ as zeros. Assume that b is a positive integer.

72. Find a quadratic function f (with integer coefficients) that has $a \pm bi$ as zeros. Assume that b is a positive integer.

Review

In Exercises 73–76, simplify the rational expression.

73. $\dfrac{x^2 - 8x + 15}{x^2 - 9}$

74. $\dfrac{x^2 - 4x - 12}{x^2 - 36}$

75. $\dfrac{x^3 - 9x^2 + 20x}{x^2 - 4x}$

76. $\dfrac{x^3 - 6x^2 - 5x + 30}{x^3 - 3x^2 - 18x}$

In Exercises 77–80, sketch the graph of the function. Identify the vertex and any intercepts. Use a graphing utility to verify your results.

77. $f(x) = x^2 - 7x - 8$

78. $f(x) = -x^2 + x + 6$

79. $f(x) = 6x^2 + 5x - 6$

80. $f(x) = 4x^2 + 2x - 12$

3.5 Rational Functions and Asymptotes

Introduction to Rational Functions

A **rational function** can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial. In this section it is assumed that $N(x)$ and $D(x)$ have no common factors.

In general, the *domain* of a rational function of x includes all real numbers except x-values that make the denominator zero. Much of the discussion of rational functions will focus on their graphical behavior near these x-values.

What You Should Learn:

- How to find domains of rational functions
- How to find horizontal and vertical asymptotes of graphs of rational functions
- How to use rational functions to model and solve real-life problems

Why You Should Learn It:

Rational functions are convenient in modeling a wide variety of real-life problems, such as environmental scenarios. For instance, Exercise 31 on page 278 shows how to determine the cost of removing pollutants from a river.

David Woodfull/Tony Stone Images

EXAMPLE 1 Finding the Domain of a Rational Function

Find the domain of $f(x) = 1/x$ and discuss the behavior of f near any excluded x-values.

Solution

Because the denominator is zero when $x = 0$, the domain of f is all real numbers except $x = 0$. To determine the behavior of f near this excluded value, evaluate $f(x)$ to the left and right of $x = 0$, as indicated in the following tables.

x	-1	-0.5	-0.1	-0.01	-0.001	$\rightarrow 0$
$f(x)$	-1	-2	-10	-100	-1000	$\rightarrow -\infty$

x	$0 \leftarrow$	0.001	0.01	0.1	0.5	1
$f(x)$	$\infty \leftarrow$	1000	100	10	2	1

Note that as x approaches 0 *from the left*, $f(x)$ decreases without bound. In contrast, as x approaches 0 *from the right*, $f(x)$ increases without bound. The graph of f is shown in Figure 3.30.

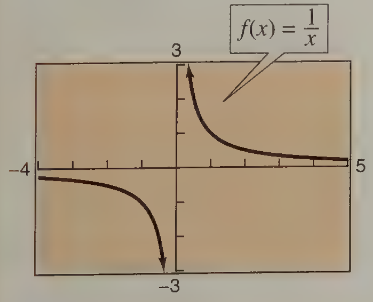


Figure 3.30

Horizontal and Vertical Asymptotes

In Example 1, the behavior of $f(x) = 1/x$ near $x = 0$ is denoted as follows.

$$\underbrace{f(x) \to -\infty \text{ as } x \to 0^-}_{f(x) \text{ decreases without bound as } x \text{ approaches 0 from the left.}} \qquad \underbrace{f(x) \to \infty \text{ as } x \to 0^+}_{f(x) \text{ increases without bound as } x \text{ approaches 0 from the right.}}$$

The line $x = 0$ is a **vertical asymptote** of the graph of f, as shown in Figure 3.31. The graph of f also has a **horizontal asymptote**—the line $y = 0$. This means the values of $f(x) = 1/x$ approach zero as x increases or decreases without bound.

$$\underbrace{f(x) \to 0 \text{ as } x \to -\infty}_{f(x) \text{ approaches 0 as } x \text{ decreases without bound.}} \qquad \underbrace{f(x) \to 0 \text{ as } x \to \infty}_{f(x) \text{ approaches 0 as } x \text{ increases without bound.}}$$

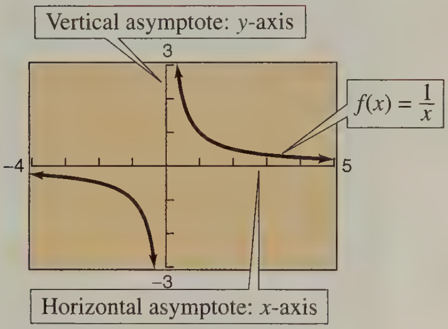


Figure 3.31

Definition of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$, either from the right or from the left.
2. The line $y = b$ is a **horizontal asymptote** of the graph of f if $f(x) \to b$ as $x \to \infty$ or $x \to -\infty$.

Eventually (as $x \to \infty$ or $x \to -\infty$), the distance between the horizontal asymptote and the points on the graph must approach zero. Figure 3.32 shows the horizontal and vertical asymptotes of the graphs of three rational functions.

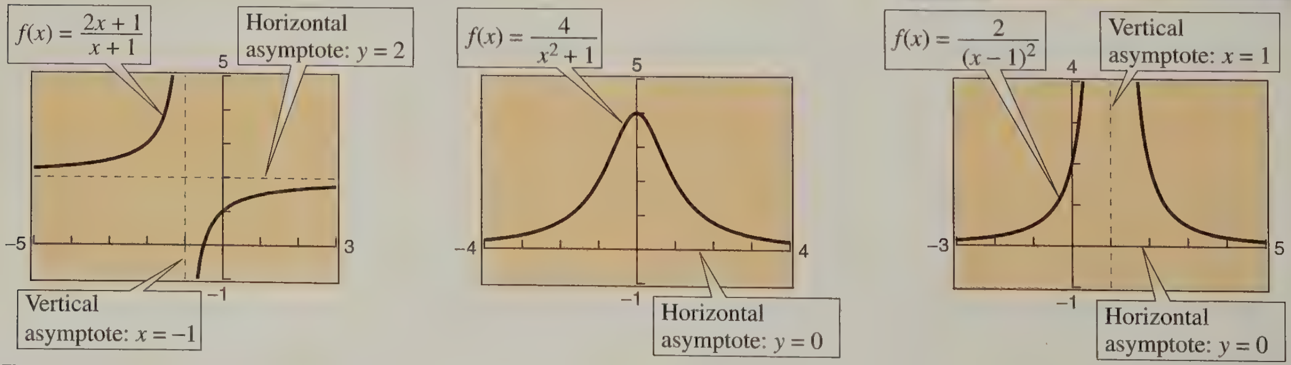


Figure 3.32

Library of Functions

A rational function $f(x)$ is the quotient of two polynomials, $f(x) = N(x)/D(x)$. A rational function is not defined at points x for which $D(x) = 0$. Near these points, the graph of the rational function may increase or decrease without bound.

Consult the Library of Functions Summary inside the front cover for a description of the rational functions.

Exploration

Use a graphing utility to graph $g(x) = (3x^2)/(x^2 + 2)$ in a standard viewing window. Graph $g(x)$ again, using a viewing window in which $-100 \le x \le 100$. Graph $g(x)$ a third time, using a viewing window in which $-1000 \le x \le 1000$. Describe the left and right behavior of the graphs. Complete the table.

x	10	100	1000
$g(x)$			

x	−10	−100	−1000
$g(x)$			

As the x-values become very large or very small, what do the values of $g(x)$ approach? What is the horizontal asymptote of the graph of g?

Asymptotes of a Rational Function

Let f be the rational function

$$f(x) = \frac{N(x)}{D(x)}$$

$$= \frac{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0}{b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of f has vertical asymptotes at the zeros of $D(x)$.
2. The graph of f has at most one horizontal asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.
   - **a.** If $n < m$, the line $y = 0$ (the x-axis) is a horizontal asymptote.
   - **b.** If $n = m$, the line $y = a_n/b_m$ is a horizontal asymptote.
   - **c.** If $n > m$, the graph of f has no horizontal asymptote.

Exploration

Use a graphing utility to compare the graphs of y_1 and y_2.

$$y_1 = \frac{3x^3 - 5x^2 + 4x - 5}{2x^2 - 6x + 7}$$

$$y_2 = \frac{3x^3}{2x^2}$$

Start with a viewing window in which $-5 \le x \le 5$ and $-10 \le y \le 10$, then zoom out. Write a convincing argument that the shape of the graph of a rational function eventually behaves like the graph of $y = a_nx^n/b_mx^m$, where a_nx^n is the leading term of the numerator and b_mx^m is the leading term of the denominator.

EXAMPLE 2 Finding Horizontal Asymptotes

a. The graph of

$$f(x) = \frac{2x}{3x^2 + 1}$$

has the line $y = 0$ (the x-axis) as a horizontal asymptote, as shown in Figure 3.33(a). Note that the degree of the numerator is *less than* the degree of the denominator.

b. The graph of

$$g(x) = \frac{2x^2}{3x^2 + 1}$$

has the line $y = \frac{2}{3}$ as a horizontal asymptote, as shown in Figure 3.33(b). Note that the degree of the numerator is *equal to* the degree of the denominator, and the horizontal asymptote is given by the ratio of the leading coefficients of the numerator and denominator.

c. The graph of

$$h(x) = \frac{2x^3}{3x^2 + 1}$$

has no horizontal asymptote because the degree of the numerator is *greater than* the degree of the denominator. See Figure 3.33(c).

Although the graph of the function in part (c) does not have a horizontal asymptote, it does have a *slant asymptote*—the line $y = \frac{2}{3}x$. You will study slant asymptotes in the next section.

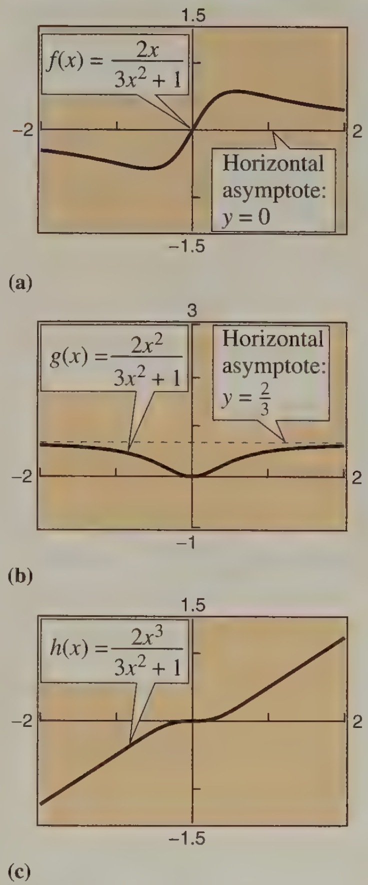


Figure 3.33

EXAMPLE 3 Finding a Function's Domain and Asymptotes

For the function f, find (a) the domain of f, (b) the vertical asymptote of f, and (c) the horizontal asymptote of f.

$$f(x) = \frac{3x^3 + 7x^2 + 2}{-4x^3 + 5}$$

Algebraic Solution

a. Because the denominator is zero when $-4x^3 + 5 = 0$, solve this equation to determine that the domain of f is all real numbers except $x = \sqrt[3]{\frac{5}{4}}$.

b. Because the denominator of f has a zero at $x = \sqrt[3]{\frac{5}{4}}$, and $\sqrt[3]{\frac{5}{4}}$ is not a zero of the numerator, the graph of f has the vertical asymptote $x = \sqrt[3]{\frac{5}{4}} \approx 1.08$, as shown in Figure 3.34.

c. Because the degrees of the numerator and denominator are the same, the horizontal asymptote is given by the ratio of the leading coefficients.

$$y = \frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = -\frac{3}{4}$$

The horizontal asymptote of f, $y = -\frac{3}{4}$, is shown in Figure 3.34.

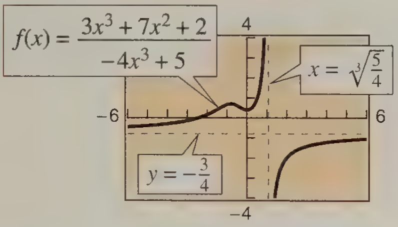


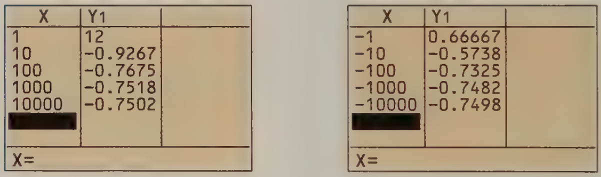
Figure 3.34

Numerical Solution

a. Because the denominator is zero when $-4x^3 + 5 = 0$, solve this equation to determine that the domain of f is all real numbers except $x = \sqrt[3]{\frac{5}{4}}$.

b. Because the denominator of f has a zero at $x = \sqrt[3]{\frac{5}{4}}$, the graph of f has the vertical asymptote $x = \sqrt[3]{\frac{5}{4}} \approx 1.08$, as shown in Figure 3.34.

c. You can create tables like those shown in Figure 3.35 to estimate that the graph of f has a horizontal asymptote at $y = -\frac{3}{4}$ because the values of $f(x)$ become closer and closer to $-\frac{3}{4}$ as x becomes increasingly large or small.



```
  X      Y1
1        12
10       -0.9267
100      -0.7675
1000     -0.7518
10000    -0.7502

X=
```

x Increases Without Bound

```
  X      Y1
-1       0.66667
-10      -0.5738
-100     -0.7325
-1000    -0.7482
-10000   -0.7498

X=
```

x Decreases Without Bound

Figure 3.35

EXAMPLE 4 A Graph with Two Horizontal Asymptotes

A function that is not rational can have two horizontal asymptotes—one to the left and one to the right. For instance, the graph of

$$f(x) = \frac{x + 10}{|x| + 2}$$

is shown in Figure 3.36. It has the line $y = -1$ as a horizontal asymptote to the left and the line $y = 1$ as a horizontal asymptote to the right. You can confirm this by rewriting the function as follows.

$$f(x) = \begin{cases} \dfrac{x + 10}{-x + 2}, & x < 0 \qquad |x| = -x \text{ for } x < 0 \\ \dfrac{x + 10}{x + 2}, & x \geq 0 \qquad |x| = x \text{ for } x \geq 0 \end{cases}$$

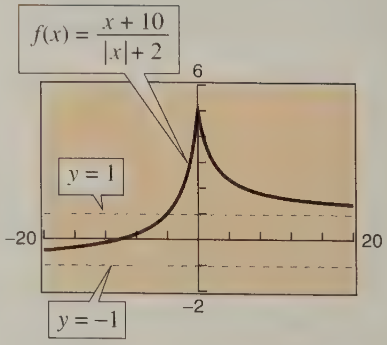


Figure 3.36

Applications

There are many examples of asymptotic behavior in real life. For instance, Example 5 shows how a vertical asymptote can be used to analyze the cost of removing pollutants from smokestack emissions.

EXAMPLE 5 Cost-Benefit Model

A utility company burns coal to generate electricity. The cost of removing a certain *percent* of the pollutants from the smokestack emissions is typically not a linear function. That is, if it costs C dollars to remove 25% of the pollutants, it would cost more than $2C$ dollars to remove 50% of the pollutants. As the percent of removed pollutants approaches 100%, the cost tends to become prohibitive. Suppose that the cost C (in dollars) of removing $p\%$ of the smokestack pollutants is

$$C = \frac{80{,}000p}{100 - p}, \quad 0 \leq p < 100.$$

Sketch the graph of this function. Suppose you are a member of a state legislature that is considering a law that would require utility companies to remove 90% of the pollutants from their smokestack emissions. If the current law requires 85% removal, how much additional cost would there be to the utility company because of the new law?

Solution

The graph of this function is shown in Figure 3.37. Note that the graph has a vertical asymptote at $p = 100$. Because the current law requires 85% removal, the current cost to the utility company is

$$C = \frac{80{,}000(85)}{100 - 85} \qquad \text{Substitute 85 for } p.$$

$$\approx \$453{,}333.$$

If the new law increased the percent removal to 90%, the cost to the utility company would be

$$C = \frac{80{,}000(90)}{100 - 90} \qquad \text{Substitute 90 for } p.$$

$$= \$720{,}000.$$

So, the new law would require the utility company to spend an additional

$$720{,}000 - 453{,}333 = \$266{,}667.$$

STUDY TIP

The *table* feature of a graphing utility can be used to estimate vertical and horizontal asymptotes of rational functions. Use the *table* feature to find any horizontal or vertical asymptotes of

$$f(x) = \frac{2x}{x + 1}.$$

Write a statement explaining how you found the asymptote(s) using the table.

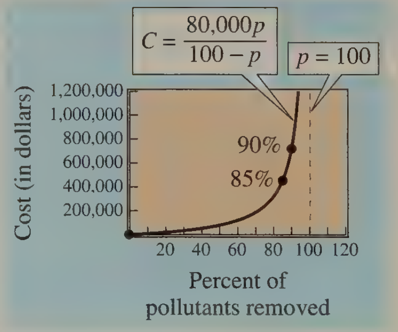


Figure 3.37

EXAMPLE 6 Average Cost of Producing a Product

A business has a cost function of $C = 0.5x + 5000$, where C is measured in dollars and x is the number of units produced. The *average cost per unit* is

$$\overline{C} = \frac{C}{x} = \frac{0.5x + 5000}{x}.$$

Find the average cost per unit when $x = 1000$, 5000, 10,000, and 100,000. What is the horizontal asymptote for this function, and what does it represent?

Algebraic Solution

When $x = 1000$,

$$\overline{C} = \frac{0.5(1000) + 5000}{1000} = \$5.50.$$

When $x = 5000$,

$$\overline{C} = \frac{0.5(5000) + 5000}{5000} = \$1.50.$$

When $x = 10{,}000$,

$$\overline{C} = \frac{0.5(10{,}000) + 5000}{10{,}000} = \$1.00.$$

When $x = 90{,}000$,

$$\overline{C} = \frac{0.5(90{,}000) + 5000}{90{,}000} \approx \$0.56.$$

Because the degree of the numerator and denominator are the same for

$$\overline{C} = \frac{0.5x + 5000}{x}$$

the horizontal asymptote is given by the ratio of the leading coefficients of the numerator and denominator. So, the graph has the line $\overline{C} = \$0.50$ as a horizontal asymptote. This line represents the least possible unit cost for the product.

Graphical Solution

Using a graphing utility to graph the function

$$y = \frac{0.5x + 5000}{x}$$

using a viewing window similar to the graph shown in Figure 3.38. Then use the *trace* or *value* feature to approximate the following.

When $x = 1000$, $y_1 = \$5.50$. When $x = 5000$, $y_1 = \$1.50$.

When $x = 10{,}000$, $y_1 = \$1.00$. When $x = 90{,}000$, $y_1 = \$0.56$.

Continue to use the *trace* or *value* feature to approximate values of $f(x)$ for larger and larger values of x. From this, you can estimate the horizontal asymptote to be $y = \$0.50$. This line represents the least possible cost for the product.

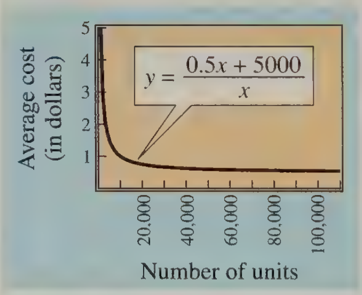


Figure 3.38

Writing About Math *Common Factors in the Numerator and Denominator*

Consider the function $f(x) = \frac{x(x - 1)}{x}$ which has a common factor of x in the numerator and denominator. Use a graphing utility to graph this function. Use the *zoom* and *trace* features to decide whether the function has a vertical asymptote at $x = 0$. Then write a short paragraph explaining why you should check for common factors in the numerator and denominator when graphing a rational function.

3.5 Exercises

In Exercises 1–6, (a) complete each table, (b) determine the vertical and horizontal asymptotes of the function, and (c) find the domain of the function.

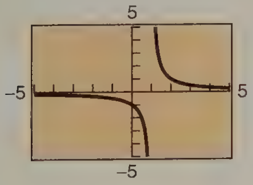
x	$f(x)$
0.5	
0.9	
0.99	
0.999	

x	$f(x)$
1.5	
1.1	
1.01	
1.001	

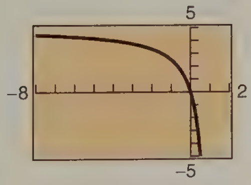
x	$f(x)$
5	
10	
100	
1000	

x	$f(x)$
-5	
-10	
-100	
-1000	

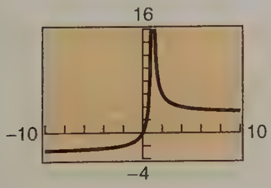
1. $f(x) = \dfrac{1}{x-1}$



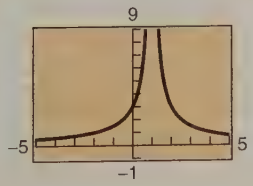
2. $f(x) = \dfrac{5x}{x-1}$



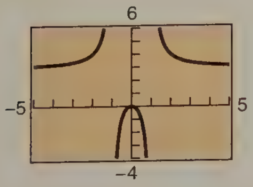
3. $f(x) = \dfrac{3x}{|x-1|}$



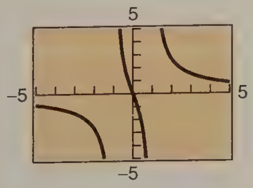
4. $f(x) = \dfrac{3}{|x-1|}$



5. $f(x) = \dfrac{3x^2}{x^2-1}$

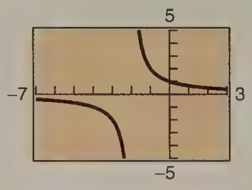


6. $f(x) = \dfrac{4x}{x^2-1}$

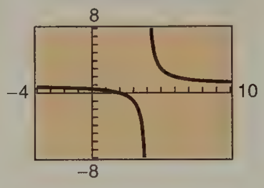


In Exercises 7–12, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

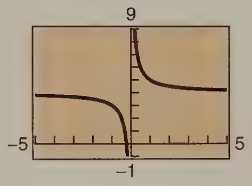
(a)



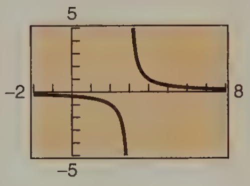
(b)



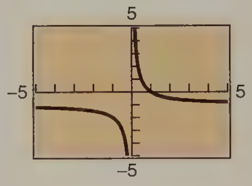
(c)



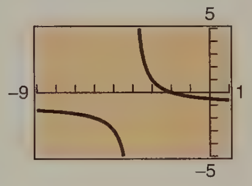
(d)



(e)



(f)



7. $f(x) = \dfrac{2}{x+2}$

8. $f(x) = \dfrac{1}{x-3}$

9. $f(x) = \dfrac{4x+1}{x}$

10. $f(x) = \dfrac{1-x}{x}$

11. $f(x) = \dfrac{x-2}{x-4}$

12. $f(x) = -\dfrac{x+2}{x+4}$

In Exercises 13–18, (a) find the domain of the function, (b) identify any horizontal and vertical asymptotes, and (c) verify your answer to part (a) both graphically by using a graphing utility and numerically by creating a table of values.

13. $f(x) = \dfrac{1}{x^2}$

14. $f(x) = \dfrac{3}{(x-2)^3}$

15. $f(x) = \dfrac{3+x}{3-x}$

16. $f(x) = \dfrac{2-5x}{2+2x}$

17. $f(x) = \dfrac{2x^3}{x^2-1}$

18. $f(x) = \dfrac{3x^2+1}{x^2+x+9}$

Analytical and Numerical Explanation **In Exercises 19–22, (a) determine the domain of f and g, (b) find any vertical asymptotes of f, (c) complete the table, and (d) explain what your results suggest.**

19. $f(x) = \dfrac{x^2 - 4}{x + 2}$, $\quad g(x) = x - 2$

x	-4	-3	-2.5	-2	-1.5	-1	0
$f(x)$							
$g(x)$							

20. $f(x) = \dfrac{x^2(x - 3)}{x^2 - 3x}$, $\quad g(x) = x$

x	-1	0	1	2	3	3.5	4
$f(x)$							
$g(x)$							

21. $f(x) = \dfrac{x - 3}{x^2 - 3x}$, $\quad g(x) = \dfrac{1}{x}$

x	-1	-0.5	0	0.5	2	3	4
$f(x)$							
$g(x)$							

22. $f(x) = \dfrac{2x - 8}{x^2 - 9x + 20}$, $\quad g(x) = \dfrac{2}{x - 5}$

x	0	1	2	3	4	5	6
$f(x)$							
$g(x)$							

Exploration **In Exercises 23–26, determine the value the function f approaches as the magnitude of x increases. Is $f(x)$ greater than or less than this functional value when x is positive and large in magnitude? What about when x is negative and large in magnitude?**

23. $f(x) = 4 - \dfrac{1}{x}$

24. $f(x) = 2 + \dfrac{1}{x - 3}$

25. $f(x) = \dfrac{2x - 1}{x - 3}$

26. $f(x) = \dfrac{2x - 1}{x^2 + 1}$

In Exercises 27–30, find the zeros (if any) of the rational function. Use a graphing utility to verify your answer.

27. $g(x) = \dfrac{x^2 - 9}{x + 1}$

28. $g(x) = \dfrac{x^3 - 8}{x^2 + 4}$

29. $f(x) = 1 - \dfrac{2}{x - 5}$

30. $h(x) = 6 + \dfrac{4}{x^2 + 2}$

31. ***Pollution*** The cost (in millions of dollars) for removing $p\%$ of the industrial and municipal pollutants discharged into a river is

$$C = \frac{255p}{100 - p}, \quad 0 \le p < 100.$$

(a) Find the cost of removing 10% of the pollutants.

(b) Find the cost of removing 40% of the pollutants.

(c) Find the cost of removing 75% of the pollutants.

(d) Use a graphing utility to graph the cost function. Be sure to choose an appropriate viewing window. Explain why you chose the values that you used in your viewing window.

(e) According to this model, would it be possible to remove 100% of the pollutants? Explain.

32. ***Recycling*** In a pilot project, a rural township was given recycling bins for separating and storing recyclable products. The cost (in dollars) for supplying bins to $p\%$ of the population is

$$C = \frac{25{,}000p}{100 - p}, \quad 0 \le p < 100.$$

(a) Find the cost of giving bins to 15% of the population.

(b) Find the cost of giving bins to 50% of the population.

(c) Find the cost of giving bins to 90% of the population.

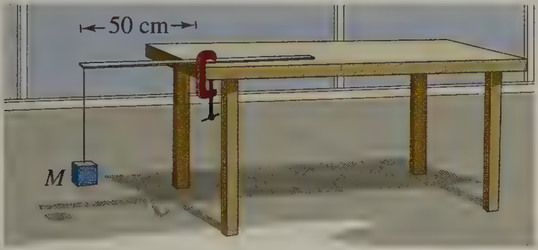
(d) Use a graphing utility to graph the cost function. Be sure to choose an appropriate viewing window. Explain why you chose the values that you used in your viewing window.

(e) According to this model, would it be possible to supply bins to 100% of the residents? Explain.

33. ***Data Analysis*** Consider a physics laboratory experiment designed to determine an unknown mass. A flexible metal meter stick is clamped to a table with 50 centimeters overhanging the edge. Known masses M ranging from 200 grams to 2000 grams are attached to the end of the meter stick. For each mass, the meter stick is displaced vertically and then allowed to oscillate. The average time t in seconds of one oscillation for each mass is recorded in the table.

M	200	400	600	800	1000
t	0.450	0.597	0.721	0.831	0.906

M	1200	1400	1600	1800	2000
t	1.003	1.088	1.168	1.218	1.338



A model for the data is

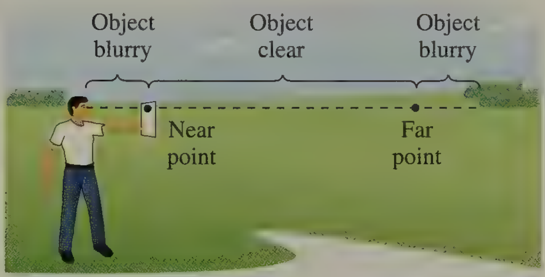
$$t = \frac{38M + 16{,}965}{10(M + 5000)}.$$

(a) Use a graphing utility to create a table showing the estimated time based on the model for each of the masses shown in the table. What can you conclude?

(b) Use the model to approximate the mass of an object if the average for one oscillation is 1.056 seconds.

34. ***Data Analysis*** The endpoints of the interval over which distinct vision is possible are called the *near point* and *far point* of the eye. With increasing age these points normally change. The table gives the approximate near points y in centimeters for various ages x.

x	10	20	30	40	50
y	7	10	14	22	40



(a) Fit a rational model to the data. Take the reciprocals of the near points to generate the points $(x, 1/y)$. Use the regression capabilities of a graphing utility to fit a line to this data. The resulting line has the form

$$\frac{1}{y} = ax + b.$$

Solve for y.

(b) Use a graphing utility to create a table giving the predicted near point based on the model for each of the ages in the given table.

(c) Do you think the model can be used to predict the near point for a person who is 60 years old? Explain.

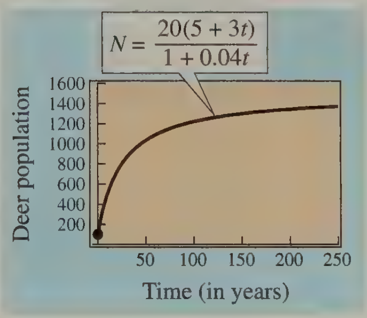
35. ***Deer Population*** The game commission introduces 100 deer into newly acquired state game lands. The population of the herd is

$$N = \frac{20(5 + 3t)}{1 + 0.04t}, \quad t \geq 0$$

where t is the time (in years).

(a) Find the population when t is 5, 10, and 25.

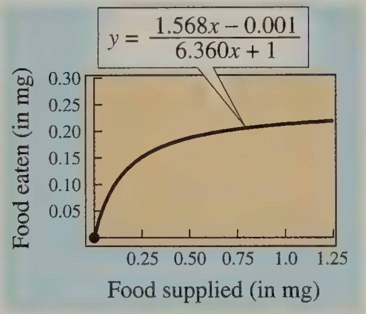
(b) What is the limiting size of the herd as time increases? Explain your reasoning.



36. ***Insects*** A biology class performs an experiment comparing the quantity of food consumed by a certain kind of moth with the quantity supplied. The model for their experimental data is

$$y = \frac{1.568x - 0.001}{6.360x + 1}, \quad x > 0$$

where x is the quantity (in milligrams) of food supplied and y is the quantity (in milligrams) eaten. At what level of consumption will the moth become satiated?



37. ***Military*** The number of United States military reserve personnel M (in thousands) for the years 1990 through 1997 is shown in the table. (Source: U.S. Department of Defense)

Year	1990	1991	1992	1993
M	1671	1786	1883	1867

Year	1994	1995	1996	1997
M	1805	1659	1550	1461

A model for the data is

$$y = \frac{1671.92 + 130.23t}{1 - 0.02t + 0.02t^2}$$

where t is time (in years), with $t = 0$ corresponding to 1990.

(a) Use a graphing utility to plot the data and graph the model in the same viewing window.

(b) Use the model to estimate the number of military reserve personnel in 2002.

(c) Would this model be useful for estimating the number of military reserve personnel for future years? Explain.

Synthesis

True or False? **In Exercises 38 and 39, determine whether the statement is true or false. Justify your answer.**

38. A rational function can have infinitely many vertical asymptotes.

39. $f(x) = x^3 - 2x^2 - 5x + 6$ is a rational function.

Think About It **In Exercises 40–43, write a rational function f having the specified characteristics. (There are many correct answers.)**

40. Vertical asymptotes: $x = -2, x = 1$

41. Vertical asymptote: None
Horizontal asymptote: $y = 0$

42. Vertical asymptote: None
Horizontal asymptote: $y = 2$

43. Vertical asymptotes: $x = 0, x = \frac{5}{2}$
Horizontal asymptote: $y = -3$

Review

In Exercises 44–49, find all solutions of the equation.

44. $225x - 50x^3 = 0$

45. $x(10 - x) = 25$

46. $2z^2 - 3z - 35 = 0$

47. $t^3 - 50t = 0$

48. $27x^3 - 147x = 0$

49. $x^4 - 225 = 0$

In Exercises 50–53, divide using synthetic division.

50. $(x^2 + 5x + 6) \div (x - 4)$

51. $(x^2 - 10x + 15) \div (x - 3)$

52. $(2x^2 + x - 11) \div (x + 5)$

53. $(4x^2 + 3x - 10) \div (x + 6)$

In Exercises 54–57, find a polynomial with integer coefficients that has the given zeros. (There are many correct answers.)

54. $8, 5i, -5i$

55. $-2, 6i, -6i$

56. $6, 3 + i, 3 - i$

57. $1, -3 + 2i, -3 - 2i$

3.6 Graphs of Rational Functions

The Graph of a Rational Function

To sketch the graph of a rational function, use the following guidelines.

Guidelines for Graphing Rational Functions

Let $f(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials with no common factors.

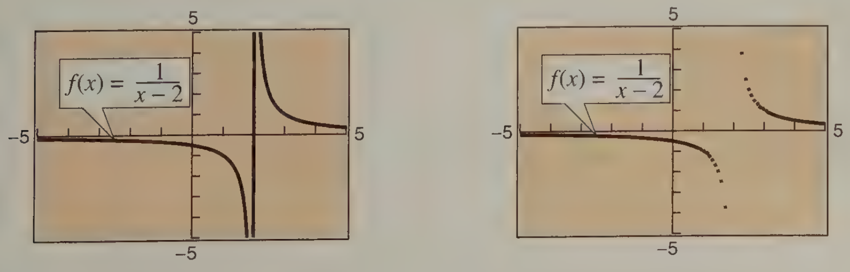
1. Find and plot the y-intercept (if any) by evaluating $f(0)$.
2. Set the numerator equal to zero and solve the equation $N(x) = 0$. The real solutions represent the x-intercepts of the graph. Plot these intercepts.
3. Set the denominator equal to zero and solve the equation $D(x) = 0$. The real solutions represent the vertical asymptotes. Sketch these asymptotes using dashed vertical lines.
4. Find and sketch the horizontal asymptote of the graph using a dashed horizontal line.
5. Plot at least one point between and one point beyond each x-intercept and vertical asymptote.
6. Use smooth curves to complete the graph between and beyond the vertical asymptotes.

Testing for symmetry can be useful, especially for simple rational functions. For example, the graph of $f(x) = 1/x$ is symmetrical with respect to the origin, and the graph of $g(x) = 1/x^2$ is symmetrical with respect to the y-axis.

Graphing utilities have difficulty sketching graphs of rational functions that have vertical asymptotes. Often, the utility will connect parts of the graph that are not supposed to be connected. For instance, Figure 3.39(a) shows the graph of

$$f(x) = \frac{1}{x-2}.$$

Notice that the graph should consist of two *unconnected* portions—one to the left of $x = 2$ and the other to the right of $x = 2$. To eliminate this problem, you can try changing the *mode* of the graphing utility to *dot mode*. The problem with this is that the graph is then represented as a collection of dots rather than as a smooth curve, as shown in Figure 3.39(b).



(a) Connected mode

(b) Dot mode

Figure 3.39

What You Should Learn:

- How to analyze and sketch graphs of rational functions
- How to decide whether graphs of rational functions have slant asymptotes
- How to use rational functions to model and solve real-life problems

Why You Should Learn It:

The graph of a rational function provides a good indication of the future behavior of a function. Exercise 75 on page 288 models the number of kidney transplants in the U.S., and allows you to estimate the number of transplants in the coming years.

Superstock

EXAMPLE 1 Sketching the Graph of a Rational Function

Sketch the graph of $g(x) = \dfrac{3}{x-2}$ by hand.

Solution

y-Intercept: $\left(0, -\frac{3}{2}\right)$, because $g(0) = -\frac{3}{2}$.

x-Intercept: None, because $3 \neq 0$.

Vertical Asymptote: $x = 2$, zero of denominator

Horizontal Asymptote: $y = 0$, degree of $N(x)$ < degree of $D(x)$

Additional Points:

x	-4	1	3	5
$g(x)$	-0.5	-3	3	1

By plotting the intercepts, asymptotes, and a few additional points, you can obtain the graph shown in Figure 3.40. Confirm this with a graphing utility.

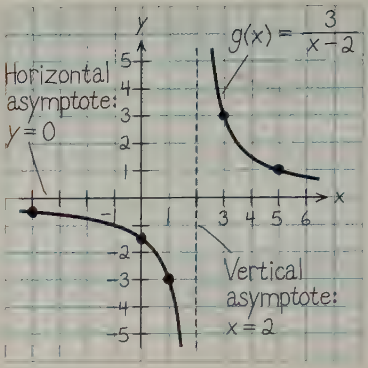


Figure 3.40

Note that the graph of g in Example 1 is a vertical stretch and a right shift of the graph of

$$f(x) = \frac{1}{x}$$

because

$$g(x) = \frac{3}{x-2} = 3\left(\frac{1}{x-2}\right) = 3f(x-2).$$

EXAMPLE 2 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{2x-1}{x}$ by hand.

Solution

y-Intercept: None, because $x = 0$ is not in the domain.

x-Intercept: $\left(\frac{1}{2}, 0\right)$, because $2x - 1 = 0$, or $x = \frac{1}{2}$.

Vertical Asymptote: $x = 0$, zero of denominator

Horizontal Asymptote: $y = 2$, degree of $N(x)$ = degree of $D(x)$

Additional Points:

x	-4	-1	$\frac{1}{4}$	4
$f(x)$	2.25	3	-2	1.75

By plotting the intercepts, asymptotes, and a few additional points, you can obtain the graph shown in Figure 3.41. Confirm this with a graphing utility.

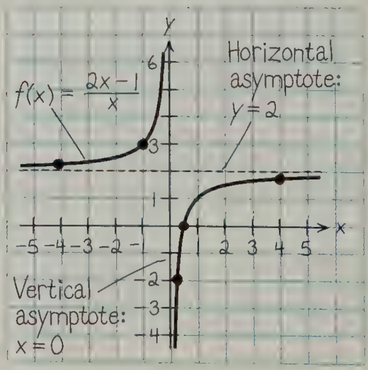


Figure 3.41

EXAMPLE 3 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{x}{x^2 - x - 2}$.

Solution

By factoring the denominator, you have

$$f(x) = \frac{x}{x^2 - x - 2}$$
$$= \frac{x}{(x + 1)(x - 2)}.$$

y-Intercept: $(0, 0)$, because $f(0) = 0$.

x-Intercept: $(0, 0)$

Vertical Asymptotes: $x = -1$, $x = 2$, zeros of denominator

Horizontal Asymptote: $y = 0$, degree of $N(x) <$ degree of $D(x)$

Additional Points:

x	-3	-0.5	1	3
$f(x)$	-0.3	0.4	-0.5	0.75

The graph is shown in Figure 3.42.

EXAMPLE 4 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$.

Solution

By factoring the numerator and denominator, you have

$$f(x) = \frac{2(x^2 - 9)}{x^2 - 4}$$
$$= \frac{2(x - 3)(x + 3)}{(x - 2)(x + 2)}.$$

y-Intercept: $\left(0, \frac{9}{2}\right)$, because $f(0) = \frac{9}{2}$.

x-Intercepts: $(-3, 0)$, $(3, 0)$

Vertical Asymptotes: $x = -2$, $x = 2$, zeros of denominator

Horizontal Asymptote: $y = 2$, degree of $N(x)$ = degree of $D(x)$

Symmetry: With respect to y-axis, because $f(-x) = f(x)$.

Additional Points:

x	0.5	2.5	6
$f(x)$	4.67	-2.44	1.69

The graph is shown in Figure 3.43.

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Exploration

Use a graphing utility to graph

$$f(x) = 1 + \frac{1}{x - \dfrac{1}{x}}.$$

Set the graphing utility to *dot* mode and use a decimal viewing window. Use the *trace* feature to find three "holes" or "breaks" in the graph. Do all three holes represent zeros of the denominator

$$x - \frac{1}{x}?$$

Explain.

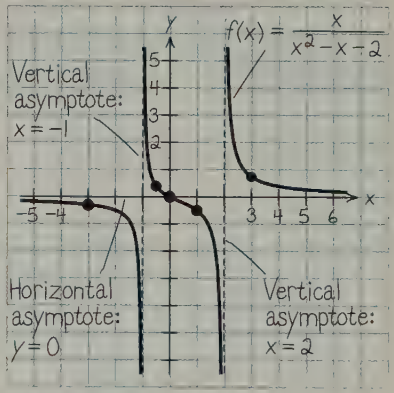


Figure 3.42

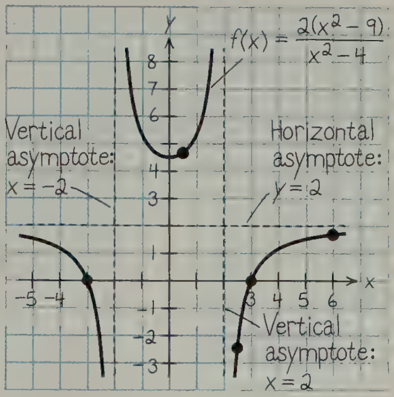


Figure 3.43

Slant Asymptotes

If the degree of the numerator of a rational function is exactly *one more* than the degree of its denominator, the graph of the function has a **slant** (or **oblique**) **asymptote.** For example, the graph of

$$f(x) = \frac{x^2 - x}{x + 1}$$

has a slant asymptote, as shown in Figure 3.44. To find the equation of a slant asymptote, use long division. For instance, by dividing $x + 1$ into $x^2 - x$, you have

$$f(x) = \frac{x^2 - x}{x + 1}$$

$$= \underbrace{x - 2}_{\substack{\text{Slant asymptote}\\ (y = x - 2)}} + \frac{2}{x + 1}. \qquad \frac{2}{x+1} \to 0 \text{ as } x \to \pm\infty$$

In Figure 3.44, notice that the graph of f approaches the line $y = x - 2$ as x moves to the right or left.

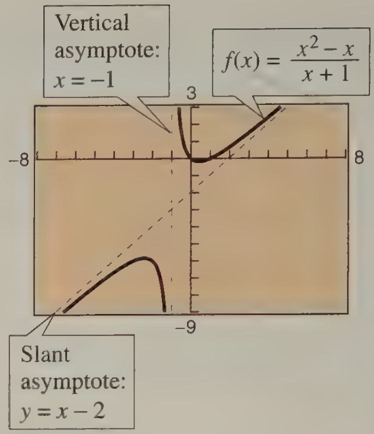


Figure 3.44

Exploration

Do you think it is possible for the graph of a rational function to cross its horizontal asymptote or its slant asymptote? Use the graphs of the following functions to investigate this question. Write a summary of your conclusion. Explain your reasoning.

$$f(x) = \frac{x}{x^2 + 1}$$

$$g(x) = \frac{2x}{3x^2 - 2x + 1}$$

$$h(x) = \frac{x^3}{x^2 + 1}$$

EXAMPLE 5 A Rational Function with a Slant Asymptote

Graph the function $f(x) = \dfrac{x^2 - x - 2}{x - 1}$.

Solution

First write $f(x)$ in two different ways. Factoring the numerator

$$f(x) = \frac{x^2 - x - 2}{x - 1} = \frac{(x - 2)(x + 1)}{x - 1}$$

allows you to recognize the x-intercepts, and long division

$$f(x) = \frac{x^2 - x - 2}{x - 1}$$

$$= x - \frac{2}{x - 1} \qquad \frac{2}{x-1} \to 0 \text{ as } x \to \pm\infty$$

allows you to recognize that the line $y = x$ is a slant asymptote of the graph.

y-Intercept: (0, 2), because $f(0) = 2$.

x-Intercepts: (−1, 0), (2, 0)

Vertical Asymptote: $x = 1$, zero of denominator

Slant Asymptote: $y = x$

Additional Points:

x	−2	0.5	1.5	3
$f(x)$	−1.33	4.5	−2.5	2

The graph is shown in Figure 3.45.

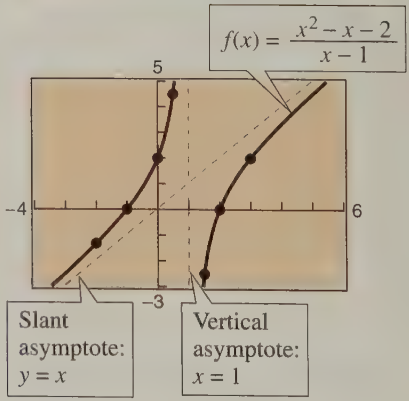


Figure 3.45

Application

EXAMPLE 6 Finding a Minimum Area

A rectangular page is designed to contain 48 square inches of print. The margins on each side of the page are each $1\frac{1}{2}$ inches. The margins at the top and bottom are each 1 inch. What should the dimensions of the page be so that the minimum amount of paper is used?

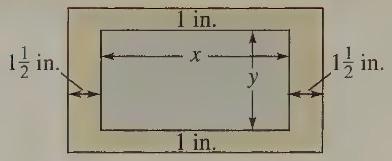


Figure 3.46

Graphical Solution

Let A be the area to be minimized. From Figure 3.46, you can write

$$A = (x + 3)(y + 2).$$

The printed area inside the margins is modeled by $48 = xy$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right)$$
$$= \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

The graph of this rational function is shown in Figure 3.47. Because x represents the width of the printed area, you need consider only the portion of the graph for which x is positive. Using a graphing utility, you can approximate the minimum value of A to occur when $x \approx 8.5$ inches. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be

$x + 3 \approx 11.5$ inches by $y + 2 \approx 7.6$ inches.

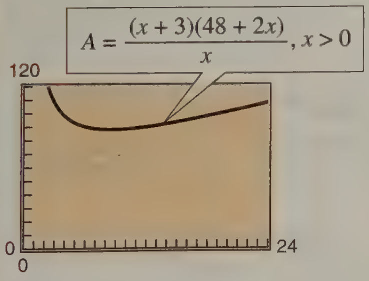


Figure 3.47

Numerical Solution

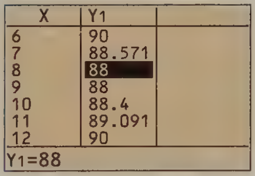
Let A be the area to be minimized. From Figure 3.46, you can write

$$A = (x + 3)(y + 2).$$

The printed area inside the margins is modeled by $48 = xy$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right) = \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

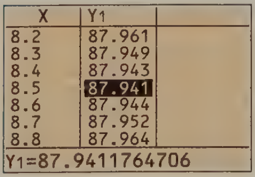
Use the *table* feature of a graphing utility to create a table of values for the function $y_1 = \frac{(x + 3)(48 + 2x)}{x}$ beginning at $x = 1$, as shown in Figure 3.48.



```
   X    | Y1     |
6       | 90     |
7       | 88.571 |
8       | 88     |
9       | 88     |
10      | 88.4   |
11      | 89.091 |
12      | 90     |
Y1=88
```

Figure 3.48

From the table, you can see that the minimum value of y_1 occurs when x is somewhere between 8 and 9. To approximate the minimum value of y_1 to one decimal place, change the table to begin at 8 with an increment of 0.1. The minimum value of y_1 occurs when $x \approx 8.5$, as shown in Figure 3.49. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be $x + 3 \approx 11.5$ inches by $y + 2 \approx 7.6$ inches.



```
   X    | Y1     |
8.2     | 87.961 |
8.3     | 87.949 |
8.4     | 87.943 |
8.5     | 87.941 |
8.6     | 87.944 |
8.7     | 87.952 |
8.8     | 87.964 |
Y1=87.9411764706
```

Figure 3.49

If you go on to take a course in calculus, you will learn an analytic technique for finding the exact value of x that produces a minimum area in Example 6. In this case, that value is $x = 6\sqrt{2} \approx 8.485$.

3.6 Exercises

In Exercises 1–4, use a graphing utility to graph $f(x) = 2/x$ and the function g in the same viewing window. Describe the relationship between the two graphs.

1. $g(x) = f(x) + 1$

2. $g(x) = f(x - 1)$

3. $g(x) = -f(x)$

4. $g(x) = \frac{1}{2}f(x + 2)$

In Exercises 5–8, use a graphing utility to graph $f(x) = 2/x^2$ and the function g in the same viewing window. Describe the relationship between the two graphs.

5. $g(x) = f(x) - 2$

6. $g(x) = -f(x)$

7. $g(x) = f(x - 2)$

8. $g(x) = \frac{1}{4}f(x)$

In Exercises 9–12, use a graphing utility to graph $f(x) = 4/x^3$ and the function g in the same viewing window. Describe the relationship between the two graphs.

9. $g(x) = f(x + 2)$

10. $g(x) = f(x) + 1$

11. $g(x) = -f(x)$

12. $g(x) = \frac{1}{4}f(x)$

In Exercises 13–30, sketch the graph of the rational function by hand. As sketching aids, check for intercepts, symmetry, vertical asymptotes, and horizontal asymptotes. Use a graphing utility to verify your graph.

13. $f(x) = \dfrac{1}{x + 2}$

14. $f(x) = \dfrac{1}{x - 5}$

15. $C(x) = \dfrac{5 + 2x}{1 + x}$

16. $P(x) = \dfrac{1 - 3x}{1 - x}$

17. $g(x) = \dfrac{1}{x + 2} + 2$

18. $f(t) = \dfrac{1 - 2t}{t}$

19. $f(x) = 2 - \dfrac{3}{x^2}$

20. $h(x) = \dfrac{x^2}{x^2 - 9}$

21. $f(x) = \dfrac{x^2}{x^2 - 4}$

22. $g(x) = \dfrac{x}{x^2 - 9}$

23. $f(x) = \dfrac{x}{x^2 - 4}$

24. $f(x) = -\dfrac{1}{(x - 2)^2}$

25. $g(x) = \dfrac{4(x + 1)}{x(x - 4)}$

26. $h(x) = \dfrac{2}{x^2(x - 2)}$

27. $f(x) = \dfrac{3x}{x^2 - x - 2}$

28. $f(x) = \dfrac{2x}{x^2 + x - 2}$

29. $f(x) = -\dfrac{4}{\dfrac{5}{x} - 5}$

30. $f(x) = 2 + \dfrac{8}{3x - \dfrac{3}{x}}$

In Exercises 31–40, use a graphing utility to obtain the graph of the function. Give its domain and identify any vertical or horizontal asymptotes.

31. $f(x) = \dfrac{2 + x}{1 - x}$

32. $f(x) = \dfrac{3 - x}{2 - x}$

33. $f(t) = \dfrac{3t + 1}{t}$

34. $h(x) = \dfrac{x - 2}{x - 3}$

35. $h(t) = \dfrac{4}{t^2 + 1}$

36. $g(x) = -\dfrac{x}{(x - 2)^2}$

37. $f(x) = \dfrac{x + 1}{x^2 - x - 6}$

38. $f(x) = \dfrac{x + 4}{x^2 + x - 6}$

39. $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x}$

40. $f(x) = 5\left(\dfrac{1}{x - 4} - \dfrac{1}{x + 2}\right)$

Exploration **In Exercises 41–46, use a graphing utility to obtain the graph of the function. What do you observe about its asymptotes?**

41. $h(x) = \dfrac{6x}{\sqrt{x^2 + 1}}$

42. $f(x) = -\dfrac{x}{\sqrt{9 + x^2}}$

43. $g(x) = \dfrac{4|x - 2|}{x + 1}$

44. $f(x) = -\dfrac{8|3 + x|}{x - 2}$

45. $f(x) = \dfrac{4(x - 1)^2}{x^2 - 4x + 5}$

46. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

In Exercises 47–54, sketch the graph of the rational function. As sketching aids, check for intercepts, symmetry, vertical asymptotes, and slant asymptotes.

47. $f(x) = \dfrac{2x^2 + 1}{x}$

48. $g(x) = \dfrac{x^2 + 1}{x}$

49. $h(x) = \dfrac{x^2}{x - 1}$

50. $f(x) = \dfrac{x^3}{x^2 - 1}$

51. $g(x) = \dfrac{x^3}{2x^2 - 8}$

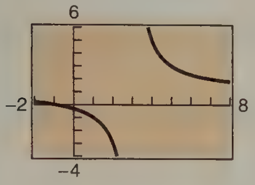
52. $f(x) = \dfrac{x^2 - 1}{x^2 + 4}$

53. $f(x) = \dfrac{x^3 + 2x^2 + 4}{2x^2 + 1}$

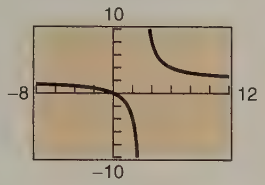
54. $f(x) = \dfrac{2x^2 - 5x + 5}{x - 2}$

Graphical Reasoning **In Exercises 55–58, (a) use the graph to estimate any x-intercepts of the rational function and (b) set $y = 0$ and solve the resulting equation to confirm your result in part (a).**

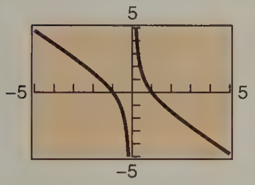
55. $y = \dfrac{x+1}{x-3}$



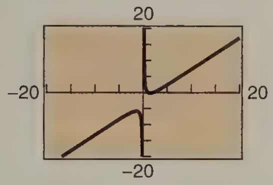
56. $y = \dfrac{2x}{x-3}$



57. $y = \dfrac{1}{x} - x$



58. $y = x - 3 + \dfrac{2}{x}$



In Exercises 59–62, use a graphing utility to graph the rational function. Give the domain of the function and identify any asymptotes. Then zoom out far enough so that the graph appears as a line. Identify the line.

59. $y = \dfrac{2x^2 + x}{x+1}$

60. $y = \dfrac{x^2 + 5x + 8}{x+3}$

61. $y = \dfrac{1 + 3x^2 - x^3}{x^2}$

62. $y = \dfrac{12 - 2x - x^2}{2(4+x)}$

Graphical Reasoning **In Exercises 63–66, (a) use a graphing utility to graph the function and determine any x-intercepts, and (b) set $y = 0$ and solve the resulting equation to confirm your result in part (a).**

63. $y = \dfrac{1}{x+5} + \dfrac{4}{x}$

64. $y = 20\left(\dfrac{2}{x+1} - \dfrac{3}{x}\right)$

65. $y = x - \dfrac{6}{x-1}$

66. $y = x - \dfrac{9}{x}$

67. ***Concentration of a Mixture*** A 1000-liter tank contains 10 liters of a 25% brine solution. You add x liters of a 75% brine solution to the tank.

(a) Show that the concentration C of the final mixture is

$$C = \frac{3x + 10}{4(x + 10)}.$$

(b) Determine the domain of the function on the basis of the physical constraints of the problem.

(c) Use a graphing utility to graph the function. As the tank is filled, what happens to the rate at which the concentration of brine increases? How close to the horizontal asymptote is the graph of C when the tank is full?

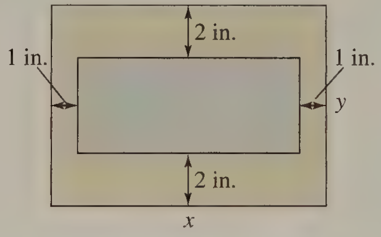
68. ***Geometry*** A rectangular region of length x and width y has an area of 500 square meters.

(a) Express the width y as a function of x.

(b) Determine the domain of the function on the basis of the physical constraints of the problem.

(c) Sketch a graph of the function and determine the width of the rectangle if $x = 30$ meters.

69. ***Page Design*** A page that is x inches wide and y inches high contains 30 square inches of print. The margins at the top and bottom are 2 inches and the margins on each side are 1 inch.



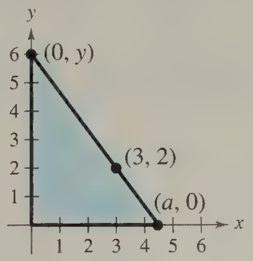
(a) Show that the total area A of the page is

$$A = \frac{2x(2x + 11)}{x - 2}.$$

(b) Determine the domain of the function on the basis of the physical constraints of the problem.

(c) Use a graphing utility to graph the area function and approximate the page size such that the minimum amount of paper will be used. Verify your answer numerically using the *table* feature of a graphing utility.

70. ***Minimum Area*** A right triangle is formed in the first quadrant by the x-axis, the y-axis, and a line segment through the point (3, 2).



(a) Show that an equation of the line segment is

$$y = \frac{2(a - x)}{a - 3}, \quad 0 \le x \le a.$$

(b) Show that the area of the triangle is $A = \dfrac{a^2}{a - 3}$.

(c) Use a graphing utility to graph the area function and estimate the value of a that yields a minimum area. Estimate the minimum area. Verify your answer numerically using the *table* feature of a graphing utility.

71. ***Ordering and Transportation Cost*** The ordering and transportation cost C for the components used in manufacturing a certain product is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \ge 1$$

where C is measured in thousands of dollars and x is the order size (in hundreds). Use a graphing utility to obtain the graph of the cost function and, from the graph, estimate the order size that minimizes cost.

72. ***Average Cost*** The cost of producing x units of a product is $C = 0.2x^2 + 10x + 5$, and therefore the average cost per unit is

$$\overline{C} = \frac{C}{x} = \frac{0.2x^2 + 10x + 5}{x}, \quad x > 0.$$

Sketch the graph of the average cost function, and estimate the number of units that should be produced to minimize the average cost per unit.

73. ***Medicine*** The concentration of a certain chemical in the bloodstream t hours after injection into muscle tissue is

$$C = \frac{3t^2 + t}{t^3 + 50}, \quad t \ge 0.$$

(a) Determine the horizontal asymptote of the function and interpret its meaning in the context of the problem.

(b) Use a graphing utility to graph the function and approximate the time when the bloodstream concentration is greatest.

(c) Use a graphing utility to determine when the concentration is less than 0.345.

74. ***Numerical and Graphical Analysis*** A driver averaged 50 miles per hour on the round trip between home and a city 100 miles away. The average speeds for going and returning were x and y miles per hour, respectively.

(a) Show that $y = \dfrac{25x}{x - 25}$.

(b) Determine the vertical and horizontal asymptotes of the function.

(c) Use a graphing utility to complete the table. What do you observe?

x	30	35	40	45	50	55	60
y							

(d) Use a graphing utility to graph the function.

(e) Is it possible to average 20 miles per hour in one direction and still average 50 miles per hour on the round trip? Explain.

75. ***Comparing Models*** The number of kidney transplants K in the United States from 1987 through 1996 is shown in the table. (Sources: U.S. Department of Health and Human Services; United Network for Organ Sharing)

Year	1987	1988	1989	1990	1991	1992	1993	1994	1995	1996
K	8967	9123	8890	9877	10,122	10,231	11,020	11,392	11,891	12,080

For each of the following, let t be the time (in years), with $t = 7$ corresponding to 1987.

(a) A model for the data is

$$K = \frac{8116.17 - 280t}{1 - 0.0447t}.$$

Use a graphing utility to plot the data points and graph the model in the same viewing window.

(b) Use the regression capabilities of a graphing utility to fit a line to the data.

(c) Use the regression capabilities of a graphing utility to fit a parabola to the data.

(d) Which of the three models would you recommend as an estimator of the number of kidney transplants for the years following 1996? Explain your reasoning.

76. ***Comparing Models*** The number N (in thousands) of insured commercial banks in the United States for the years 1988 through 1997 is shown in the table. (Source: U.S. Federal Deposit Insurance Corporation)

Year	1988	1989	1990	1991	1992
N	13.1	12.7	12.3	11.9	11.5

Year	1993	1994	1995	1996	1997
N	11.0	10.5	9.9	9.5	9.1

For each of the following, let t be the time (in years), with $t = 8$ corresponding to 1988.

(a) Use the regression capabilities of a graphing utility to fit a line to the data. Use a graphing utility to plot the data points and graph the model in the same viewing window.

(b) Fit a rational model to the data. Take the reciprocal of N to generate the points $(t, 1/N)$. Use the regression capabilities of a graphing utility to fit a line to this data. The resulting line has the form

$$\frac{1}{N} = at + b.$$

Solve for N. Use a graphing utility to plot the data points and graph the rational model in the same viewing window.

(c) Use a graphing utility to create a table showing the predicted number of banks based on each model for each of the years in the given table. Which model do you prefer? Why?

Synthesis

True or False? **In Exercises 77 and 78, determine whether the statement is true or false. Justify your answer.**

77. If the graph of a rational function f has a vertical asymptote at $x = 5$, it is possible to sketch the graph without lifting your pencil from the paper.

78. A rational function can never cross one of its asymptotes.

Think About It **In Exercises 79 and 80, use a graphing utility to obtain the graph of the function. Explain why there is no vertical asymptote when a superficial examination of the function may indicate that there should be one.**

79. $h(x) = \dfrac{6 - 2x}{3 - x}$

80. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

81. ***Writing*** Write a paragraph discussing whether every rational function has a vertical asymptote.

Think About It **In Exercises 82–85, write a rational function satisfying the following criteria.**

82. Vertical asymptote: $x = 2$
Slant asymptote: $y = x + 1$
Zero of the function: $x = -2$

83. Vertical asymptote: $x = -4$
Slant asymptote: $y = x - 2$
Zero of the function: $x = 3$

84. Vertical asymptote: $x = -1$
Horizontal asymptote: $y = 2$
Zero of the function: $x = 3$

85. Vertical asymptote: $x = 3$
Horizontal asymptote: $y = -2$
Zero of the function: $x = -6$

Review

In Exercises 86–91, simplify the expression.

86. $\left(\dfrac{x}{8}\right)^{-3}$

87. $(4x^2)^{-2}$

88. $\dfrac{3x^3y^2}{15xy^4}$

89. $\dfrac{(4x^2)^{3/2}}{8x^5}$

90. $\dfrac{3^{7/6}}{3^{1/6}}$

91. $\dfrac{x^{-2} \cdot x^{1/2}}{x^{-1} \cdot x^{5/2}}$

In Exercises 92–97, sketch a graph of the equation by hand. Use a graphing utility to verify your graph.

92. $-y + 3x + 8 = 0$

93. $4x + 5y - 2 = 0$

94. $7x + 3 = 0$

95. $4y - 10 = 0$

96. $x - y - 1 = 0$

97. $-7 + 8x - 2y = 0$

In Exercises 98–101, use a graphing utility to graph the function and find its domain and range.

98. $f(x) = \sqrt{6 + x^2}$

99. $f(x) = \sqrt{121 - x^2}$

100. $f(x) = -|x + 9|$

101. $f(x) = -x^2 + 9$

3 Chapter Summary

What did you learn?

	Review Exercises
Section 3.1	
☐ How to analyze graphs of quadratic functions	1, 2
☐ How to write quadratic functions in standard form and use the results to sketch graphs of functions	3–16
☐ How to use quadratic functions to model and solve real-life problems	17, 18
Section 3.2	
☐ How to use transformations to sketch graphs of polynomial functions	19–22
☐ How to use the Leading Coefficient Test to determine the end behavior of graphs of polynomial functions	23–28
☐ How to find and use zeros of polynomial functions to sketch their graphs	29–36
☐ How to use the Intermediate Value Theorem to help locate zeros of polynomial functions	37–40
Section 3.3	
☐ How to use long division to divide polynomials by other polynomials	41–48
☐ How to use synthetic division to divide polynomials by binomials of the form $(x - k)$	49–52
☐ How to use the Remainder Theorem and the Factor Theorem	53–58
☐ How to use the Rational Zero Test to determine possible rational zeros of polynomial functions	59–64
☐ How to determine upper and lower bounds for zeros of polynomial functions	65, 66
Section 3.4	
☐ How to use the Fundamental Theorem of Algebra to determine the number of zeros of polynomial functions	67–70
☐ How to find all zeros of polynomial functions, including complex zeros	71–76
☐ How to find conjugate pairs of complex zeros	77–82
☐ How to find zeros of polynomials by factoring	83–86
Section 3.5	
☐ How to find domains of rational functions	87–90
☐ How to find horizontal and vertical asymptotes of graphs of rational functions	91–98
☐ How to use rational functions to model and solve real-life problems	99, 100
Section 3.6	
☐ How to analyze and sketch graphs of rational functions	101–110
☐ How to decide whether graphs of rational functions have slant asymptotes	111–116
☐ How to use rational functions to model and solve real-life problems	117, 118

3 Review Exercises

3.1 ***Graphical Reasoning*** **In Exercises 1 and 2, use a graphing utility to graph each equation in the same viewing window. Describe how each graph differs from the graph of $y = x^2$.**

1. (a) $y = 2x^2$ (b) $y = -2x^2$
   (c) $y = x^2 + 2$ (d) $y = (x + 2)^2$
2. (a) $y = x^2 - 4$ (b) $y = 4 - x^2$
   (c) $y = (x - 3)^2$ (d) $y = \frac{1}{2}x^2 - 1$

In Exercises 3–6, sketch the graph of the quadratic function. Identify the vertex and the intercepts.

3. $f(x) = \left(x + \frac{3}{2}\right)^2 + 1$
4. $f(x) = (x - 4)^2 - 4$
5. $f(x) = \frac{1}{3}(x^2 + 5x - 4)$
6. $f(x) = 3x^2 - 12x + 11$

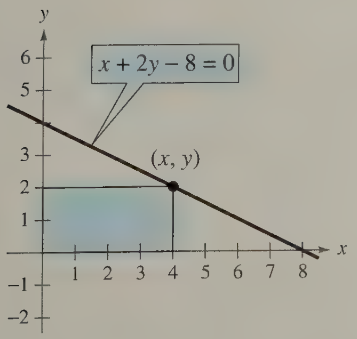
In Exercises 7 and 8, find the quadratic function that has the indicated vertex and whose graph passes through the given point.

7. Vertex: $(1, -4)$; Point: $(2, -3)$
8. Vertex: $(2, 3)$; Point: $(-1, 6)$

In Exercises 9–16, find the maximum or minimum value of the quadratic function.

9. $g(x) = x^2 - 2x$
10. $f(x) = x^2 + 8x + 10$
11. $f(x) = 6x - x^2$
12. $h(x) = 3 + 4x - x^2$
13. $f(t) = -2t^2 + 4t + 1$
14. $h(x) = 4x^2 + 4x + 13$
15. $h(x) = x^2 + 5x - 4$
16. $f(x) = 4x^2 + 4x + 5$

17. ***Numerical, Graphical, and Analytical Analysis*** A rectangle is inscribed in the region bounded by the x-axis, the y-axis, and the graph of $x + 2y - 8 = 0$.



(a) Complete six rows of a table like the one below.

x	y	Area
1	$4 - \frac{1}{2}(1)$	$(1)\left[4 - \frac{1}{2}(1)\right] = \frac{7}{2}$
2	$4 - \frac{1}{2}(2)$	$(2)\left[4 - \frac{1}{2}(2)\right] = 6$

(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the dimensions that will produce the maximum area.

(c) Write the area A as a function of x. Determine the domain of the function in the context of the problem.

(d) Use a graphing utility to graph the function. Use the graph to approximate the dimensions that will produce the maximum area.

(e) Write the area function in standard form to find algebraically the dimensions that will produce the maximum area.

18. ***Maximum Profit*** Let x be the amount (in hundreds of dollars) a company spends on advertising, and let P be the profit, where

$$P = 320 + 15x - \frac{1}{2}x^2.$$

(a) Use a graphing utility to graph the function.

(b) Use the *zoom* and *trace* features of a graphing utility to estimate the vertex of the function.

(c) Verify your answer in part (b) both algebraically and numerically by creating a table of values.

(d) Explain what the vertex represents.

3.2 **In Exercises 19–22, sketch the graphs of $y = x^n$ and the specified transformations.**

19. $y = x^4$
    (a) $f(x) = (x + 5)^4$ (b) $f(x) = x^4 - 4$
    (c) $f(x) = 3 + x^4$ (d) $f(x) = \frac{1}{4}(x - 2)^4$
20. $y = x^5$
    (a) $f(x) = (x + 4)^5$ (b) $f(x) = 6 + x^5$
    (c) $f(x) = 3 - \frac{1}{2}x^5$ (d) $f(x) = 2(x + 3)^5$
21. $y = x^6$
    (a) $f(x) = x^6 - 2$ (b) $f(x) = -\frac{1}{4}x^6$
    (c) $f(x) = -\frac{1}{2}x^6 - 5$ (d) $f(x) = -(x + 7)^6 - 5$

22. $y = x^7$

(a) $f(x) = -x^7 + 4$ (b) $f(x) = (x + 2)^7 - 1$

(c) $f(x) = -\frac{1}{3}x^7 + 1$ (d) $f(x) = -(x + 8)^7$

In Exercises 23–26, determine the right-hand and left-hand behavior of the graph of the polynomial function.

23. $f(x) = -x^2 + 6x + 9$

24. $f(x) = \frac{1}{2}x^3 + 2x$

25. $g(x) = \frac{3}{4}(x^4 + 3x^2 + 2)$

26. $h(x) = -x^5 - 7x^2 + 10x$

Graphical Analysis **In Exercises 27 and 28, use a graphing utility to graph the functions f and g in the same viewing window. Zoom out far enough so that the right-hand and left-hand behavior of f and g appear identical.**

27. $f(x) = \frac{1}{2}x^3 - 2x + 1,\quad g(x) = \frac{1}{2}x^3$

28. $f(x) = -x^4 + 2x^3,\quad g(x) = -x^4$

In Exercises 29–34, (a) find the zeros of the function and (b) sketch its graph.

29. $g(x) = x^4 - x^3 - 2x^2$

30. $h(x) = -2x^3 - x^2 + x$

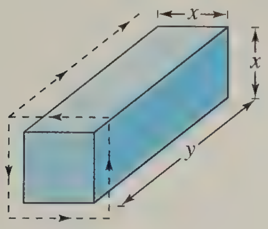
31. $f(t) = t^3 - 3t$ 32. $f(x) = -(x + 6)^3 - 8$

33. $f(x) = x(x + 3)^2$ 34. $f(t) = t^4 - 4t^2$

35. ***Volume*** A rectangular package can have a maximum combined length and girth (perimeter of a cross section) of 216 centimeters.

(a) Write the volume V as a function of x.

(b) Use a graphing utility to graph the volume function, and then use the graph to estimate the dimensions of the package of maximum volume.



36. ***Volume*** Rework Exercise 35 for a cylindrical package. (The cross sections are circular.)

In Exercises 37–40, (a) use the Intermediate Value Theorem and a graphing utility to find intervals of length 1 in which the polynomial function is guaranteed to have a zero, and (b) use the *zero* or *root* feature of a graphing utility to approximate the zeros of the function.

37. $f(x) = x^3 + 2x^2 - x - 1$

38. $f(x) = 0.24x^3 - 2.6x - 1.4$

39. $f(x) = x^4 - 6x^2 - 4$

40. $f(x) = 2x^4 + \frac{7}{2}x^3 - 2$

3.3 ***Graphical Analysis*** **In Exercises 41 and 42, use a graphing utility to graph the two equations in the same viewing window. Use the graphs to verify that the expressions are equivalent. Verify your results algebraically.**

41. $y_1 = \dfrac{x^2}{x - 2},\quad y_2 = x + 2 + \dfrac{4}{x - 2}$

42. $y_1 = \dfrac{x^4 + 1}{x^2 + 2},\quad y_2 = x^2 - 2 + \dfrac{5}{x^2 + 2}$

In Exercises 43–48, divide by long division.

43. $\dfrac{24x^2 - x - 8}{3x - 2}$ 44. $\dfrac{4x + 7}{3x - 2}$

45. $\dfrac{x^4 - 3x^2 + 2}{x^2 - 1}$ 46. $\dfrac{3x^4}{x^2 - 1}$

47. $\dfrac{x^4 + x^3 - x^2 + 2x}{x^2 + 2x}$

48. $\dfrac{6x^4 + 10x^3 + 13x^2 - 5x + 2}{2x^2 - 1}$

In Exercises 49–52, divide by synthetic division.

49. $(0.25x^4 - 4x^3) \div (x + 2)$

50. $(0.1x^3 + 0.3x^2 - 0.5) \div (x - 5)$

51. $(6x^4 - 4x^3 - 27x^2 + 18x) \div \left(x - \frac{2}{3}\right)$

52. $(2x^3 + 2x^2 - x + 2) \div \left(x - \frac{1}{2}\right)$

In Exercises 53 and 54, use synthetic division to decide whether the x-values are zeros of the function.

53. $f(x) = 2x^3 + 3x^2 - 20x - 21$

(a) $x = 4$ (b) $x = -1$

(c) $x = -\frac{7}{2}$ (d) $x = 0$

54. $f(x) = 20x^4 + 9x^3 - 14x^2 - 3x$

(a) $x = -1$ (b) $x = \frac{3}{4}$

(c) $x = 0$ (d) $x = 1$

In Exercises 55–58, find a polynomial with integer coefficients that has the given zeros.

55. $-1, -1, \frac{1}{3}, -\frac{1}{2}$

56. $5, 1 - \sqrt{2}, 1 + \sqrt{2}$

57. $\frac{2}{3}, 4, \sqrt{3}i, -\sqrt{3}i$

58. $2, -3, 1 - 2i, 1 + 2i$

In Exercises 59–64, find all the zeros of the function.

59. $f(x) = 4x^3 - 11x^2 + 10x - 3$

60. $f(x) = 10x^3 + 21x^2 - x - 6$

61. $f(x) = 6x^3 - 5x^2 + 24x - 20$

62. $f(x) = x^3 - 1.3x^2 - 1.7x + 0.6$

63. $f(x) = 6x^4 - 25x^3 + 14x^2 + 27x - 18$

64. $f(x) = 5x^4 + 126x^2 + 25$

In Exercises 65 and 66, use synthetic division to verify the upper bound and lower bound of the zeros of *f*.

65. $f(x) = 4x^3 - 3x^2 + 4x - 3$

Upper bound: $x = 1$; Lower bound: $x = -\frac{1}{4}$

66. $f(x) = 2x^3 - 5x^2 - 14x + 8$

Upper bound: $x = 8$; Lower bound: $x = -4$

3.4 **In Exercises 67–70, use a graphing utility to (a) graph the function, (b) determine the number of real zeros of the function, and (c) approximate the real zeros of the function to the nearest hundredth.**

67. $f(x) = x^4 + 2x + 1$

68. $g(x) = x^3 - 3x^2 + 3x + 2$

69. $h(x) = x^3 - 6x^2 + 12x - 10$

70. $f(x) = x^5 + 2x^3 - 3x - 20$

In Exercises 71–76, find all the zeros of the function and write the polynomial as a product of linear factors. Use a graphing utility to verify your answer.

71. $f(x) = x^3 - 4x^2 + 6x - 4$

72. $f(x) = x^3 - 5x^2 - 7x + 51$

73. $f(x) = x^3 + 6x^2 + 11x + 12$

74. $f(x) = 2x^3 - 9x^2 + 22x - 30$

75. $f(x) = x^4 + 34x^2 + 225$

76. $f(x) = x^4 + 10x^3 + 26x^2 + 10x + 25$

In Exercises 77–82, find a polynomial function with integer coefficients that has the given zeros. (There are many correct answers.)

77. $-2, -2, -5i$

78. $4, 4, 2i$

79. $1, -4, -3 + 5i$

80. $3, 6 + i$

81. $-\frac{2}{3}, -1, 3 + \sqrt{2}i$

82. $-4, -4, 1 + \sqrt{3}i$

In Exercises 83–86, write the polynomial (a) as the product of factors that are irreducible over the rationals, (b) as the product of linear and quadratic factors that are irreducible over the reals, and (c) in completely factored form.

83. $f(x) = x^4 + 2x^2 - 8$

84. $f(x) = x^4 - x^3 - x^2 + 5x - 20$

(*Hint:* One factor is $x^2 - 5$.)

85. $f(x) = x^4 - 2x^3 + 8x^2 - 18x - 9$

(*Hint:* One factor is $x^2 + 9$.)

86. $f(x) = x^4 - 4x^3 + 3x^2 + 8x - 16$

(*Hint:* One factor is $x^2 - x - 4$.)

3.5 **In Exercises 87–90, find the domain of the function and identify any horizontal and vertical asymptotes.**

87. $f(x) = \dfrac{x - 8}{1 - x}$

88. $f(x) = \dfrac{5x}{x + 12}$

89. $f(x) = \dfrac{2}{x^2 - 3x - 18}$

90. $f(x) = \dfrac{2x^2 + 3}{x^2 + x + 3}$

In Exercises 91–98, determine the horizontal asymptotes of the function.

91. $f(x) = \dfrac{7 + x}{7 - x}$

92. $f(x) = \dfrac{6x}{x^2 - 1}$

93. $f(x) = \dfrac{4x^2}{2x^2 - 3}$

94. $f(x) = \dfrac{3x^2 - 11x - 4}{x^2 + 2}$

95. $f(x) = \dfrac{2x^3}{x^2 + 2x - 8}$

96. $f(x) = \dfrac{3x^2}{2x + 3}$

97. $f(x) = \dfrac{x - 2}{|x| + 2}$

98. $f(x) = \dfrac{2x}{|2x - 1|}$

99. ***Average Cost*** A business has a cost of $C = 0.5x + 500$ for producing x units. The average cost per unit is

$$\overline{C} = \frac{C}{x} = \frac{0.5x + 500}{x}, \quad x > 0.$$

(a) Use a graphing utility to graph the function.

(b) Find the average cost of producing $x = 50$, 100, 1000, and 10,000 units.

(c) Determine the average cost per unit as x increases without bound. (Find the horizontal asymptote.)

100. ***Seizure of Illegal Drugs*** The cost in millions of dollars for the U.S. government to seize $p\%$ of a certain illegal drug as it enters the country is

$$C = \frac{528p}{100 - p}, \quad 0 \le p < 100.$$

(a) Find the cost of seizing 25%.

(b) Find the cost of seizing 50%.

(c) Find the cost of seizing 75%.

(d) Use a graphing utility to graph the function.

(e) According to this model, would it be possible to seize 100% of the drug?

3.6 In Exercises 101–110, sketch the graph of the rational function. As a sketching aid, check for intercepts, symmetry, vertical asymptotes, and horizontal asymptotes. Use a graphing utility to verify your graph.

101. $f(x) = \dfrac{2x - 1}{x - 5}$

102. $f(x) = \dfrac{x - 3}{x - 2}$

103. $f(x) = \dfrac{2x}{x^2 + 4}$

104. $f(x) = \dfrac{2x^2}{x^2 - 4}$

105. $f(x) = \dfrac{x^2}{x^2 + 1}$

106. $f(x) = \dfrac{5x}{x^2 + 1}$

107. $f(x) = \dfrac{2}{(x + 1)^2}$

108. $f(x) = \dfrac{4}{(x - 1)^2}$

109. $f(x) = \dfrac{2x}{x^2 + x - 12}$

110. $f(x) = \dfrac{3x - 1}{x^2 + 5x + 4}$

In Exercises 111–116, sketch the graph of the rational function. As a sketching aid, check for intercepts, symmetry, vertical asymptotes, horizontal asymptotes, and slant asymptotes. Use a graphing utility to verify your graph.

111. $f(x) = \dfrac{2x^3}{x^2 + 1}$

112. $f(x) = \dfrac{x^3}{3x^2 - 6}$

113. $f(x) = \dfrac{1}{x + 3} + 2$

114. $f(x) = \dfrac{5x}{x^2 - 4}$

115. $f(x) = \dfrac{x^2 - x + 1}{x - 3}$

116. $f(x) = \dfrac{2x^2 + 7x + 3}{x + 1}$

117. ***Population of Fish*** The Parks and Wildlife Commission introduces 80,000 fish into a large human-made lake. The population of the fish in thousands is

$$N = \frac{20(4 + 3t)}{1 + 0.05t}, \quad t \ge 0$$

where t is time in years.

(a) Sketch the graph of the function by hand. Use a graphing utility to verify your graph.

(b) Find the populations when t is 5, 10, and 25.

(c) What is the maximum number of fish in the lake as time increases? Explain your reasoning.

118. ***Numerical and Graphical Analysis*** A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(2, 3)$.

(a) Draw a diagram that illustrates the problem. Label the known and unknown quantities.

(b) Verify that the area of the triangle is

$$A = \frac{3x^2}{2(x - 2)}, \quad x > 2.$$

(c) Use a graphing utility to generate a table giving the area for values of x. Start the table with $x = 2.5$ and x-increments of 0.5. Continue until you can approximate the dimensions of the triangle of minimum area.

(d) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions of the triangle of minimum area.

(e) Determine the slant asymptote of the area function. Explain its meaning.

Synthesis

True or False? **In Exercises 119 and 120, determine whether the statement is true or false. Justify your answer.**

119. The graph of $f(x) = \dfrac{2x^3}{x + 1}$ has a slant asymptote.

120. The graphs of $f(x) = (3x^2 - 10)/x^2$ and $f(x) = (3x^2 - 10)/|x^2|$ are the same, with both having a horizontal asymptote at $x = 3$.

Chapter Project *Finding Points of Intersection*

You can use the *zoom* feature of a graphing utility to approximate the points of intersection of the graphs of equations.

In this project you will find the points of intersection of the circle and parabola given by

$$x^2 + y^2 - 3x + 5y - 11 = 0 \quad \text{and} \quad y = x^2 - 4x + 5.$$

a. Begin by writing the circle as the union of two functions. Identify the functions that represent the top half and the bottom half of the circle.

b. Use a graphing utility to graph all three functions in the same viewing window, as shown in Figure 3.50. Use the *intersect* feature of the graphing utility to estimate the points of intersection.

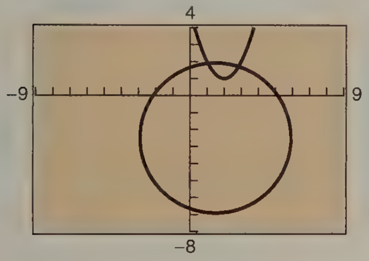


Figure 3.50

Questions for Further Exploration

1. Using a window setting of $1.05 \le x \le 1.06$ and $1.89 \le y \le 1.90$, graph the top half of the circle and the parabola in the example on the same screen. Then use the *trace* or *value* feature to approximate (accurate to three decimal places) the y-coordinate of the point of intersection that is shown on the screen.
2. Another method for finding the points of intersection is to substitute $x^2 - 4x + 5$ for y in the equation of the circle to get a fourth-degree polynomial equation. Graph this polynomial function.
   - (a) Find a setting that allows you to approximate the solution $x \approx 1.055$ of the polynomial equation to two more decimal places.
   - (b) Find a setting that allows you to approximate the solution $x \approx 2.841$ to two more decimal places.
3. Use a graphing utility to find the points of intersection of the circle and the parabola given by

   $$x^2 + y^2 - 5x + 4y - 13 = 0$$

   $$y = x^2 - 3x + 2.$$
4. ***Market Equilibrium*** The *market equilibrium* of a commodity is the quantity (and corresponding price) at which the supply of the commodity and the demand for the commodity are equal. The supply and demand curves for a business dealing with wheat are

   Supply: $p = 1.45 + 0.00014x^2$

   Demand: $p = (2.388 - 0.007x)^2$

   where p is the price (in dollars) per bushel and x is the quantity (in bushels per day). Use a graphing utility to graph the supply and demand equations and find the market equilibrium. (*Hint:* The *market equilibrium* is the point of intersection of the graphs for $x > 0$.)

3 Chapter Test

Take this test as you would take a test in class. After you are done, check your work against the answers in the back of the book.

The *Interactive* CD-ROM and *Internet* verions of this text provide answers to the Chapter Tests and Cumulative Tests. They also offer Chapter Pre-Tests (that test key skills and concepts covered in previous chapters) and Chapter Post-Tests, both of which have randomly generated exercises with diagnostic capabilities.

1. Describe how the graph of g differs from the graph of $f(x) = x^2$.

(a) $g(x) = 2 - x^2$ (b) $g(x) = \left(x - \frac{3}{2}\right)^2$

2. Identify the vertex and intercepts of the graph of $y = x^2 + 4x + 3$.

3. Find an equation of the parabola shown at the right.

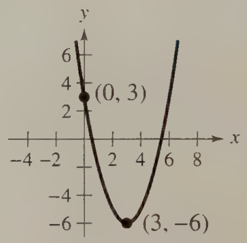


FIGURE FOR 3

4. The path of a ball is given by $y = -\frac{1}{20}x^2 + 3x + 5$, where y is the height in feet and x is the horizontal distance in feet.

(a) Find the maximum height of the ball.

(b) Which term determines the height at which the ball was thrown? Does changing this term change the coordinates of the maximum height of the ball? Explain.

5. Divide using long division: $(3x^3 + 4x - 1) \div (x^2 + 1)$.

6. Divide using synthetic division: $(2x^4 - 5x^2 - 3) \div (x - 2)$.

In Exercises 7 and 8, list all the possible rational zeros of the function. Use a graphing utility to graph the function and find all the rational zeros.

7. $g(t) = 2t^4 - 3t^3 + 16t - 24$ **8.** $h(x) = 3x^5 + 2x^4 - 3x - 2$

In Exercises 9 and 10, use the *zero* or *root* feature of a graphing utility to approximate (accurate to three decimal places) the real zeros of the function.

9. $f(x) = x^4 - x^3 - 1$ **10.** $f(x) = 3x^5 + 2x^4 - 12x - 8$

In Exercises 11–13, find a polynomial function with integer coefficients that has the given zeros.

11. $0, 3, 3 + i, 3 - i$ **12.** $1 - \sqrt{3}i, 2, 2$ **13.** $0, -5, 1 + i$

In Exercises 14–16, sketch the graph of the rational function. As a sketching aid, check for intercepts, symmetry, vertical asymptotes, horizontal asymptotes, and slant asymptotes. Use a graphing utility to verify your graph.

14. $h(x) = \dfrac{4}{x^2} - 1$ **15.** $g(x) = \dfrac{x^2 + 2}{x - 1}$ **16.** $f(x) = \dfrac{2x^2 + 9}{5x^2 + 2}$

17. Find a rational function with vertical asymptotes at $x = \pm 3$ and a horizontal asymptote at $y = 4$.

18. The average age of the groom in a wedding for a given age of the bride can be approximated by the model

$$y = -0.00428x^2 + 1.442x - 3.136, \quad 20 \le x \le 55$$

where y is the age of the groom and x is the age of the bride. For what age of the bride is the average age of the groom 30? (Source: U.S. National Center for Health Statistics)

Exponential and Logarithmic Functions

4.1 Exponential Functions and Their Graphs
4.2 Logarithmic Functions and Their Graphs
4.3 Properties of Logarithms
4.4 Solving Exponential and Logarithmic Equations
4.5 Exponential and Logarithmic Models

Louise Gubb/The Image Works

Personal savings as a percent of disposable income was 3.9% in 1997 and disposable per capita income was $21,969. So, the per capita personal savings was $856.79. (Source: U.S. Bureau of Economic Analysis)

The Big Picture

In this chapter you will learn how to

- ☐ recognize, evaluate, and graph exponential and logarithmic functions.
- ☐ rewrite logarithmic functions with different bases.
- ☐ use properties of logarithms to evaluate, rewrite, expand, or condense logarithmic expressions.
- ☐ solve exponential and logarithmic equations.
- ☐ use exponential growth models, exponential decay models, Gaussian models, logistic models, and logarithmic models to solve real-life problems.
- ☐ fit exponential and logarithmic models to sets of data.

Important Vocabulary

As you encounter each new vocabulary term in this chapter, add the term and its definition to your notebook glossary.

- algebraic functions (p. 298)
- transcendental functions (p. 298)
- exponential function with base *a* (p. 298)
- natural base (p. 302)
- natural exponential function (p. 302)
- logarithmic function with base *a* (p. 311)
- common logarithmic function (p. 312)
- natural logarithmic function (p. 315)
- change-of-base formula (p. 322)
- exponential growth model (p. 340)
- exponential decay model (p. 340)
- Gaussian model (p. 340)
- logistic growth model (p. 340)
- logarithmic models (p. 340)
- normally distributed (p. 344)
- bell-shaped curve (p. 344)
- logistic curve (p. 345)
- sigmoidal curve (p. 345)

Additional Resources Text-specific additional resources are available to help you do well in this course. See page xvi for details.

4.1 Exponential Functions and Their Graphs

Exponential Functions

So far, this text has dealt mainly with **algebraic functions,** which include polynomial functions and rational functions. In this chapter you will study two types of nonalgebraic functions—*exponential* functions and *logarithmic* functions. These functions are examples of **transcendental functions.**

Definition of Exponential Function

The **exponential function f with base a** is denoted by

$$f(x) = a^x$$

where $a > 0$, $a \neq 1$, and x is any real number.

Note that in the definition of an exponential function, the base $a = 1$ is excluded because it yields $f(x) = 1^x = 1$. This is a constant function, not an exponential function.

You already know how to evaluate a^x for integer and rational values of x. For example, you know that $4^3 = 64$ and $4^{1/2} = 2$. However, to evaluate 4^x for any real number x, you need to interpret forms with *irrational* exponents. For the purposes of this text, it is sufficient to think of

$$a^{\sqrt{2}} \text{ (where } \sqrt{2} \approx 1.41421356)$$

as the number that has the successively closer approximations

$$a^{1.4}, a^{1.41}, a^{1.414}, a^{1.4142}, a^{1.41421}, \ldots .$$

Example 1 shows how to use a calculator to evaluate an exponential expression.

EXAMPLE 1 Evaluating Exponential Expressions

Use a calculator to evaluate each expression.

a. $2^{-3.1}$ **b.** $2^{-\pi}$ **c.** $12^{5/7}$ **d.** $(0.6)^{3/2}$

Solution

Number	*Graphing Calculator Keystrokes*	*Display*
a. $2^{-3.1}$	2 $\boxed{\wedge}$ $\boxed{(-)}$ 3.1 $\boxed{\text{ENTER}}$	0.1166291
b. $2^{-\pi}$	2 $\boxed{\wedge}$ $\boxed{(-)}$ π $\boxed{\text{ENTER}}$	0.1133147
c. $12^{5/7}$	12 $\boxed{\wedge}$ $\boxed{(}$ 5 $\boxed{\div}$ 7 $\boxed{)}$ $\boxed{\text{ENTER}}$	5.8998877
d. $(0.6)^{3/2}$	.6 $\boxed{\wedge}$ $\boxed{(}$ 3 $\boxed{\div}$ 2 $\boxed{)}$ $\boxed{\text{ENTER}}$	0.4647580

What You Should Learn:

- How to recognize and evaluate exponential functions with base *a*
- How to graph exponential functions
- How to recognize, evaluate, and graph exponential functions with base *e*
- How to use exponential functions to model and solve real-life problems

Why You Should Learn It:

Exponential functions are useful in modeling data that increase or decrease quickly. For instance, Exercise 77 on page 309 shows how to use an exponential function to model the amount of defoliation caused by a gypsy moth.

Jenny Hager/The Image Works

Graphs of Exponential Functions

The graphs of all exponential functions have similar characteristics, as shown in Examples 2, 3, and 4.

Library of Functions

The exponential function $f(x) = a^x$ is different from all the functions you have studied so far because the variable x is an *exponent*. The domain, like those of polynomial functions, is the set of all real numbers.

Consult the Library of Functions Summary inside the front cover for a description of the exponential function.

EXAMPLE 2 Graphs of $y = a^x$

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 2^x$ **b.** $g(x) = 4^x$

Solution

The table below lists some values for each function, and Figure 4.1 shows the graphs of both functions. Note that both graphs are increasing. Moreover, the graph of $g(x) = 4^x$ is increasing more rapidly than the graph of $f(x) = 2^x$.

x	-2	-1	0	1	2	3
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
4^x	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64

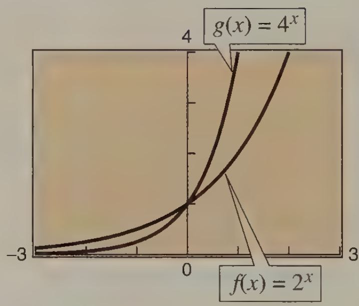


Figure 4.1

The table in Example 2 was evaluated by hand. You could, of course, use a graphing utility to construct tables with even more values.

EXAMPLE 3 Graphs of $y = a^{-x}$

In the same coordinate plane, sketch the graph of each function.

a. $F(x) = 2^{-x}$ **b.** $G(x) = 4^{-x}$

Solution

The table below lists some values for each function, and Figure 4.2 shows the graphs of both functions. Note that both graphs are decreasing. Moreover, the graph of $G(x) = 4^{-x}$ is decreasing more rapidly than the graph of $F(x) = 2^{-x}$.

x	-3	-2	-1	0	1	2
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
4^{-x}	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$

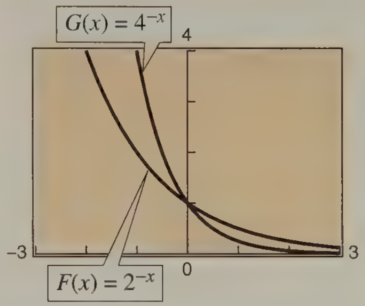


Figure 4.2

In Example 3, note that the functions $F(x) = 2^{-x}$ and $G(x) = 4^{-x}$ can be rewritten with positive exponents.

$$F(x) = 2^{-x} = \left(\frac{1}{2}\right)^x \quad \text{and} \quad G(x) = 4^{-x} = \left(\frac{1}{4}\right)^x$$

In general,

$$a^{-x} = \left(\frac{1}{a}\right)^x.$$

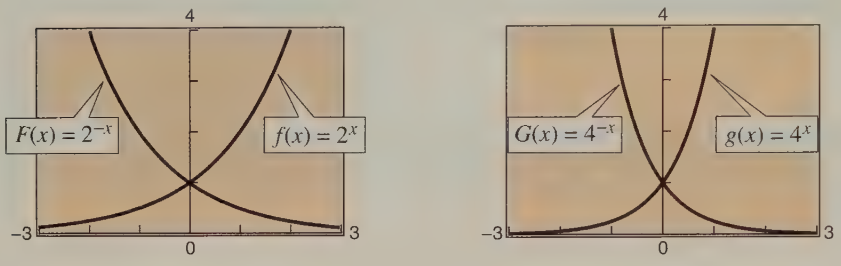
Comparing the functions in Examples 2 and 3, observe that

$$F(x) = 2^{-x} = f(-x) \quad \text{and} \quad G(x) = 4^{-x} = g(-x).$$

Consequently, the graph of F is a reflection (in the y-axis) of the graph of f, as shown in Figure 4.3(a). The graph of G and g have the same relationship, as shown in Figure 4.3(b).

STUDY T!P

Notice that the range of an exponential function is $(0, \infty)$, which means that $a^x > 0$ for all values of x.



(a) **(b)**

Figure 4.3

The graphs in Figures 4.1 and 4.2 are typical of the exponential functions a^x and a^{-x}. They have one y-intercept and one horizontal asymptote (the x-axis), and they are continuous. The basic characteristics of these exponential functions are summarized in Figure 4.4.

Graph of $y = a^x, a > 1$

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Intercept: $(0, 1)$
- Increasing
- x-axis is a horizontal asymptote ($a^x \rightarrow 0$ as $x \rightarrow -\infty$)
- Continuous

Graph of $y = a^{-x}, a > 1$

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Intercept: $(0, 1)$
- Decreasing
- x-axis is a horizontal asymptote ($a^{-x} \rightarrow 0$ as $x \rightarrow \infty$)
- Continuous

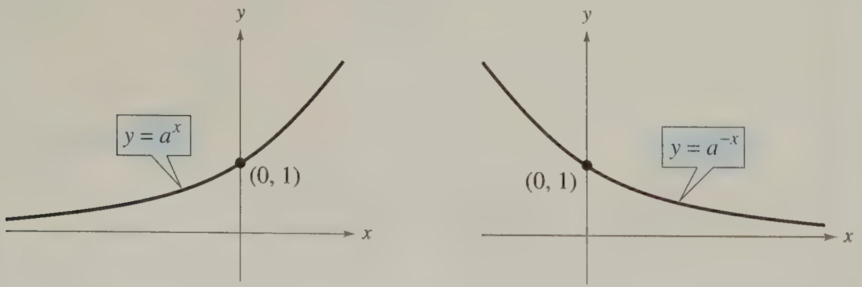


Figure 4.4

Exploration

Use a graphing utility to graph $y = a^x$ with $a = 3, 5$, and 7 in the same viewing window. (Use a viewing window in which $-2 \leq x \leq 1$ and $0 \leq y \leq 2$.) How do the graphs compare with each other? Which graph is on the top in the interval $(-\infty, 0)$? Which is on the bottom? Which graph is on the top in the interval $(0, \infty)$? Which is on the bottom? Repeat this experiment with the graphs of $y = a^x$ for $a = \frac{1}{3}, \frac{1}{5}$, and $\frac{1}{7}$. (Use a viewing window in which $-1 \leq x \leq 2$ and $0 \leq y \leq 2$.) What can you conclude about the shape of the graph of $y = a^x$ and the value of a?

In the following example, notice how the graph of $y = a^x$ can be used to sketch the graphs of functions of the form

$$f(x) = b \pm a^{x+c}.$$

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 4 Transformations of Graphs of Exponential Functions

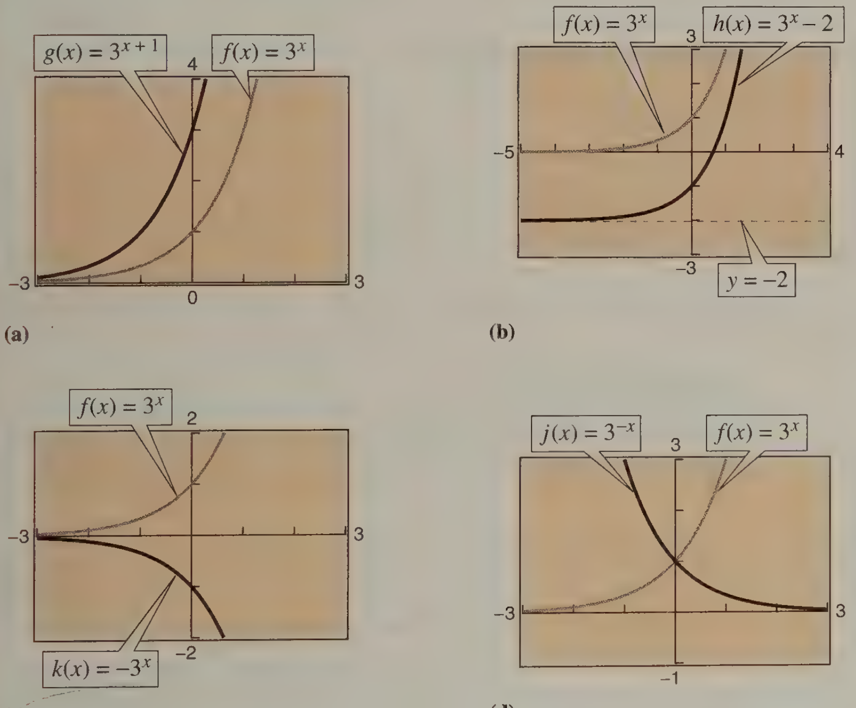
Each of the following graphs is a transformation of the graph of $f(x) = 3^x$, as shown in Figure 4.5.

a. Because $g(x) = 3^{x+1} = f(x + 1)$, the graph of g can be obtained by shifting the graph of f one unit to the left.

b. Because $h(x) = 3^x - 2 = f(x) - 2$, the graph of h can be obtained by shifting the graph of f down two units.

c. Because $k(x) = -3^x = -f(x)$, the graph of k can be obtained by reflecting the graph of f in the x-axis.

d. Because $j(x) = 3^{-x} = f(-x)$, the graph of j can be obtained by reflecting the graph of f in the y-axis.



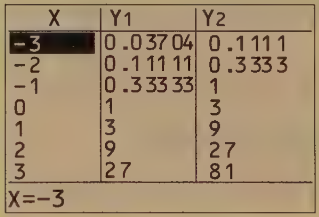
(a) (b) (c) (d)

Figure 4.5

In Figure 4.5, notice that the transformations in parts (a), (c), and (d) keep the x-axis as a horizontal asymptote, but the transformation in part (b) yields a new horizontal asymptote of $y = -2$. Also, be sure to note how the y-intercept is affected by each transformation.

STUDY T!P

The following table shows some points of the graphs in Figure 4.5(a). The functions $f(x)$ and $g(x)$ are represented by Y1 and Y2, respectively. Explain how you can use the table to describe the transformation.



X	Y1	Y2
-3	0.03704	0.1111
-2	0.11111	0.3333
-1	0.33333	1
0	1	3
1	3	9
2	9	27
3	27	81

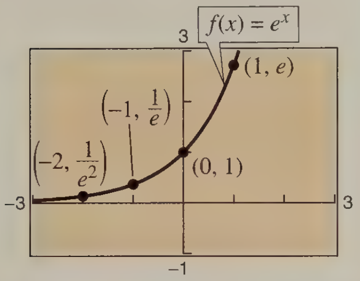
X=-3

The Natural Base e

For many applications, the convenient choice for a base is the irrational number

$$e \approx 2.71828 \ldots .$$

This number is called the **natural base.** The function $f(x) = e^x$ is the **natural exponential function.** Its graph is shown in Figure 4.6. Be sure you see that for the exponential function $f(x) = e^x$, e is the constant $2.71828 \ldots$, whereas x is the variable.



Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Intercept: $(0, 1)$

Figure 4.6 *The Natural Exponential Function*

Exploration

Use your graphing utility to graph the functions

$$y_1 = 2^x$$
$$y_2 = e^x$$
$$y_3 = 3^x$$

in the same viewing window. From the relative positions of these graphs, make a guess as to the value of the real number e. Then try to find a number a such that the graphs of $y_2 = e^x$ and $y_4 = a^x$ are as close as possible.

In Example 5, you will see that the number e can be approximated by the expression

$$\left(1 + \frac{1}{x}\right)^x \text{ for large values of } x.$$

EXAMPLE 5 Approximation of the Number e

Evaluate the expression $[1 + (1/x)]^x$ for several large values of x to see that the values approach $e \approx 2.71828$ as x increases without bound.

Graphical Solution

Use a graphing utility to graph

$$y_1 = [1 + (1/x)]^x \quad \text{and} \quad y_2 = e$$

in the same viewing window, as shown in Figure 4.7. Use the *trace* feature of the graphing utility to verify that as x increases, the graph of y_1 gets closer and closer to the line $y_2 = e$.

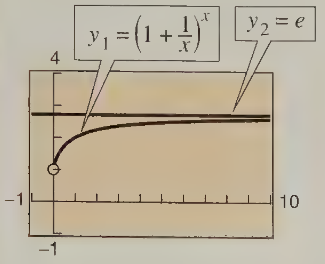


Figure 4.7

Numerical Solution

Create a table of values for the function $y = [1 + (1/x)]^x$, beginning at $x = 10$ and increasing the x-values as shown in the table below.

x	10	100	1000
$\left(1 + \frac{1}{x}\right)^x$	2.59374	2.70481	2.71692

x	10,000	100,000	1,000,000
$\left(1 + \frac{1}{x}\right)^x$	2.71815	2.71827	2.71828

From the table, it seems reasonable to conclude that

$$\left(1 + \frac{1}{x}\right)^x \to e \text{ as } x \to \infty.$$

EXAMPLE 6 Evaluating the Natural Exponential Function

Use a calculator to evaluate each expression.

a. e^{-2} **b.** e^{-1} **c.** e^{1} **d.** e^{2}

Solution

Number	*Graphing Calculator Keystrokes*	*Display*
a. e^{-2}	e^x (−) 2 ENTER	0.1353353
b. e^{-1}	e^x (−) 1 ENTER	0.3678794
c. e^{1}	e^x 1 ENTER	2.7182818
d. e^{2}	e^x 2 ENTER	7.3890561

EXAMPLE 7 Graphing Natural Exponential Functions

Sketch the graph of each natural exponential function.

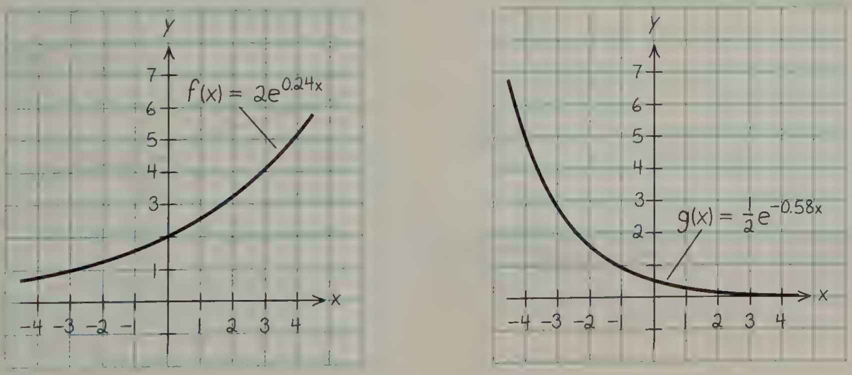
a. $f(x) = 2e^{0.24x}$ **b.** $g(x) = \frac{1}{2}e^{-0.58x}$

Solution

To sketch these two graphs, you can use a calculator to construct a table of values, as shown below.

x	-3	-2	-1	0	1	2	3
$f(x)$	0.974	1.238	1.573	2.000	2.542	3.232	4.109
$g(x)$	2.849	1.595	0.893	0.500	0.280	0.157	0.088

After constructing the table, plot the points and connect them with smooth curves, as shown in Figure 4.8. Note that the graph in part (a) is increasing, whereas the graph in part (b) is decreasing. Use a graphing calculator to verify these graphs.



(a) (b)

Figure 4.8

Exploration

Use a graphing utility to graph $y = (1 + x)^{1/x}$. Describe the behavior of the graph near $x = 0$. Is there a y-intercept? How does the behavior of the graph near $x = 0$ relate to the result of Example 5? Create a table that shows values of y for values of x near $x = 0$ to help you describe the behavior of the graph near this point.

The *Interactive* CD-ROM and *Internet* versions of this text show every example with its solution; clicking on the *Try It!* button brings up similar problems. Guided Examples and Integrated Examples show step-by-step solutions to additional examples. Integrated Examples are related to several concepts in the section.

Compound Interest

One of the most familiar examples of exponential growth is that of an investment earning *continuously compounded interest.* Suppose a principal P is invested at an annual interest rate r, compounded once a year. If the interest is added to the principal at the end of the year, the balance is $P_1 = P + Pr = P(1 + r)$. This pattern of multiplying the previous principal by $1 + r$ is then repeated each successive year, as shown in the table.

Time in Years	Balance After Each Compounding
0	$P = P$
1	$P_1 = P(1 + r)$
2	$P_2 = P_1(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$
$\vdots$	$\vdots$
n	$P_n = P(1 + r)^n$

To accommodate more frequent (quarterly, monthly, or daily) compounding of interest, let n be the number of compoundings per year and let t be the number of years. (The product nt represents the total number of times the interest will be compounded.) Then the interest rate per compounding period is r/n, and the account balance after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}. \qquad \text{Amount with } n \text{ compoundings per year}$$

If you let the number of compoundings n increase without bound, you approach **continuous compounding.** In the formula for n compoundings per year, let $m = n/r$. This produces

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left(1 + \frac{1}{m}\right)^{mrt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}.$$

As m increases without bound, you know from Example 5 that $[1 + (1/m)]^m$ approaches e. So, for continuous compounding, it follows that

$$P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt} \rightarrow P[e]^{rt}$$

and you can write $A = Pe^{rt}$. This result is part of the reason that e is the "natural" choice for a base of an exponential function.

Formulas for Compound Interest

After t years, the balance A in an account with principal P and annual interest rate r (expressed as a decimal) is given by the following formulas.

1. For n compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$
2. For continuous compounding: $A = Pe^{rt}$

Exploration

Use the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

to calculate the amount in an account when $P = \$3000$, $r = 6\%$, $t = 10$ years, and the number of compoundings is (1) by the day, (2) by the hour, (3) by the minute, and (4) by the second. Use these results to present an argument that increasing the number of compoundings does not mean unlimited growth of the amount in the account.

STUDY TIP

The interest rate r in the formula for compound interest should be written as a decimal. For example, an interest rate of 7% would be written $r = 0.07$.

A computer simulation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 8 Finding the Balance for Compound Interest

A sum of \$9000 is invested at an annual interest rate of 8.5%, compounded annually. Find the balance in the account after 3 years.

Algebraic Solution

In this case,

$$P = 9000,\ r = 8.5\% = 0.085,\ n = 1,\ t = 3.$$

Using the formula for compound interest with n compoundings per year, you have

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} && \text{Formula for compound interest} \\ A &= 9000\left(1 + \frac{0.085}{1}\right)^{1(3)} && \text{Substitute values for } P, r, n, \text{ and } t. \\ &= 9000(1.085)^3 && \text{Simplify.} \\ &\approx \$11{,}495.60. && \text{Use a calculator.} \end{aligned}$$

So, the balance in the account after 3 years will be about \$11,495.60.

Graphical Solution

Substitute values for P, r, and n into the formula for compound interest with n compoundings per year as follows.

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} && \text{Formula for compound interest} \\ &= 9000\left(1 + \frac{0.085}{1}\right)^{(1)t} && \text{Substitute values for } P, r, \text{ and } n. \\ &= 9000(1 + 0.085)^t && \text{Simplify.} \end{aligned}$$

Use a graphing utility to graph $y = 9000(1 + 0.085)^x$, as shown in Figure 4.9. Using the *value* feature or *zoom* and *trace* features, you can approximate the value of y when $x = 3$ to be about 11,495.60. So, the balance in the account after 3 years will be about \$11,495.60.

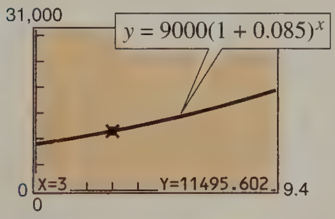


Figure 4.9

EXAMPLE 9 Finding Compound Interest

A total of \$12,000 is invested at an annual interest rate of 9%. Find the balance after 5 years if it is compounded

a. quarterly. **b.** monthly. **c.** continuously.

Solution

a. For quarterly compoundings, $n = 4$. So, in 5 years at 9%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = 12{,}000\left(1 + \frac{0.09}{4}\right)^{4(5)} = \$18{,}726.11.$$

b. For monthly compoundings, $n = 12$. So, in 5 years at 9%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = 12{,}000\left(1 + \frac{0.09}{12}\right)^{12(5)} = \$18{,}788.17.$$

c. For continuous compounding, the balance is

$$A = Pe^{rt} = 12{,}000e^{0.09(5)} = \$18{,}819.75.$$

Note that continuous compounding yields more than quarterly or monthly compounding.

Other Applications

Exponential functions are used in various other applications.

EXAMPLE 10 Radioactive Decay

Let y represent the mass of a quantity of a radioactive element whose half-life is 25 years. After t years, the mass (in grams) is $y = 10\left(\frac{1}{2}\right)^{t/25}$.

a. What is the initial mass (when $t = 0$)?

b. How much of the initial mass is present after 80 years?

Algebraic Solution

a. $y = 10\left(\dfrac{1}{2}\right)^{t/25}$ Write original equation.

$= 10\left(\dfrac{1}{2}\right)^{0/25}$ Substitute 0 for t.

$= 10$ Simplify.

So, the initial mass is 10 grams.

b. $y = 10\left(\dfrac{1}{2}\right)^{t/25}$ Write original equation.

$= 10\left(\dfrac{1}{2}\right)^{80/25}$ Substitute 80 for t.

$= 10\left(\dfrac{1}{2}\right)^{3.2}$ Simplify.

≈ 1.088 Use a calculator.

So, about 1.088 grams is present after 80 years.

Graphical Solution

Use a graphing utility to graph $y = 10\left(\frac{1}{2}\right)^{x/25}$.

a. Use the *value* feature or *zoom* and *trace* features of the graphing utility to determine that the value of y when $x = 0$ is 10, as shown in Figure 4.10. So, the initial mass is 10 grams.

b. Use the *value* feature or *zoom* and *trace* features of the graphing utility to determine that the value of y when $x = 80$ is about 1.088, as shown in Figure 4.11. So, about 1.088 grams is present after 80 years.

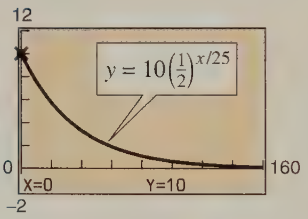


Figure 4.10

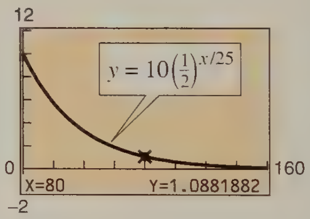


Figure 4.11

EXAMPLE 11 Population Growth

The approximate number of fruit flies in an experimental population after t hours is

$$Q(t) = 20e^{0.03t}, \qquad t \geq 0.$$

a. Find the initial number of fruit flies in the population.

b. How large is the population of fruit flies after 72 hours?

c. Sketch the graph of Q.

Solution

a. To find the initial population, evaluate $Q(t)$ at $t = 0$.

$$Q(0) = 20e^{0.03(0)} = 20e^0 = 20(1) = 20 \text{ flies}$$

b. After 72 hours, the population size is

$$Q(72) = 20e^{0.03(72)} = 20e^{2.16} \approx 173 \text{ flies.}$$

c. The graph of Q is shown in Figure 4.12.

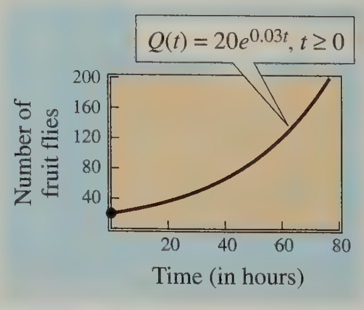


Figure 4.12

4.1 Exercises

In Exercises 1–10, use a calculator to evaluate the expression. Round your result to three decimal places.

1. $(3.4)^{6.8}$
2. $5000(2^{-1.5})$
3. $6^{2\pi}$
4. $5^{-\pi}$
5. $\sqrt[3]{7493}$
6. $100^{\sqrt{2}}$
7. $e^{1/2}$
8. $e^{-3/4}$
9. $e^{9.2}$
10. $e^{3.78}$

Think About It **In Exercises 11–14, use properties of exponents to determine which functions (if any) are the same.**

11. $f(x) = 3^{x-2}$
$g(x) = 3^x - 9$
$h(x) = \frac{1}{9}(3^x)$

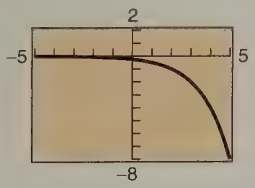
12. $f(x) = 4^x + 12$
$g(x) = 2^{2x+6}$
$h(x) = 64(4^x)$

13. $f(x) = 16(4^{-x})$
$g(x) = \left(\frac{1}{4}\right)^{x-2}$
$h(x) = 16(2^{-2x})$

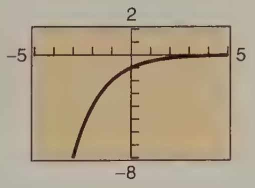
14. $f(x) = 5^{-x} + 3$
$g(x) = 5^{3-x}$
$h(x) = -5^{x-3}$

In Exercises 15–22, match the exponential function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

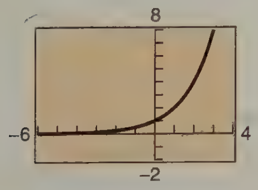
(a)



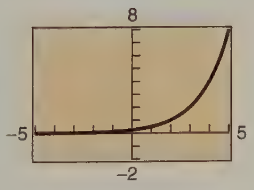
(b)



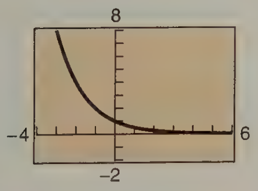
(c)



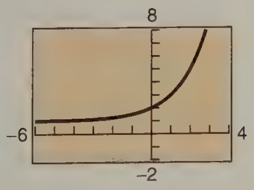
(d)



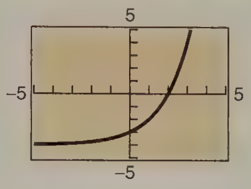
(e)



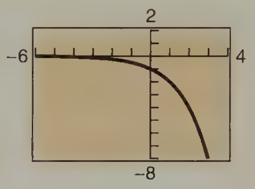
(f)



(g)



(h)



15. $f(x) = 2^x$
16. $f(x) = -2^x$
17. $f(x) = 2^{-x}$
18. $f(x) = -2^{-x}$
19. $f(x) = 2^x - 4$
20. $f(x) = 2^x + 1$
21. $f(x) = -2^{x-2}$
22. $f(x) = 2^{x-2}$

In Exercises 23–26, use the graph of f to describe the transformation that yields the graph of g.

23. $f(x) = 3^x,\ g(x) = 3^{x-5}$
24. $f(x) = -2^x,\ g(x) = 5 - 2^x$
25. $f(x) = \left(\frac{3}{5}\right)^x,\ g(x) = -\left(\frac{3}{5}\right)^{x+4}$
26. $f(x) = 0.3^x,\ g(x) = -0.3^x + 5$

In Exercises 27–34, graph the exponential function *by hand*. Identify the following features of the graph.

(a) **Asymptotes**
(b) **Intercepts**
(c) **Increasing or decreasing**

27. $g(x) = 5^x$
28. $f(x) = \left(\frac{3}{2}\right)^x$
29. $f(x) = \left(\frac{1}{5}\right)^x = 5^{-x}$
30. $h(x) = \left(\frac{3}{2}\right)^{-x}$
31. $h(x) = 5^{x-2}$
32. $g(x) = \left(\frac{3}{2}\right)^{x+2}$
33. $g(x) = 5^{-x} - 3$
34. $f(x) = \left(\frac{3}{2}\right)^{-x} + 2$

In Exercises 35–44, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function.

35. $f(x) = \left(\frac{5}{2}\right)^x$
36. $f(x) = \left(\frac{5}{2}\right)^{-x}$
37. $f(x) = 6^x$
38. $f(x) = 2^{x-1}$
39. $f(x) = 3^{x+2}$
40. $f(x) = e^{-x}$
41. $f(x) = 3e^{x+4}$
42. $f(x) = 2e^{-0.5x}$
43. $f(x) = 2 + e^{x-5}$
44. $f(x) = 4^{x-3} + 3$

The *Interactive* CD-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered Section and Review Exercises. They also provide Tutorial Exercises, which link to Guided Examples for additional help.

In Exercises 45–52, use a graphing utility to graph the exponential function. Identify any asymptotes of the graph.

45. $y = 2^{-x^2}$

46. $y = 3^{-|x|}$

47. $y = 3^{x-2} + 1$

48. $y = 4^{x+1} - 2$

49. $y = 1.08^{-5x}$

50. $s(t) = 2e^{0.12t}$

51. $s(t) = 3e^{-0.2t}$

52. $g(x) = 1 + e^{-x}$

53. ***Exploration*** Consider the functions $f(x) = 3^x$ and $g(x) = 4^x$.

(a) Use a graphing utility to complete the table, and use the table to estimate the solution of the inequality $4^x < 3^x$.

x	-1	-0.5	0	0.5	1
$f(x)$					
$g(x)$					

(b) Use a graphing utility to graph $f(x)$ and $g(x)$ in the same viewing window. Use the graphs to solve the inequalities.

(i) $4^x < 3^x$ (ii) $4^x > 3^x$

54. ***Exploration*** Consider the functions $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{4}\right)^x$.

(a) Use a graphing utility to complete the table, and use the table to estimate the solution of the inequality $\left(\frac{1}{4}\right)^x < \left(\frac{1}{2}\right)^x$.

x	-1	-0.5	0	0.5	1
$f(x)$					
$g(x)$					

(b) Use a graphing utility to graph $f(x)$ and $g(x)$ in the same viewing window. Use the graphs to solve the inequalities.

(i) $\left(\frac{1}{4}\right)^x < \left(\frac{1}{2}\right)^x$ (ii) $\left(\frac{1}{4}\right)^x > \left(\frac{1}{2}\right)^x$

In Exercises 55–58, use a graphing utility to (a) graph the function and (b) find any asymptotes numerically by creating a table of values for the function.

55. $f(x) = \dfrac{8}{1 + e^{-0.5x}}$

56. $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

57. $f(x) = -\dfrac{6}{2 - e^{0.2x}}$

58. $f(x) = \dfrac{6}{2 - e^{0.2/x}}$

In Exercises 59–62, (a) use a graphing utility to graph the function, (b) use the graph to find the open intervals on which the function is increasing and decreasing, and (c) approximate any relative maximum or minimum values.

59. $f(x) = x^2e^{-x}$

60. $f(x) = 2x^2e^{x+1}$

61. $f(x) = x(2^{3-x})$

62. $f(x) = -\left(\frac{1}{2}x\right)3^{x+4}$

Compound Interest **In Exercises 63–66, complete the table to determine the balance *A* for *P* dollars invested at rate *r* for *t* years and compounded *n* times per year.**

n	1	2	4	12	365	Continuous
A						

63. $P = \$2500, r = 8\%, t = 10$ years

64. $P = \$1000, r = 6\%, t = 10$ years

65. $P = \$2500, r = 8\%, t = 20$ years

66. $P = \$1000, r = 6\%, t = 40$ years

Compound Interest **In Exercises 67–70, complete the table to determine the balance *A* for $12,000 invested at a rate *r* for *t* years.**

t	1	10	20	30	40	50
A						

67. $r = 8\%$, compounded continuously

68. $r = 6\%$, compounded continuously

69. $r = 6.5\%$, compounded monthly

70. $r = 7.5\%$, compounded daily

71. ***Demand Function*** The demand equation for a certain product is

$$p = 5000\left(1 - \frac{4}{4 + e^{-0.002x}}\right)$$

where p is the price and x is the number of units.

(a) Use a graphing utility to graph the demand function for $x > 0$ and $p > 0$.

(b) Find the price p for a demand of $x = 500$ units.

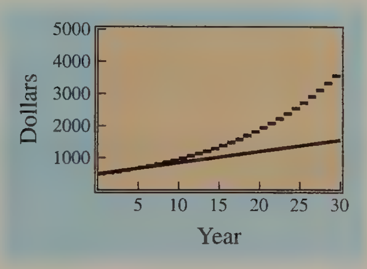
(c) Use the graph in part (a) to approximate the highest price that will still yield a demand of at least 600 units.

(d) Verify your answers to parts (b) and (c) numerically by creating a table of values for the function.

72. ***Graphical Reasoning*** There are two options for investing \$500. The first earns 7% compounded annually, and the second earns 7% simple interest. The figure shows the growth of each investment over a 30-year period.

(a) Identify the two types of investments in the figure. Explain your reasoning.

(b) Verify your answer in part (a) by finding the equations that model the investment growth and using a graphing utility to graph the models.



73. ***Bacteria Growth*** A certain type of bacteria increases according to the model

$P(t) = 100e^{0.2197t}$

where t is the time in hours.

(a) Use a graphing utility to graph the model.

(b) Use a graphing utility to approximate $P(0)$, $P(5)$, and $P(10)$.

(c) Verify your answers in part (b) algebraically.

74. ***Population Growth*** The population of a town increases according to the model

$P(t) = 2500e^{0.0293t}$

where t is the time in years, with $t = 0$ corresponding to 1990.

(a) Find the population in 1992, 1995, and 1998.

(b) Use a graphing utility to graph the function for the years 1990 through 2015.

(c) Use a graphing utility to approximate the population in 2005 and 2010.

(d) Verify your answers in part (c) algebraically.

75. ***Radioactive Decay*** Let Q (in grams) represent the mass of a quantity of radium 226, which has a half-life of 1620 years. The quantity of radium present after t years is

$Q = 25\left(\frac{1}{2}\right)^{t/1620}$.

(a) Determine the initial quantity (when $t = 0$).

(b) Determine the quantity present after 1000 years.

(c) Use a graphing utility to graph the function over the interval $t = 0$ to $t = 5000$.

(d) When will the quantity of radium be 0 grams? Explain.

76. ***Radioactive Decay*** Let Q (in grams) represent the mass of a quantity of carbon 14, which has a half-life of 5730 years. The quantity present after t years is $Q = 10\left(\frac{1}{2}\right)^{t/5730}$.

(a) Determine the initial quantity (when $t = 0$).

(b) Determine the quantity present after 2000 years.

(c) Sketch the graph of the function over the interval $t = 0$ to $t = 10{,}000$.

77. ***Data Analysis*** To estimate the amount of defoliation caused by gypsy moths, a forester counts the number x of egg masses on $\frac{1}{40}$ of an acre (circle of radius 18.6 feet) during the fall. The percent of defoliation y the next spring is shown in the table. (Source: USDA Forest Service)

x	0	25	50	75	100
y	12	44	81	96	99

(a) Use a graphing utility to plot the data points.

(b) A model for the data is $y = \dfrac{300}{3 + 17e^{-0.065x}}$.

Use a graphing utility to graph the model in the same viewing window used in part (a). How well does the model fit the data?

(c) Use a graphing utility to create a table comparing the model with the sample data.

(d) Estimate the percent of defoliation if 36 egg masses are counted on $\frac{1}{40}$ acre.

(e) Use the graph to estimate the number of egg masses per $\frac{1}{40}$ acre if approximately $\frac{2}{3}$ of the forest is defoliated the next spring.

78. ***Data Analysis*** A cup of water at an initial temperature of 78°C is placed in a room at a constant temperature of 21°C. The temperature of the water is measured every 5 minutes for a period of $\frac{1}{2}$ hour. The results are recorded in the table, where t is the time (in minutes) and T is the temperature (in degrees Celsius).

t	0	5	10	15	20	25	30
T	78.0°	66.0°	57.5°	51.2°	46.3°	42.5°	39.6°

(a) Use the regression capabilities of a graphing utility to fit a line to the data. Use the graphing utility to plot the data points and the regression line in the same viewing window. Does the data appear linear? Explain.

(b) Use the regression capabilities of a graphing utility to fit a parabola to the data. Use the graphing utility to plot the data points and the regression parabola in the same viewing window. Does the data appear quadratic? Even though the quadratic model appears to be a "good" fit, explain why it may not be a good model for predicting the temperature of the water when $t = 60$.

(c) The graph of the model should be asymptotic with the temperature of the room. Subtract the room temperature from each of the temperatures in the table. Use a graphing utility to fit an exponential model to the revised data. Add the room temperature to this regression model. Use a graphing utility to plot the original data points and the model in the same viewing window.

(d) Explain why the procedure in part (c) was necessary for finding the exponential model.

79. ***Inflation*** If the annual rate of inflation averages 4% over the next 10 years, the approximate cost C of goods or services during any year in that decade will be $C(t) = P(1.04)^t$, where t is the time (in years) and P is the present cost. Assume the price of an oil change for your car is presently \$23.95.

(a) Use a graphing utility to graph the function.

(b) Use the graph in part (a) to approximate the price of an oil change 10 years from now.

(c) Verify your answer in part (b) algebraically.

80. ***Depreciation*** After t years, the value of a car that costs \$20,000 is $V(t) = 20{,}000\left(\frac{3}{4}\right)^t$.

(a) Use a graphing utility to graph the function.

(b) Use a graphing utility to create a table of values that shows the value V for $t = 1$ to 10 years.

Synthesis

True or False? **In Exercises 81 and 82, determine whether the statement is true or false. Justify your answer.**

81. The x-axis is an asymptote for the graph of $f(x) = 10^x$.

82. $e = \dfrac{271{,}801}{99{,}990}$

83. ***Exploration*** Use a graphing utility to graph $y_1 = e^x$ and each of the functions $y_2 = x^2$, $y_3 = x^3$, $y_4 = \sqrt{x}$, and $y_5 = |x|$.

(a) Which function increases at the fastest rate for "large" values of x?

(b) Use the result of part (a) to make a conjecture about the rates of growth of $y_1 = e^x$ and $y = x^n$ where n is a natural number and x is "large."

(c) Use the results of parts (a) and (b) to describe what is implied when it is stated that a quantity is growing exponentially.

84. ***Conjecture*** Use a graphing utility to graph $f(x) = (1 + 0.5/x)^x$ and $g(x) = e^{0.5}$ in the same viewing window.

(a) What is the relationship between f and g as x increases without bound?

(b) Use the result of part (a) to make a conjecture about the value of $(1 + r/x)^x$ as x increases without bound.

85. ***Think About It*** Without a graphing utility, explain why you know $2^{\sqrt{2}}$ is greater than 2, but less than 4.

86. ***Think About It*** Which functions are exponential? Why?

(a) $3x$ (b) $3x^2$ (c) 3^x (d) 2^{-x}

87. ***Pattern Recognition*** Use a graphing utility to compare the graph of the function $y = e^x$ with the graphs of the following functions.

(a) $y_1 = 1 + \dfrac{x}{1!}$ (b) $y_2 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!}$

(c) $y_3 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$

In your opinion, which function is the best approximation of $y = e^x$?

Review

In Exercises 88–91, determine whether the function has an inverse. If it does, find f^{-1}.

88. $f(x) = 5x - 7$ **89.** $f(x) = -\frac{2}{3}x + \frac{5}{2}$

90. $f(x) = \sqrt[3]{x + 8}$ **91.** $f(x) = \sqrt{x^2 + 6}$

In Exercises 92–95, sketch the graph of the rational function.

92. $f(x) = \dfrac{2x}{x - 7}$ **93.** $f(x) = \dfrac{x^2 + 3}{x + 1}$

94. $f(x) = \dfrac{4x}{x^2 + 11x + 24}$ **95.** $f(x) = \dfrac{2x^3}{(x - 2)^2}$

4.2 Logarithmic Functions and Their Graphs

Logarithmic Functions

In Section 1.7, you studied the concept of the inverse of a function. There, you learned if a function has the property such that no horizontal line intersects its graph more than once, the function must have an inverse. By looking back at the graphs of the exponential functions introduced in Section 4.1, you will see that every function of the form

$$f(x) = a^x, \qquad a > 0, a \neq 1$$

passes the Horizontal Line Test and therefore must have an inverse. This inverse function is called the **logarithmic function with base *a*.**

Definition of Logarithmic Function

For $x > 0$ and $0 < a \neq 1$,

$$y = \log_a x \qquad \text{if and only if} \qquad x = a^y.$$

The function

$$f(x) = \log_a x$$

is called the **logarithmic function with base *a*.**

The equations

$y = \log_a x$ Logarithmic form

and

$x = a^y$ Exponential form

are equivalent. The first equation is in logarithmic form and the second is in exponential form.

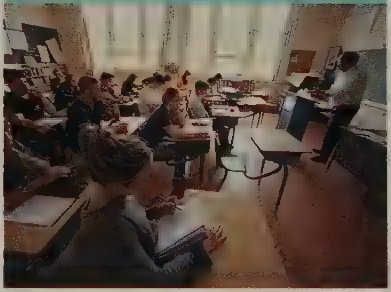
When evaluating logarithms, remember that *a logarithm is an exponent.* This means that $\log_a x$ is the exponent to which a must be raised to obtain x. For instance, $\log_2 8 = 3$ because 2 must be raised to the third power to get 8.

What You Should Learn:

- How to recognize and evaluate logarithmic functions with base *a*
- How to graph logarithmic functions
- How to recognize, evaluate, and graph natural logarithmic functions
- How to use logarithmic functions to model and solve real-life problems

Why You Should Learn It:

Logarithmic functions are useful in modeling data that increase or decrease slowly. For instance, Exercise 81 on page 320 shows how to use a logarithmic function to model the minimum required ventilation rates in public school classrooms.



Mark Richards/PhotoEdit

EXAMPLE 1 Evaluating Logarithms

Evaluate each expression.

a. $\log_2 32$ **b.** $\log_3 27$ **c.** $\log_4 2$

d. $\log_{10} \frac{1}{100}$ **e.** $\log_3 1$ **f.** $\log_2 2$

Solution

a. $\log_2 32 = 5$ because $2^5 = 32.$

b. $\log_3 27 = 3$ because $3^3 = 27.$

c. $\log_4 2 = \frac{1}{2}$ because $4^{1/2} = \sqrt{4} = 2.$

d. $\log_{10} \frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}.$

e. $\log_3 1 = 0$ because $3^0 = 1.$

f. $\log_2 2 = 1$ because $2^1 = 2.$

The logarithmic function with base 10 is called the **common logarithmic function.** On most calculators, this function is denoted by $\boxed{\text{LOG}}$. Because $\log_a x$ is the inverse function of a^x, it follows that the domain of $\log_a x$ is the range of a^x, $(0, \infty)$. In other words, $\log_a x$ is defined only if x is positive.

EXAMPLE 2 Evaluating Logarithms on a Calculator

Use a calculator to evaluate each expression.

a. $\log_{10} 10$ **b.** $2 \log_{10} 2.5$ **c.** $\log_{10}(-2)$

Solution

Number	*Graphing Calculator Keystrokes*	*Display*
a. $\log_{10} 10$	$\boxed{\text{LOG}}$ 10 $\boxed{\text{ENTER}}$	1
b. $2\log_{10} 2.5$	2 $\boxed{\text{LOG}}$ 2.5 $\boxed{\text{ENTER}}$	0.7958800
c. $\log_{10}(-2)$	$\boxed{\text{LOG}}$ $\boxed{(-)}$ 2 $\boxed{\text{ENTER}}$	ERROR

Note that the calculator displays an error message when you try to evaluate $\log_{10}(-2)$. The reason for this is that the domain of every logarithmic function is the set of *positive* real numbers. In this case, there is no *real* power to which 10 can be raised to get -2.

STUDY T!P

Some graphing utilities do not give an error message for $\log_{10}(-2)$. Instead, the graphing utility will display a complex number. For the purpose of this text, however, it will still be said that the domain of a logarithmic function is the set of positive *real* numbers.

The following properties follow directly from the definition of the logarithmic function with base *a*.

Properties of Logarithms

1. $\log_a 1 = 0$ because $a^0 = 1$.
2. $\log_a a = 1$ because $a^1 = a$.
3. $\log_a a^x = x$ and $a^{\log_a x} = x$. Inverse Properties
4. If $\log_a x = \log_a y$, then $x = y$. One-to-One Property

Library of Functions

The logarithmic function is the inverse of the exponential function. Its domain is the set of positive real numbers and its range is the set of all real numbers. Because of the inverse properties of logarithms and exponents, the exponential equation $a^0 = 1$ implies that $\log_a 1 = 0$.

Consult the Library of Functions Summary inside the front cover for a description of the logarithmic function.

EXAMPLE 3 Using Properties of Logarithms

Solve each equation for x.

a. $\log_2 x = \log_2 3$

b. $\log_4 4 = x$

Solution

a. Using the One-to-One Property (Property 4), you can conclude that $x = 3$.

b. Using Property 2, you can conclude that $x = 1$.

Graphs of Logarithmic Functions

To sketch the graph of $y = \log_a x$, you can use the fact that the graphs of inverse functions are reflections of each other in the line $y = x$.

EXAMPLE 4 Graphs of Exponential and Logarithmic Functions

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 2^x$ **b.** $g(x) = \log_2 x$

Solution

a. For $f(x) = 2^x$, construct a table of values.

x	-2	-1	0	1	2	3
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

By plotting these points and connecting them with a smooth curve, you obtain the graph of $f(x)$ shown in Figure 4.13.

b. Because $g(x) = \log_2 x$ is the inverse of $f(x) = 2^x$, the graph of g is obtained by plotting the points $(2^x, x)$ and connecting them with a smooth curve. The graph of g is a reflection of the graph of f in the line $y = x$, as shown in Figure 4.13.

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

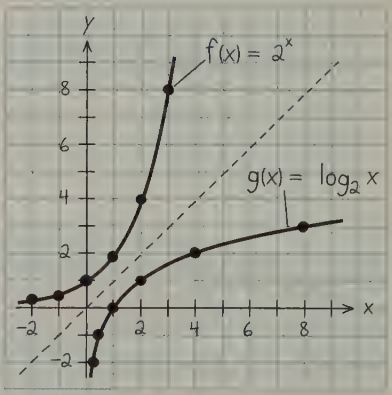


Figure 4.13

Before you can confirm the result of Example 4 using a graphing utility, you need to know how to enter $\log_2 x$. You will learn how to do this using the *change-of-base formula* discussed in Section 4.3.

EXAMPLE 5 Sketching the Graph of a Logarithmic Function

Sketch the graph of the common logarithmic function $f(x) = \log_{10} x$.

Solution

Begin by constructing a table of values. Note that some of the values can be obtained without a calculator using the Inverse Property of logarithms. Others require a calculator. Next, plot the points and connect them with a smooth curve, as shown in Figure 4.14.

	Without Calculator				With Calculator		
x	$\frac{1}{100}$	$\frac{1}{10}$	1	10	2	5	8
$\log_{10} x$	-2	-1	0	1	0.301	0.699	0.903

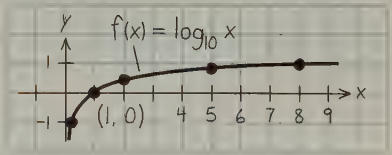


Figure 4.14

STUDY TIP

Compare Figure 4.14 with that obtained using a graphing utility. Note that the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

In Example 5, you can also sketch the graph of $f(x) = \log_{10} x$ as follows. Evaluate the inverse of f, $g(x) = 10^x$ for several values of x. Plot the points, sketch the graph of g, then reflect the graph in the line $y = x$ to obtain the graph of f.

The nature of the graph in Figure 4.14 is typical of functions of the form $f(x) = \log_a x$, $a > 1$. They have one x-intercept and one vertical asymptote. Notice how slowly the graph rises for $x > 1$. In Figure 4.14, you would need to move out to $x = 100$ before the graph rose to $y = 2$, and $x = 1000$ before the graph rose to $y = 3$. The basic characteristics of logarithmic graphs are summarized in Figure 4.15.

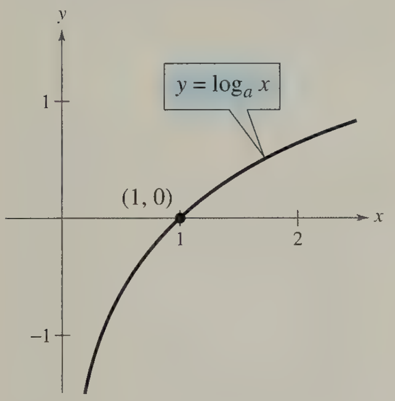


Figure 4.15

Graph of $y = \log_a x,\ a > 1$

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Intercept: $(1, 0)$
- Increasing
- y-axis is a vertical asymptote ($\log_a x \to -\infty$ as $x \to 0^+$)
- Continuous
- Reflection of graph of $y = a^x$ in the line $y = x$

In Figure 4.15, the vertical asymptote occurs at $x = 0$, where $\log_a x$ is *undefined.*

Exploration

Use a graphing utility to graph $y = \log_{10} x$ and $y = 8$ in the same viewing window. Do the graphs intersect? If so, find a viewing window that shows the point of intersection. What is the point of intersection? Use the point of intersection to complete the equation below.

$\log_{10} \quad\quad = 8$

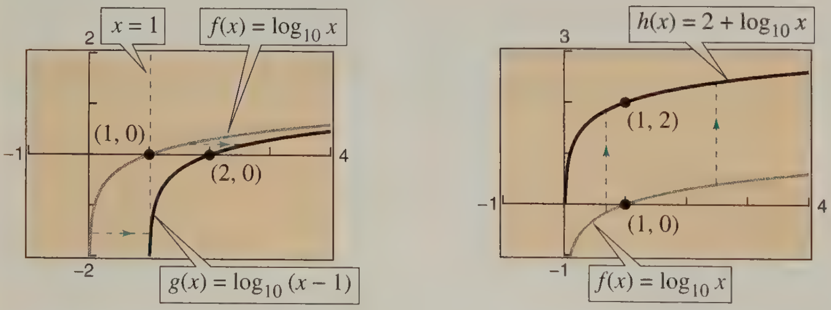
EXAMPLE 6 Transformations of Graphs of Logarithmic Functions

The graph of each of the following functions is similar to the graph of $f(x) = \log_{10} x$, as shown in Figure 4.16.

a. Because $g(x) = \log_{10}(x - 1) = f(x - 1)$, the graph of g can be obtained by shifting the graph of f one unit to the right.

b. Because $h(x) = 2 + \log_{10} x = 2 + f(x)$, the graph of h can be obtained by shifting the graph of f two units up.

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.



(a) (b)

Figure 4.16

In Figure 4.16, notice that the transformation in part (b) keeps the y-axis as a vertical asymptote, but the transformation in part (a) yields the new vertical asymptote $x = 1$.

The Natural Logarithmic Function

The most widely used base for logarithmic functions is the number e, where

$$e \approx 2.718281828 \ldots .$$

The logarithmic function with base e is the **natural logarithmic function** and is denoted by the special symbol $\ln x$, read as "the natural log of x" or "el en of x."

The Natural Logarithmic Function

The function defined by

$$f(x) = \log_e x = \ln x, \qquad x > 0$$

is called the **natural logarithmic function.**

The *Interactive* CD-ROM and *Internet* versions of this text offer a built-in graphing calculator, which can be used with the Examples, Explorations, and Exercises.

On most calculators, the natural logarithm is denoted by $\boxed{\text{LN}}$, as illustrated in Example 7.

EXAMPLE 7 Evaluating the Natural Logarithmic Function

Use a calculator to evaluate each expression.

a. $\ln 2$ **b.** $\ln 0.3$ **c.** $\ln e^2$ **d.** $\ln(-1)$

Solution

Number	*Graphing Calculator Keystrokes*	*Display*
a. $\ln 2$	$\boxed{\text{LN}}$ 2 $\boxed{\text{ENTER}}$	0.6931472
b. $\ln 0.3$	$\boxed{\text{LN}}$ .3 $\boxed{\text{ENTER}}$	-1.2039728
c. $\ln e^2$	$\boxed{\text{LN}}$ $\boxed{e^x}$ 2 $\boxed{\text{ENTER}}$	2
d. $\ln(-1)$	$\boxed{\text{LN}}$ $\boxed{(-)}$ 1 $\boxed{\text{ENTER}}$	ERROR

Exploration

Because the natural exponential function $y = e^x$ passes the Horizontal Line Test, it has an inverse. Use the *draw inverse* feature of your graphing utility to graph $y = e^x$ and its inverse. Compare this graph with that of the natural logarithmic function. What can you conclude?

In Example 7, be sure you see that $\ln(-1)$ gives an error message on most calculators. This occurs because the domain of $\ln x$ is the set of *positive* real numbers. So, $\ln(-1)$ is undefined.

The four properties of logarithms listed on page 312 are also valid for natural logarithms.

Properties of Natural Logarithms

1. $\ln 1 = 0$ because $e^0 = 1$.
2. $\ln e = 1$ because $e^1 = e$.
3. $\ln e^x = x$ and $e^{\ln x} = x$. Inverse Properties
4. If $\ln x = \ln y$, then $x = y$. One-to-One Property

EXAMPLE 8 Using Properties of Natural Logarithms

Use the properties of natural logarithms to rewrite each expression.

a. $\ln \frac{1}{e}$ **b.** $\ln e^2$ **c.** $\ln e^0$ **d.** $2 \ln e$

Solution

a. $\ln \frac{1}{e} = \ln e^{-1} = -1$	Inverse Property
b. $\ln e^2 = 2$	Inverse Property
c. $\ln e^0 = 0$	Property 1
d. $2 \ln e = 2(1) = 2$	Property 2

The graph of the natural logarithmic function is shown in Figure 4.17. Try using a graphing utility to confirm this graph and to verify that the domain of the natural logarithmic function is $(0, \infty)$.

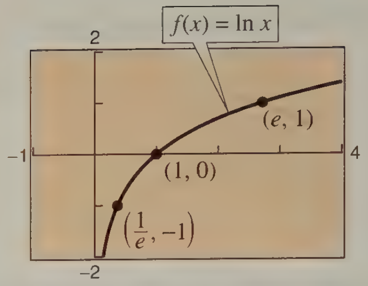


Figure 4.17

EXAMPLE 9 Finding the Domains of Logarithmic Functions

Find the domain of each function.

a. $f(x) = \ln(x - 2)$ **b.** $g(x) = \ln(2 - x)$ **c.** $h(x) = \ln x^2$

Algebraic Solution

a. Because $\ln(x - 2)$ is defined only if

$$x - 2 > 0,$$

it follows that the domain of f is $(2, \infty)$.

b. Because $\ln(2 - x)$ is defined only if

$$2 - x > 0,$$

it follows that the domain of g is $(-\infty, 2)$.

c. Because $\ln x^2$ is defined only if

$$x^2 > 0,$$

it follows that the domain of h is all real numbers except $x = 0$.

Graphical Solution

a. Use a graphing utility to graph the equation $y = \ln(x - 2)$ using a decimal setting, as shown in Figure 4.18. Use the *trace* feature to see that the x-coordinates of the points on the graph appear to extend from the right of 2 to $+\infty$. So, you can estimate the domain to be $(2, \infty)$.

b. Use a graphing utility to graph the equation $y = \ln(2 - x)$ using a decimal setting, as shown in Figure 4.19. Use the *trace* feature to see that the x-coordinates of the points on the graph appear to extend from $-\infty$ to the left of 2. So, you can estimate the domain to be $(-\infty, 2)$.

c. Use a graphing utility to graph the equation $y = \ln x^2$ using a decimal setting, as shown in Figure 4.20. Use the *trace* feature to see that the x-coordinates of the points on the graph appear to include all real numbers except $x = 0$. So, you can estimate the domain to be all real numbers except $x = 0$.

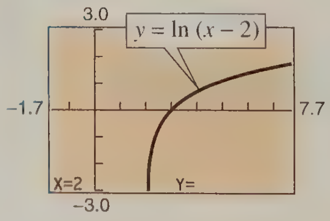


Figure 4.18

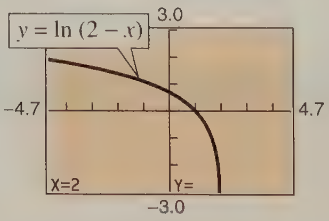


Figure 4.19

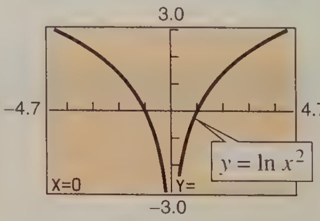


Figure 4.20

In Example 9, suppose you had been asked to analyze the function $h(x) = \ln|x - 2|$. How would the domain of this function compare with the domains of the functions given in parts (a) and (b) of the example?

Application

Exponential and logarithmic functions are used to model many situations in real life, as shown in the next example.

EXAMPLE 10 Human Memory Model

Students participating in a psychological experiment attended several lectures on a subject. At the end of the last lecture, and every month for the next year, the students were tested to see how much of the material they remembered. The average scores for the group were given by the *human memory model*

$$f(t) = 75 - 6\ln(t + 1), \qquad 0 \le t \le 12$$

where t is the time in months.

a. What was the average score on the original ($t = 0$) exam?

b. What was the average score at the end of $t = 2$ months?

c. What was the average score at the end of $t = 6$ months?

Algebraic Solution

a. The original average score was

$$\begin{aligned} f(0) &= 75 - 6\ln 1 \\ &= 75 - 6(0) \\ &= 75. \end{aligned}$$

b. After 2 months, the average score was

$$\begin{aligned} f(2) &= 75 - 6\ln 3 \\ &\approx 75 - 6(1.0986) \\ &\approx 68.4. \end{aligned}$$

c. After 6 months, the average score was

$$\begin{aligned} f(6) &= 75 - 6\ln 7 \\ &\approx 75 - 6(1.9459) \\ &\approx 63.3. \end{aligned}$$

Graphical Solution

Use a graphing utility to graph the function $y = 75 - 6\ln(x + 1)$ as shown in Figure 4.21. Then use the *value* or *trace* feature to approximate the following.

a. When $x = 0$, $y = 75$. So, the original average score was 75.

b. When $x = 2$, $y \approx 68.4$. So, the average score after 2 months was about 68.4.

c. When $x = 6$, $y \approx 63.3$. So, the average score after 6 months was about 63.3.

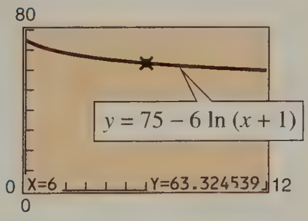


Figure 4.21

Writing About Math *Analyzing a Human Memory Model*

Use a graphing utility to determine how many months it would take for the average score in Example 10 to decrease to 60. Explain your method of solving the problem. Describe another way that you can use a graphing utility to determine the answer. Also, make a statement about the general shape of the model. Would the students forget more quickly soon after the test or as time passes? Explain your reasoning.

4.2 Exercises

In Exercises 1–8, write the logarithmic equation in exponential form. For example, the exponential form of $\log_5 25 = 2$ is $5^2 = 25$.

1. $\log_4 64 = 3$

2. $\log_3 81 = 4$

3. $\log_7 \frac{1}{49} = -2$

4. $\log_{10} \frac{1}{1000} = -3$

5. $\log_{32} 4 = \frac{2}{5}$

6. $\log_{16} 8 = \frac{3}{4}$

7. $\ln 1 = 0$

8. $\ln 4 = 1.386 \ldots$

In Exercises 9–18, write the exponential equation in logarithmic form.

9. $5^3 = 125$

10. $8^2 = 64$

11. $81^{1/4} = 3$

12. $9^{3/2} = 27$

13. $6^{-2} = \frac{1}{36}$

14. $10^{-3} = 0.001$

15. $e^3 = 20.0855 \ldots$

16. $e^x = 4$

17. $e^{2.6} = 13.463 \ldots$

18. $e^{\pi} = 23.140 \ldots$

In Exercises 19–24, evaluate the expression without using a calculator.

19. $\log_2 16$

20. $\log_{27} 9$

21. $\log_{16}\left(\frac{1}{4}\right)$

22. $\log_2\left(\frac{1}{8}\right)$

23. $\log_{10} 0.01$

24. $\log_{10} 0.1$

In Exercises 25–30, solve the equation for x.

25. $\log_7 x = \log_7 9$

26. $\log_5 5 = x$

27. $\ln e^8 = x$

28. $\ln 1 = \ln x$

29. $\log_6 6^2 = x$

30. $\log_2 2^{-1} = x$

In Exercises 31–42, use a calculator to evaluate the logarithm. Round to three decimal places.

31. $\log_{10} 345$

32. $\log_{10} 145$

33. $\log_{10}\left(\frac{4}{5}\right)$

34. $\log_{10} \frac{25}{2}$

35. $\ln\left(4 + \sqrt{3}\right)$

36. $\ln\left(\sqrt{5} - 2\right)$

37. $\ln \sqrt{42}$

38. $\ln \sqrt{752}$

39. $6 \log_{10} 14.8$

40. $-3 \log_{10} 0.09$

41. $12 \ln 6.4$

42. $-5.5 \ln 34$

In Exercises 43–46, describe the relationship between the graphs of f and g. What is the relationship between the functions f and g?

43. $f(x) = 3^x$
$g(x) = \log_3 x$

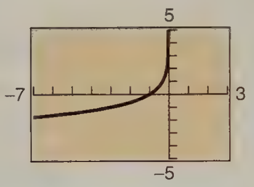
44. $f(x) = 5^x$
$g(x) = \log_5 x$

45. $f(x) = e^x$
$g(x) = \ln x$

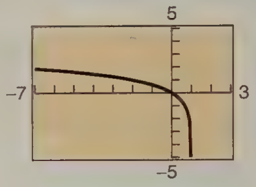
46. $f(x) = 10^x$
$g(x) = \log_{10} x$

In Exercises 47–52, use the graph of $y = \log_3 x$ to match the given function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

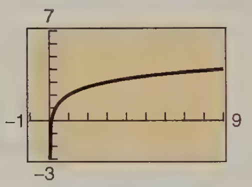
(a)



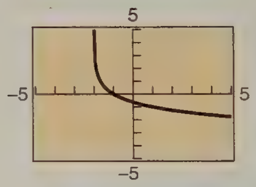
(b)



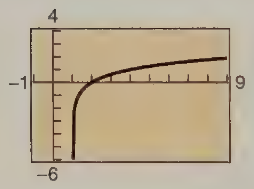
(c)



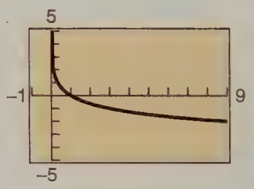
(d)



(e)



(f)



47. $f(x) = \log_3 x + 2$

48. $f(x) = -\log_3 x$

49. $f(x) = -\log_3(x + 2)$

50. $f(x) = \log_3(x - 1)$

51. $f(x) = \log_3(1 - x)$

52. $f(x) = -\log_3(-x)$

In Exercises 53–60, find the domain, vertical asymptote, and x-intercept of the logarithmic function, and sketch its graph by hand. Verify using a graphing utility.

53. $f(x) = \log_4 x$

54. $g(x) = \log_6 x$

55. $h(x) = \log_4(x - 3)$

56. $f(x) = -\log_6(x + 2)$

57. $y = -\log_3 x + 2$

58. $y = \log_5(x - 1) + 4$

59. $f(x) = 6 + \log_6(x - 3)$

60. $f(x) = -\log_3(x + 2) - 4$

In Exercises 61–66, use a graphing utility to graph the logarithmic function. Determine the domain and identify any vertical asymptote and x-intercept.

61. $y = \log_{10}\left(\frac{x}{5}\right)$

62. $y = \log_{10}(-x)$

63. $f(x) = \ln(x - 2)$

64. $h(x) = \ln(x + 1)$

65. $g(x) = \ln(-x)$

66. $f(x) = \ln(3 - x)$

In Exercises 67–70, (a) use a graphing utility to graph the function, (b) find the domain, (c) use the graph to determine the intervals in which the function is increasing and decreasing, and (d) approximate any relative maximum or minimum values of the function.

67. $f(x) = \frac{x}{2} - \ln\frac{x}{4}$

68. $g(x) = \frac{12 \ln x}{x}$

69. $h(x) = 4x \ln x$

70. $f(x) = \frac{x}{\ln x}$

71. ***Population Growth*** The population of a town will double in

$$t = \frac{10 \ln 2}{\ln 67 - \ln 50} \text{ years.}$$

Find t.

72. ***Work*** The work (in foot-pounds) done in compressing an initial volume of 9 cubic feet of air at a pressure of 15 pounds per square inch to a volume of 3 cubic feet is $W = 19{,}440(\ln 9 - \ln 3)$. Find W.

73. ***Human Memory Model*** Students in a mathematics class were given an exam and then tested monthly with an equivalent exam. The average score for the class was given by the human memory model

$$f(t) = 80 - 17 \log_{10}(t + 1), \quad 0 \le t \le 12$$

where t is the time in months.

(a) What was the average score on the original exam ($t = 0$)?

(b) What was the average score after 4 months?

(c) What was the average score after 10 months?

(d) Verify your answers in parts (a), (b), and (c) using a graphing utility.

74. ***Investment Time*** A principal P, invested at $5\frac{1}{2}\%$ and compounded continuously, increases to an amount K times the original principal after t years, where $t = (\ln K)/0.055$.

(a) Complete the table and interpret your results.

K	1	2	4	6	8	10	12
t							

(b) Use a graphing utility to graph the function.

75. ***Data Analysis*** The table shows the temperatures T (in °F) at which water boils at selected pressures p (in pounds per square inch). (Source: Standard Handbook of Mechanical Engineers)

p	5	10	14.696 (1 atm)
T	162.24°	193.21°	212.00°

p	20	30	40
T	227.96°	250.33°	267.25°

p	60	80	100
T	292.71°	312.03°	327.81°

A model that approximates this data is

$$T = 87.97 + 34.96 \ln p + 7.91\sqrt{p}.$$

(a) Use a graphing utility to plot the data points and graph the model in the same viewing window. How well does the model fit the data?

(b) Use the graph to estimate the pressure required for the boiling point of water to exceed 300°F.

(c) Calculate T when the pressure is 74 pounds per square inch. Verify your answer graphically.

76. ***Data Analysis*** A meteorologist measures the atmospheric pressure P (in Pascals) at altitude h (in kilometers). The data is shown below.

h	0	5	10	15	20
P	101,293	54,735	23,294	12,157	5069

A model for the data is

$$P = 102{,}303e^{-0.137h}.$$

(a) Use a graphing utility to plot the data points and graph the model in the same viewing window.

(b) Use a graphing utility to plot the points $(h, \ln P)$. Use the regression capabilities of the graphing utility to fit a regression line to the revised data points.

(c) The line in part (b) has the form $\ln P = ah + b$. Write the line in exponential form.

(d) Verify graphically and algebraically that the result of part (c) is equivalent to the given exponential model for the data.

77. ***World Population Growth*** The time t in years for the world population to double if it is increasing at a continuous rate of r is

$$t = \frac{\ln 2}{r}.$$

Complete the table. What do your results imply?

r	0.005	0.010	0.015	0.020	0.025	0.030
t						

78. ***Tractrix*** A person walking along a dock (the y-axis) drags a boat by a 10-foot rope. The boat travels along a path known as a tractrix. The equation of this path is

$$y = 10 \ln\left(\frac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2}.$$

(a) Use a graphing utility to obtain a graph of the function. What is the domain of the function?

(b) Identify any asymptotes of the graph.

(c) Determine the position of the person when the x-coordinate of the position of the boat is $x = 2$.

(d) Let $(0, p)$ be the position of the person. Determine p as a function of x, the x-coordinate of the position of the boat.

(e) Use a graphing utility to graph the function p. When does the position of the person change most for a small change in the position of the boat? Explain.

79. ***Sound Intensity*** The relationship between the number of decibels β and the intensity of a sound I in watts per square meter is

$$\beta = 10 \log_{10}\left(\frac{I}{10^{-12}}\right).$$

(a) Determine the number of decibels of a sound with an intensity of 1 watt per square meter.

(b) Determine the number of decibels of a sound with an intensity of 10^{-2} watt per square meter.

(c) The intensity of the sound in part (a) is 100 times as great as that in part (b). Is the number of decibels 100 times as great? Explain.

Ventilation Rates **In Exercises 80 and 81, use the model**

$$y = 80.4 - 11 \ln x, \quad 100 \le x \le 1500$$

which approximates the minimum required ventilation rate in terms of the air space per child in a public school classroom. In the model, x is the air space per child (in cubic feet) and y is the ventilation rate (in cubic feet per minute).

80. Use a graphing utility to graph the function and approximate the required ventilation rate if there is 300 cubic feet of air space per child.

81. A classroom is designed for 30 students. The air-conditioning system in the room has the capacity to move 450 cubic feet of air per minute.

(a) Determine the ventilation rate per child, assuming that the room is filled to capacity.

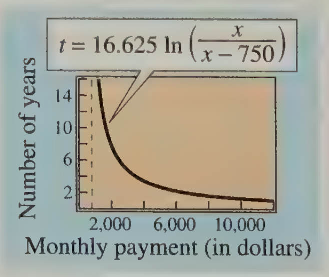
(b) Use the graph of Exercise 80 to estimate the air space required per child.

(c) Determine the minimum number of square feet of floor space required for the room if the ceiling height is 30 feet.

Monthly Payment **In Exercises 82–85, use the model**

$$t = 16.625 \ln\left(\frac{x}{x - 750}\right), \quad x > 750$$

which approximates the length of a home mortgage of \$150,000 at 6% in terms of the monthly payment. In the model, t is the length of the mortgage in years and x is the monthly payment in dollars.



82. Use the model to approximate the length of a \$150,000 mortgage (at 6%) if the monthly payment is \$897.72.

83. Use the model to approximate the length of a \$150,000 mortgage (at 6%) if the monthly payment is \$1659.24.

84. Approximate the total amount paid over the term of the mortgage in Exercise 82 with a monthly payment of \$897.72. What amount of the total is interest costs?

85. Approximate the total amount paid over the term of the mortgage in Exercise 83 with a monthly payment of \$1659.24. What amount of the total is interest costs?

Synthesis

True or False? **In Exercises 86 and 87, determine whether the statement is true or false. Justify your answer.**

86. You can determine the graph of $f(x) = \log_6 x$ by graphing $g(x) = 6^x$ and reflecting it in the x-axis.

87. The graph of $f(x) = \log_3 x$ contains the point $(27, 3)$.

88. ***Exploration*** The table of values was obtained by evaluating a function. Determine which of the statements may be true and which must be false.

x	1	2	8
y	0	1	3

(a) y is an exponential function of x.

(b) y is a logarithmic function of x.

(c) x is an exponential function of y.

(d) y is a linear function of x.

89. ***Exploration*** Use a graphing utility to compare the graph of the function $y = \ln x$ with the graphs of the following functions.

(a) $y = x - 1$

(b) $y = (x - 1) - \frac{1}{2}(x - 1)^2$

(c) $y = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3$

90. ***Finding a Pattern*** Identify the pattern of successive polynomials given in Exercise 89. Extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of $y = \ln x$. What do you think the pattern implies?

91. ***Numerical and Graphical Analysis*** (a) Use a graphing utility to complete the table for the function

$$f(x) = \frac{\ln x}{x}.$$

x	1	5	10	10^2	10^4	10^6
$f(x)$						

(b) Use the table to determine what $f(x)$ approaches as x increases without bound.

(c) Use a graphing utility to check the result of part (b).

Review

In Exercises 92–95, find the vertical and horizontal asymptotes of the rational function.

92. $f(x) = \dfrac{4}{-8 - x}$

93. $f(x) = \dfrac{2x^3 - 3}{x^2}$

94. $f(x) = \dfrac{x + 5}{2x^2 + x - 15}$

95. $f(x) = \dfrac{2x^2(x - 5)}{x - 7}$

In Exercises 96–99, evaluate the expression. Round your answer to three decimal places.

96. e^{12}

97. $e^{7/2}$

98. e^{-5}

99. $6e^{-8}$

4.3 Properties of Logarithms

What You Should Learn:

- How to rewrite logarithmic functions with different bases
- How to use properties of logarithms to evaluate or rewrite logarithmic expressions
- How to use properties of logarithms to expand or condense logarithmic expressions
- How to use logarithmic functions to model and solve real-life problems

Why You Should Learn It:

Logarithmic functions can be used to model and solve real-life problems, such as human memory in Exercise 96 on page 327.

Gary Conner/PhotoEdit

Change of Base

Most calculators have only two types of log keys, one for common logarithms (base 10) and one for natural logarithms (base e). Although common logs and natural logs are the most frequently used, you may occasionally need to evaluate logarithms to other bases. To do this, you can use the following **change-of-base formula.**

Change-of-Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base using any of the following formulas.

Base b	*Base 10*	*Base e*
$\log_a x = \dfrac{\log_b x}{\log_b a}$	$\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$	$\log_a x = \dfrac{\ln x}{\ln a}$

One way to look at the change-of-base formula is that logarithms to base a are simply *constant multiples* of logarithms to base b. The constant multiplier is $1/(\log_b a)$.

EXAMPLE 1 Changing Bases Using Common Logarithms

a. $\log_4 30 = \dfrac{\log_{10} 30}{\log_{10} 4}$ $\qquad \log_a x = \dfrac{\log_{10} x}{\log_{10} a}$

$\approx \dfrac{1.47712}{0.60206}$ Use a calculator.

≈ 2.4534 Use a calculator.

b. $\log_2 x = \dfrac{\log_{10} x}{\log_{10} 2} \approx \dfrac{\log_{10} x}{0.30103} \approx 3.3219 \log_{10} x$

EXAMPLE 2 Changing Bases Using Natural Logarithms

a. $\log_4 30 = \dfrac{\ln 30}{\ln 4}$ $\qquad \log_a x = \dfrac{\ln x}{\ln a}$

$\approx \dfrac{3.40120}{1.38629}$ Use a calculator.

≈ 2.4535 Use a calculator.

b. $\log_2 x = \dfrac{\ln x}{\ln 2} \approx \dfrac{\ln x}{0.693147} \approx 1.4427 \ln x$

In Examples 1 and 2, the result is the same whether common logarithms or natural logarithms are used in the change-of-base formula.

Properties of Logarithms

You know from the previous section that the logarithmic function with base a is the *inverse* of the exponential function with base a. So, it makes sense that the properties of exponents should have corresponding properties involving logarithms. For instance, the exponential property $a^0 = 1$ has the corresponding logarithmic property $\log_a 1 = 0$.

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There is no general property that can be used to rewrite $\log_a(u \pm v)$. Specifically, $\log_a(x + y)$ is *not* equal to $\log_a x + \log_a y$.

Properties of Logarithms

Let a be a positive number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, the following properties are true.

1. $\log_a(uv) = \log_a u + \log_a v$ 1. $\ln(uv) = \ln u + \ln v$
2. $\log_a \dfrac{u}{v} = \log_a u - \log_a v$ 2. $\ln \dfrac{u}{v} = \ln u - \ln v$
3. $\log_a u^n = n \log_a u$ 3. $\ln u^n = n \ln u$

For a proof of Property 1, see Appendix A.

EXAMPLE 3 Using Properties of Logarithms

Write the logarithm in terms of ln 2 and ln 3.

a. $\ln 6$ **b.** $\ln \dfrac{2}{27}$

Solution

a. $\ln 6 = \ln(2 \cdot 3)$ Rewrite 6 as $2 \cdot 3$.

$= \ln 2 + \ln 3$ Property 1

b. $\ln \dfrac{2}{27} = \ln 2 - \ln 27$ Property 2

$= \ln 2 - \ln 3^3$ Rewrite 27 as 3^3.

$= \ln 2 - 3 \ln 3$ Property 3

EXAMPLE 4 Using Properties of Logarithms

Use the properties of logarithms to verify that $-\ln \frac{1}{2} = \ln 2$.

Solution

$-\ln \frac{1}{2} = -\ln(2^{-1})$ Rewrite $\frac{1}{2}$ as 2^{-1}.

$= -(-1) \ln 2$ Property 3

$= \ln 2$ Simplify.

Try checking this result on your calculator.

The Granger Collection

John Napier (1550–1617), a Scottish mathematician, developed logarithms as a way to simplify some of the tedious calculations of his day. Beginning in 1594, Napier worked about 20 years on the invention of logarithms. Napier was only partially successful in his quest to simplify tedious calculations. Nonetheless, the development of logarithms was a step forward and received immediate recognition.

Rewriting Logarithmic Expressions

The properties of logarithms are useful for rewriting logarithmic expressions in forms that simplify the operations of algebra. This is true because they convert complicated products, quotients, and exponential forms into simpler sums, differences, and products, respectively.

EXAMPLE 5 Expanding the Logarithm of a Product

Use the properties of logarithms to expand $\log_{10} 5x^3y$.

Solution

$$\begin{aligned} \log_{10} 5x^3y &= \log_{10} 5 + \log_{10} x^3y && \text{Property 1} \\ &= \log_{10} 5 + \log_{10} x^3 + \log_{10} y && \text{Property 1} \\ &= \log_{10} 5 + 3\log_{10} x + \log_{10} y && \text{Property 3} \end{aligned}$$

EXAMPLE 6 Expanding the Logarithm of a Quotient

Use the properties of logarithms to expand $\ln \dfrac{\sqrt{3x-5}}{7}$.

Solution

$$\begin{aligned} \ln \frac{\sqrt{3x-5}}{7} &= \ln\left[\frac{(3x-5)^{1/2}}{7}\right] && \text{Rewrite rational exponent.} \\ &= \ln(3x-5)^{1/2} - \ln 7 && \text{Property 2} \\ &= \frac{1}{2}\ln(3x-5) - \ln 7 && \text{Property 3} \end{aligned}$$

Exploration

Use a graphing utility to graph the functions

$$y = \ln x - \ln(x-3)$$

and

$$y = \ln \frac{x}{x-3}$$

in the same viewing window. Does the graphing utility show the functions with the same domain? If so, should it? Explain your reasoning.

In Examples 5 and 6, the properties of logarithms were used to *expand* logarithmic expressions. In Example 7, this procedure is reversed and the properties of logarithms are used to *condense* logarithmic expressions.

EXAMPLE 7 Condensing a Logarithmic Expression

Use the properties of logarithms to condense each logarithmic expression.

a. $\frac{1}{2}\log_{10} x + 3\log_{10}(x+1)$ **b.** $2\ln(x+2) - \ln x$

Solution

a. $$\begin{aligned} \tfrac{1}{2}\log_{10} x + 3\log_{10}(x+1) &= \log_{10} x^{1/2} + \log_{10}(x+1)^3 && \text{Property 3} \\ &= \log_{10}\left[\sqrt{x}\cdot(x+1)^3\right] && \text{Property 1} \end{aligned}$$

b. $$\begin{aligned} 2\ln(x+2) - \ln x &= \ln(x+2)^2 - \ln x && \text{Property 3} \\ &= \ln\frac{(x+2)^2}{x} && \text{Property 2} \end{aligned}$$

Application

EXAMPLE 8 Finding a Mathematical Model

The table shows the mean distance from the sun x and the orbital period y of the six planets that are closest to the sun. In the table, the mean distance is given in terms of astronomical units (where the earth's mean distance is defined as 1.0), and the period is given in terms of years. Find an equation that expresses y as a function of x.

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
Mean Distance, x	0.387	0.723	1.0	1.524	5.203	9.539
Period, y	0.241	0.615	1.0	1.881	11.862	29.458

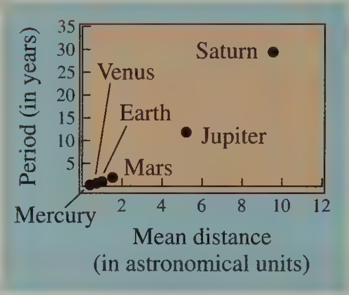


Figure 4.22

Algebraic Solution

The points in the table are plotted in Figure 4.22. From this figure it is not clear how to find an equation that relates y and x. To solve this problem, take the natural log of each of the x- and y-values shown in the table. This produces the following results.

Planet	Mercury	Venus	Earth
$\ln x = X$	−0.949	−0.324	0.0
$\ln y = Y$	−1.423	−0.486	0.0

Planet	Mars	Jupiter	Saturn
$\ln x = X$	0.421	1.649	2.255
$\ln y = Y$	0.632	2.473	3.383

Now, by plotting the points in the table, you can see that all six of the points appear to lie in a line.

Choose any two points to determine the slope of the line. Using the two points (0.421, 0.632) and (0, 0), you can determine the slope of the line.

$$m = \frac{0.632 - 0}{0.421 - 0} \approx 1.5 = \frac{3}{2}$$

By the point-slope form, the equation of the line is $Y = \frac{3}{2}X$, where $Y = \ln y$ and $X = \ln x$. You can therefore conclude that $\ln y = \frac{3}{2}\ln x$.

Graphical Solution

The points in the table are plotted in Figure 4.22. From this figure it is not clear how to find an equation that relates y and x. To solve this problem, take the natural log of each of the x- and y-values given in the table. This produces the following results.

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
$\ln x = X$	−0.949	−0.324	0.0	0.421	1.649	2.255
$\ln y = Y$	−1.423	−0.486	0.0	0.632	2.473	3.383

Now, by plotting the points in the table, you can see that all six of the points appear to lie in a line, as shown in Figure 4.23.

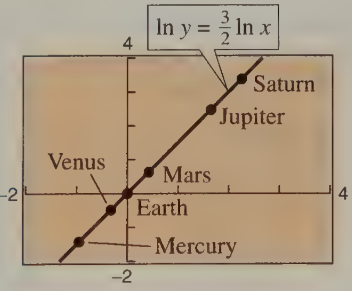


Figure 4.23

Using the regression features of a graphing utility, you can find a linear model for the data $Y = 1.5X = \frac{3}{2}X$, where $Y = \ln y$ and $X = \ln x$. From the model, you can see that the slope of the line is $\frac{3}{2}$. So, you can conclude that $\ln y = \frac{3}{2}\ln x$.

In Example 8, try to convert this to $y = f(x)$ form. You will get a function of the form $y = ax^b$, which is a power model.

4.3 Exercises

In Exercises 1 and 2, use a graphing utility to graph the two functions in the same viewing window. What do the graphs suggest? Explain your reasoning.

1. $f(x) = \log_{10} x$

$g(x) = \dfrac{\ln x}{\ln 10}$

2. $f(x) = \ln x$

$g(x) = \dfrac{\log_{10} x}{\log_{10} e}$

In Exercises 3–10, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

3. $\log_3 7$

4. $\log_7 4$

5. $\log_{1/2} 4$

6. $\log_{1/8} 64$

7. $\log_9(0.8)$

8. $\log_{1/3}(0.015)$

9. $\log_{15} 1460$

10. $\log_{20} 135$

In Exercises 11–18, rewrite the logarithm as a multiple of (a) a common logarithm and (b) a natural logarithm.

11. $\log_5 x$

12. $\log_3 x$

13. $\log_{1/5} x$

14. $\log_{1/3} x$

15. $\log_x \frac{3}{10}$

16. $\log_x \frac{3}{4}$

17. $\log_{2.6} x$

18. $\log_{7.1} x$

In Exercises 19–26, use the change-of-base formula to rewrite the logarithm as a multiple of a logarithm. Then use a graphing utility to sketch the graph.

19. $f(x) = \log_2 x$

20. $f(x) = \log_4 x$

21. $f(x) = \log_{1/2} x$

22. $f(x) = \log_{1/4} x$

23. $f(x) = \log_{11.8} x$

24. $f(x) = \log_{12.4} x$

25. $f(x) = \log_3 x^{1/2}$

26. $f(x) = \log_5 \dfrac{x}{3}$

In Exercises 27–46, use the properties of logarithms to write the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)

27. $\log_{10} 5x$

28. $\log_{10} 10z$

29. $\log_{10} \dfrac{5}{x}$

30. $\log_{10} \dfrac{y}{2}$

31. $\log_8 x^4$

32. $\log_6 z^{-3}$

33. $\ln \sqrt{z}$

34. $\ln \sqrt[3]{t}$

35. $\ln xyz$

36. $\ln \dfrac{xy}{z}$

37. $\ln \sqrt{a-1}, \quad a > 1$

38. $\ln\left(\dfrac{x^2-1}{x^3}\right), \quad x > 1$

39. $\ln z(z-1)^2, \quad z > 1$

40. $\ln \sqrt{\dfrac{x^2}{y^3}}$

41. $\ln \sqrt[3]{\dfrac{x}{y}}$

42. $\ln \dfrac{x}{\sqrt{x^2+1}}$

43. $\ln \dfrac{x^4\sqrt{y}}{z^5}$

44. $\ln \sqrt{x^2(x+2)}$

45. $\log_b \dfrac{x^2}{y^2z^3}$

46. $\log_b \dfrac{\sqrt{x}y^4}{z^4}$

Graphical Analysis **In Exercises 47 and 48, use a graphing utility to graph the two equations in the same viewing window. What do the graphs suggest? Explain your reasoning.**

47. $y_1 = \ln[x^3(x+4)], \qquad y_2 = 3\ln x + \ln(x+4)$

48. $y_1 = \ln\left(\dfrac{\sqrt{x}}{x-2}\right), \qquad y_2 = \frac{1}{2}\ln x - \ln(x-2)$

In Exercises 49–68, write the expression as the logarithm of a single quantity.

49. $\ln x + \ln 4$

50. $\ln y + \ln z$

51. $\log_4 z - \log_4 y$

52. $\log_5 8 - \log_5 t$

53. $2\log_2(x+3)$

54. $-6\log_6 2x$

55. $\frac{1}{3}\log_3 7x$

56. $\frac{5}{2}\log_7(z-4)$

57. $\ln x - 3\ln(x+1)$

58. $2\ln 8 + 5\ln z$

59. $\ln(x-2) - \ln(x+2)$

60. $3\ln x + 2\ln y - 4\ln z$

61. $\ln x - 2[\ln(x+2) + \ln(x-2)]$

62. $4[\ln z + \ln(z+5)] - 2\ln(z-5)$

63. $\frac{1}{3}[2\ln(x+3) + \ln x - \ln(x^2-1)]$

64. $2[\ln x - \ln(x+1) - \ln(x-1)]$

65. $\frac{1}{3}[\ln y + 2\ln(y+4)] - \ln(y-1)$

66. $\frac{1}{2}[\ln(x+1) + 2\ln(x-1)] + 3\ln x$

67. $2\ln 3 - \frac{1}{2}\ln(x^2+1)$

68. $\frac{3}{2}\ln 5t^6 - \frac{3}{4}\ln t^4$

Graphical Analysis **In Exercises 69 and 70, use a graphing utility to graph the two equations in the same viewing window. What do the graphs suggest? Verify your conclusion algebraically.**

69. $y_1 = 2[\ln 8 - \ln(x^2 + 1)]$, $y_2 = \ln\left[\dfrac{64}{(x^2 + 1)^2}\right]$

70. $y_1 = \ln x + \frac{1}{3}\ln(x + 1)$, $y_2 = \ln\left(x\sqrt[3]{x + 1}\right)$

Think About It **In Exercises 71 and 72, use a graphing utility to graph the two equations in the same viewing window. Are the expressions equivalent? Explain.**

71. $y_1 = \ln x^2$, $y_2 = 2\ln x$

72. $y_1 = \frac{1}{4}\ln[x^4(x^2 + 1)]$, $y_2 = \ln x + \frac{1}{4}\ln(x^2 + 1)$

In Exercises 73–86, find the exact value of the logarithm without using a calculator. (If this is not possible, state the reason.)

73. $\log_3 9$

74. $\log_6 \sqrt[3]{6}$

75. $\log_4 16^{3.4}$

76. $\log_5\left(\frac{1}{125}\right)$

77. $\log_2(-4)$

78. $\log_4(-16)$

79. $\log_5 75 - \log_5 3$

80. $\log_4 2 + \log_4 32$

81. $\ln e^3 - \ln e^7$

82. $3\ln e^4$

83. $\log_{10} 0$

84. $\ln 1$

85. $\ln e^{8.5}$

86. $\ln \sqrt[5]{e^3}$

In Exercises 87–94, use the properties of logarithms to simplify the logarithmic expression.

87. $\log_4 8$

88. $\log_5\left(\frac{1}{15}\right)$

89. $\log_7 \sqrt{70}$

90. $\log_2(4^2 \cdot 3^4)$

91. $\log_5\left(\frac{1}{250}\right)$

92. $\log_{10}\left(\frac{9}{300}\right)$

93. $\ln(5e^6)$

94. $\ln \dfrac{6}{e^2}$

95. ***Sound Intensity*** The relationship between the number of decibels β and the intensity of a sound I in watts per square meter is

$$\beta = 10\log_{10}\left(\frac{I}{10^{-12}}\right).$$

(a) Use the properties of logarithms to write the formula in a simpler form.

(b) Use a graphing utility to complete the table.

I	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}	10^{-14}
β						

(c) Verify your answers in part (b) algebraically.

96. ***Human Memory Model*** Students participating in a psychological experiment attended several lectures. After the last lecture, and every month for the next year, the students were tested to see how much of the material they remembered. The average scores for the group were given by the memory model

$$f(t) = 90 - 15\log_{10}(t + 1), \quad 0 \le t \le 12$$

where t is the time (in months).

(a) Use a graphing utility to graph the function over the specified domain.

(b) What was the average score on the original exam $(t = 0)$?

(c) What was the average score after 6 months?

(d) What was the average score after 12 months?

(e) When would the average score have decreased to 75?

97. ***Comparing Models*** A cup of water at an initial temperature of 78°C is placed in a room at a constant temperature of 21°C. The temperature of the water is measured every 5 minutes during a half-hour period. The results are recorded as ordered pairs of the form (t, T), where t is the time (in minutes) and T is the temperature (in degrees Celsius).

$(0, 78.0°), (5, 66.0°), (10, 57.5°), (15, 51.2°),$
$(20, 46.3°), (25, 42.5°), (30, 39.6°)$

(a) The graph of the model for the data should be asymptotic with the temperature of the room. Subtract the room temperature from each of the temperatures in the ordered pairs. Use a graphing utility to plot the data points (t, T) and $(t, T - 21)$.

(b) Use the regression capabilities of a graphing utility to fit an exponential model to the revised data. This model will be of the form

$$T - 21 = ab^x.$$

Solve for T and graph the model. Compare the result with the plot of the original data.

(c) Take the natural logarithms of the revised temperatures. Use a graphing utility to plot the points $(t, \ln(T-21))$ and observe that the points appear linear. Use the regression capabilities of a graphing utility to fit a line to this data. The resulting line has the form

$$\ln(T-21) = at + b.$$

Use the properties of logarithms to solve for T. Verify that the result is equivalent to the model in part (b).

(d) Fit a rational model to the data. Take the reciprocals of the y-coordinates of the revised data to generate the points

$$\left(t, \frac{1}{T-21}\right).$$

Use a graphing utility to plot these points and observe that they appear linear. Use the regression capabilities of a graphing utility to fit a line to this data. The resulting line has the form

$$\frac{1}{T-21} = at + b.$$

Solve for T, and use a graphing utility to graph the rational function and the original data points.

98. ***Writing*** Write a short paragraph explaining why the transformations of the data in Exercise 97 were necessary to obtain the models. Why did taking the logarithms of the temperatures lead to a linear scatter plot? Why did taking the reciprocals of the temperatures lead to a linear scatter plot?

Synthesis

True or False? **In Exercises 99–104, determine whether the statement is true or false given that $f(x) = \ln x$. Justify your answer.**

99. $f(0) = 0$

100. $f(ax) = f(a) + f(x), \quad a > 0, x > 0$

101. $f(x-2) = f(x) - f(2), \quad x > 2$

102. $\sqrt{f(x)} = \frac{1}{2}f(x)$

103. If $f(u) = 2f(v)$, then $v = u^2$.

104. If $f(x) < 0$, then $0 < x < 1$.

105. ***Think About It*** Use a graphing utility to graph

$$f(x) = \ln\frac{x}{2}, \quad g(x) = \frac{\ln x}{\ln 2}, \quad h(x) = \ln x - \ln 2$$

in the same viewing window. Which two functions have identical graphs? Explain why.

106. ***Exploration*** Approximate the natural logarithms of as many integers as possible between 1 and 20, given that $\ln 2 \approx 0.6931$, $\ln 3 \approx 1.0986$, and $\ln 5 \approx 1.6094$. (Do not use a calculator.)

107. Prove that $\log_b \frac{u}{v} = \log_b u - \log_b v$.

108. Prove that $\log_b u^n = n \log_b u$.

Review

In Exercises 109–112, simplify the expression.

109. $\dfrac{24xy^{-2}}{16x^{-3}y}$

110. $\left(\dfrac{2x^2}{3y}\right)^{-3}$

111. $(18x^3y^4)^{-3}(18x^3y^4)^3$

112. $xy(x^{-1} + y^{-1})^{-1}$

In Exercises 113–118, find all solutions of the equation. Be sure to check all your solutions.

113. $x^2 - 6x + 2 = 0$

114. $2x^3 + 20x^2 + 50x = 0$

115. $x^4 - 19x^2 + 48 = 0$

116. $9x^4 - 37x^2 + 4 = 0$

117. $x^3 - 6x^2 - 4x + 24 = 0$

118. $9x^4 - 226x^2 + 25 = 0$

In Exercises 119–126, use a calculator to evaluate the expression. Round the result to three decimal places.

119. $1.6^{-2\pi}$

120. $\sqrt[5]{8251}$

121. $260^{\sqrt{3}}$

122. $170(4^{-1.1})$

123. $\log_{10}(220)$

124. $\log_{10}\left(\frac{7}{5}\right)$

125. $\ln 2.008$

126. $\ln\left(5 - \sqrt{7}\right)$

4.4 Solving Exponential and Logarithmic Equations

Introduction

So far in this chapter, you have studied the definitions, graphs, and properties of exponential and logarithmic functions. In this section, you will study procedures for *solving equations* involving exponential and logarithmic functions.

There are two basic strategies for solving exponential or logarithmic equations. The first is based on the One-to-One Properties and the second is based on the Inverse Properties. For $a > 0$ and $a \neq 1$, the following properties are true for all x and y for which $\log_a x$ and $\log_a y$ are defined.

One-to-One Properties

$a^x = a^y$ if and only if $x = y$.

$\log_a x = \log_a y$ if and only if $x = y$.

Inverse Properties

$$a^{\log_a x} = x$$

$$\log_a a^x = x$$

What You Should Learn:

- How to solve simple exponential and logarithmic equations
- How to solve more complicated exponential equations
- How to solve more complicated logarithmic equations
- How to use exponential and logarithmic equations to model and solve real-life problems

Why You Should Learn It:

Exponential and logarithmic equations can be used to model and solve real-life problems. For instance, Exercise 126 on page 339 shows how to use a logarithmic function to model crumple zones for automobile crash tests.

Index Stock

EXAMPLE 1 Solving Simple Exponential and Logarithmic Equations

	Original Equation	*Rewritten Equation*	*Solution*	*Property*
a.	$2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
b.	$\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
c.	$\left(\frac{1}{3}\right)^x = 9$	$3^{-x} = 3^2$	$x = -2$	One-to-One
d.	$e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
e.	$\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
f.	$\log x = -1$	$10^{\log x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse

The strategies used in Example 1 are summarized as follows.

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the given equation in a form to use the One-to-One Properties of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

Solving Exponential Equations

EXAMPLE 2 Solving Exponential Equations

Solve each equation. **a.** $e^x = 72$ **b.** $3(2^x) = 42$

Algebraic Solution

a. $e^x = 72$ Write original equation.

$\ln e^x = \ln 72$ Take natural log of each side.

$x = \ln 72$ Inverse Property

$x \approx 4.277$ Use a calculator.

The solution is $\ln 72 \approx 4.277$. Check this solution in the original equation.

b. $3(2^x) = 42$ Write original equation.

$2^x = 14$ Divide each side by 3.

$\log_2 2^x = \log_2 14$ Take log (base 2) of each side.

$x = \log_2 14$ Inverse Property

$x = \dfrac{\ln 14}{\ln 2}$ Change-of-base formula

$x \approx 3.807$ Use a calculator.

The solution is $\log_2 14 \approx 3.807$. Check this solution in the original equation.

Graphical Solution

a. Use a graphing utility to graph the left- and right-hand sides of the equation as $y_1 = e^x$ and $y_2 = 72$ in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features of the graphing utility to approximate the intersection point, as shown in Figure 4.24. So, the approximate solution is 4.277.

b. Use a graphing utility to graph $y_1 = 3(2^x)$ and $y_2 = 42$ in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features to approximate the intersection point, as shown in Figure 4.25. So, the approximate solution is 3.807.

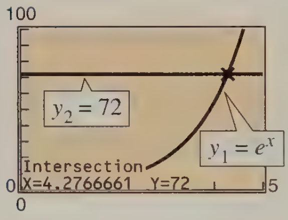


Figure 4.24

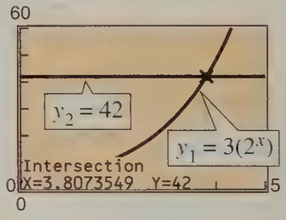


Figure 4.25

EXAMPLE 3 Solving an Exponential Equation

Solve $4e^{2x} = 5$.

Algebraic Solution

$4e^{2x} = 5$ Write original equation.

$e^{2x} = \dfrac{5}{4}$ Divide each side by 4.

$\ln e^{2x} = \ln \dfrac{5}{4}$ Take logarithm of each side.

$2x = \ln \dfrac{5}{4}$ Inverse Property

$x = \dfrac{1}{2} \ln \dfrac{5}{4}$ Solve for x.

$x \approx 0.112$ Use a calculator.

The solution is $\frac{1}{2} \ln \frac{5}{4} \approx 0.112$. Check this solution in the original equation.

Graphical Solution

Rather than graph both sides of the equation you are solving as separate graphs, as you did in Example 2, another way to graphically solve the equation is to first rewrite the equation as $4e^{2x} - 5 = 0$, then use a graphing utility to graph $y = 4e^{2x} - 5$. Use the *zero* or *root* feature or the *zoom* and *trace* features of the graphing utility to approximate the value of x for which $y = 0$. From Figure 4.26, you can see that the zero occurs when $x \approx 0.112$. So, the approximate solution is 0.112.

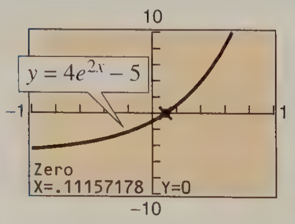


Figure 4.26

EXAMPLE 4 Solving an Exponential Equation

Solve $2(3^{2t-5}) - 4 = 11$.

Solution

$$\begin{aligned} 2(3^{2t-5}) - 4 &= 11 && \text{Write original equation.} \\ 2(3^{2t-5}) &= 15 && \text{Add 4 to each side.} \\ 3^{2t-5} &= \tfrac{15}{2} && \text{Divide each side by 2.} \\ \log_3 3^{2t-5} &= \log_3 \tfrac{15}{2} && \text{Take log (base 3) of each side.} \\ 2t - 5 &= \log_3 \tfrac{15}{2} && \text{Inverse Property} \\ 2t &= 5 + \log_3 7.5 && \text{Add 5 to each side.} \\ t &= \tfrac{5}{2} + \tfrac{1}{2}\log_3 7.5 && \text{Divide each side by 2.} \\ t &\approx 3.417 && \text{Use a calculator.} \end{aligned}$$

The solution is $\frac{5}{2} + \frac{1}{2}\log_3 7.5 \approx 3.417$. Check this in the original equation.

When an equation involves two or more exponential expressions, you can still use a procedure similar to that demonstrated in the previous three examples. However, the algebra is a bit more complicated.

EXAMPLE 5 Solving an Exponential Equation in Quadratic Form

Solve $e^{2x} - 3e^{x} + 2 = 0$.

Algebraic Solution

$$\begin{aligned} e^{2x} - 3e^{x} + 2 &= 0 && \text{Write original equation.} \\ (e^{x})^2 - 3e^{x} + 2 &= 0 && \text{Write in quadratic form.} \\ (e^{x} - 2)(e^{x} - 1) &= 0 && \text{Factor.} \\ e^{x} - 2 &= 0 && \text{Set 1st factor equal to 0.} \\ e^{x} &= 2 && \text{Add 2 to each side.} \\ x &= \ln 2 && \text{Inverse Property} \\ e^{x} - 1 &= 0 && \text{Set 2nd factor equal to 0.} \\ e^{x} &= 1 && \text{Add 1 to each side.} \\ x &= \ln 1 && \text{Inverse Property} \\ x &= 0 && \text{Simplify.} \end{aligned}$$

The solutions are $\ln 2 \approx 0.693$ and 0. Check these in the original equation.

Graphical Solution

Use a graphing utility to graph $y = e^{2x} - 3e^{x} + 2$. Use the *zero* or *root* feature or the *zoom* and *trace* features of the graphing utility to approximate the values of x for which $y = 0$. In Figure 4.27, you can see that the zeros occur when $x = 0$ and when $x \approx 0.693$. So, the approximate solutions are 0 and 0.693.

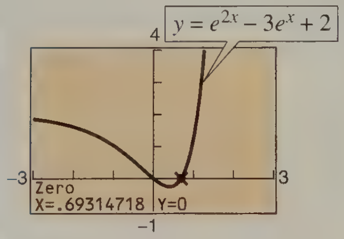


Figure 4.27

Solving Logarithmic Equations

To solve a logarithmic equation, you can write it in exponential form.

$$\begin{aligned} \ln x &= 3 && \text{Logarithmic form} \\ e^{\ln x} &= e^3 && \text{Exponentiate each side.} \\ x &= e^3 && \text{Exponential form} \end{aligned}$$

This procedure is called *exponentiating* both sides of an equation. It is applied after the logarithmic expression has been isolated.

EXAMPLE 6 Solving Logarithmic Equations

Solve each logarithmic equation.

a. $\ln x = 2$ **b.** $\log_3(5x - 1) = \log_3(x + 7)$

Solution

a.

$$\begin{aligned} \ln x &= 2 && \text{Write original equation.} \\ e^{\ln x} &= e^2 && \text{Exponentiate each side.} \\ x &= e^2 && \text{Inverse Property} \\ x &\approx 7.389 && \text{Use a calculator.} \end{aligned}$$

The solution is $e^2 \approx 7.389$. Check this in the original equation.

b.

$$\begin{aligned} \log_3(5x - 1) &= \log_3(x + 7) && \text{Write original equation.} \\ 5x - 1 &= x + 7 && \text{One-to-One Property} \\ 4x &= 8 && \text{Add } -x \text{ and 1 to each side.} \\ x &= 2 && \text{Divide each side by 4.} \end{aligned}$$

The solution is 2. Check this in the original equation.

EXAMPLE 7 Solving a Logarithmic Equation

Solve $5 + 2 \ln x = 4$.

Algebraic Solution

$$\begin{aligned} 5 + 2\ln x &= 4 && \text{Write original equation.} \\ 2\ln x &= -1 && \text{Subtract 5 from each side.} \\ \ln x &= -\tfrac{1}{2} && \text{Divide each side by 2.} \\ e^{\ln x} &= e^{-1/2} && \text{Exponentiate each side.} \\ x &= e^{-1/2} && \text{Inverse property} \\ x &\approx 0.607 && \text{Use a calculator.} \end{aligned}$$

The solution is $e^{-1/2} \approx 0.607$. Check this in the original equation.

Graphical Solution

Use a graphing utility to graph $y_1 = 5 + 2 \ln x$ and $y_2 = 4$ in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features to approximate the intersection point, as shown in Figure 4.28. So, the approximate solution is 0.607.

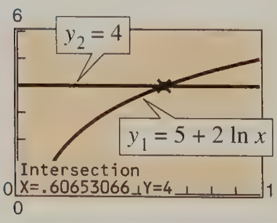


Figure 4.28

EXAMPLE 8 Solving a Logarithmic Equation

Solve $2 \log_5 3x = 4$.

Solution

$2 \log_5 3x = 4$ Write original equation.

$\log_5 3x = 2$ Divide each side by 2.

$5^{\log_5 3x} = 5^2$ Exponentiate each side (base 5).

$3x = 25$ Inverse Property.

$x = \frac{25}{3}$ Divide each side by 3.

The solution is $\frac{25}{3}$. Check this in the original equation. Or, perform a graphical check by graphing

$$y_1 = 2 \log_5 3x = 2\left(\frac{\log_{10} 3x}{\log_{10} 5}\right) \quad \text{and} \quad y_2 = 4$$

in the same viewing window. The two graphs should intersect when $x = \frac{25}{3}$ and $y = 4$, as shown in Figure 4.29.

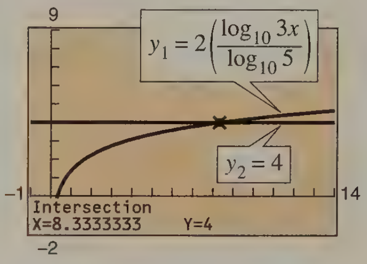


Figure 4.29

Because the domain of a logarithmic function generally does not include all real numbers, you should be sure to check for extraneous solutions of logarithmic equations, as shown in the next example.

EXAMPLE 9 Checking for Extraneous Solutions

Solve for x in the equation $\ln(x - 2) + \ln(2x - 3) = 2 \ln x$.

Algebraic Solution

$\ln(x - 2) + \ln(2x - 3) = 2 \ln x$ Write original equation.

$\ln[(x - 2)(2x - 3)] = \ln x^2$ Use properties of logarithms.

$\ln(2x^2 - 7x + 6) = \ln x^2$

$2x^2 - 7x + 6 = x^2$ One-to-One Property

$x^2 - 7x + 6 = 0$ Write in general form.

$(x - 6)(x - 1) = 0$ Factor.

$x - 6 = 0 \Rightarrow x = 6$ Set 1st factor equal to 0.

$x - 1 = 0 \Rightarrow x = 1$ Set 2nd factor equal to 0.

Finally, by checking these two "solutions" in the original equation, you can conclude that $x = 1$ is not valid. This is because when $x = 1$, $\ln(x - 2) + \ln(2x - 3) = \ln(-1) + \ln(-1)$, which is invalid because -1 is not in the domain of the natural log function. So, the only solution is 6.

Graphical Solution

First rewrite the original equation as $\ln(x - 2) + \ln(2x - 3) - 2 \ln x = 0$. Then use a graphing utility to graph $y = \ln(x - 2) + \ln(2x - 3) - 2 \ln x$, as shown in Figure 4.30. Use the *zero* or *root* feature or the *zoom* and *trace* features of the graphing utility to determine that 6 is an approximate solution. You can verify that 6 is an exact solution by substituting $x = 6$ in the original equation.

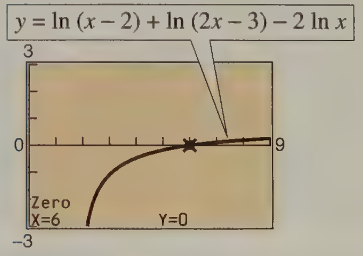


Figure 4.30

EXAMPLE 10 The Change-of-Base Formula

Prove the change-of-base formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$.

Solution

Begin by letting $y = \log_a x$ and writing the equivalent exponential form $a^y = x$. Now, taking the logarithms *with base b* of both sides produces the following.

$$\begin{aligned} \log_b a^y &= \log_b x \\ y \log_b a &= \log_b x \\ y &= \frac{\log_b x}{\log_b a} \\ \log_a x &= \frac{\log_b x}{\log_a a} \end{aligned}$$

STUDY TIP

To solve exponential equations, it is useful to first isolate the exponential expression, then take the logarithm of both sides and solve for the variable.

To solve logarithmic equations, condense the logarithmic part into a single logarithm, then rewrite in exponential form and solve for the variable.

Approximating Solutions

Equations that involve combinations of algebraic functions, exponential functions, and/or logarithmic functions can be very difficult to solve by algebraic procedures. Here again, you can take advantage of a graphing utility.

EXAMPLE 11 Approximating the Solution of an Equation

Approximate the solution of $\ln x = x^2 - 2$.

Solution

To begin, write the equation so that all terms on one side are equal to 0.

$$\ln x - x^2 + 2 = 0$$

Then use a graphing utility to graph

$$y = -x^2 + 2 + \ln x$$

as shown in Figure 4.31. From this graph, you can see that the equation has two solutions. Next, using the *zero* or *root* feature or the *zoom* and *trace* features, you can approximate the two solutions to be 0.138 and 1.564.

Check

$$\begin{aligned} \ln x &= x^2 - 2 && \text{Write original equation.} \\ \ln(0.138) &\overset{?}{\approx} (0.138)^2 - 2 && \text{Substitute 0.138 for } x. \\ -1.9805 &\approx -1.9810 && \text{Solution checks. } \checkmark \\ \ln(1.564) &\overset{?}{\approx} (1.564)^2 - 2 && \text{Substitute 1.564 for } x. \\ 0.4472 &\approx 0.4461 && \text{Solution checks. } \checkmark \end{aligned}$$

So, the two solutions 0.138 and 1.564 seem reasonable.

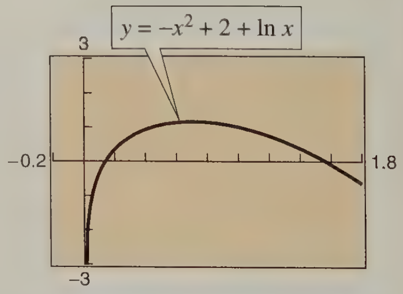


Figure 4.31

Applications

EXAMPLE 12 Doubling an Investment

You have deposited \$500 in an account that pays 6.75% interest, compounded continuously. How long will it take your money to double?

Solution

Using the formula for continuous compounding, you can find that the balance in the account is

$$\begin{aligned} A &= Pe^{rt} \\ &= 500e^{0.0675t}. \end{aligned}$$

To find the time required for the balance to double, let $A = 1000$, and solve the resulting equation for t.

$$\begin{aligned} 500e^{0.0675t} &= 1000 && \text{Substitute 1000 for } A. \\ e^{0.0675t} &= 2 && \text{Divide each side by 500.} \\ \ln e^{0.0675t} &= \ln 2 && \text{Take natural log of each side.} \\ 0.0675t &= \ln 2 && \text{Inverse Property} \\ t &= \frac{\ln 2}{0.0675} && \text{Divide each side by 0.0675.} \\ t &\approx 10.27 && \text{Use a calculator.} \end{aligned}$$

The balance in the account will double after approximately 10.27 years. This result is demonstrated graphically in Figure 4.32.

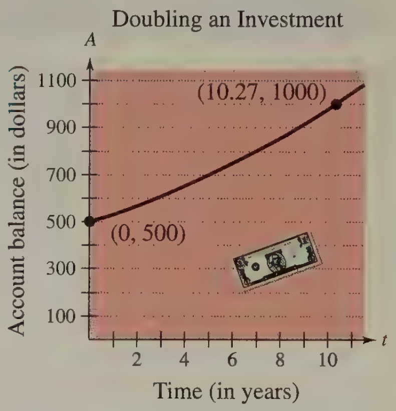


Figure 4.32

EXAMPLE 13 Consumer Price Index for Sugar

From 1970 to 1997, the Consumer Price Index (CPI) value y for a fixed amount of sugar for the year t can be modeled by the equation

$$y = -171.8 + 87.1 \ln t$$

where $t = 10$ represents 1970 (see Figure 4.33). During which year did the price of sugar reach 4.5 times its 1970 price of 28.8 on the CPI? (Source: U.S. Bureau of Labor Statistics)

Solution

$$\begin{aligned} -171.8 + 87.1 \ln t &= y && \text{Write original equation.} \\ -171.8 + 87.1 \ln t &= 129.6 && \text{Substitute } (4.5)(28.8) = 129.6 \text{ for } y. \\ 87.1 \ln t &= 301.4 && \text{Add 171.8 to each side.} \\ \ln t &= 3.460 && \text{Divide each side by 87.1.} \\ e^{\ln t} &= e^{3.460} && \text{Exponentiate each side.} \\ t &\approx 31.8 && \text{Inverse Property} \end{aligned}$$

The solution is 31.8 years. Because $t = 10$ represents 1970, it follows that the price of sugar reached 4.5 times its 1970 price in late 1991.

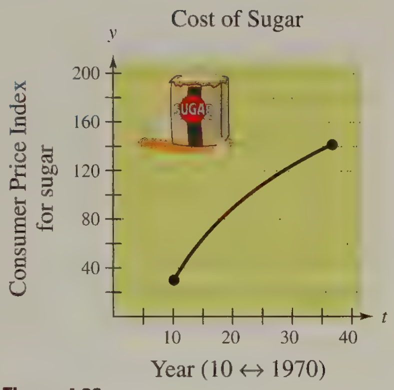


Figure 4.33

4.4 Exercises

In Exercises 1–8, determine whether each x-value is a solution of the equation.

1. $4^{2x-7} = 64$

(a) $x = 5$

(b) $x = 2$

2. $2^{3x+1} = 32$

(a) $x = -1$

(b) $x = 2$

3. $3e^{x+2} = 75$

(a) $x = -2 + e^{25}$

(b) $x = -2 + \ln 25$

(c) $x \approx 1.2189$

4. $-4e^{x-1} = -60$

(a) $x = 1 + \ln 15$

(b) $x \approx 3.7081$

(c) $x = \ln 16$

5. $\log_4(3x) = 3$

(a) $x \approx 20.3560$

(b) $x = -4$

(c) $x = \frac{64}{3}$

6. $\log_6\left(\frac{5}{3}x\right) = 2$

(a) $x \approx 20.2882$

(b) $x = \frac{108}{5}$

(c) $x = 7.2$

7. $\ln(x - 1) = 3.8$

(a) $x = 1 + e^{3.8}$

(b) $x \approx 45.7012$

(c) $x = 1 + \ln 3.8$

8. $\frac{1}{5}\ln(2 + x) = \frac{1}{2}$

(a) $x = e^{2.5} - 2$

(b) $x \approx \frac{4073}{400}$

(c) $x = \frac{1}{2}$

In Exercises 9–14, use a graphing utility to graph f and g in the same viewing window. Approximate the point of intersection of the graphs of f and g. Then solve the equation $f(x) = g(x)$ algebraically.

9. $f(x) = 2^x$
$g(x) = 8$

10. $f(x) = 27^x$
$g(x) = 9$

11. $f(x) = \log_3 x$
$g(x) = 2$

12. $f(x) = 3\log_5 x$
$g(x) = 6$

13. $f(x) = \ln(x - 4)$
$g(x) = 0$

14. $f(x) = \ln e^{x-2}$
$g(x) = 3x + 2$

In Exercises 15–36, solve for x.

15. $4^x = 16$

16. $3^x = 243$

17. $5^x = 625$

18. $7^x = \frac{1}{49}$

19. $8^x = 4$

20. $\left(\frac{1}{2}\right)^x = 32$

21. $\left(\frac{1}{4}\right)^x = 64$

22. $\left(\frac{3}{4}\right)^x = \frac{27}{64}$

23. $3^{x-1} = 27$

24. $\ln x - \ln 2 = 0$

25. $\ln x - \ln 5 = 0$

26. $e^x = 2$

27. $e^x = 4$

28. $\ln x = -1$

29. $\ln x = -7$

30. $\log_4 x = 3$

31. $\log_x 625 = 4$

32. $\log_{10} x - 2 = 0$

33. $\log_{10} x = -1$

34. $\log_{10} x = -\frac{1}{2}$

35. $\ln(2x - 1) = 0$

36. $\ln(3x + 5) = 8$

In Exercises 37–42, simplify the expression.

37. $\ln e^{x^2}$

38. $\ln e^{2x-1}$

39. $e^{\ln(5x+2)}$

40. $-1 + \ln e^{2x}$

41. $e^{\ln x^2}$

42. $-8 + e^{\ln x^3}$

In Exercises 43–60, solve the exponential equation algebraically. Round your result to three decimal places. Use a graphing utility to verify your answer.

43. $10^x = 570$

44. $6^{5x} = 3000$

45. $e^x = 10$

46. $4e^{2x} = 40$

47. $5^{-t/2} = 0.20$

48. $4^{-3t} = 0.10$

49. $2^{3-x} = 565$

50. $8^{-2-x} = 431$

51. $500e^{-x} = 300$

52. $1000e^{-4x} = 75$

53. $7 - 2e^x = 5$

54. $-14 + 3e^x = 11$

55. $e^{2x} - 4e^x - 5 = 0$

56. $e^{2x} - 5e^x + 6 = 0$

57. $50(120 - e^{x/2}) = 600$

58. $\dfrac{525}{1 + e^{-x}} = 275$

59. $\left(1 + \dfrac{0.10}{12}\right)^{12t} = 2$

60. $\left(16 + \dfrac{0.878}{26}\right)^{3t} = 30$

In Exercises 61–64, let $f(x)$ represent the left side of the equation. Complete the table to obtain a rough estimate of the solution, and then use a graphing utility to graph both sides of the equation to obtain a better estimate of the solution. Round your result to three decimal places.

61. $e^{3x} = 12$

x	0.6	0.7	0.8	0.9	1.0
$f(x)$					

62. $e^{2x} = 50$

x	1.6	1.7	1.8	1.9	2.0
$f(x)$					

63. $20(100 - e^{x/2}) = 500$

x	5	6	7	8	9
$f(x)$					

64. $\dfrac{400}{1 + e^{-x}} = 350$

x	0	1	2	3	4
$f(x)$					

In Exercises 65–74, use the *zero* or *root* feature or the *zoom* and *trace* features of a graphing utility to find an approximate solution of the exponential equation accurate to three decimal places.

65. $2^{3x} = 50$

66. $4^{-x/2} = 0.10$

67. $2^{-3x} = 0.90$

68. $8(10^{3x}) = 12$

69. $5(10^{x-6}) = 7$

70. $8(3^{6-x}) = 40$

71. $\left(1 + \dfrac{0.065}{365}\right)^{365t} = 4$

72. $\left(4 - \dfrac{2.471}{40}\right)^{9t} = 21$

73. $\dfrac{3000}{2 + e^{2x}} = 2$

74. $\dfrac{119}{e^{6x} - 14} = 7$

In Exercises 75–78, use a graphing utility to graph the function and approximate its zero accurate to three decimal places.

75. $g(x) = 6e^{1-x} - 25$

76. $f(x) = 3e^{3x/2} - 962$

77. $g(t) = e^{0.09t} - 3$

78. $h(t) = e^{0.125t} - 8$

In Exercises 79–98, solve the logarithmic equation algebraically. Round the result to three decimal places. Verify your answer using a graphing utility.

79. $\ln x = -3$

80. $\ln x = 2$

81. $\ln 4x = 2.1$

82. $\ln 4x = 1$

83. $2 \ln 3x = 19$

84. $2 \ln x = 7$

85. $\log_{10}(z - 3) = 2$

86. $\log_{10} x^2 = 6$

87. $7 \log_4(0.6x) = 12$

88. $4 \log_{10}(x - 6) = 11$

89. $\ln \sqrt{x + 2} = 1$

90. $\ln \sqrt{x - 8} = 5$

91. $\ln(x + 1)^2 = 2$

92. $\ln(x^2 + 1) = 8$

93. $\log_4 x - \log_4(x - 1) = \frac{1}{2}$

94. $\log_3 x + \log_3(x^2 - 8) = \log_3 8x$

95. $\ln(x + 5) = \ln(x - 1) - \ln(x + 1)$

96. $\ln(x + 1) - \ln(x - 2) = \ln x^2$

97. $\log_{10} 8x - \log_{10}(1 + \sqrt{x}) = 2$

98. $\log_{10} 4x - \log_{10}(12 + \sqrt{x}) = 2$

In Exercises 99–102, let $f(x)$ represent the left side of the equation. Complete the table to obtain a rough estimate of the solution, and then use a graphing utility to graph both sides of the equation to obtain a better estimate of the solution. Round your result to three decimal places.

99. $\ln 2x = 2.4$

x	2	3	4	5	6
$f(x)$					

100. $3 \ln 5x = 10$

x	4	5	6	7	8
$f(x)$					

101. $6 \log_3(0.5x) = 11$

x	12	13	14	15	16
$f(x)$					

102. $5 \log_{10}(x - 2) = 11$

x	150	155	160	165	170
$f(x)$					

In Exercises 103–110, use the *zero* or *root* feature or the *zoom* and *trace* features of a graphing utility to find an approximate solution of the logarithmic equation accurate to three decimal places.

103. $\log_{10}(z - 4) = 1$

104. $\log_{10} x^2 = 4$

105. $3 \ln x = 5$

106. $\ln 4x = 3$

107. $\ln x + \ln(x - 3) = 1$

108. $\log_2 x + \log_2(x + 5) = \log_2(x + 4)$

109. $\ln(x - 5) = \ln(x - 3) - \ln(x + 3)$

110. $\ln x + \ln(x^2 + 4) = 10$

In Exercises 111–116, use a graphing utility to approximate the point of intersection of the graphs. Round your result to three decimal places.

111. $y_1 = 7$
$y_2 = 2^x$

112. $y_1 = -4$
$y_2 = -3^{x+1} - 2$

113. $y_1 = 8$
$y_2 = 4e^{-0.2x}$

114. $y_1 = 500$
$y_2 = 1500e^{-x/2}$

115. $y_1 = 3$
$y_2 = \ln x$

116. $y_1 = 10$
$y_2 = 4\ln(x - 2)$

Compound Interest **In Exercises 117 and 118, find the time required for a \$1000 investment to (a) double at interest rate r, compounded continuously, and (b) triple at interest rate r, compounded continuously.**

117. $r = 0.085$

118. $r = 0.12$

119. ***Average Heights*** The percentages of American males and females between the ages of 18 and 24 who are at least x inches tall are

$$m(x) = \frac{100}{1 + e^{-0.6114(x-69.71)}} \qquad \text{Males}$$

$$f(x) = \frac{100}{1 + e^{-0.66607(x-64.51)}} \qquad \text{Females}$$

where m and f are the percentages and x is the height in inches. (Source: U.S. National Center for Health Statistics)

(a) Use a graphing utility to graph the two functions in the same viewing window.

(b) Use the graph to determine the horizontal asymptotes of the functions.

(c) What is the median height for each sex?

120. ***Human Memory Model*** In a group project in learning theory, a mathematical model for the proportion P of correct responses after n trials was found to be

$$P = \frac{0.83}{1 + e^{-0.2n}}.$$

(a) Use a graphing utility to graph the function.

(b) Use the graph to determine the horizontal asymptotes of the function. Interpret the meaning of the upper asymptote in the context of the problem.

(c) After how many trials will 60% of the responses be correct?

121. ***Demand Function*** The demand equation for a camera is

$$p = 500 - 0.5(e^{0.004x}).$$

Find the demands x for prices of (a) $p = \$350$ and (b) $p = \$300$.

122. ***Demand Function*** The demand equation for a camcorder is

$$p = 5000\left(1 - \frac{4}{4 + e^{-0.002x}}\right).$$

Find the demands x for prices of (a) $p = \$600$ and (b) $p = \$400$.

123. ***Forest Yield*** The yield V (in millions of cubic feet per acre) for a forest at age t years is

$$V = 6.7e^{-48.1/t}.$$

(a) Use a graphing utility to graph the function.

(b) Determine the horizontal asymptote of the function. Interpret its meaning in the context of the problem.

(c) Find the time necessary to obtain a yield of 1.3 million cubic feet.

124. ***Trees per Acre*** The number of trees per acre N of a certain species is approximated by the model

$$N = 68(10^{-0.04x}), \quad 5 \le x \le 40$$

where x is the average diameter of the trees (in inches) three feet above the ground. Use the model to approximate the average diameter of the trees in a test plot when $N = 21$.

125. ***Data Analysis*** An object at a temperature of 160°C is removed from a furnace and placed in a room at 20°C. The temperature T of the object is measured each hour h and recorded in the table.

h	0	1	2	3	4	5
T	160°	90°	56°	38°	29°	24°

A model for this data is

$$T = 20[1 + 7(2^{-h})].$$

(a) Use a graphing utility to plot the data points and graph the model in the same viewing window.

(b) Use the graph to identify the horizontal asymptote of the model and interpret the asymptote in the context of the problem.

(c) Approximate the time when the temperature of the object is 100°C.

126. ***Automobiles*** Automobiles are designed with crumple zones that help protect their occupants in crashes. The crumple zones allow the occupants to move short distances when the automobiles come to abrupt stops. The greater the distance moved, the less g's the crash victims experience. (One g is equal to the acceleration due to gravity. For very short periods of time, humans have withstood as much as 40 g's.) In crash tests with vehicles moving at 90 kilometers per hour, analysts measured the number of g's experienced during deceleration y by crash dummies that were permitted to move x meters during impact. The data is shown in the table.

x	0.2	0.4	0.6	0.8	1
y	158	80	53	40	32

(a) Use the regression capabilities of a graphing utility to find a natural logarithmic model for the data. A second model for the data is

$$y = -3.00 + 11.88 \ln x + (36.94/x).$$

Graph both models with the data points in the same viewing window. Which model do you think is a better choice? Comment on accuracy and simplicity.

(b) Use both models to estimate the distance traveled during impact if the passenger deceleration must not exceed 30 g's. Comment on the difference in distances traveled given by the two models and the practical implications of choosing a model when designing the crumple zones.

127. ***Comparing Mathematical Models*** The table shows the number y (in millions) of single compact discs (CDs) shipped annually by manufacturers from 1993 through 1997, where $x = 3$ represents 1993. (Source: Recording Industry Association of America)

x	3	4	5	6	7
y	7.8	9.3	21.5	43.2	66.7

(a) Create a scatter plot of the data. Find a linear model for the data, and add its graph to your scatter plot. Using this model, approximate the year that corresponds to an annual shipment of single CDs of 100 million.

(b) Create a new table giving values for $\ln x$ and $\ln y$ and create a scatter plot of this transformed data. Use the method illustrated in Example 8 in Section 4.3 to find a model for the transformed data, and add its graph to your scatter plot. According to this model, approximate the year that corresponds to an annual shipment of single CDs of 100 million.

(c) Solve the model in part (b) for y, and add its graph to your scatter plot in part (a). Which model better fits the original data? Which model will better predict future shipping levels? Explain.

Synthesis

True or False? **In Exercises 128 and 129, determine whether the statement is true or false. Justify your answer.**

128. You can approximate the solution to the equation $\frac{2}{3}e^x = 42$ by graphing $y = \frac{2}{3}e^x - 42$ and finding its x-intercept.

129. A logarithmic equation can have, at most, one extraneous solution.

130. ***Think About It*** Is the time required for an investment to quadruple twice as long as the time for it to double? Give a reason for your answer and give an example to verify your answer algebraically.

131. ***Writing*** Write a paragraph explaining whether or not the time required for an investment to double is dependent on the size of the investment.

Review

In Exercises 132–135, sketch the graph of the function.

132. $f(x) = 3^{x+3} - 5$

133. $f(x) = -3^{-x-3} + 5$

134. $f(x) = -2^{x+2} - 6$

135. $f(x) = \left(\frac{1}{2}\right)^{-x} + 3$

In Exercises 136–139, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

136. $\log_7 11$

137. $\log_3 22$

138. $\log_9 6$

139. $\log_{21} 140$

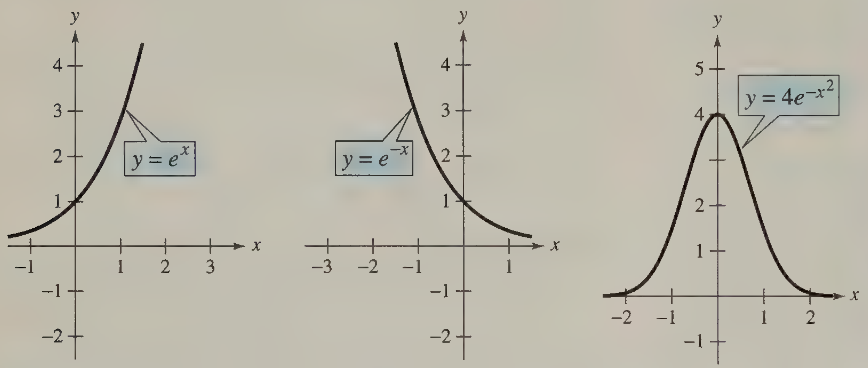
4.5 Exponential and Logarithmic Models

Introduction

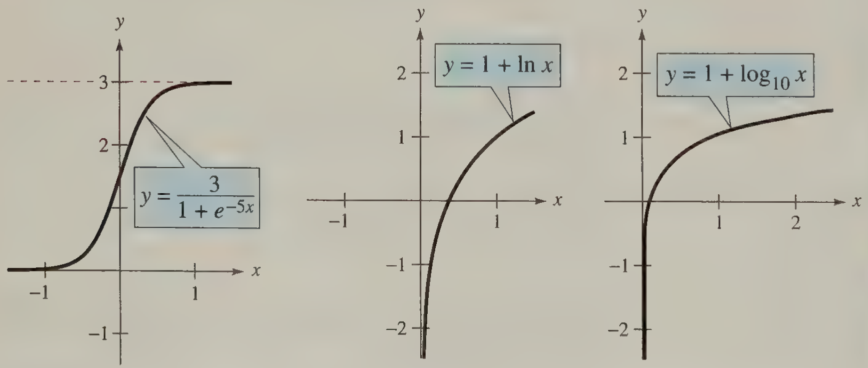
The five most common types of mathematical models involving exponential functions and logarithmic functions are as follows.

1. **Exponential growth model:** $y = ae^{bx}, \quad b > 0$
2. **Exponential decay model:** $y = ae^{-bx}, \quad b > 0$
3. **Gaussian model:** $y = ae^{-(x-b)^2/c}$
4. **Logistic growth model:** $y = \dfrac{a}{1 + be^{-rx}}$
5. **Logarithmic models:** $y = a + b \ln x, \quad y = a + b \log_{10} x$

The basic shapes of these graphs are shown in Figure 4.34.



Exponential Growth Model *Exponential Decay Model* *Gaussian Model*



Logistic Growth Model *Natural Logarithmic Model* *Common Logarithmic Model*

Figure 4.34

You can often gain quite a bit of insight into a situation modeled by an exponential or logarithmic function by identifying and interpreting the function's asymptotes. Use the graphs in Figure 4.34 to identify the asymptotes of each function.

What You Should Learn:

- How to recognize the five most common types of models involving exponential or logarithmic functions
- How to use exponential growth and decay functions to model and solve real-life problems
- How to use Gaussian functions to model and solve real-life problems
- How to use logistic growth functions to model and solve real-life problems
- How to use logarithmic functions to model and solve real-life problems
- How to fit exponential and logarithmic models to sets of data

Why You Should Learn It:

Exponential and logarithmic functions can be used to model and solve a variety of business applications. In Exercise 44 on page 350, you will compare an exponential decay model and a linear model for the depreciation of a computer over three years.

Spencer Grant/PhotoEdit

Exponential Growth and Decay

EXAMPLE 1 Population Growth

Estimates of the world population (in millions) from 1992 through 2000 are shown in the table. The scatter plot of the data is shown in Figure 4.35. (Source: U.S. Bureau of the Census)

Year	1992	1993	1994	1995	1996	1997	1998	1999	2000
Population	5445	5527	5607	5688	5767	5847	5926	6005	6083

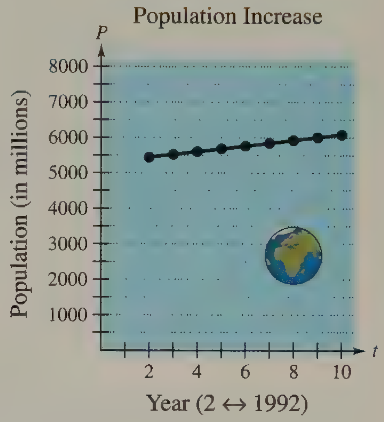


Figure 4.35

An exponential growth model that approximates this data is

$$P = 5303e^{0.013838t}, \qquad 2 \le t \le 10$$

where P is the population (in millions) and $t = 2$ represents 1992. Compare the values given by the model with the estimates given by the U.S. Bureau of the Census. According to this model, what year corresponds to a world population of 6.5 billion?

Algebraic Solution

The following table compares the two sets of population figures.

Year	1992	1993	1994	1995	1996	1997	1998	1999	2000
Population	5445	5527	5607	5688	5767	5847	5926	6005	6083
Model	5452	5528	5605	5683	5762	5842	5924	6006	6090

To find the year for which the world population is 6.5 billion, let $P = 6500$ in the model and solve for t.

$$5303e^{0.013838t} = P \qquad \text{Write original model.}$$

$$5303e^{0.013838t} = 6500 \qquad \text{Substitute 6500 for } P.$$

$$e^{0.013838t} \approx 1.2257 \qquad \text{Divide each side by 5303.}$$

$$\ln e^{0.013838t} \approx \ln 1.2257 \qquad \text{Take natural log of each side.}$$

$$0.013838t \approx 0.2035 \qquad \text{Inverse Property}$$

$$t \approx 14.71 \qquad \text{Divide each side by 0.013838.}$$

According to the model, the year 2004 corresponds to a population of 6.5 billion.

Graphical Solution

Use a graphing utility to graph the model $y = 5303e^{0.013838x}$ and the data in the same viewing window. You can see in Figure 4.36 that the model appears to closely fit the data.

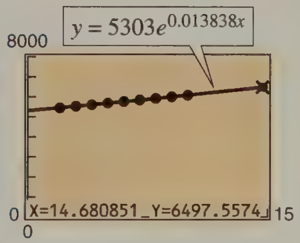


Figure 4.36

Use the *zoom* and *trace* features of the graphing utility to find that the approximate value of x for which $y = 6500$ is $x \approx 14.71$. So, according to the model, the year 2004 corresponds to a world population of 6.5 billion.

An exponential model increases (or decreases) by the same percent each year. What is the annual percent increase for the model above?

In Example 1, you were given the exponential growth model. But suppose this model had not been given; how could you have found such a model? One technique for doing this is demonstrated in Example 2.

EXAMPLE 2 Modeling Population Growth

In a research experiment, a population of fruit flies is increasing in accordance with the law of exponential growth. After 2 days there are 100 flies, and after 4 days there are 300 flies. How many flies will there be after 5 days?

Solution

Let y be the number of flies at time t. From the given information, you know that $y = 100$ when $t = 2$ and $y = 300$ when $t = 4$. Substituting this information into the model $y = ae^{bt}$ produces

$$100 = ae^{2b} \quad \text{and} \quad 300 = ae^{4b}.$$

To solve for b, solve for a in the first equation.

$$100 = ae^{2b} \quad \Rightarrow \quad a = \frac{100}{e^{2b}} \qquad \text{Solve for } a \text{ in the first equation.}$$

Then substitute the result into the second equation.

$$\begin{aligned}
300 &= ae^{4b} && \text{Write second equation.}\\
300 &= \left(\frac{100}{e^{2b}}\right)e^{4b} && \text{Substitute } 100/e^{2b} \text{ for } a.\\
\frac{300}{100} &= e^{2b} && \text{Divide each side by 100.}\\
\ln\frac{300}{100} &= \ln e^{2b} && \text{Take natural log of each side.}\\
\ln 3 &= 2b && \text{Inverse Property}\\
\frac{1}{2}\ln 3 &= b && \text{Solve for } b.
\end{aligned}$$

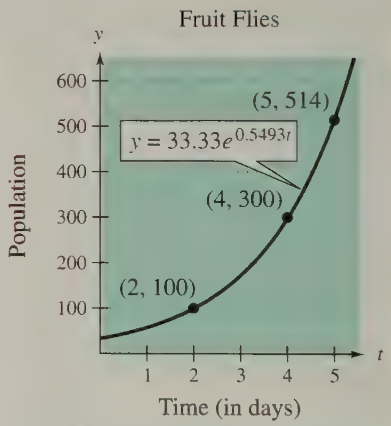


Figure 4.37

Using $b = \frac{1}{2}\ln 3 \approx 0.5493$, you can determine that

$$\begin{aligned}
a &= \frac{100}{e^{2[(1/2)\ln 3]}} && \text{Substitute } \tfrac{1}{2}\ln 3 \text{ for } b.\\
&= \frac{100}{e^{\ln 3}} && \text{Simplify.}\\
&= \frac{100}{3} && \text{Inverse Property}\\
&\approx 33.33 && \text{Simplify.}
\end{aligned}$$

So, the exponential growth model is

$$y = 33.33e^{0.5493t},$$

as shown in Figure 4.37. This implies that after 5 days, the population is

$$y = 33.33e^{0.5493(5)} \approx 520 \text{ flies.}$$

In living organic material, the ratio of the content of radioactive carbon (carbon 14) to the content of nonradioactive carbon (carbon 12) is about 1 to 10^{12}. When organic material dies, its carbon 12 content remains fixed, whereas its radioactive carbon 14 begins to decay with a half-life of about 5700 years. To estimate the age of dead organic material, scientists use the following formula, which denotes the ratio of carbon 14 to carbon 12 present at any time t (in years).

$$R = \frac{1}{10^{12}}e^{-t/8223}$$

The graph of R is shown in Figure 4.38. Note that R decreases as t increases.

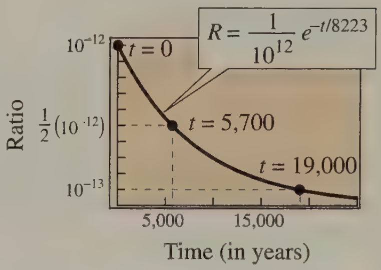


Figure 4.38

EXAMPLE 3 Carbon Dating

The ratio of carbon 14 to carbon 12 in a newly discovered fossil is

$$R = \frac{1}{10^{13}}.$$

Estimate the age of the fossil.

Algebraic Solution

In the carbon dating model, substitute the given value of R to obtain the following.

$$\frac{1}{10^{12}}e^{-t/8223} = R \qquad \text{Write original formula.}$$

$$\frac{e^{-t/8223}}{10^{12}} = \frac{1}{10^{13}} \qquad \text{Substitute } \frac{1}{10^{13}} \text{ for } R.$$

$$e^{-t/8223} = \frac{1}{10} \qquad \text{Multiply each side by } 10^{12}.$$

$$\ln e^{-t/8223} = \ln\frac{1}{10} \qquad \text{Take natural log of each side.}$$

$$-\frac{t}{8223} \approx -2.3026 \qquad \text{Inverse Property}$$

$$t \approx 18{,}934 \qquad \text{Multiply each side by } -8223.$$

So, to the nearest thousand years, you can estimate the age of the fossil to be 19,000 years.

Graphical Solution

Use a graphing utility to graph the formula for the ratio of carbon 14 to carbon 12 at any time t as

$$y_1 = \frac{1}{10^{12}}e^{-x/8223}.$$

In the same viewing window, graph $y_2 = 1/(10^{13})$. Use the *intersect* feature or the *zoom* and *trace* features of the graphing utility to estimate that $x \approx 18{,}934$ when $y = 1/(10^{13})$, as shown in Figure 4.39.

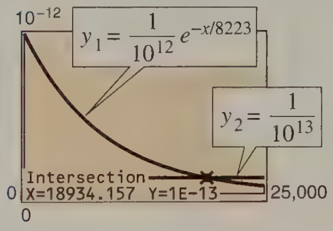

10^{-12}
$y_1 = \frac{1}{10^{12}}e^{-x/8223}$
$y_2 = \frac{1}{10^{13}}$
Intersection
X=18934.157 Y=1E-13
0
25,000
0

Figure 4.39

So, to the nearest thousand years, you can estimate the age of the fossil to be 19,000 years.

The carbon dating model in Example 3 assumed that the carbon 14 to carbon 12 ratio was 1 part in 10,000,000,000,000. Suppose an error in measurement occurred and the actual ratio was only 1 part in 8,000,000,000,000. The fossil age corresponding to the actual ratio would then be approximately 17,000 years. Try checking this result.

Gaussian Models

As mentioned at the beginning of this section, Gaussian models are of the form

$$y = ae^{-(x-b)^2/c}.$$

This type of model is commonly used in probability and statistics to represent populations that are **normally distributed.** For *standard* normal distributions, the model takes the form

$$y = \frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}$$

where σ is the standard deviation (σ is the lowercase Greek letter sigma). The graph of a Gaussian model is called a **bell-shaped curve.** Try assigning a value to σ and sketching a normal distribution curve with a graphing utility. Can you see why it is called a bell-shaped curve?

EXAMPLE 4 SAT Scores

In 1997, the Scholastic Aptitude Test (SAT) math scores for college-bound seniors roughly followed the normal distribution

$$y = 0.0036e^{-(x-511)^2/25{,}088}, \qquad 200 \le x \le 800$$

where x is the SAT score for mathematics. Use a graphing utility to graph this function. From the graph, estimate the average SAT score. (Source: College Board)

Solution

The graph of the function is given in Figure 4.40. On this bell-shaped curve, the maximum value of the curve represents the average score. Using the *maximum* feature or the *zoom* and *trace* features, you can see that the average mathematics score for college-bound seniors in 1997 was 511.

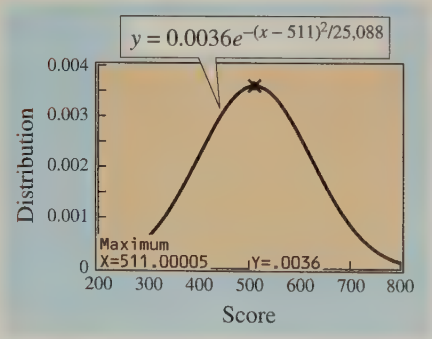


Figure 4.40

In Example 4, note that 50% of the seniors who took the test received a score lower than 511.

Logistic Growth Models

Some populations initially have rapid growth, followed by a declining rate of growth, as indicated by the graph in Figure 4.41. One model for describing this type of growth pattern is the **logistic curve** given by the function

$$y = \frac{a}{1 + be^{-rx}}$$

where y is the population size and x is the time. An example is a bacteria culture allowed to grow initially under ideal conditions and then under less favorable conditions that inhibit growth. A logistic growth curve is also called a **sigmoidal curve.**

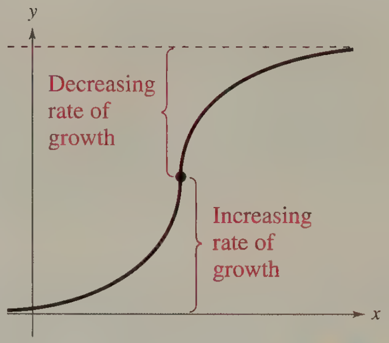


Figure 4.41 *Logistic Curve*

EXAMPLE 5 Spread of a Virus

On a college campus of 5000 students, one student returns from vacation with a contagious flu virus. The spread of the virus is modeled by

$$y = \frac{5000}{1 + 4999e^{-0.8t}}, \qquad t \geq 0$$

where y is the total number infected after t days. The college will cancel classes when 40% or more of the students are ill.

a. How many students are infected after 5 days?

b. After how many days will the college cancel classes?

Algebraic Solution

a. After 5 days, the number infected is

$$y = \frac{5000}{1 + 4999e^{-0.8(5)}} = \frac{5000}{1 + 4999e^{-4}} \approx 54.$$

b. In this case, the number of infected students is $(0.40)(5000) = 2000$. Therefore, you solve for t in the following equation.

$$2000 = \frac{5000}{1 + 4999e^{-0.8t}}$$

$$1 + 4999e^{-0.8t} = 2.5$$

$$e^{-0.8t} \approx \frac{1.5}{4999}$$

$$\ln e^{-0.8t} \approx \ln \frac{1.5}{4999}$$

$$-0.8t \approx \ln \frac{1.5}{4999}$$

$$t = -\frac{1}{0.8} \ln \frac{1.5}{4999}$$

$$t \approx 10.14$$

So, after about 10 days, at least 40% of the students will be infected, and classes will be canceled.

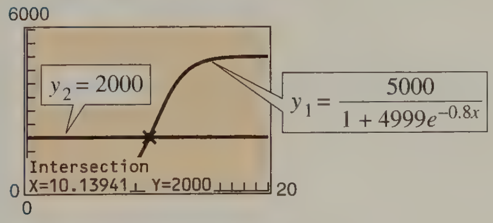
Graphical Solution

a. Use a graphing utility to graph $y = \frac{5000}{1 + 4999e^{-0.8x}}$. Use the *value* feature or *zoom* and *trace* features of the graphing utility to estimate that $y = 54$ when $x = 5$. So, after 5 days, about 54 people will be infected.

b. In this case, the number infected must be $(0.40)(5000) = 2000$ to cancel classes. Use a graphing utility to graph

$$y_1 = \frac{5000}{1 + 4999e^{-0.8x}} \quad \text{and} \quad y_2 = 2000$$

in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features of the graphing utility to find the point of intersection of the graphs. In Figure 4.42, you can see that the point of intersection occurs near $x \approx 10.14$. So, a short time after 10 days, at least 40% of the students will be infected, and the college will cancel classes.



```
Intersection
X=10.13941  Y=2000
```

Figure 4.42

Logarithmic Models

EXAMPLE 6 Magnitudes of Earthquakes

On the Richter scale, the magnitude R of an earthquake of intensity I is

$$R = \log_{10} \frac{I}{I_0}$$

where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensities per unit of area for the following earthquakes. (Intensity is a measure of the wave energy of the earthquake.)

a. Tokyo and Yokohama, Japan, in 1923; $R = 8.3$

b. Izmit, Turkey, in 1999; $R = 7.4$

Chip Hires/Liaison International

More than 15,000 people were killed in the 7.4 magnitude earthquake in Izmit, Turkey, on August 17, 1999.

Solution

a. Because $I_0 = 1$ and $R = 8.3$, you have

$$8.3 = \log_{10} I$$

$$I = 10^{8.3} \approx 199{,}526{,}000. \qquad \text{Intensity}$$

b. For $R = 7.4$, you have $7.4 = \log_{10} I$, and $I = 10^{7.4} \approx 25{,}119{,}000$.

Note that an increase of 0.9 units on the Richter scale (from 7.4 to 8.3) represents an intensity change by a factor of

$$\frac{199{,}526{,}000}{25{,}119{,}000} \approx 8.$$

In other words, the earthquake in 1923 had a magnitude about 8 times greater than that of the 1999 quake.

Fitting Models to Data

EXAMPLE 7 Fitting a Logarithmic Model

The data in the table gives the yield y (in milligrams) of a chemical reaction after x minutes.

x	1	2	3	4	5	6	7	8
y	1.5	7.4	10.2	13.4	15.8	16.3	18.2	18.3

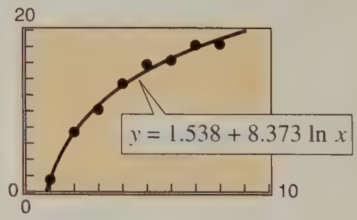


Figure 4.43

Use a graphing utility to fit a logarithmic model to the data.

Solution

Enter the data in the table into a graphing utility. Use the regression capabilities of the graphing utility to find the following logarithmic model.

$$y = 1.538 + 8.373 \ln x$$

When you graph the data and the model in the same viewing window, as shown in Figure 4.43, you can see that the model is a good fit to the data.

EXAMPLE 8 Fitting an Exponential Model

The total amount A (in billions of dollars) spent on health care in the United States in the years 1970 through 1996 are shown below. Find a model for the data, and use the model to estimate the amount spent in 2004. In the list of data points (t, A), t represents the year, with $t = 0$ corresponding to 1970. (Source: U.S. Health Care Financing Administration)

(0, 74.3), (1, 82.2), (2, 92.3), (3, 102.4), (4, 115.9), (5, 132.6),
(6, 151.9), (7, 172.6), (8, 193.2), (9, 218.3), (10, 247.3), (11, 291.4),
(12, 328.2), (13, 360.8), (14, 396.0), (15, 428.7), (16, 466.0), (17, 506.2),
(18, 562.3), (19, 623.9), (20, 699.5), (21, 766.8), (22, 836.6), (23, 895.1),
(24, 945.7), (25, 991.4), (26, 1035.1)

Solution

Enter the data into a graphing utility. Then graph the data, as shown in Figure 4.44. From the scatter plot, it appears that an exponential model is a good fit. Use the regression capabilities of the graphing utility to find the following exponential model.

$$A = 81.10(1.11)^t \quad \text{or} \quad A = 81.10e^{0.1044t}$$

From the model, you can see that the amount spent on health care from 1970 through 1996 had an average annual increase of 10%. From this model, you can estimate the 2004 amount to be

$$A = 81.10(1.11)^{34} \approx 2818.4 \text{ billion dollars}$$

which is more than twice the amount spent in 1996.

STUDY T!P

You can change an exponential model of the form $y = ab^x$ to one of the form $y = ae^{cx}$ by rewriting b in the form

$$b = e^{\ln b}.$$

For instance, $y = 3(2^x)$ can be written as

$$\begin{aligned} y &= 3(2^x) \\ &= 3e^{(\ln 2)x} \\ &\approx 3e^{0.693x}. \end{aligned}$$

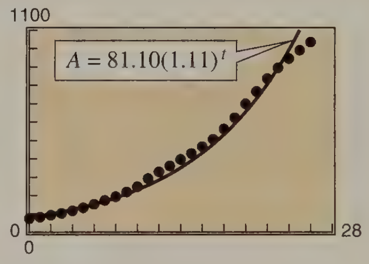


Figure 4.44

Writing About Math *Fitting a Model to Data*

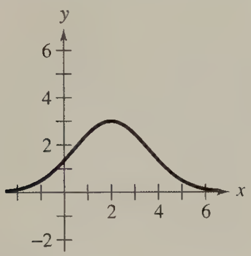
The numbers y (in millions) of vinyl single records sold in the United States in the years 1984 through 1997 are listed below. The data is given as ordered pairs of the form (t, y), where t is the year, with $t = 4$ representing 1984. Create a scatter plot of the data. Decide which type of model best fits this data. Then find the model. Write a paragraph explaining why you think the model you chose is a good fit to the data.

(4, 131.5), (5, 120.7), (6, 93.9), (7, 82.0),
(8, 65.6), (9, 36.6), (10, 27.6), (11, 22.0),
(12, 19.8), (13, 15.1), (14, 11.7), (15, 10.2),
(16, 10.1), (17, 7.5)

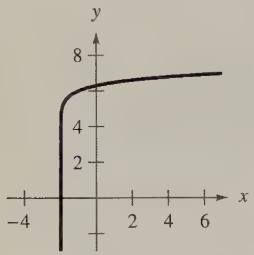
4.5 Exercises

In Exercises 1–6, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

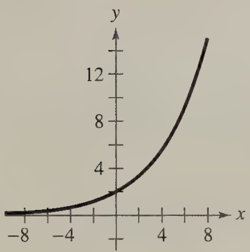
(a)



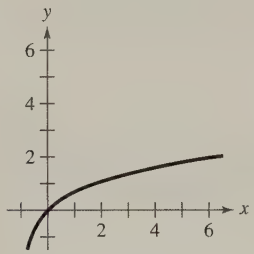
(b)



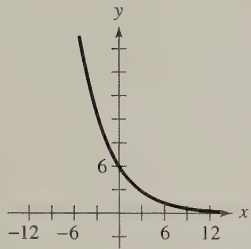
(c)



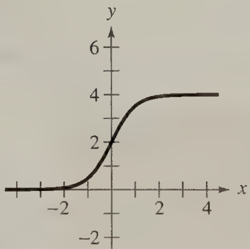
(d)



(e)



(f)



1. $y = 2e^{x/4}$

2. $y = 6e^{-x/4}$

3. $y = 6 + \log_{10}(x + 2)$

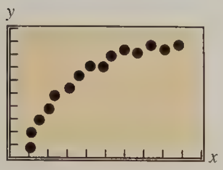
4. $y = 3e^{-(x-2)^2/5}$

5. $y = \ln(x + 1)$

6. $y = \dfrac{4}{1 + e^{-2x}}$

In Exercises 7–14, determine whether the scatter plot could best be modeled by a linear model, a quadratic model, an exponential model, a logarithmic model, a Gaussian model, or a logistic model.

7.



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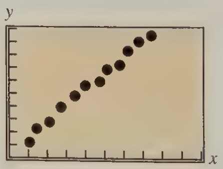

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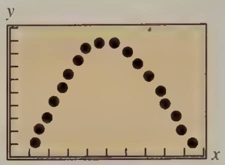

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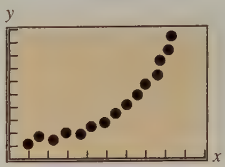

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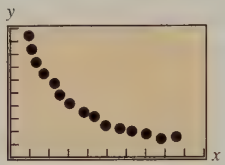

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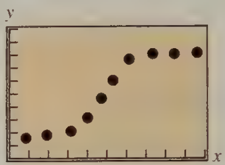
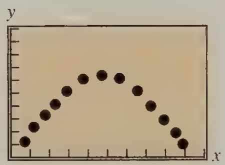
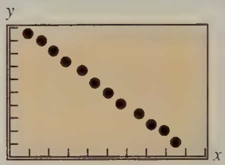


13.



14.



In Exercises 15–18, sketch a scatter plot of the data. Decide whether the data could best be modeled by a linear, an exponential, or a logarithmic model.

15. (1, 2.0), (1.5, 3.5), (2, 4.0), (4, 5.8), (6, 7.0), (8, 7.8)

16. (1, 11.0), (1.5, 9.6), (2, 8.2), (4, 4.5), (6, 2.5), (8, 1.4)

17. (1, 7.5), (1.5, 7.0), (2, 6.8), (4, 5.0), (6, 3.5), (8, 2.0)

18. (1, 5.0), (1.5, 6.0), (2, 6.4), (4, 7.8), (6, 8.6), (8, 9.0)

Compound Interest **In Exercises 19–26, complete the table for a savings account in which interest is compounded continuously.**

	Initial Investment	*Annual % Rate*	*Time to Double*	*Amount After 10 Years*
19.	\$1000	12%		
20.	\$20,000	$10\frac{1}{2}$%		
21.	\$750		$7\frac{3}{4}$ yr	
22.	\$10,000		12 yr	
23.	\$500			\$1292.85
24.	\$600			\$1505.00
25.		4.5%		\$10,000.00
26.		8%		\$20,000.00

27. ***Compound Interest*** (a) Complete the table for the time t necessary for P dollars to triple if interest is compounded continuously at rate r.

r	2%	4%	6%	8%	10%	12%
t						

(b) Draw a scatter plot of the data in part (a). Use the regression capabilities of a graphing utility to find a model for the data.

28. ***Compound Interest*** (a) Complete the table for the time t necessary for P dollars to triple if interest is compounded annually at rate r.

r	2%	4%	6%	8%	10%	12%
t						

(b) Draw a scatter plot of the data in part (a). Use the regression capabilities of a graphing utility to find a model for the data.

29. ***Comparing Investments*** If \$1 is invested in an account over a 10-year period, the amount in the account, where t represents the time in years, is

$$A = 1 + 0.075[\![t]\!] \quad \text{or} \quad A = e^{0.07t}$$

depending on whether the account pays simple interest at $7\frac{1}{2}\%$ or continuous compound interest at 7%. Use a graphing utility to graph each function in the same viewing window. Which grows at a faster rate?

30. ***Comparing Investments*** If \$1 is invested in an account over a 10-year period, the amount in the account, where t represents the time in years, is

$$A = 1 + 0.06[\![t]\!] \quad \text{or} \quad A = \left(1 + \frac{0.055}{365}\right)^{[\![365t]\!]}$$

depending on whether the account pays simple interest at 6% or compound interest at $5\frac{1}{2}\%$ compounded daily. Use a graphing utility to graph each function in the same viewing window. Which grows at a faster rate?

In Exercises 31–34, complete the table for the given radioactive isotope.

	Isotope	Half-Life (years)	Initial Quantity	Amount After 1000 Years
31.	^{226}Ra	1620	10 g	
32.	^{226}Ra	1620		1.5 g
33.	^{14}C	5730	3 g	
34.	^{230}Pu	24,360		0.4 g

35. ***Population*** The population P of a city is

$$P = 105{,}300e^{0.015t}$$

where $t = 0$ represents 2000. According to this model, when will the population reach 150,000?

36. ***Population*** The population P of a city is

$$P = 240{,}360e^{0.012t}$$

where $t = 0$ represents 2000. According to this model, when will the population reach 275,000?

37. ***Population*** The population P of a city is

$$P = 2500e^{kt}$$

where $t = 0$ represents 2000. In 1945, the population was 1350. Find the value of k, and use this value to estimate the population in the year 2010.

38. ***Population*** The table shows the population (in millions) of a country in 1997 and the projected population (in millions) for the year 2020. (Source: U.S. Bureau of the Census)

Country	*1997*	*2020*
Croatia	5.0	4.8
Mali	9.9	20.4
Singapore	3.5	4.3
Sweden	8.9	9.5

(a) Find the exponential growth model $y = ae^{bt}$ for the population in each country by letting $t = 0$ correspond to 1997. Use the model to estimate the population of each country in 2030.

(b) You can see that the populations of Mali and Sweden are growing at different rates. What constant in the equation $y = ae^{bt}$ is determined by these different growth rates? Discuss the relationship between the different growth rates and the magnitude of the constant.

(c) You can see that the population of Singapore is increasing while the population of Croatia is decreasing. What constant in the equation $y = ae^{bt}$ reflects this difference? Explain.

39. ***Bacteria Growth*** The number of bacteria N in a culture is given by the model

$$N = 100e^{kt}$$

where t is the time (in hours). If $N = 300$ when $t = 5$, estimate the time required for the population to double in size. Verify your estimate graphically.

40. ***Bacteria Growth*** The number of bacteria N in a culture is given by the model

$$N = 250e^{kt}$$

where t is the time (in hours). If $N = 280$ when $t = 10$, estimate the time required for the population to double in size. Verify your estimate graphically.

41. ***Radioactive Decay*** The half-life of radioactive radium (radium 226) is 1620 years. What percent of a present amount of radioactive radium will remain after 100 years?

42. ***Radioactive Decay*** Carbon 14 dating assumes that the carbon dioxide on earth today has the same radioactive content as it did centuries ago. If this is true, the amount of carbon 14 absorbed by a tree that grew several centuries ago should be the same as the amount of carbon 14 absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal if the half-life of carbon 14 is 5730 years?

43. ***Depreciation*** A car that cost \$22,000 new has a book value of \$13,000 after 2 years.

(a) Find the linear model $V = mt + b$.

(b) Find the exponential model $V = ae^{kt}$.

(c) Use a graphing utility to graph the two models in the same viewing window. Which model depreciates faster in the first 2 years?

(d) Use each model to find the book values of the car after 1 year and after 3 years.

(e) Interpret the slope of the linear model.

44. ***Depreciation*** A computer that cost \$4600 new has a book value of \$3000 after 2 years.

(a) Find the linear model $V = mt + b$.

(b) Find the exponential model $V = ae^{kt}$.

(c) Use a graphing utility to graph the two models in the same viewing window. Which model depreciates faster in the first year?

(d) Use each model to find the book values of the computer after 1 year and after 3 years.

(e) Interpret the slope of the linear model.

45. ***Sales*** The sales S (in thousands of units) of a new product after it has been on the market t years are

$$S(t) = 100(1 - e^{kt}).$$

Fifteen thousand units of the new product were sold the first year.

(a) Complete the model by solving for k.

(b) Use a graphing utility to graph the model.

(c) Use the graph to estimate the number of units sold after 5 years.

46. ***Sales and Advertising*** The sales S (in thousands of units) of a product after x hundred dollars is spent on advertising is

$$S = 10(1 - e^{kx}).$$

When \$500 is spent on advertising, 2500 units are sold.

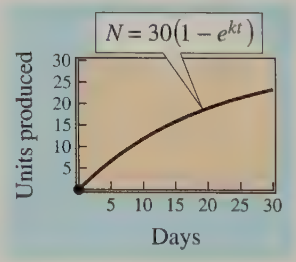
(a) Complete the model by solving for k.

(b) Estimate the number of units that will be sold if advertising expenditures are raised to \$700.

47. ***Learning Curve*** The management at a factory has found that the maximum number of units a worker can produce in a day is 30. The learning curve for the number of units N produced per day after a new employee has worked t days is

$$N = 30(1 - e^{kt}).$$

After 20 days on the job, a new employee produces 19 units.



(a) Find the learning curve for this employee (first find the value of k).

(b) How many days should pass before this employee is producing 25 units per day?

(c) Is the employee's production increasing at a linear rate? Explain your reasoning.

48. ***Endangered Species*** A conservation organization releases 100 animals of an endangered species into a game preserve. The organization believes that the preserve has a carrying capacity of 1000 animals and that the growth of the herd will follow the logistic curve

$$p(t) = \frac{1000}{1 + 9e^{-0.1656t}}$$

where t is measured in months.

(a) Use a graphing utility to graph the function. Use the graph to determine the horizontal asymptotes and interpret the meaning of the larger asymptote in the context of the problem.

(b) Estimate the population after 5 months.

(c) When will the population reach 500?

Earthquake Magnitudes **In Exercises 49 and 50, use the Richter scale (see page 346) for measuring the magnitudes of earthquakes.**

49. Find the magnitude R of an earthquake of intensity I (let $I_0 = 1$).

(a) Taiwan in 1999, $I = 39{,}811{,}000$

(b) Hector Mine, California, in 1999, $I = 12{,}589{,}000$

50. Find the intensity I of an earthquake measuring R on the Richter scale (let $I_0 = 1$).

(a) Chile in 1906, $R = 8.6$

(b) Los Angeles in 1971, $R = 6.7$

Intensity of Sound **In Exercises 51–54, use the following information. For Exercises 51 and 52, determine the level of sound (in decibels) for the given sound intensity.**

The level of sound β (in decibels) with an intensity I is $\beta(I) = 10 \log_{10}(I/I_0)$, where I_0 is an intensity of 10^{-12} watt per square meter, corresponding roughly to the faintest sound that can be heard by the human ear.

51. (a) $I = 10^{-10}$ watt per m^2 (faint whisper)

(b) $I = 10^{-5}$ watt per m^2 (busy street corner)

(c) $I = 10^{0}$ watt per m^2 (threshold of pain)

52. (a) $I = 10^{-3.5}$ watt per m^2 (jet 4 miles from takeoff)

(b) $I = 10^{-3}$ watt per m^2 (diesel truck at 25 feet)

(c) $I = 10^{-1.5}$ watt per m^2 (auto horn at 3 feet)

53. ***Noise Level*** As a result of the installation of noise suppression materials, the noise level in an auditorium was reduced from 93 to 80 decibels. Find the percent decrease in the intensity level of the noise due to the installation of these materials.

54. ***Noise Level*** As a result of the installation of a muffler, the level of noise emitted by an engine was reduced from 88 to 72 decibels. Find the percent decrease in the intensity level of the noise due to the installation of the muffler.

Acidity **In Exercises 55–58, use the acidity model given by $\mathrm{pH} = -\log_{10}[H^+]$, where acidity (pH) is a measure of the hydrogen ion concentration $[H^+]$ (measured in moles of hydrogen per liter) of a solution.**

55. Find the pH if $[H^+] = 2.3 \times 10^{-5}$.

56. Compute $[H^+]$ for a solution in which pH $= 5.8$.

57. A certain fruit has a pH of 2.5, and an antacid tablet has a pH of 9.5. The hydrogen ion concentration of the fruit is how many times that of the tablet?

58. If the pH of a solution is decreased by one unit, the hydrogen ion concentration is increased by what factor?

59. ***Finance*** A \$120,000 home mortgage for 30 years at $7\frac{1}{2}\%$ has a monthly payment of \$839.06. Part of the monthly payment goes for the interest charge on the unpaid balance, and the remainder of the payment is used to reduce the principal. The amount that goes for interest is

$$u = M - \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}$$

and the amount that goes toward reduction of the principal is

$$v = \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}.$$

In these formulas, P is the size of the mortgage, r is the interest rate, M is the monthly payment, and t is the time (in years).

(a) Use a graphing utility to graph each function in the same viewing window. (The viewing window should show all 30 years of mortgage payments.)

(b) In the early years of the mortgage, the larger part of the monthly payment goes for what purpose? Approximate the time when the monthly payment is evenly divided between interest and principal reduction.

(c) Repeat (a) and (b) for a repayment period of 20 years ($M = \$966.71$). What can you conclude?

60. ***Finance*** The total interest u paid on a home mortgage of P dollars at interest rate r for t years is

$$u = P\left[\frac{rt}{1 - \left(\dfrac{1}{1 + r/12}\right)^{12t}} - 1\right].$$

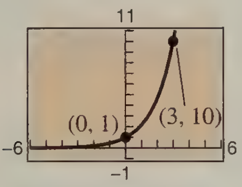
Consider a \$120,000 home mortgage at $7\frac{1}{2}\%$.

(a) Use a graphing utility to graph the total interest function.

(b) Approximate the length of the mortgage when the total interest paid is the same as the size of the mortgage. Is it possible that a person could pay twice as much in interest charges as the size of one's mortgage?

In Exercises 61–64, find the exponential model $y = ae^{bx}$ that fits the points given in the graph or table.

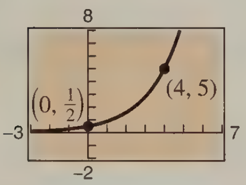
61.



62.

x	y
0	4
5	1

63.



64.

x	y
0	1
3	$\frac{1}{4}$

65. ***Data Analysis*** The time t (in seconds) required to attain a speed of s miles per hour from a standing start for a particular car is given in the table. (Source: *Road & Track*, March 1995)

s	30	40	50	60	70	80	90
t	3.4	5.0	7.0	9.3	12.0	15.8	20.0

Two models for this data are as follows.

$$t_1 = 40.757 + 0.556s - 15.817 \ln s$$

$$t_2 = 1.2259 + 0.0023s^2$$

(a) Use a graphing utility to fit a linear model t_3 and an exponential model t_4 to the data.

(b) Use a graphing utility to graph the data points and each model.

(c) Create a table to compare the data with estimates obtained from each model.

(d) Use the results of part (c) to find the sum of the absolute values of the differences between the data and the estimated values given by each model. Based on the four sums, which model do you think best fits the data? Explain.

66. ***Comparing Models*** Suppose that in 1995 a car had a retail price of \$21,800. A local dealership used the following guide for the approximate value of the car for the years 1995 through 2000.

Year	1995	1996	1997
Value	\$21,800	\$19,530	\$19,210

Year	1998	1999	2000
Value	\$17,945	\$17,120	\$16,080

Let V represent the value of the automobile in the year t, with $t = 0$ corresponding to 1995.

(a) Use the regression capabilities of a graphing utility to find linear and quadratic models for the data. Use the graphing utility to plot the data and graph the models in the same viewing window.

(b) What does the slope represent in the linear model in part (a)?

(c) Do you think the quadratic model is realistic? Explain.

(d) Use the regression capabilities of a graphing utility to fit an exponential model to the data. Use the graphing utility to plot the data and graph the model in the same viewing window. How well does the model fit the data?

(e) Fit a rational model to the data. Take the reciprocal of the depreciated values of the automobile to generate the points $(t, 1/V)$. Use the regression capabilities of a graphing utility to fit a line to this data. The resulting line has the form

$$\frac{1}{V} = at + b.$$

Solve for V and use the graphing utility to graph the rational function and the original data points in the same viewing window. How well does the model fit the data?

(f) Determine the horizontal asymptotes of the exponential and rational models. Interpret their meanings in the context of the problem.

67. ***Forensics*** At 8:30 A.M., a coroner was called to the home of a person who had died during the night. In order to estimate the time of death, the coroner took the person's temperature twice. At 9:00 A.M. the temperature was 85.7°F, and at 9:30 A.M. the temperature was 82.8°F. From these two temperatures the coroner was able to determine that the time elapsed since death and the body temperature were related by the formula

$$t = -2.5 \ln \frac{T - 70}{98.6 - 70}$$

where t is the time (in hours elapsed since the person died) and T is the temperature (in degrees Fahrenheit) of the person's body. Assume the person had a normal body temperature of 98.6°F at death, and the room temperature was a constant 70°F. (This formula is derived from a general cooling principle called Newton's Law of Cooling.) Use the formula to estimate the time of death of the person.

68. ***Foreign Travel*** The numbers of United States citizens y (in millions) who traveled to foreign countries in the years 1988 through 1996 are shown in the table, where $t = 8$ represents the year 1988. (Source: U.S. Travel and Tourism Administration, U.S. Department of Commerce)

t	8	9	10	11	12
y	40.7	41.1	44.6	41.6	43.9

t	13	14	15	16
y	44.4	46.5	50.8	52.3

(a) Use the regression capabilities of a graphing utility to find an exponential model that fits the data.

(b) Use the graphing utility to plot the data and graph the model in the same viewing window.

(c) Use the model to estimate the number of foreign travelers in the year 2005.

69. ***World Population*** The world population y (in billions) for the years 1987 through 1998 is shown in the table, where $x = 7$ corresponds to 1987. (Source: U.S. Bureau of the Census)

x	7	8	9	10	11	12
y	5.02	5.11	5.19	5.28	5.37	5.45

x	13	14	15	16	17	18
y	5.53	5.61	5.69	5.77	5.85	5.93

(a) Use the regression capabilities of a graphing utility to fit a linear model to the data.

(b) Use the regression capabilities of a graphing utility to fit an exponential model to the data.

(c) Population growth is often exponential. For the 12 years of data in the table, is the exponential model better than the linear model? Explain.

(d) Use each model to estimate the population in the year 2005.

70. ***Atmospheric Pressure*** The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is 1.033227 kilograms per square centimeter, and this pressure is called one atmosphere. Variations in weather conditions cause changes in the atmospheric pressure of up to ±5 percent. The table shows the pressures p (in atmospheres) at given altitudes h (in kilometers).

h	0	5	10	15	20	25
p	1	0.55	0.25	0.12	0.06	0.02

(a) Use a graphing utility to attempt to find the logarithmic model $p = a + b \ln h$ for the data. Explain why the result is an error message.

(b) Use a graphing utility to find the logarithmic model $h = a + b \ln p$ for the data.

(c) Use a graphing utility to plot the data and graph the logarithmic model.

(d) Use the model to estimate the altitude at which the pressure is 0.75 atmosphere.

(e) Use the graph to estimate the pressure at an altitude of 13 kilometers.

71. ***National Health Expenditures*** The table shows the national health expenditures y (in billions of dollars) for the years 1987 through 1996. The years are represented by x, where $x = 7$ corresponds to 1987. (Source: U.S. Health Care Financing Administration)

x	7	8	9	10	11
y	500.1	559.6	622.0	699.5	766.8

x	12	13	14	15	16
y	836.6	895.1	945.7	991.4	1035.1

(a) Use a graphing utility to find an exponential model for the data. Use the graphing utility to plot the data and graph the exponential model in the same viewing window.

(b) Use a graphing utility to find a logarithmic model for the data. Use the graphing utility to plot the data and graph the logarithmic model in the same viewing window.

(c) Use the graphs plotted in parts (a) and (b) to determine which model is better. If the rate of growth of health care costs could be slowed, which model may be better for the future? Explain.

72. ***Comparing Models*** The amounts y (in billions of dollars) donated to charity (by individuals, foundations, corporations, and charitable bequests) in the years 1987 through 1996 in the United States are shown in the table, where $x = 7$ corresponds to 1987. (Source: AAFRC Trust for Philanthropy)

x	7	8	9	10	11
y	90.0	98.1	106.7	111.5	117.2

x	12	13	14	15	16
y	121.1	126.5	129.3	140.5	150.7

(a) Use the regression capabilities of a graphing utility to find the following models for the data.

$y_1 = ax + b$

$y_2 = a + b \ln x$

$y_3 = ab^x$

(b) Use the graphing utility to graph the data and each of the models. Use the graphs to select the model that you think best fits the data.

(c) For each of the models y_i ($i = 1, 2,$ and 3), complete the table.

x	y	$y - y_i$	$(y - y_i)^2$
7	90.0		
8	98.1		
9	106.7		
10	111.5		
11	117.2		
12	121.1		
13	126.5		
14	129.3		
15	140.5		
16	150.7		

(d) For each model, find the sum of the entries in the last column of the table in part (c). Use the results to select the best model for the data.

(e) Explain what the sums in part (d) represent.

73. ***Comparing Models*** A V-8 car engine is coupled to a dynamometer, and the horsepower y is measured at different engine speeds x (in thousands of revolutions per minute). The results are shown in the table.

x	1	2	3	4	5	6
y	40	85	140	200	225	245

(a) Use the regression capabilities of a graphing utility to find the following models for the data.

$y_1 = ax^3 + bx^2 + cx + d$

$y_2 = a + b \ln x$

$y_3 = ab^x$

(b) Use the graphing utility to graph the data and each of the models. Use the graphs to select the model that you think best fits the data.

(c) For each of the models y_i ($i = 1, 2$, and 3), complete the table.

x	y	$y - y_i$	$(y - y_i)^2$
1	40		
2	85		
3	140		
4	200		
5	225		
6	245		

(d) For each model, find the sum of the entries in the last column of the table in part (c). Use the results to select the best model for the data.

(e) Explain what the sums in part (d) represent.

Synthesis

True or False? **In Exercises 74–78, determine whether the statement is true or false. Justify your answer.**

74. The domain of a logistic growth function cannot be the set of real numbers.

75. The graph of a logistic growth function will always have an x-intercept.

76. The graph of a Gaussian model will never have an x-intercept.

77. The graph of a Gaussian model will always have a maximum point.

78. The graph of

$$f(x) = \frac{4}{1 + 6e^{-2x}} + 5$$

is the graph of

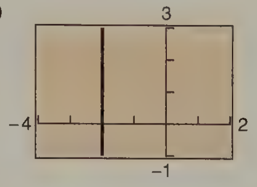
$$f(x) = \frac{4}{1 + 6e^{-2x}}$$

shifted to the right five units.

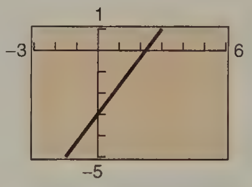
Review

In Exercises 79–84, match the equation with its graph, and identify any intercepts. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

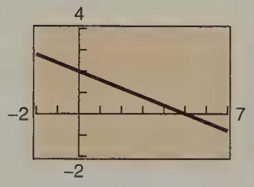
(a)



(b)



(c)



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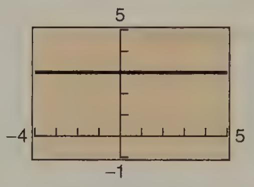

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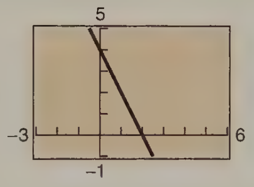
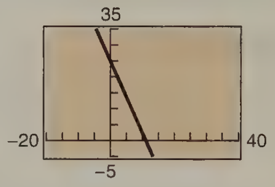


(f)



79. $4x - 3y - 9 = 0$

80. $2x + 5y - 10 = 0$

81. $y = 25 - 2.25x$

82. $\frac{x}{2} + \frac{y}{4} = 1$

83. $y - 3 = 0$

84. $x + 2 = 0$

In Exercises 85–88, divide using synthetic division.

85. $\dfrac{4x^3 + 4x^2 - 39x + 36}{x + 4}$

86. $\dfrac{8x^3 - 36x^2 + 54x - 27}{x - \frac{3}{2}}$

87. $(2x^3 - 8x^2 + 3x - 9) \div (x - 4)$

88. $(x^4 - 3x + 1) \div (x + 5)$

In Exercises 89–92, graph the exponential function.

89. $f(x) = 2^{x-1} + 5$

90. $f(x) = -2^{-x-1} - 1$

91. $f(x) = 3^x - 4$

92. $f(x) = -3^x + 4$

4 Chapter Summary

What did you learn?

Section 4.1

	Review Exercises
☐ How to recognize and evaluate exponential functions with base *a*	1–4
☐ How to graph exponential functions	5–26
☐ How to recognize, evaluate, and graph exponential functions with base *e*	27–30
☐ How to use exponential functions to model and solve real-life problems	31–34

Section 4.2

☐ How to recognize and evaluate logarithmic functions with base *a*	35–44
☐ How to graph logarithmic functions	45–56
☐ How to recognize, evaluate, and graph natural logarithmic functions	57–60
☐ How to use logarithmic functions to model and solve real-life problems	61, 62

Section 4.3

☐ How to rewrite logarithmic functions with different bases	63–66
☐ How to use properties of logarithms to evaluate or rewrite logarithmic expressions	67–74
☐ How to use properties of logarithms to expand or condense logarithmic expressions	75–86
☐ How to use logarithmic functions to model and solve real-life problems	87, 88

Section 4.4

☐ How to solve simple exponential and logarithmic equations	89–94
☐ How to solve more complicated exponential equations	95–104
☐ How to solve more complicated logarithmic equations	105–116
☐ How to use exponential and logarithmic equations to model and solve real-life problems	117, 118

Section 4.5

☐ How to recognize the five most common types of models involving exponential or logarithmic functions	119–124
☐ How to use exponential growth and decay functions to model and solve real-life problems	125–127
☐ How to use Gaussian functions to model and solve real-life problems	128
☐ How to use logistic growth functions to model and solve real-life problems	129
☐ How to use logarithmic functions to model and solve real-life problems	130
☐ How to fit exponential and logarithmic models to sets of data	131–136

4 Review Exercises

4.1 **In Exercises 1–4, use a calculator to evaluate the expression. Round your answer to three decimal places.**

1. $(1.45)^{2\pi}$

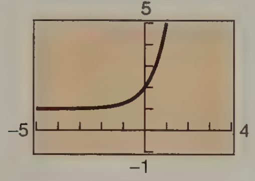
2. $\sqrt[5]{6240}$

3. $1.59^{-2\sqrt{3}}$

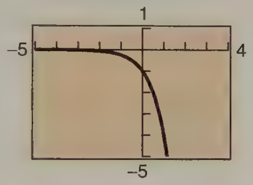
4. $136(5^{-1.1})$

In Exercises 5–10, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

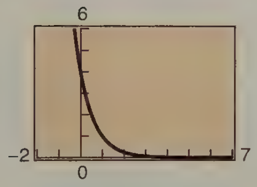
(a)



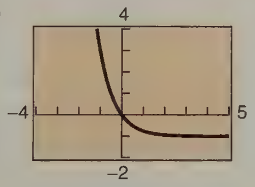
(b)



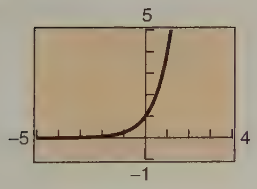
(c)



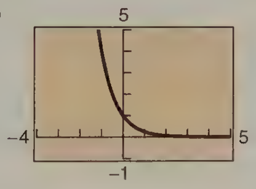
(d)



(e)



(f)



5. $f(x) = 4^x$

6. $f(x) = 4^{-x}$

7. $f(x) = -4^x$

8. $f(x) = 4^x + 1$

9. $f(x) = 4^{-x} - 1$

10. $f(x) = 4^{-x+1}$

In Exercises 11–20, sketch the graph of the function. Use a graphing utility to verify your graph.

11. $f(x) = 6^x$

12. $f(x) = 0.3^x$

13. $g(x) = 6^{-x}$

14. $g(x) = 0.3^{-x}$

15. $h(x) = e^{-x/2}$

16. $f(x) = e^{x+2}$

17. $h(x) = 2 - e^{-x/2}$

18. $f(x) = 3 - e^{-x-2}$

19. $s(t) = 4e^{-2/t},\ t > 0$

20. $f(x) = 6e^{-3/x},\ x > 0$

In Exercises 21–26, use a graphing utility to graph the function. Identify any asymptotes.

21. $g(t) = 8 - 0.5e^{-t/4}$

22. $h(x) = 12(1 + e^{-x/2})$

23. $g(x) = 200e^{4/x}$

24. $f(x) = -8e^{-4/x}$

25. $f(x) = \dfrac{10}{1 + 2^{-0.05x}}$

26. $f(x) = -\dfrac{12}{1 + 4^{-x}}$

In Exercises 27–30, use a calculator to evaluate the expression. Round your result to three decimal places.

27. $e^{-8/3}$

28. $e^{\sqrt{5}}$

29. $6e^{4.12}$

30. $-\frac{5}{3}e^{-3/5}$

In Exercises 31 and 32, complete the table to determine the amount of money *P* that should be invested at rate *r* to produce a final balance of $200,000 in *t* years.

t	1	10	20	30	40	50
P						

31. $r = 8\%$, compounded continuously

32. $r = 10\%$, compounded monthly

33. ***Depreciation*** After t years, the value of a car that originally cost $26,000 is

$$V(t) = 26{,}000\left(\frac{3}{4}\right)^t.$$

(a) Use a graphing utility to graph the function.

(b) Find the value of the car 2 years after it was purchased.

(c) According to the model, when does the car depreciate most rapidly? Is this realistic? Explain.

34. ***Fuel Efficiency*** A certain automobile gets 28 miles per gallon of gasoline for speeds up to 50 miles per hour. Over 50 miles per hour, the number of miles per gallon drops at the rate of 12% for each additional 10 miles per hour. If s is the speed and y is the number of miles per gallon, then

$$y = 28e^{0.6-0.012s}, \quad s \ge 50.$$

Use the model to complete the table.

s	50	55	60	65	70
y	28				

4.2 In Exercises 35–38, write the exponential equation in logarithmic form.

35. $4^3 = 64$
36. $3^5 = 243$
37. $25^{3/2} = 125$
38. $12^{-1} = \frac{1}{12}$

In Exercises 39–44, evaluate the expression without using a graphing utility.

39. $\log_8 8^{-0.34}$
40. $\log_7 1$
41. $\log_6 216$
42. $\log_4 1024$
43. $\log_{36} \frac{1}{6}$
44. $\log_{10} 0.001$

In Exercises 45–52, sketch the graph of the function. Use a graphing utility to verify your graph.

45. $g(x) = -\log_2 x + 5$
46. $g(x) = \log_5(x - 3)$
47. $f(x) = \log_2(x - 1) + 6$
48. $f(x) = \log_5(x + 2) - 3$
49. $f(x) = \ln x + 3$
50. $f(x) = \ln(x - 3)$
51. $h(x) = \frac{1}{2} \ln x$
52. $f(x) = \frac{1}{4} \ln x$

In Exercises 53–56, use a graphing utility to graph the function and determine its domain.

53. $y = \log_{10}(x^2 + 1)$
54. $y = -2 \log_{10}(4 - x^2)$
55. $y = \sqrt{x} \ln(x + 1)$
56. $y = \dfrac{14 \ln x}{x}$

In Exercises 57–60, evaluate the expression without using a graphing utility.

57. $\ln e^7$
58. $\ln 1$
59. $6 \ln e^{-3}$
60. $-\frac{3}{2} \ln e^{-10/11}$

61. ***Climb Rate*** The time t (in minutes) for a small plane to climb to an altitude of h feet is

$$t = 50 \log_{10} \frac{18{,}000}{18{,}000 - h}$$

where 18,000 feet is the plane's absolute ceiling.

(a) Determine the domain of the function appropriate for the context of the problem.

(b) Use a graphing utility to graph the time function and identify any asymptotes.

(c) As the plane approaches its absolute ceiling, what can be said about the time required to further increase its altitude?

(d) Find the time for the plane to climb to an altitude of 4000 feet.

62. ***Earthquake Magnitudes*** The magnitude M of an earthquake measured on the Richter scale can be found using the model

$$M = \tfrac{2}{3}(\log_{10} E - 11.4)$$

where E is the energy of the earthquake (in ergs). Use a graphing utility and the table shown below to find the ratio of the magnitudes of the earthquakes.

Earthquake	E (ergs)
San Francisco, 1906	7.079×10^{23}
Japan, 1933	5.623×10^{24}
Armenia, 1988	3.981×10^{21}
Russia, 1995	6.310×10^{22}
Indonesia, 1996	4.467×10^{22}

(a) Japan and San Francisco

(b) Russia and Indonesia

(c) Indonesia and Armenia

4.3 In Exercises 63–66, evaluate the logarithm using the change-of-base formula. Do each problem twice, once with common logarithms and once with natural logarithms. Round the result to three decimal places.

63. $\log_4 9$
64. $\log_{1/2} 5$
65. $\log_{12} 200$
66. $\log_3 0.28$

In Exercises 67–70, rewrite the expression in terms of ln 4 and ln 5.

67. $\ln 20$
68. $\ln 45 - 2 \ln 3$
69. $\ln \frac{5}{64}$
70. $\ln \frac{2}{5}$

In Exercises 71–74, approximate the logarithm using the properties of logarithms, given that $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$.

71. $\log_b 25$
72. $\log_b\left(\frac{25}{9}\right)$
73. $\log_b \sqrt{3}$
74. $\log_b 30$

In Exercises 75–80, rewrite the expression as a sum, difference, and/or multiple of logarithms.

75. $\log_5 5x^2$
76. $\log_7 \dfrac{\sqrt{x}}{4}$
77. $\log_{10} \dfrac{5\sqrt{y}}{x^2}$
78. $\ln \left|\dfrac{x - 1}{x + 1}\right|$

79. $\ln[(x^2+1)(x-1)]$ **80.** $\ln\sqrt[5]{\dfrac{4x^2-1}{4x^2+1}}$

In Exercises 81–86, write the expression as the logarithm of a single quantity.

81. $\log_2 5 + \log_2 x$ **82.** $\log_6 y - 2\log_6 z$

83. $\frac{1}{2}\ln|2x-1| - 2\ln|x+1|$

84. $5\ln|x-2| - \ln|x+2| - 3\ln|x|$

85. $\ln 3 + \frac{1}{3}\ln(4-x^2) - \ln x$

86. $3[\ln x - 2\ln(x^2+1)] + 2\ln 5$

87. ***Snow Removal*** The number of miles s of roads cleared of snow is approximated by the model

$$s = 25 - \frac{13\ln(h/12)}{\ln 3}, \quad 2 \le h \le 15$$

where h is the depth of the snow (in inches).

(a) Use a graphing utility to graph the function.

(b) Complete the table.

h	4	6	8	10	12	14
s						

(c) Using the graph of the function and the table, what conclusion can you make as the depth of the snow increases?

88. ***Human Memory Model*** Students in a sociology class were given an exam and then were retested monthly with an equivalent exam. The average score for the class was given by the human memory model, $f(t) = 85 - 14\log_{10}(t+1)$, $0 \le t \le 4$, where t is the time in months. How did the average score change over the 4-month period?

4.4 **In Exercises 89–94, solve for x.**

89. $8^x = 512$ **90.** $3^x = 729$

91. $6^x = \frac{1}{216}$ **92.** $6^{x-2} = 1296$

93. $\log_7 x = 4$ **94.** $\log_x 243 = 5$

In Exercises 95–104, solve the exponential equation. Round the result to three decimal places.

95. $e^x = 12$ **96.** $e^{3x} = 25$

97. $3e^{-5x} = 132$ **98.** $14e^{3x+2} = 560$

99. $e^x + 13 = 35$ **100.** $e^x - 28 = -8$

101. $-4(5^x) = -68$ **102.** $2(12^x) = 190$

103. $e^{2x} - 7e^x + 10 = 0$ **104.** $e^{2x} - 6e^x + 8 = 0$

In Exercises 105–116, solve the logarithmic equation. Round the result to three decimal places.

105. $\ln 3x = 8.2$ **106.** $\ln 5x = 7.2$

107. $2\ln 4x = 15$ **108.** $4\ln 3x = 15$

109. $\ln x - \ln 3 = 2$ **110.** $\ln\sqrt{x+8} = 3$

111. $\ln\sqrt{x+1} = 2$ **112.** $\ln x - \ln 5 = 4$

113. $\log_{10}(x-1) = \log_{10}(x-2) - \log_{10}(x+2)$

114. $\log_{10}(x+2) - \log_{10} x = \log_{10}(x+5)$

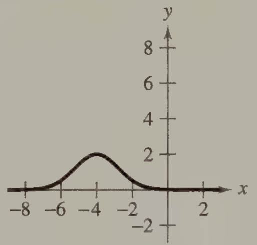
115. $\log_{10}(1-x) = -1$ **116.** $\log_{10}(-x-4) = 2$

117. ***Finance*** You deposit \$7550 into an account that pays 7.25% interest, compounded continuously. How long will it take the money to triple?

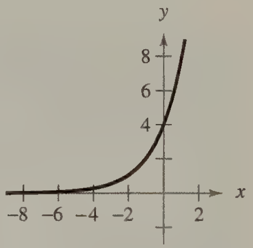
118. ***Economics*** The demand equation for a certain product is modeled by $p = 500 - 0.5e^{0.004x}$. Find the demand x that corresponds to a price of (a) $p = \$450$ and (b) $p = \$400$.

4.5 **In Exercises 119–124, match the function on page 360 with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

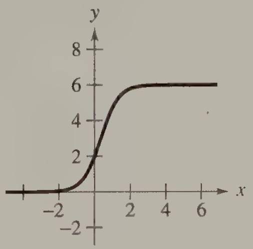
(a)



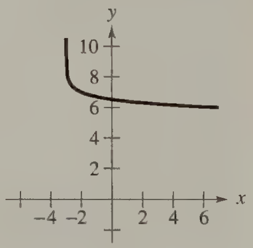
(b)



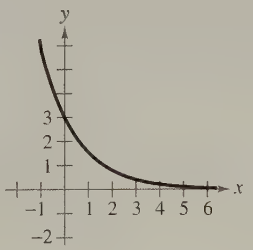
(c)



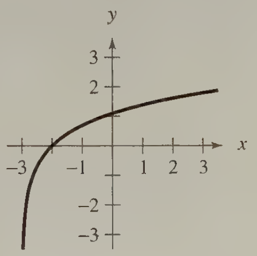
(d)



(e)



(f)



119. $y = 3e^{-2x/3}$

120. $y = 4e^{2x/3}$

121. $y = \ln(x + 3)$

122. $y = 7 - \log_{10}(x + 3)$

123. $y = 2e^{-(x+4)^2/3}$

124. $y = \dfrac{6}{1 + 2e^{-2x}}$

125. ***Population*** The population of a town is modeled by

$$P = 12{,}620e^{0.0118t}$$

where $t = 0$ represents 2000. According to this model, when will the population reach 17,000?

126. ***Radioactive Decay*** The half-life of radioactive uranium II (^{234}U) is 250,000 years. What percent of the present amount of radioactive uranium II will remain after 5000 years?

127. ***Compound Interest*** A deposit of $10,000 is made in a savings account for which the interest is compounded continuously. The balance will double in 12 years.

(a) What is the annual interest rate for this account?

(b) Find the balance after 1 year.

(c) The *effective yield* of a savings plan is the percent increase in the balance after 1 year. Find the effective yield.

128. ***Test Scores*** The test scores for a biology test follow a normal distribution modeled by

$$y = 0.0499e^{-(x-71)^2/128}.$$

(a) Use a graphing utility to graph the equation.

(b) From the graph, estimate the average test score.

129. ***Typing Speed*** In a typing class, the average number of words per minute N typed after t weeks of lessons was found to be

$$N = \frac{157}{1 + 5.4e^{-0.12t}}.$$

Find the time necessary to type (a) 50 words per minute and (b) 75 words per minute.

130. ***Earthquake Magnitude*** On the Richter scale, the magnitude R of an earthquake I is

$$R = \log_{10}\frac{I}{I_0}$$

where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensity per unit of area for the following values of R.

(a) $R = 8.4$ (b) $R = 6.85$ (c) $R = 9.1$

In Exercises 131–134, find the exponential function $y = Ae^{bx}$ that passes through the two points.

131. $(0, 2), (4, 3)$

132. $(0, 2), (5, 1)$

133. $\left(0, \frac{1}{2}\right), (5, 5)$

134. $(0, 4), \left(5, \frac{1}{2}\right)$

135. ***Exponential Model*** Use a graphing utility to find an exponential model $y = ab^x$ through the points $(0, 250)$, $(4, 135)$, $(6, 92)$, and $(10, 67)$. Graph the data and the exponential model in the same viewing window.

136. ***Net Profit*** The table shows the net profit P (in millions of dollars) of the Bristol-Myers Squibb Company for 1990 through 1998. (Source: Bristol-Myers Squibb Company)

Year	1990	1991	1992	1993	1994
P	1748.0	2056.0	2108.0	2269.0	2330.6

Year	1995	1996	1997	1998
P	2600.0	2850.0	3205.0	3141.0

(a) Make a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 1990.

(b) Use a graphing utility to find an exponential model $P = ab^t$ for the data. Let $t = 0$ represent 1990.

Synthesis

True or False? **In Exercises 137–142, determine whether the equation or statement is true or false. Justify your answer.**

137. $\log_b b^{2x} = 2x$

138. $e^{x-1} = \dfrac{e^x}{e}$

139. $\ln(x + y) = \ln x + \ln y$

140. $\ln(x + y) = \ln(xy)$

141. $\log_{10}\left(\dfrac{10}{x}\right) = 1 - \log_{10} x$

142. The domain of the function $f(x) = \ln x$ is the set of all real numbers.

Chapter Project A Graphical Approach to Compound Interest

In this project, you will use a graphing utility to compare savings plans. For instance, suppose you are depositing \$1000 in a savings account and are given the following options.

- 6.2% annual interest rate, compounded annually
- 6.1% annual interest rate, compounded quarterly
- 6.0% annual interest rate, compounded continuously

a. For each option, write a function that gives the balance as a function of the time t (in years).

b. Graph all three functions in the same viewing window. Can you find a viewing window that distinguishes among the graphs of the three functions? If so, describe the viewing window.

c. Find the balances for the three options after 25, 50, 75, and 100 years. Is the option that yields the greatest balance after 25 years the same option that yields the greatest balances after 50, 75, and 100 years? Explain.

d. The *effective yield* of a savings plan is the percent increase in the balance after 1 year. Find the effective yields for the three options listed above. How can the effective yield be used to decide which option is best?

Questions for Further Exploration

1. You deposit \$25,000 in an account to accrue interest for 40 years. The account pays 4% compounded annually. Assume that the income tax on the earned interest is 30%. Which of the following plans produces a larger balance after all income tax is paid?

(a) ***Deferred*** The income tax on the interest that is earned is paid in one lump sum at the end of 40 years.

(b) ***Not Deferred*** The income tax on the interest that is earned each year is paid at the end of each year.

2. Which of the following would produce a larger balance? Explain.

(a) 4.02% annual interest rate, compounded monthly

(b) 4% annual interest rate, compounded continuously

3. You deposit \$1000 in each of two savings accounts. The interest for the accounts is paid according to the two options described in Question 2. How long would it take for the balance in one of the accounts to exceed the balance in the other account by \$100? By \$100,000?

4. No income tax is due on the interest earned in some types of investments. You deposit \$25,000 into an account. Which of the following plans is better? Explain.

(a) ***Tax-Free*** The account pays 5%, compounded annually. There is no income tax on the earned interest.

(b) ***Tax-Deferred*** The account pays 7%, compounded annually. At maturity, the earned interest is taxable at a rate of 40%.

4 Chapter Test

Take this test as you would take a test in class. After you are done, check your work against the answers in the back of the book.

The *Interactive* CD-ROM and *Internet* versions of this text provide answers to the Chapter Tests and Cumulative Tests. They also offer Chapter Pre-Tests (that test key skills and concepts covered in previous chapters) and Chapter Post-Tests, both of which have randomly generated exercises with diagnostic capabilities.

1. Sketch the graph of the function $f(x) = 2^{-x+1} - 3$ by hand.

2. Use a graphing utility to graph f and determine its horizontal asymptotes.

$$f(x) = \frac{1000}{1 + 4e^{-0.2x}}$$

3. Determine the principal that will yield \$200,000 when invested at 8% compounded daily for 20 years.

4. Write the logarithmic equation $\log_4 64 = 3$ in exponential form.

5. Sketch the graph of the function $g(x) = \log_3(x - 2)$ by hand.

6. Use the properties of logarithms to expand $\ln\left(\dfrac{6x^2}{\sqrt{x^2+1}}\right)$.

In Exercises 7 and 8, evaluate the expression without using a calculator.

7. $\log_5 25$ **8.** $-2 \ln e^2 + 1$

In Exercises 9–11, solve the equation. Round to three decimal places.

9. $8 + \frac{1}{4}e^{x/2} = 450$ **10.** $\left(1 + \dfrac{0.06}{4}\right)^{4t} = 3$ **11.** $3.6 - 5\ln(x + 4) = 22$

12. A truck that costs \$28,000 new has a depreciated value of \$20,000 after 1 year. Find the value of the truck when it is 3 years old by using the exponential model $y = ae^{bx}$.

13. The average time between incoming calls at a switchboard is 3 minutes. The probability of waiting less than t minutes for the next incoming call is approximated by the model $F(t) = 1 - e^{-t/3}$. If a call has just come in, find the probability that the next call will come within (a) $\frac{1}{2}$ minute, (b) 2 minutes, and (c) 5 minutes.

14. The population of a certain species t years after it is introduced into a new habitat is $p(t) = \dfrac{1200}{1 + 3e^{-t/5}}$. (a) Determine the initial size of the population. (b) Determine the population after 5 years. (c) After how many years will the population be 800?

15. By observation, identify the equation that corresponds to the graph shown at the right. Explain your reasoning.

(a) $y = 6e^{-x^2/2}$ (b) $y = \dfrac{6}{1 + e^{-x/2}}$ (c) $y = 6(1 - e^{-x^2/2})$

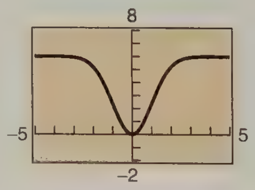


FIGURE FOR 15

16. The numbers of cellular-phone subscribers y (in millions) for the years 1993 through 1997 are (3, 16.0), (4, 24.1), (5, 33.8), (6, 44.0), and (7, 55.3), where x is the time (in years) and $x = 3$ corresponds to 1993. Use a graphing utility to fit an exponential model to the data. Sketch a scatter plot of the data and graph the model in the same viewing window. (Source: Cellular Telecommunications Industry Association)

3–4 Cumulative Test

Take this test to review the material from earlier chapters. After you are done, check your work against the answers in the back of the book.

In Exercises 1–3, sketch the graph of the function. Use a graphing utility to verify the graph.

1. $f(x) = -\frac{1}{2}(x^2 + 4x)$ **2.** $f(x) = \frac{1}{4}x(x - 2)^2$ **3.** $f(x) = x^3 + 2x^2 - 9x - 18$

4. Find all the zeros of $f(x) = x^3 + 2x^2 + 4x + 8$.

5. Use a graphing utility to approximate any real zeros of $g(x) = x^3 + 4x^2 - 11$ to the nearest hundredth.

6. Divide $(4x^2 + 14x - 9)$ by $(x + 3)$ using long division.

7. Divide $(2x^3 - 5x^2 + 6x - 20)$ by $(x - 6)$ using synthetic division.

8. Find a polynomial function with integer coefficients that has the zeros $0, -3, 1 + \sqrt{5}i$.

In Exercises 9–11, sketch the graph of the rational function. Be sure to label all asymptotes. Use a graphing utility to verify the graph.

9. $f(x) = \dfrac{2x}{x - 3}$ **10.** $f(x) = \dfrac{5x}{x^2 + x - 6}$ **11.** $f(x) = \dfrac{x^2 - 3x + 8}{x - 2}$

12. Write a rational function that has no vertical asymptotes and a horizontal asymptote at $y = 4$.

In Exercises 13–16, evaluate the expression without using a calculator.

13. $\log_2 64$ **14.** $\log_2\left(\frac{1}{16}\right)$ **15.** $\ln e^{10}$ **16.** $\ln \dfrac{1}{e^3}$

In Exercises 17–20, use a calculator to evaluate the expression. Round your answer to three decimal places.

17. $(1.85)^{3.1}$ **18.** $58^{\sqrt{5}}$ **19.** $e^{-20/11}$ **20.** $4e^{2.56}$

In Exercises 21–24, sketch the graph of the function by hand. Use a graphing utility to verify the graph.

21. $f(x) = -3^{x+4} - 5$ **22.** $f(x) = -\left(\frac{1}{2}\right)^{-x} - 3$

23. $f(x) = 4 + \log_{10}(x - 3)$ **24.** $f(x) = \ln(4 - x)$

In Exercises 25–28, evaluate the logarithm using the change-of-base formula. Round your answer to three decimal places.

25. $\log_5 21$ **26.** $\log_9 6.8$ **27.** $\log_{3/4}(8.61)$ **28.** $\log_{7/8}\left(\frac{3}{2}\right)$

29. Write $2\ln x - \frac{1}{2}\ln(x + 5)$ as a logarithm of a single quantity.

30. Expand the logarithmic expression: $\ln\left(x\sqrt[3]{x - 5}\right)$.

In Exercises 31–36, solve the equation algebraically. Round your result to three decimal places. Use a graphing utility to verify your answer graphically.

31. $6e^{2x} = 72$	**32.** $4e^{x-3} + 21 = 30$	**33.** $\log_2 x + \log_2 5 = 6$
34. $-3 + \ln 4x = 0$	**35.** $\ln \sqrt{x+2} = 3$	**36.** $\ln 4x^2 = 7$

37. Find the exponential function $y = ae^{bx}$ that passes through the points $\left(0, \frac{5}{2}\right)$ and $(6, 14)$.

38. The profit of a company is given by

$$P = 230 + 20x - \tfrac{1}{2}x^2$$

where P is the profit (in thousands of dollars), and x is the amount (in hundreds of dollars) the company spends on advertising. What amount will yield a maximum profit?

39. The average cost (in dollars) of recycling a waste product x (in pounds) is

$$\overline{C}(x) = \frac{450{,}000 + 5x}{x}, \; x > 0.$$

Find the average cost of recycling $x = 10{,}000$ pounds, $x = 100{,}000$ pounds, and $x = 1{,}000{,}000$ pounds. According to this model, what is the limiting average cost as the number of pounds increases?

40. You deposit \$2500 in an account earning 7.5% interest, compounded continuously. Find the balance after 25 years.

41. If the inflation rate averages 4.5% over the next 10 years, the approximate cost C of goods or services t years from now is

$$C(t) = P(1.045)^t$$

where P is the present cost. If the price of a tire is presently \$69.95, estimate the price 10 years from now.

42. The population P of a city is $P = 135{,}200e^{0.018t}$ where $t = 0$ represents 1990. Use a graphing utility to graph the model and determine when the population reaches 190,000. Verify your answer algebraically.

43. The population of a certain species t years after it is introduced into a new habitat is given by $p(t) = 1200/(1 + 3e^{-t/5})$.

(a) Determine the population size that was introduced into the habitat.

(b) Determine the population after 5 years.

(c) After how many years will the population be 800?

44. The numbers of subscribers of cellular phones y (in millions) for the years 1990 through 1997 are given by

$(0, 5.3), (1, 7.6), (2, 11.0), (3, 16.0), (4, 24.1), (5, 33.8), (6, 44.0), (7, 55.3)$

where x is the time (in years) with $x = 0$ corresponding to 1990. Use the regression capabilities of a graphing utility to fit an exponential model to the data. Sketch a scatter plot and graph the model in the same viewing window. (Source: Cellular Telecommunications Industry Association)

Trigonometric Functions

5

5.1 Angles and Their Measure
5.2 Right Triangle Trigonometry
5.3 Trigonometric Functions of Any Angle
5.4 Graphs of Sine and Cosine Functions
5.5 Graphs of Other Trigonometric Functions
5.6 Inverse Trigonometric Functions
5.7 Applications and Models

Ron Watts/CORBIS

The Mauna Loa Observatory in Hawaii conducts research to understand the global carbon cycle. It is located far from pollution sources that would affect the gases being measured. (Source: NOAA/Climate Monitoring and Diagnostic Laboratory)

The Big Picture

In this chapter you will learn how to

- ☐ describe an angle and convert between degree and radian measures.
- ☐ evaluate trigonometric functions of any angle.
- ☐ use fundamental trigonometric identities.
- ☐ sketch graphs of trigonometric functions.
- ☐ evaluate inverse trigonometric functions.
- ☐ evaluate the composition of trigonometric functions.
- ☐ use trigonometric functions to model and solve real-life problems.

Important Vocabulary

As you encounter each new vocabulary term in this chapter, add the term and its definition to your notebook glossary.

- angle (p. 366)
- initial side (p. 366)
- terminal side (p. 366)
- vertex (p. 366)
- positive angles (p. 366)
- negative angles (p. 366)
- coterminal angles (p. 366)
- degree (p. 367)
- acute angles (p. 367)
- obtuse angles (p. 367)
- complementary angles (p. 368)
- supplementary angles (p. 368)
- radian (p. 369)
- linear speed (p. 371)
- angular speed (p. 371)
- hypotenuse (p. 377)
- opposite side (p. 377)
- adjacent side (p. 377)
- sine (p. 377)
- cosecant (p. 377)
- cosine (p. 377)
- secant (p. 377)
- tangent (p. 377)
- cotangent (p. 377)
- angle of elevation (p. 382)
- angle of depression (p. 382)
- reference angles (p. 390)
- period (pp. 395, 404)
- sine curve (p. 401)
- one cycle (p. 401)
- amplitude (p. 403)
- phase shift (p. 405)
- damping factor (p. 417)
- inverse sine function (p. 423)
- inverse cosine function (p. 425)
- inverse tangent function (p. 425)

Additional Resources Text-specific additional resources are available to help you do well in this course. See page xvi for details.

5.1 Angles and Their Measure

What You Should Learn:

- How to describe angles
- How to use degree measure
- How to use radian measure
- How to convert between degree and radian measure
- How to use angles to model and solve real-life problems

Why You Should Learn It:

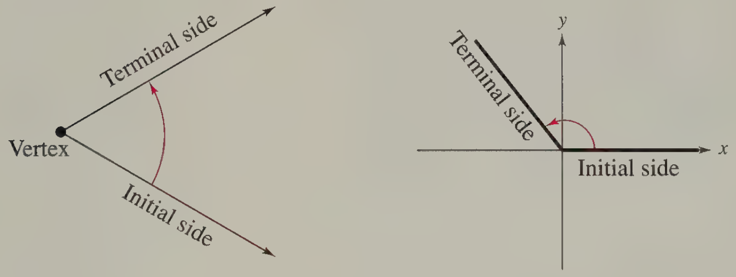
Radian measures of angles are involved in numerous aspects of our daily lives. For instance, in Exercise 89 on page 375, you are asked to determine measures of angles of certain figure-skating jumps.

Bob Martin/Allsport

Angles

As derived from the Greek language, the word **trigonometry** means "measurement of triangles." Initially, trigonometry dealt with relationships among the sides and angles of triangles and was used in the development of astronomy, navigation, and surveying. With the development of calculus and the physical sciences in the 17th century, a different perspective arose—one that viewed the classic trigonometric relationships as *functions* having the set of real numbers as their domains. Consequently, the applications of trigonometry expanded to include a vast number of physical phenomena involving rotations and vibrations, including sound waves, light rays, planetary orbits, vibrating strings, pendulums, and orbits of atomic particles.

The approach in this text incorporates *both* perspectives, starting with angles and their measure.



(a) (b)

Figure 5.1

An **angle** is determined by rotating a ray (half-line) about its endpoint. The starting position of the ray is the **initial side** of the angle, and the position after rotation is the **terminal side,** as shown in Figure 5.1(a). The endpoint of the ray is the **vertex** of the angle. This perception of an angle fits a coordinate system in which the origin is the vertex and the initial side coincides with the positive x-axis. Such an angle is in **standard position,** as shown in Figure 5.1(b). **Positive angles** are generated by counterclockwise rotation, and **negative angles** by clockwise rotation, as shown in Figure 5.2. Angles are labeled with Greek letters α (alpha), β (beta), and θ (theta), as well as uppercase letters A, B, and C. In Figure 5.3, note that angles α and β have the same initial and terminal sides. Such angles are **coterminal.**

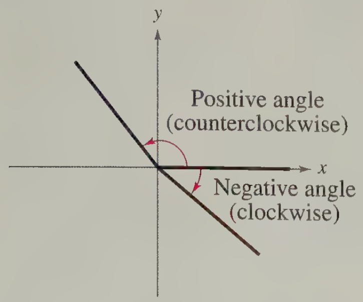


Figure 5.2

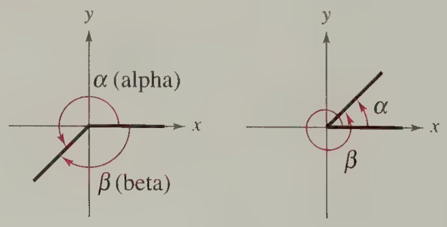


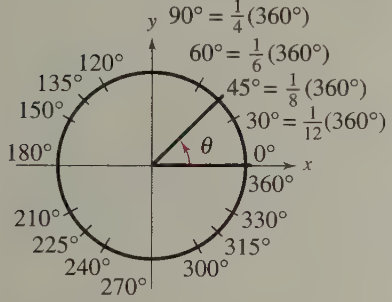
Figure 5.3

Degree Measure

The measure of an angle is determined by the amount of rotation from the initial side to the terminal side. The most common unit of angle measure is the **degree,** denoted by the symbol °. A measure of one degree (1°) is equivalent to a rotation of 1/360 of a complete revolution about the vertex. To measure angles, it is convenient to mark degrees on the circumference of a circle, as shown in Figure 5.4. So, a full revolution (counterclockwise) corresponds to 360°, a half revolution to 180°, a quarter revolution to 90°, and so on.

STUDY T!P

The phrase "the terminal side of θ lies in a quadrant" is often abbreviated by simply saying that "θ lies in a quadrant." The terminal sides of the "quadrant angles" 0°, 90°, 180°, and 270° do not lie within quadrants.



y
$90° = \frac{1}{4}(360°)$
$60° = \frac{1}{6}(360°)$
$45° = \frac{1}{8}(360°)$
$30° = \frac{1}{12}(360°)$
120°
135°
150°
180°
θ
0°
x
360°
210°
225°
240°
270°
300°
315°
330°

Figure 5.4

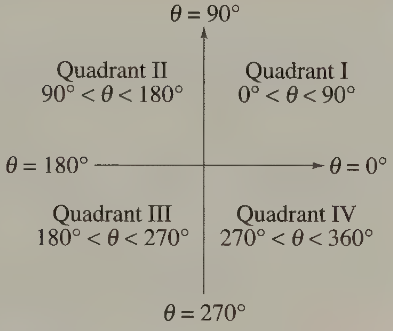
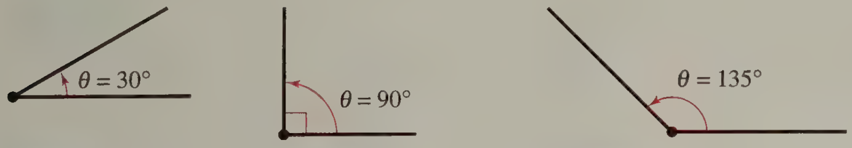


Figure 5.5

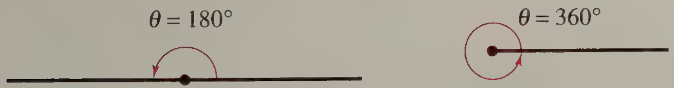
Recall that the four quadrants in a coordinate system are numbered I, II, III, and IV. Figure 5.5 shows which angles between 0° and 360° lie in each of the four quadrants. Figure 5.6 shows several common angles with their degree measures. Note that angles between 0° and 90° are **acute** and angles between 90° and 180° are **obtuse.**



Acute angle: between 0° and 90°

Right angle: quarter revolution

Obtuse angle: between 90° and 180°



Straight angle: half revolution

Full revolution

Figure 5.6

Two angles are coterminal if they have the same initial and terminal sides. For instance, the angles 0° and 360° are coterminal, as are the angles 30° and 390°. You can find an angle that is coterminal to a given angle θ by adding or subtracting 360° (one revolution), as demonstrated in Example 1. A given angle θ has many coterminal angles. For instance, $\theta = 30°$ is coterminal with

$$30° + n(360°)$$

where n is an integer.

EXAMPLE 1 Sketching and Finding Coterminal Angles

Find two coterminal angles (one positive and one negative) for the given angle.

a. 390° **b.** 135° **c.** −120°

Solution

a. For the positive angle 390°, subtract 360° to obtain a positive coterminal angle

$$390° - 360° = 30°. \quad \text{See Figure 5.7(a).}$$

Subtract 2(360°) = 720° to obtain a negative coterminal angle

$$390° - 720° = -330°.$$

b. For the positive angle 135°, add 360° to obtain a positive coterminal angle

$$135° + 360° = 495°.$$

Subtract 360° to obtain a negative coterminal angle

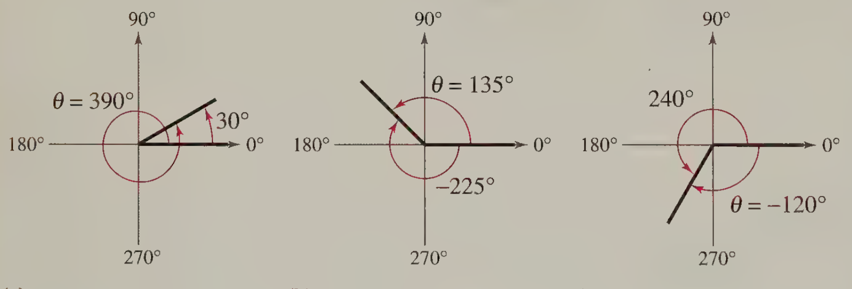
$$135° - 360° = -225°. \quad \text{See Figure 5.7(b).}$$

c. For the negative angle −120°, add 360° to obtain a positive coterminal angle

$$-120° + 360° = 240°. \quad \text{See Figure 5.7(c).}$$

Subtract 360° to obtain a negative coterminal angle

$$-120° - 360° = -480°.$$



(a) (b) (c)

Figure 5.7

STUDY T!P

With calculators it is convenient to use *decimal* degrees to denote fractional parts of degrees. Historically, however, fractional parts of degrees were expressed in *minutes* and *seconds*, using the prime (′) and double prime (″) notations, respectively. That is,

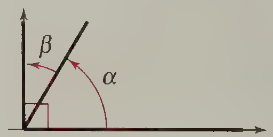
$$1' = \text{one minute} = \tfrac{1}{60}(1°)$$

$$1'' = \text{one second} = \tfrac{1}{3600}(1°).$$

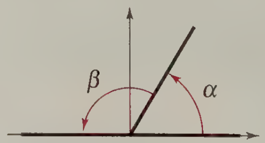
Consequently, an angle of 64 degrees, 32 minutes, and 47 seconds was represented by the notation $\theta = 64°\,32'\,47''$.

Many calculators have special keys for converting angles in degrees, minutes, and seconds (D° M′ S″) to decimal degree form, and vice versa.

Two positive angles α and β are **complementary** (complements of each other) if their sum is 90°. Two positive angles are **supplementary** (supplements of each other) if their sum is 180°. See Figure 5.8.



Complimentary angles



Supplementary angles

Figure 5.8

EXAMPLE 2 Complementary and Supplementary Angles

If possible, find the complement and the supplement of (a) 72° and (b) 148°.

Solution

a. The complement of $\theta = 72°$ is $90° - \theta = 90° - 72° = 18°$.

The supplement of $\theta = 72°$ is $180° - \theta = 180° - 72° = 108°$.

b. Because $\theta = 148°$ is greater than 90°, it has no complement. (Remember that complements are *positive* angles.) The supplement is

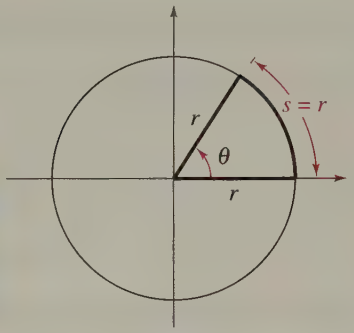
$$180° - \theta = 180° - 148° = 32°.$$

Radian Measure

A second way to measure angles is in radians. This type of measure is especially useful in calculus. To define a radian, you can use a **central angle** of a circle, one whose vertex is the center of the circle, as shown in Figure 5.9.

Definition of a Radian

One **radian** is the measure of a central angle θ that intercepts an arc s equal in length to the radius r of the circle. See Figure 5.9.



Arc length = radius when $\theta = 1$ radian.

Figure 5.9

Because the circumference of a circle is $2\pi r$, it follows that a central angle of one full revolution (counterclockwise) corresponds to an arc length of $s = 2\pi r$. Therefore, 2π radians corresponds to 360°, π radians corresponds to 180°, and $\pi/2$ radians corresponds to 90°. Moreover, because $2\pi \approx 6.28$, there are just over six radius lengths in a full circle, as shown in Figure 5.10. In general, the radian measure of a central angle θ is obtained by dividing the arc length s by r. That is, $s/r = \theta$, where θ is *measured in radians*. Because the units of measure for s and r are the same, this ratio is unitless—it is simply a real number.

STUDY TIP

One revolution around a circle of radius r corresponds to an angle of 2π radians because

$$\frac{s}{r} = \frac{2\pi r}{r} = 2\pi \text{ radians.}$$

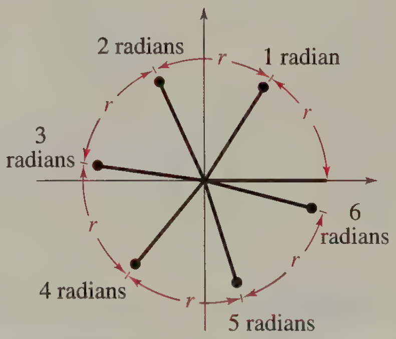


Figure 5.10

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 3 Complementary, Supplementary, and Coterminal Angles

Find each of the following.

a. The angle complementary to $\pi/12$.

b. The angle supplementary to $5\pi/6$.

c. An angle coterminal with $17\pi/6$.

Solution

a. The complement of $\theta = \pi/12$ is $\pi/2 - \theta$, which is

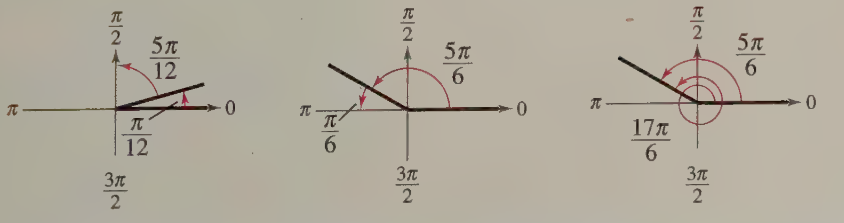
$$\frac{\pi}{2} - \frac{\pi}{12} = \frac{6\pi}{12} - \frac{\pi}{12} = \frac{5\pi}{12}.$$ See Figure 5.11(a).

b. The supplement of $\theta = 5\pi/6$ is $\pi - \theta$, which is

$$\pi - \frac{5\pi}{6} = \frac{6\pi}{6} - \frac{5\pi}{6} = \frac{\pi}{6}.$$ See Figure 5.11(b).

c. In radian measure, a coterminal angle is found by adding or subtracting 2π. For $\theta = 17\pi/6$, subtract 2π to obtain a coterminal angle.

$$\frac{17\pi}{6} - 2\pi = \frac{17\pi}{6} - \frac{12\pi}{6} = \frac{5\pi}{6}$$ See Figure 5.11(c).



(a) (b) (c)

Figure 5.11

Conversion of Angle Measure

Because 2π radians corresponds to one complete revolution, degrees and radians are related by the equations

$$360° = 2\pi \text{ rad} \qquad \text{and} \qquad 180° = \pi \text{ rad}.$$

From the second equation, you obtain

$$1° = \frac{\pi}{180} \text{ rad} \qquad \text{and} \qquad 1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ}$$

which lead to the following conversion rules.

Conversions Between Degrees and Radians

1. To convert degrees to radians, multiply degrees by $\dfrac{\pi \text{ rad}}{180°}$.
2. To convert radians to degrees, multiply radians by $\dfrac{180°}{\pi \text{ rad}}$.

To apply these two conversion rules, use the basic relationship π rad $= 180°$. (See Figure 5.12.)

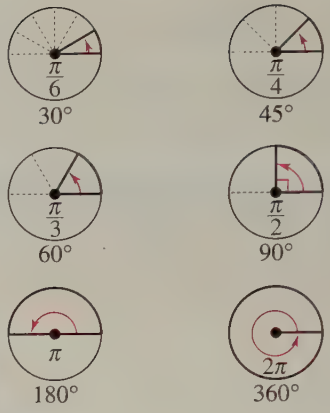


Figure 5.12

EXAMPLE 4 Converting from Degrees to Radians

Express each angle in radian measure.

a. 135° **b.** 540° **c.** −270°

Solution

a. $135° = (135 \text{ deg})\left(\dfrac{\pi \text{ rad}}{180 \text{ deg}}\right) = \dfrac{3\pi}{4} \text{ rad}$ Multiply by $\frac{\pi}{180}$.

b. $540° = (540 \text{ deg})\left(\dfrac{\pi \text{ rad}}{180 \text{ deg}}\right) = 3\pi \text{ rad}$ Multiply by $\frac{\pi}{180}$.

c. $-270° = (-270 \text{ deg})\left(\dfrac{\pi \text{ rad}}{180 \text{ deg}}\right) = -\dfrac{3\pi}{2} \text{ rad}$ Multiply by $\frac{\pi}{180}$.

EXAMPLE 5 Converting from Radians to Degrees

Express each angle in degrees.

a. $-\dfrac{\pi}{2}$ rad **b.** 2 rad

Solution

a. $-\dfrac{\pi}{2} \text{ rad} = \left(-\dfrac{\pi}{2} \text{ rad}\right)\left(\dfrac{180 \text{ deg}}{\pi \text{ rad}}\right) = -90°$ Multiply by $\frac{180}{\pi}$.

b. $2 \text{ rad} = (2 \text{ rad})\left(\dfrac{180 \text{ deg}}{\pi \text{ rad}}\right) = \dfrac{360}{\pi} \approx 114.59°$ Multiply by $\frac{180}{\pi}$.

If you have a calculator with a "radian-to-degree" conversion key, try using it to verify the result shown in part (b) of Example 5.

STUDY TIP

Note that when no units of angle measure are specified, *radian measure is implied.* For instance, if you write $\theta = \pi$ or $\theta = 2$, you should mean $\theta = \pi$ radians or $\theta = 2$ radians.

The *Interactive* CD-ROM and *Internet* versions of this text show every example with its solution; clicking on the *Try It!* button brings up similar problems. Guided Examples and Integrated Examples show step-by-step solutions to additional examples. Integrated Examples are related to several concepts in the section.

Applications

The *radian measure* formula $\theta = s/r$ can be used to measure arc length along a circle. Specifically, for a circle of radius r, a central angle θ intercepts an arc of length s given by

$\frac{\theta}{2\pi} \cdot 2\pi r = S$

$s = r\theta$ Length of circular arc

where θ is measured in radians.

A computer simulation to accompany this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 6 Finding Arc Length

A circle has a radius of 4 inches. Find the length of the arc intercepted by a central angle of 240°, as shown in Figure 5.13.

Solution

To use the formula $s = r\theta$, first convert 240° to radian measure.

$$240° = (240\ \cancel{\text{deg}})\left(\frac{\pi \text{ rad}}{180\ \cancel{\text{deg}}}\right) \quad \text{Convert from degrees to radians.}$$

$$= \frac{4\pi}{3} \text{ radians} \quad \text{Simplify.}$$

Then, using a radius of $r = 4$ inches, you can find the arc length to be

$$s = r\theta \quad \text{Length of circular arc}$$

$$= 4\left(\frac{4\pi}{3}\right) \quad \text{Substitute for } r \text{ and } \theta.$$

$$= \frac{16\pi}{3} \quad \text{Simplify.}$$

$$\approx 16.76 \text{ inches.} \quad \text{Use a calculator.}$$

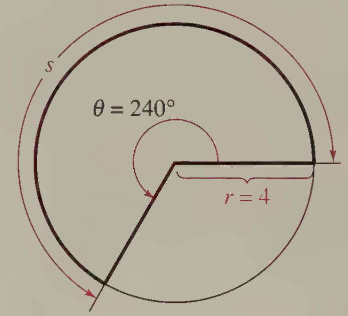


Figure 5.13

$\theta = \frac{s}{r}$, $S = \theta \cdot r$

linear speed $= \frac{S}{t} = \frac{\theta \cdot r}{t}$

ang. speed $= \frac{\theta}{t}$

Note that the units for $r\theta$ are determined by the units for r because θ is given in radian measure and therefore has no units.

The formula for the length of a circular arc can be used to analyze the motion of a particle moving at a *constant speed* along a circular path.

Linear and Angular Speed

Consider a particle moving at a constant speed along a circular arc of radius r. If s is the length of the arc traveled in time t, then the **linear speed** of the particle is

$$\text{Linear speed} = \frac{\text{arc length}}{\text{time}} = \frac{s}{t}.$$

Moreover, if θ is the angle (in radian measure) corresponding to the arc length s, the **angular speed** of the particle is

$$\text{Angular speed} = \frac{\text{central angle}}{\text{time}} = \frac{\theta}{t}.$$

$\frac{\theta}{t} \cdot r = \frac{\theta r}{t} = \frac{S}{t}$

EXAMPLE 7 Finding Linear Speed

how fast is it moving

The second hand of a clock is 10.2 centimeters long, as shown in Figure 5.14. Find the linear speed of the tip of this second hand.

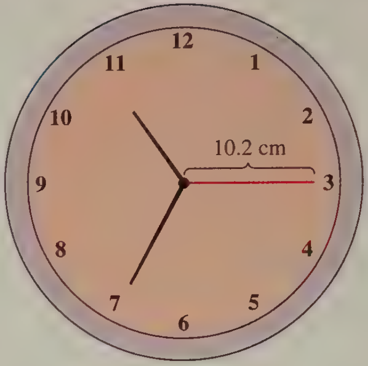


Figure 5.14

Solution

In one revolution, the arc length traveled is

$$\begin{aligned} s &= 2\pi r \\ &= 2\pi(10.2) \\ &= 20.4\pi \text{ centimeters.} \end{aligned}$$

The time required for the second hand to travel this distance is

$$\begin{aligned} t &= 60 \text{ seconds} \\ &= 1 \text{ minute.} \end{aligned}$$

So, the linear speed of the tip of the second hand is

$$\begin{aligned} \text{Linear speed} &= \frac{s}{t} \\ &= \frac{20.4\pi \text{ centimeters}}{60 \text{ seconds}} \\ &\approx 1.068 \text{ cm/sec.} \end{aligned}$$

EXAMPLE 8 Finding Angular and Linear Speed

A 10-inch radius lawn roller makes 1.2 revolutions per second, as shown in Figure 5.15.

a. Find the angular speed of the roller in radians per second.

b. Find the speed of the tractor that is pulling the roller.

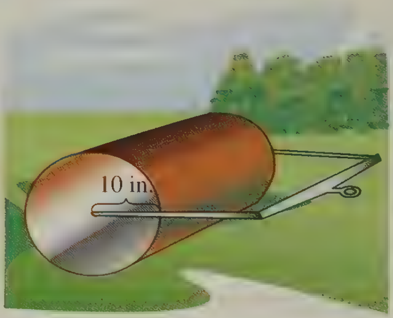


Figure 5.15

Solution

a. Because each revolution generates 2π radians, it follows that the roller turns $(1.2)(2\pi) = 2.4\pi$ radians per second. In other words, the angular speed is

$$\begin{aligned} \text{Angular speed} &= \frac{\theta}{t} \\ &= \frac{2.4\pi \text{ radians}}{1 \text{ second}} \\ &= 2.4\pi \text{ rad/sec.} \end{aligned}$$

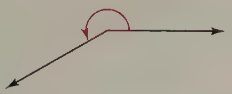
b. The linear speed is

$$\begin{aligned} \text{Linear speed} &= \frac{s}{t} = \frac{r\theta}{t} \\ &= \frac{10(2.4\pi) \text{ inches}}{1 \text{ second}} \\ &\approx 75.4 \text{ in./sec.} \end{aligned}$$

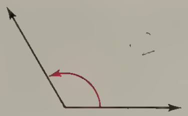
5.1 Exercises

In Exercises 1–4, estimate the number of degrees in the angle.

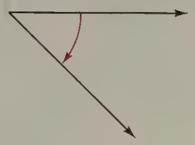
1.



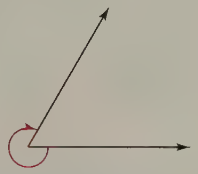
2.



3.



4.



In Exercises 5–8, determine the quadrant in which the angle lies.

5. (a) 150° 2 (b) 282° 4

6. (a) 7.9° 1 (b) 257.5° 3

7. (a) −132° 50′ 3 (b) −336° 30′

8. (a) −260.25° 2 (b) −2.4° 4

In Exercises 9–12, sketch the angle in standard position.

9. (a) 60° (b) 150°

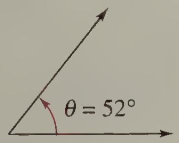
10. (a) −270° (b) −120°

11. (a) 405° (b) 780°

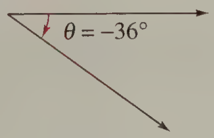
12. (a) −450° (b) −570°

In Exercises 13–16, determine two coterminal angles in degree measure (one positive and one negative) for the given angle.

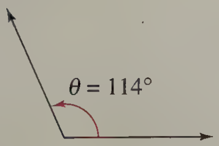
13. (a)



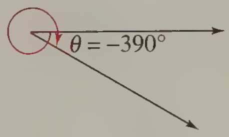
(b)



14. (a)



(b)



15. (a) 300° (b) 230°

16. (a) −445° (b) −740°

In Exercises 17–20, use the angle-conversion capabilities of a graphing utility to convert the angle measure to decimal degree form. Round your answer to three decimal places if necessary.

17. (a) 64° 45′ (b) −124° 30′

18. (a) 275° 10′ (b) 9° 12′

19. (a) 85° 18′ 30″ (b) −408° 16′ 25″

20. (a) −125° 36″ (b) 330° 25″

In Exercises 21–24, use the angle-conversion capabilities of a graphing utility to convert the angle measure to D° M′ S″ form.

21. (a) 280.6° (b) −115.8°

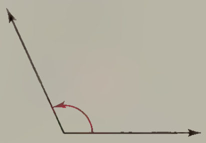
22. (a) −345.12° (b) 310.75°

23. (a) 4.5 (b) −3.58

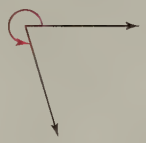
24. (a) −0.355 (b) 0.7865

In Exercises 25–28, estimate the angle to the nearest one-half radian.

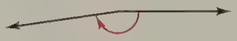
25.



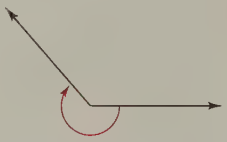
26.



27.



28.



In Exercises 29–34, determine the quadrant in which the angle lies. (The angle is given in radian measure.)

29. (a) $\frac{\pi}{5}$ (b) $\frac{7\pi}{5}$

30. (a) $\frac{7\pi}{4}$ (b) $\frac{11\pi}{4}$

31. (a) $-\frac{\pi}{12}$ (b) $-\frac{11\pi}{9}$

32. (a) −1 (b) −2

33. (a) 3.5 (b) 2.25

34. (a) 5.63 (b) −2.25

The *Interactive* CR-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered Section and Review Exercises. They also provide Tutorial Exercises, which link to Guided Examples for additional help.

In Exercises 35–38, sketch the angle in standard position.

35. (a) $\frac{3\pi}{4}$ (b) $\frac{5\pi}{3}$

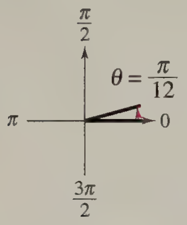
36. (a) $-\frac{7\pi}{4}$ (b) $-\frac{7\pi}{2}$

37. (a) $\frac{11\pi}{6}$ (b) 9π

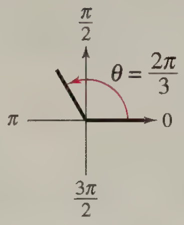
38. (a) 4 (b) -3

In Exercises 39–42, determine two coterminal angles in radian measure (one positive and one negative) for the given angle.

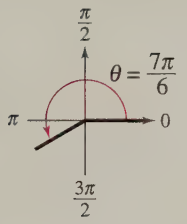
39. (a)



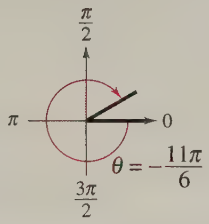
(b)



40. (a)



(b)



41. (a) $-\frac{11\pi}{4}$ (b) $-\frac{2\pi}{15}$

42. (a) $\frac{7\pi}{8}$ (b) $\frac{8\pi}{35}$

In Exercises 43–46, find (if possible) the complement and supplement of the angle.

43. (a) 24° (b) 126°

44. (a) 87° (b) 167°

45. (a) $\frac{\pi}{3}$ (b) $\frac{3\pi}{4}$

46. (a) 1 (b) 2

In Exercises 47–50, express the angle in radian measure as a multiple of π. (Do not use a calculator.)

47. (a) 30° (b) 150°

48. (a) 315° (b) 120°

49. (a) $-20°$ (b) $-240°$

50. (a) $-270°$ (b) 144°

In Exercises 51–54, express the angle in degree measure. (Do not use a calculator.)

51. (a) $\frac{3\pi}{2}$ (b) $-\frac{7\pi}{6}$

52. (a) $-\frac{7\pi}{12}$ (b) $\frac{\pi}{9}$

53. (a) $\frac{7\pi}{3}$ (b) $-\frac{13\pi}{60}$

54. (a) $\frac{15\pi}{6}$ (b) $\frac{28\pi}{15}$

In Exercises 55–62, convert the angle measure from degrees to radians. Round your answer to three decimal places.

55. 115° **56.** 83.7°

57. $-216.35°$ **58.** $-46.52°$

59. 642° **60.** 0.54°

61. $-0.78°$ **62.** 395°

In Exercises 63–70, convert the angle measure from radians to degrees. Round your answer to three decimal places.

63. $\frac{\pi}{7}$ **64.** $\frac{8\pi}{13}$

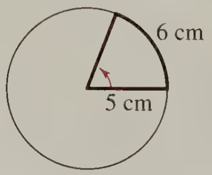
65. $\frac{25\pi}{8}$ **66.** 6.5π

67. -4.2π **68.** 4.8

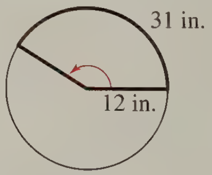
69. -2 **70.** -0.48

In Exercises 71–74, find the angle in radians.

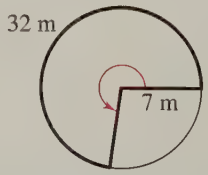
71.



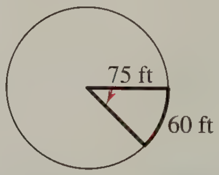
72.



73.



74.



In Exercises 75–78, find the radian measure of the central angle of a circle of radius r that intercepts an arc of length s.

	Radius r	*Arc Length s*
75.	15 inches	8 inches
76.	22 feet	10 feet
77.	14.5 centimeters	35 centimeters
78.	80 kilometers	160 kilometers

In Exercises 79–82, find the length of the arc on a circle of radius r intercepted by a central angle θ.

	Radius r	*Central Angle θ*
79.	14 inches	180°
80.	9 feet	60°
81.	6 meters	$\frac{2\pi}{3}$ radians
82.	40 centimeters	$\frac{3\pi}{4}$ radians

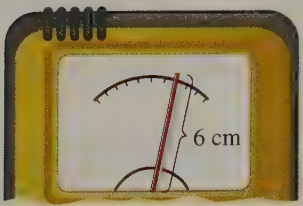
Distance Between Cities **In Exercises 83 and 84, find the distance between the cities. Assume that the earth is a sphere of radius 4000 miles and the cities are on the same meridian (one city is due north of the other).**

	City	*Latitude*
83.	Miami	$25^\circ 46' 37''$N
	Erie	$42^\circ 7' 15''$N
84.	Johannesburg, South Africa	$26^\circ 10'$S
	Jerusalem, Israel	$31^\circ 47'$N

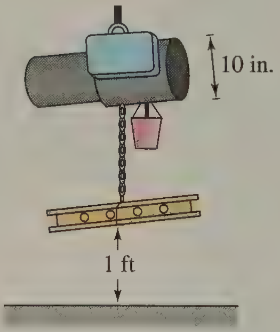
85. ***Difference in Latitudes*** Assuming that the earth is a sphere of radius 6378 kilometers, what is the difference in latitude of two cities, one of which is 600 kilometers due north of the other?

86. ***Difference in Latitudes*** Assuming that the earth is a sphere of radius 6378 kilometers, what is the difference in latitude of two cities, one of which is 800 kilometers due north of the other?

87. ***Instrumentation*** A voltmeter's pointer is 6 centimeters in length. Find the angle through which it rotates when it moves 2.5 centimeters on the scale.



88. ***Electric Hoist*** An electric hoist is used to lift a piece of equipment 1 foot. The diameter of the drum on the hoist is 10 inches. Find the number of degrees through which the drum must rotate.



89. ***Figure Skating*** The number of revolutions made by a figure skater for each type of axel jump is given. Determine the measure of the angle generated as the skater performs each jump. Give the answer in both degrees and radians.

(a) single axel: $1\frac{1}{2}$

(b) double axel: $2\frac{1}{2}$

(c) triple axel: $3\frac{1}{2}$

90. ***Linear Speed of an Earth Satellite*** An earth satellite in circular orbit 1250 kilometers high makes one complete revolution every 90 minutes. What is its linear speed? Use 6400 kilometers for the length of a radius of the earth.

91. ***Angular Speed*** A car is moving at a rate of 40 miles per hour, and the diameter of its wheels is 2.5 feet.

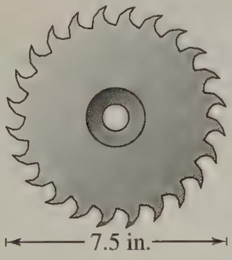
(a) Find the rotational speed of the wheels in revolutions per minute.

(b) Find the angular speed of the wheels in radians per minute.

92. ***Circular Saw Speed*** The circular blade on a saw has a diameter of 7.5 inches and rotates at 2400 revolutions per minute.

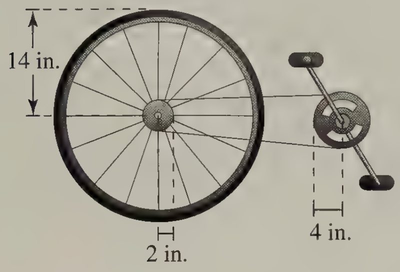
(a) Find the angular speed in radians per second.

(b) Find the linear speed of the saw teeth (in feet per second) as they contact the wood being cut.



93. ***Floppy Disk*** The radius of the magnetic disk in a 3.5-inch diskette is 1.68 inches. Find the linear speed of a point on the circumference of the disk if it is rotating at a speed of 360 revolutions per minute.

94. ***Speed of a Bicycle*** The radii of the sprocket assemblies on the pedal and the wheel of a bicycle are 4 inches and 2 inches, respectively. The radius of the rear wheel of the bicycle is 14 inches. If the cyclist is pedaling at a rate of 1 revolution per second, find the speed of the bicycle in (a) feet per second and (b) miles per hour.



Synthesis

True or False? **In Exercises 95–97, determine whether the statement is true or false. Justify your answer.**

95. A degree is a larger unit of measure than a radian.

96. An angle that measures $-1260°$ lies in Quadrant III.

97. The angles of a triangle can have radian measures

$$\frac{2\pi}{3}, \frac{\pi}{4}, \text{ and } \frac{\pi}{12}.$$

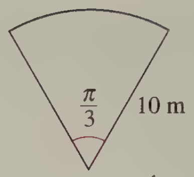
98. ***Writing*** If the radius of a circle is increasing and the magnitude of a central angle is held constant, how is the length of the intercepted arc changing? Explain your reasoning.

99. ***Writing*** Write a short paragraph defining and giving examples of coterminal angles.

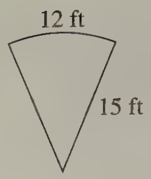
100. ***Geometry*** Show that the area of a circular sector of radius r with central angle θ is $A = \frac{1}{2}r^2\theta$, where θ is measured in radians.

Area of a Circular Sector **In Exercises 101 and 102, use the result of Exercise 100 to find the area of the sector.**

101.



102.



103. ***Graphical Reasoning*** The formulas for the area of a circular sector and arc length are $A = \frac{1}{2}r^2\theta$ and $s = r\theta$, respectively. (r is the radius and θ is the angle measured in radians.)

(a) If $\theta = 0.8$, express the area and arc length as functions of r. What is the domain of each function? Use a graphing utility to graph the functions. Use the graphs to determine which function changes more rapidly as r increases. Explain.

(b) If $r = 10$ centimeters, express the area and arc length as functions of θ. What is the domain of each function? Use a graphing utility to graph and identify the functions.

Review

In Exercises 104–107, use a graphing utility to find a linear model that approximates the set of data.

104.

x	2	3	4	5	6	7
y	25	31	33	40	45	47

105.

x	-3	-2	-1	0	1	2
y	58.3	49.9	45.0	38.1	33.2	23.6

106.

x	10	11	12	13	14	15
y	0.05	1.24	3.33	4.14	4.98	5.82

107.

x	-1	0	1	2	3	4
y	246.22	174.58	142	112.95	82.46	52.61

In Exercises 108–111, find all the real zeros of the polynomial function.

108. $f(x) = x^2 + 11x + 28$

109. $f(x) = 54x^2 - 6x^4$

110. $f(x) = x^3 + 3x^2 - 10x$

111. $f(x) = 4x^4 + 44x^3 + 96x^2$

5.2 Right Triangle Trigonometry

The Six Trigonometric Functions

Our first look at the trigonometric functions is from a *right triangle* perspective. Consider a right triangle, one of whose acute angles is labeled θ, as shown in Figure 5.16. Relative to the angle θ, the three sides of the triangle are the **hypotenuse,** the **opposite side** (the side opposite the angle θ), and the **adjacent side** (the side adjacent to the angle θ).

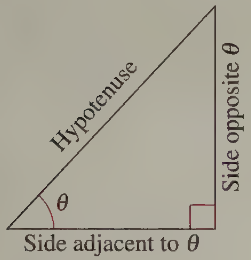


Figure 5.16

Using the lengths of these three sides, you can form six ratios that define the six trigonometric functions of the acute angle θ.

sine **cosecant** **cosine** **secant** **tangent** **cotangent**

These six functions are normally abbreviated as sin, csc, cos, sec, tan, and cot, respectively. In the following definitions it is important to see that $0° < \theta < 90°$ and that for such angles the value of each trigonometric function is *positive.*

Right Triangle Definitions of Trigonometric Functions

Let θ be an *acute* angle of a right triangle. Then the six trigonometric functions *of the angle* θ are defined as follows.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

The abbreviations "opp," "adj," and "hyp" represent the lengths of the three sides of a right triangle.

opp = the length of the side *opposite* θ

adj = the length of the side *adjacent* to θ

hyp = the length of the *hypotenuse*

Note that the functions in the second row above are the *reciprocals* of the corresponding functions in the first row.

What You Should Learn:

- How to evaluate trigonometric functions of acute angles
- How to use the fundamental trigonometric identities
- How to use a calculator to evaluate trigonometric functions
- How to use trigonometric functions to model and solve real-life problems

Why You Should Learn It:

You can use trigonometry to analyze all aspects of a geometric figure. For instance, Exercise 69 on page 386 show you how trigonometric functions can be used to approximate the height of a building.

Chen Yixin/China Stock

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 1 Evaluating Trigonometric Functions

Find the values of the six trigonometric functions of θ, as shown in Figure 5.17.

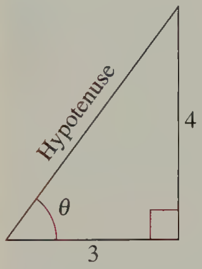


Figure 5.17

Solution

By the Pythagorean Theorem, $(\text{hyp})^2 = (\text{opp})^2 + (\text{adj})^2$, it follows that

$$\text{hyp} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5.$$

So, the six trigonometric functions of θ are

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{4} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{3} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{4}.$$

Georg Joachim Rhaeticus (1514–1576), the leading Teutonic mathematical astronomer of the 16th century, was the first to define the trigonometric functions as ratios of the sides of a right triangle.

In Example 1, you were given the lengths of the sides of the right triangle but not the angle θ. A more common problem in trigonometry is finding the trigonometric functions for a *given* acute angle θ. To do this, you can construct a right triangle having θ as one of its angles.

EXAMPLE 2 Evaluating Trigonometric Functions of 45°

Find the values of sin 45°, cos 45°, and tan 45°.

Solution

Construct a right triangle having 45° as one of its acute angles, as shown in Figure 5.18. Choose 1 as the length of the adjacent side. From geometry, you know that the other acute angle is also 45°. So, the triangle is isosceles, and the length of the opposite side is also 1. Using the Pythagorean Theorem, you find the length of the hypotenuse to be $\sqrt{2}$.

$$\sin 45^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \cos 45^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \frac{\text{opp}}{\text{adj}} = \frac{1}{1} = 1$$

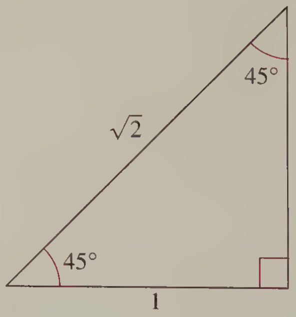


Figure 5.18

EXAMPLE 3 Evaluating Trigonometric Functions of 30° and 60°

Use the equilateral triangle shown in Figure 5.19 to find the values of sin 60°, cos 60°, sin 30°, and cos 30°.

Solution

Try using the Pythagorean Theorem and the equilateral triangle to verify the lengths of the sides given in Figure 5.19. For $\theta = 60°$, you have $\text{adj} = 1$, $\text{opp} = \sqrt{3}$, and $\text{hyp} = 2$. Therefore,

$$\sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}$$

and

$$\cos 60° = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}.$$

For $\theta = 30°$, $\text{adj} = \sqrt{3}$, $\text{opp} = 1$, and $\text{hyp} = 2$. So,

$$\sin 30° = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2}$$

and

$$\cos 30° = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2}.$$

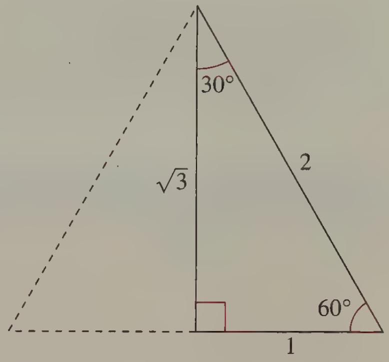


Figure 5.19

Because the angles 30°, 45°, and 60° ($\pi/6$, $\pi/4$, and $\pi/3$) occur frequently in trigonometry, you should learn to construct the triangles shown in Figures 5.18 and 5.19.

Sines, Cosines, and Tangents of Special Angles

$$\sin 30° = \sin\frac{\pi}{6} = \frac{1}{2} \qquad \cos 30° = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \tan 30° = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$$

$$\sin 45° = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \cos 45° = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \tan 45° = \tan\frac{\pi}{4} = 1$$

$$\sin 60° = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos 60° = \cos\frac{\pi}{3} = \frac{1}{2} \qquad \tan 60° = \tan\frac{\pi}{3} = \sqrt{3}$$

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

In the box, note that $\sin 30° = \frac{1}{2} = \cos 60°$. This occurs because 30° and 60° are complementary angles, and, in general, it can be shown from the right triangle definitions that *cofunctions of complementary angles are equal.* That is, if θ is an acute angle, the following relationships are true.

$$\sin(90° - \theta) = \cos\theta \qquad \cos(90° - \theta) = \sin\theta$$

$$\tan(90° - \theta) = \cot\theta \qquad \cot(90° - \theta) = \tan\theta$$

$$\sec(90° - \theta) = \csc\theta \qquad \csc(90° - \theta) = \sec\theta$$

Trigonometric Identities

In trigonometry, a great deal of time is spent studying relationships between trigonometric functions (identities).

Fundamental Trigonometric Identities

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Exploration

Select a number t and use your graphing utility to calculate $(\cos t)^2 + (\sin t)^2$. Repeat this experiment for other values of t and explain why the answer is always the same. Is the result true in both radian and degree modes?

Note that $\sin^2\theta$ represents $(\sin\theta)^2$ and not $\sin(\theta)^2$, $\cos^2\theta$ represents $(\cos\theta)^2$ and not $\cos(\theta^2)$, and so on.

EXAMPLE 4 Applying Trigonometric Identities

Let θ be an acute angle such that $\sin\theta = 0.6$. Using trigonometric identities, find the values of (a) $\cos\theta$ and (b) $\tan\theta$.

Solution

a. To find the value of $\cos\theta$, use the Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1.$$

So, you have

$$(0.6)^2 + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - (0.6)^2 = 0.64$$

$$\cos\theta = \sqrt{0.64} = 0.8.$$

b. Now, knowing the sine and cosine of θ, you can find the tangent of θ to be

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{0.6}{0.8} = 0.75.$$

Try using the definitions of $\cos\theta$ and $\tan\theta$, and the triangle shown in Figure 5.20, to check these results.

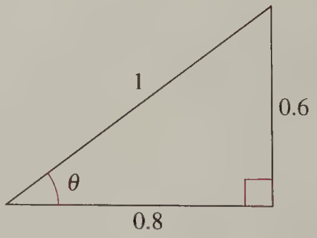


Figure 5.20

EXAMPLE 5 Using Trigonometric Identities

Use trigonometric identities to transform one side of the equation into the other.

a. $\cos\theta\sec\theta = 1$ **b.** $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

Solution

a.

$\cos\theta\sec\theta = 1$	Write original equation.
$\left(\dfrac{1}{\cancel{\sec\theta}}\right)\cancel{\sec}\,\theta = 1$	Reciprocal identity
$1 = 1$	Divide out common factor.

b.

$(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$	Write original equation.
$\sec^2\theta - \sec\theta\tan\theta + \sec\theta\tan\theta - \tan^2\theta = 1$	Distributive Property
$\sec^2\theta - \tan^2\theta = 1$	Simplify.
$1 = 1$	Pythagorean identity

Evaluating Trigonometric Functions with a Calculator

When evaluating a trigonometric function with a calculator, you need to set the calculator to the desired *mode* of measurement (degrees or radians).

Most calculators do not have keys for the cosecant, secant, or cotangent functions. To evaluate these functions, you can use the $\boxed{x^{-1}}$ key with their respective reciprocal functions sine, cosine, and tangent. For example, to evaluate $\csc(\pi/8)$, use the fact that

$$\csc\frac{\pi}{8} = \frac{1}{\sin(\pi/8)}$$

and enter the following keystroke sequence in *radian* mode.

$\boxed{(}\ \boxed{\text{SIN}}\ \boxed{(}\ \pi\ \boxed{\div}\ 8\ \boxed{)}\ \boxed{)}\ \boxed{x^{-1}}\ \boxed{\text{ENTER}}$ Display 2.6131259

EXAMPLE 6 Using a Calculator

Function	*Mode*	*Graphing Calculator Keystrokes*	*Display*
a. $\sin 76.4°$	Degree	$\boxed{\text{SIN}}\ 76.4\ \boxed{\text{ENTER}}$	0.9719610
b. $\cot 1.5$	Radian	$\boxed{(}\ \boxed{\text{TAN}}\ 1.5\ \boxed{)}\ \boxed{x^{-1}}\ \boxed{\text{ENTER}}$	0.0709148

When evaluating trigonometric functions with a calculator, remember to enclose all fractional angle measures in parentheses. For instance, if you want to evaluate $\sin\theta$ for $\theta = \pi/6$, you should enter

$\boxed{\text{SIN}}\ \boxed{(}\ \boxed{\pi}\ \boxed{\div}\ 6\ \boxed{)}\ \boxed{\text{ENTER}}$

rather than

$\boxed{\text{SIN}}\ \boxed{\pi}\ \boxed{\div}\ 6\ \boxed{\text{ENTER}}$.

The first set of keystrokes yields the correct value of 0.5. The second set yields the incorrect value of 0.

Applications Involving Right Triangles

Many applications of trigonometry involve a process called **solving right triangles.** In this type of application, you are usually given one side of a right triangle and one of the acute angles and asked to find one of the other sides, *or* you are given two sides and asked to find one of the acute angles. In Example 7, the angle you are given is the **angle of elevation,** which denotes the angle from the horizontal upward to the object. In other applications you may be given the **angle of depression,** which denotes the angle from the horizontal downward to the object.

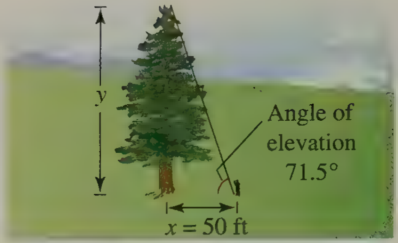


Figure 5.21

EXAMPLE 7 Solving a Right Triangle

A surveyor is standing 50 feet from the base of a large tree, as shown in Figure 5.21. The surveyor measures the angle of elevation to the top of the tree as 71.5°. How tall is the tree?

Algebraic Solution

From Figure 5.21, you can see that

$$\tan 71.5° = \frac{\text{opp}}{\text{adj}}$$

$$= \frac{y}{x}$$

where $x = 50$ and y is the height of the tree. So, the height of the tree is

$$y = x \tan 71.5°$$

$$\approx 50(2.98868)$$

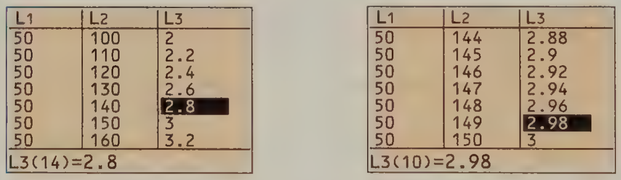
$$\approx 149.4 \text{ feet.}$$

Numerical Solution

Use a graphing utility to create a table, as shown in Figure 5.22(a). The first column is the base of the triangle, $x = 50$. The second column is the height y, which varies from 10 to 200. The quotient y/x is shown in the third column. Because

$$\tan 71.5 = \frac{y}{x} \approx 2.9887$$

you can compare 2.9887 to the value of y/x in the table. Because 2.9887 is between 2.8 and 3, the height y is between 140 and 150. Create a second table that shows the value of y/x when the values of y increase from 140 to 150. Because 2.9887 is between 2.98 and 3, the height is between 149 and 150 feet, as shown in Figure 5.22(b). Repeat the process and create a table that shows values of y/x when the values of y increase from 149.1 to 150. The result is an estimate of the tree height of 149.4 feet.



```
L1    L2    L3
50    100   2
50    110   2.2
50    120   2.4
50    130   2.6
50    140   2.8
50    150   3
50    160   3.2
L3(14)=2.8
```

(a)

```
L1    L2    L3
50    144   2.88
50    145   2.9
50    146   2.92
50    147   2.94
50    148   2.96
50    149   2.98
50    150   3
L3(10)=2.98
```

(b)

Figure 5.22

EXAMPLE 8 Solving a Right Triangle

A person is 200 yards from a river. Rather than walk directly to the river, the person walks 400 yards along a straight path to the river's edge. Find the acute angle θ between this path and the river's edge, as indicated in Figure 5.23.

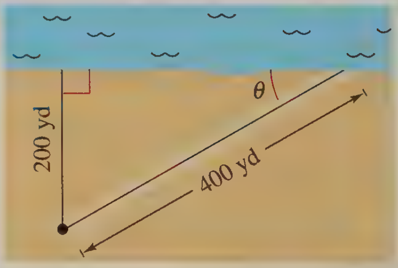


Figure 5.23

Solution

From Figure 5.23, you can see that the sine of the angle θ is

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{200}{400} = \frac{1}{2}.$$

Now, you should recognize that $\theta = 30°$.

In Example 8, you were able to recognize that $\theta = 30°$ is the acute angle that satisfies the equation $\sin\theta = \frac{1}{2}$. Suppose, however, that you were given the equation $\sin\theta = 0.6$ and asked to find the acute angle θ. Because

$$\sin 30° = \frac{1}{2} = 0.5000$$

and

$$\sin 45° = \frac{1}{\sqrt{2}} \approx 0.7071,$$

you might guess that θ lies somewhere between 30° and 45°. A more precise value of θ can be found by using the *inverse* key on a calculator. To do this, you can use the following keystroke sequence in *degree* mode.

[SIN⁻¹] .6 [ENTER]   Display 36.8699

So, you can conclude that if $\sin\theta = 0.6$, then $\theta \approx 36.87°$.

STUDY T!P

Calculators and graphing utilities have both *degree* and *radian* modes. As you progress through this chapter, be sure you use the correct mode.

EXAMPLE 9 Solving a Right Triangle

Specifications for a skateboard ramp require a rise of one foot for each three feet of horizontal length. In Figure 5.24, find the lengths of sides b and c and find the measure of θ.

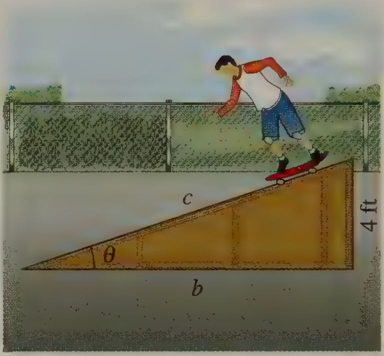


Figure 5.24

Solution

From the given specifications, you can write

$$\frac{\text{rise}}{\text{run}} = \frac{1}{3} = \frac{4}{b}\text{ ft}$$

which implies that $b = 12$ feet. Using the Pythagorean Theorem, you can write

$$c^2 = a^2 + b^2 \qquad \text{Pythagorean Theorem}$$

$$c^2 = 4^2 + 12^2 \qquad \text{Substitute for } a \text{ and } b.$$

$$c^2 = 160 \qquad \text{Simplify.}$$

$$c = 4\sqrt{10}. \qquad \text{Extract positive square root.}$$

So, $c = 4\sqrt{10} \approx 12.65$ feet. To solve for θ, you can write

$$\tan\theta = \frac{4}{12} = \frac{1}{3}.$$

Then, using the calculator keystrokes

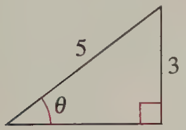
[tan⁻¹] [(] 1 [÷] 3 [)] [ENTER]

you obtain $\theta \approx 18.43°$.

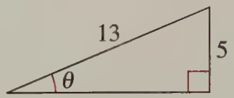
5.2 Exercises

In Exercises 1–6, find the exact values of the six trigonometric functions of the angle θ given in the figure. (Use the Pythagorean Theorem to find the third side of the triangle.)

1.



2.



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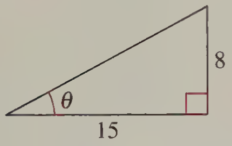
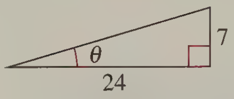
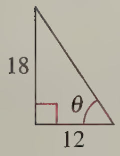


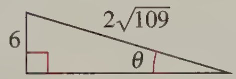
4.



5.

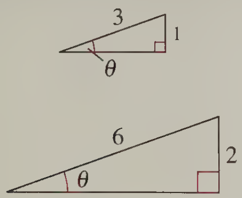


6.



In Exercises 7–10, find the exact values of the six trigonometric functions of the angle θ for each of the triangles. Explain why the function values are the same.

7.



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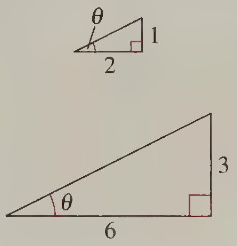
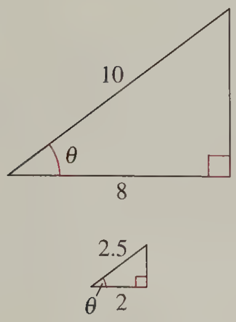
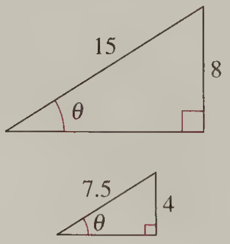


9.



10.



In Exercises 11–18, sketch a right triangle corresponding to the trigonometric function of the acute angle θ. Use the Pythagorean Theorem to determine the third side and then find the other five trigonometric functions of θ.

11. $\sin\theta = \frac{5}{6}$

12. $\cot\theta = 5$

13. $\sec\theta = 4$

14. $\cos\theta = \frac{3}{7}$

15. $\tan\theta = 3$

16. $\csc\theta = \frac{17}{4}$

17. $\cot\theta = \frac{9}{4}$

18. $\sin\theta = \frac{3}{8}$

In Exercises 19–24, use the given function value(s) and trigonometric identities to find the indicated trigonometric functions.

19. $\sin 60° = \dfrac{\sqrt{3}}{2}, \quad \cos 60° = \dfrac{1}{2}$

(a) $\tan 60°$ (b) $\sin 30°$

(c) $\cos 30°$ (d) $\cot 60°$

20. $\sin 30° = \dfrac{1}{2}, \quad \tan 30° = \dfrac{\sqrt{3}}{3}$

(a) $\csc 30°$ (b) $\cot 60°$

(c) $\cos 30°$ (d) $\cot 30°$

21. $\csc\theta = 3, \quad \sec\theta = \dfrac{3\sqrt{2}}{4}$

(a) $\sin\theta$ (b) $\cos\theta$

(c) $\tan\theta$ (d) $\sec(90° - \theta)$

22. $\sec\theta = 5, \quad \tan\theta = 2\sqrt{6}$

(a) $\cos\theta$ (b) $\cot\theta$

(c) $\cot(90° - \theta)$ (d) $\sin\theta$

23. $\cos\alpha = \frac{1}{4}$

(a) $\sec\alpha$ (b) $\sin\alpha$

(c) $\cot\alpha$ (d) $\sin(90° - \alpha)$

24. $\tan\beta = 5$

(a) $\cot\beta$ (b) $\cos\beta$

(c) $\tan(90° - \beta)$ (d) $\csc\beta$

In Exercises 25–32, use trigonometric identities to transform one side of the equation into the other.

25. $\tan\theta\cot\theta = 1$

26. $\csc\theta\tan\theta = \sec\theta$

27. $\tan\alpha\cos\alpha = \sin\alpha$

28. $\cot\alpha\sin\alpha = \cos\alpha$

29. $(1 + \cos\theta)(1 - \cos\theta) = \sin^2\theta$

30. $(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = 1$

31. $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \csc\theta\sec\theta$

32. $\dfrac{\tan\beta + \cot\beta}{\tan\beta} = \csc^2\beta$

In Exercises 33–38, evaluate the trigonometric function by hand.

33. (a) $\cos 60°$ (b) $\tan\dfrac{\pi}{6}$

34. (a) $\csc 30°$ (b) $\sin\dfrac{\pi}{4}$

35. (a) $\cot\dfrac{\pi}{4}$ (b) $\cos 45°$

36. (a) $\sin\dfrac{\pi}{3}$ (b) $\csc 45°$

37. (a) $\cos\dfrac{\pi}{6}$ (b) $\sec 60°$

38. (a) $\tan\dfrac{\pi}{3}$ (b) $\cot 30°$

In Exercises 39–46, use a calculator to evaluate each function. Round your answers to four decimal places. (Be sure the calculator is in the correct mode.)

39. (a) $\sin 25°$ (b) $\cos 65°$

40. (a) $\tan 18.5°$ (b) $\cot 71.5°$

41. (a) $\sec 42°\,12'$ (b) $\csc 48°\,7'$

42. (a) $\cos 8°\,50'\,25''$ (b) $\sec 8°\,50'\,25''$

43. (a) $\cot\dfrac{\pi}{16}$ (b) $\tan\dfrac{\pi}{16}$

44. (a) $\sec 1.25$ (b) $\cos 1.25$

45. (a) $\csc 1$ (b) $\tan\frac{1}{2}$

46. (a) $\sec\left(\dfrac{\pi}{2} - 1\right)$ (b) $\cot\left(\dfrac{\pi}{2} - \dfrac{1}{2}\right)$

In Exercises 47–52, find the value of θ in degrees ($0° < \theta < 90°$) and radians ($0 < \theta < \pi/2$) without using a calculator.

47. (a) $\sin\theta = \frac{1}{2}$ (b) $\csc\theta = 2$

48. (a) $\cos\theta = \dfrac{\sqrt{2}}{2}$ (b) $\tan\theta = 1$

49. (a) $\sec\theta = 2$ (b) $\cot\theta = 1$

50. (a) $\tan\theta = \sqrt{3}$ (b) $\cos\theta = \frac{1}{2}$

51. (a) $\csc\theta = \dfrac{2\sqrt{3}}{3}$ (b) $\sin\theta = \dfrac{\sqrt{2}}{2}$

52. (a) $\cot\theta = \dfrac{\sqrt{3}}{3}$ (b) $\sec\theta = \sqrt{2}$

In Exercises 53–56, find the value of θ in degrees ($0° < \theta < 90°$) and radians ($0 < \theta < \pi/2$) by using a calculator.

53. (a) $\sin\theta = 0.8191$ (b) $\cos\theta = 0.0175$

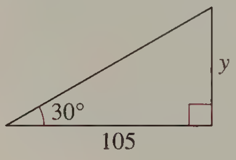
54. (a) $\cos\theta = 0.9848$ (b) $\cos\theta = 0.8746$

55. (a) $\tan\theta = 1.1920$ (b) $\tan\theta = 0.4663$

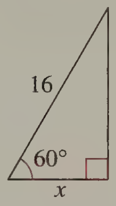
56. (a) $\sin\theta = 0.3746$ (b) $\cos\theta = 0.3746$

In Exercises 57–62, solve for x, y, or r, as indicated.

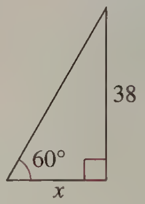
57. Solve for y.



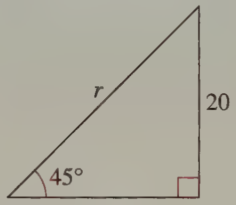
58. Solve for x.



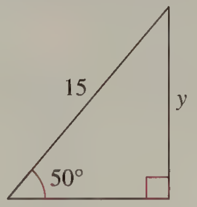
59. Solve for x.



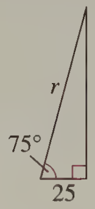
60. Solve for r.



61. Solve for y.



62. Solve for r.



63. ***Height*** A 6-foot person walked from the base of a broadcasting tower directly toward the tip of the shadow cast by the tower. When the person was 132 feet from the tower and 3 feet from the tip of the tower's shadow, the person's shadow started to appear beyond the tower's shadow.

(a) Draw a right triangle to represent the problem. Show the known quantities on the triangle and use a variable to indicate the unknown height of the tower.

(b) Write an equation for the unknown quantity.

(c) What is the height of the tower?

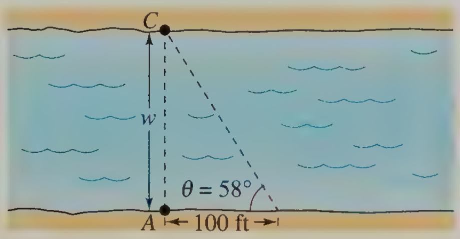
64. ***Length*** A 30-meter line is used to tether a helium-filled balloon. Because of a breeze, the line makes an angle of approximately 75° with the ground.

(a) Draw a right triangle to represent the problem. Show the known quantities on the triangle and use a variable to indicate the unknown height of the balloon.

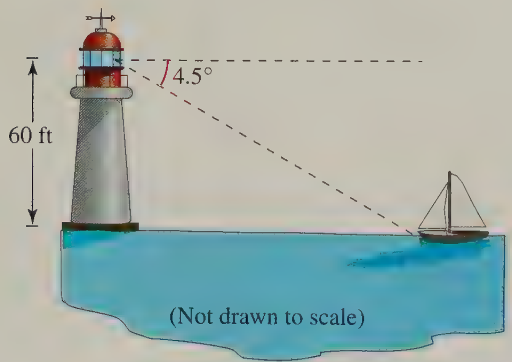
(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) What is the height of the balloon?

65. ***Width of a River*** A biologist wants to know the width w of a river in order to properly set instruments for studying the pollutants in the water. From point A, the biologist walks downstream 100 feet and sights to point C. From this sighting, it is determined that $\theta = 58°$. How wide is the river? Verify your result numerically using a graphing utility.



66. ***Distance*** From a 60-foot observation tower on the coast, a Coast Guard officer sights a boat in difficulty. The angle of depression of the boat is 4.5°. How far is the boat from the shoreline? Verify your result numerically using a graphing utility.



(Not drawn to scale)

67. ***Angle of Elevation*** A ramp 20 feet in length rises to a loading platform that is $3\frac{1}{3}$ feet off the ground.

(a) Draw a right triangle to represent the problem. Show the known quantities on the triangle and use a variable to indicate the unknown angle of elevation of the ramp.

(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) Find the angle of elevation of the ramp algebraically.

(d) Use the *table* feature of a graphing utility to approximate the angle of elevation numerically.

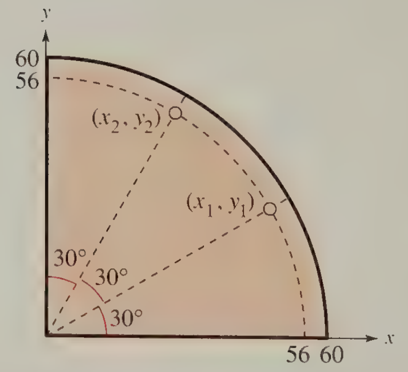
68. ***Duquesne Incline*** The track of the Duquesne Incline in Pittsburgh, Pennsylvania, is about 800 feet long, and the angle of elevation is 30°. The average speed of the cable cars is about 320 feet per minute.

(a) How high does the Duquesne Incline rise?

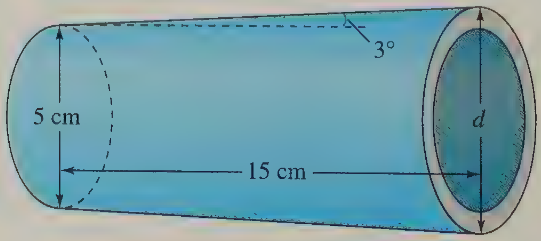
(b) What is the vertical speed of the cable cars (in feet per minute)?

69. ***Jin Mao Building*** You are standing 75 meters from the base of the Jin Mao Building in Shanghai, China. You estimate that the angle of elevation to the top of the building is 80°. What is the approximate height of the building? Suppose one of your friends is at the top of the building. What is the distance between you and your friend?

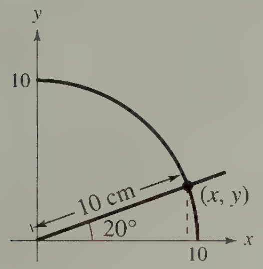
70. ***Machine Shop Calculations*** A steel plate has the form of one-fourth of a circle with a radius of 60 centimeters. Two 2-centimeter holes are to be drilled in the plate, positioned as shown in the figure. Find the coordinates of the center of each hole.



71. ***Machine Shop Calculations*** A tapered shaft has a diameter of 5 centimeters at the small end and is 15 centimeters long . If the taper is 3°, find the diameter d of the large end of the shaft.



72. ***Geometry*** Use a compass to sketch a quarter of a circle of radius 10 centimeters. Using a protractor, construct an angle of 20° in standard position. Drop a perpendicular from the point of intersection of the terminal side of the angle and the arc of the circle. By actual measurement, calculate the coordinates (x, y) of the point of intersection and use these measurements to approximate the six trigonometric functions of a 20° angle.



73. ***Geometry*** Repeat Exercise 72 using a 75° angle.

Synthesis

True or False? **In Exercises 74–77, determine whether the statement is true or false. Justify your answer.**

74. $\sin 60° \csc 60° = 1$ **75.** $\sec 30° = \csc 60°$

76. $\sin 45° + \cos 45° = 1$

77. $\cot^2 10° - \csc^2 10° = -1$

78. ***Exploration***

(a) Use a graphing utility to complete the table. Round your results to four decimal places.

θ	0°	20°	40°	60°	80°
$\sin \theta$					
$\cos \theta$					
$\tan \theta$					

(b) Classify each of the three trigonometric functions as increasing or decreasing for the table values.

(c) For the values in the table, verify that the tangent function is the quotient of the sine and cosine functions.

79. ***Exploration*** Use a graphing utility to complete the table and make a conjecture about the relationship between $\cos \theta$ and $\sin(90° - \theta)$. What are the angles θ and $90° - \theta$ called?

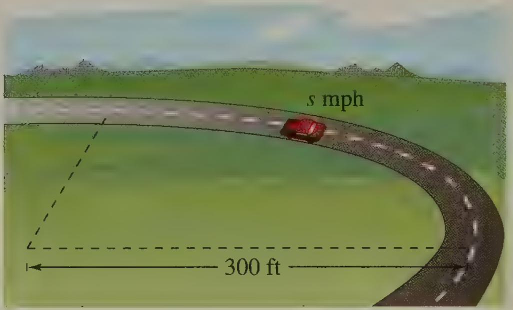
θ	0°	20°	40°	60°	80°
$\cos \theta$					
$\sin(90° - \theta)$					

80. ***Numerical Analysis*** A 3000-pound automobile is negotiating a circular interchange of radius 300 feet at a speed of s miles per hour. The relationship between the speed and the angle θ (in degrees) at which the roadway should be banked so that no lateral frictional force is exerted on the tires is $\tan \theta = 0.672s^2/3000$.

(a) Use a graphing utility to complete the table.

s	10	20	30	40	50	60
θ						

(b) The speeds are incremented by 10 miles per hour in the table, but θ does not increase by equal increments. Explain.



Review

In Exercises 81–84, sketch the graph of the equation and identify all x- and y-intercepts.

81. $y = -x - 9$ **82.** $2x + y = 10$

83. $-3x + 8y = 16$ **84.** $12x - 7y = 22$

In Exercises 85–88, use a calculator to evaluate the expression. Round your result to three decimal places.

85. $2.16^{3.8}$ **86.** $4^{2\pi}$

87. $\sqrt[3]{5286}$ **88.** $\sqrt[5]{10,321}$

In Exercises 89–92, determine the quadrant in which the angle lies.

89. 146° **90.** −290.8°

91. −310° 30′ **92.** −122° 50′

5.3 Trigonometric Functions of Any Angle

Introduction

In Section 5.2, the definitions of trigonometric functions were restricted to acute angles. In this section, the definitions are extended to cover *any* angle. If θ is an *acute* angle, the definitions here coincide with those given in the previous section.

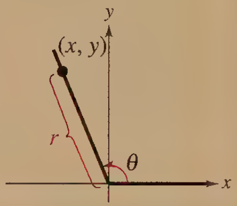
Definitions of Trigonometric Functions of Any Angle

Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}, \quad x \neq 0 \qquad \cot\theta = \frac{x}{y}, \quad y \neq 0$$

$$\sec\theta = \frac{r}{x}, \quad x \neq 0 \qquad \csc\theta = \frac{r}{y}, \quad y \neq 0$$



Because $r = \sqrt{x^2 + y^2}$ *cannot* be zero, it follows that the sine and cosine functions are defined for any real value of θ. However, if $x = 0$, the tangent and secant of θ are undefined. For example, the tangent of 90° is undefined. Similarly, if $y = 0$, the cotangent and cosecant of θ are undefined.

EXAMPLE 1 Evaluating Trigonometric Functions

Let $(-3, 4)$ be a point on the terminal side of θ. Find the sine, cosine, and tangent of θ.

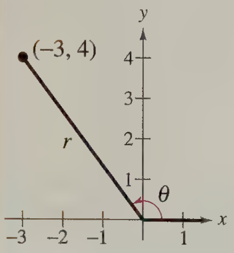


Figure 5.25

Solution

Referring to Figure 5.25, you see that $x = -3$, $y = 4$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5.$$

So, you have $\sin\theta = \frac{y}{r} = \frac{4}{5}$, $\cos\theta = \frac{x}{r} = -\frac{3}{5}$, and $\tan\theta = \frac{y}{x} = -\frac{4}{3}$.

What You Should Learn:

- How to evaluate trigonometric functions of any angle
- How to use reference angles to evaluate trigonometric functions
- How to evaluate trigonometric functions of real numbers

Why You Should Learn It:

You can use trigonometric functions to model and solve real-life problems. For instance, Exercise 115 on page 399 shows you how trigonometric functions can be used to model the average daily temperature in a city.

Henryk Kaiser/Leo de Wys

The *signs* of the trigonometric functions in the four quadrants can be determined easily from the definitions of the functions. For instance, because $\cos\theta = x/r$, it follows that $\cos\theta$ is positive wherever $x > 0$, which is in Quadrants I and IV. (Remember, r is always positive.) You can verify the results shown in Figure 5.26 in a similar manner.

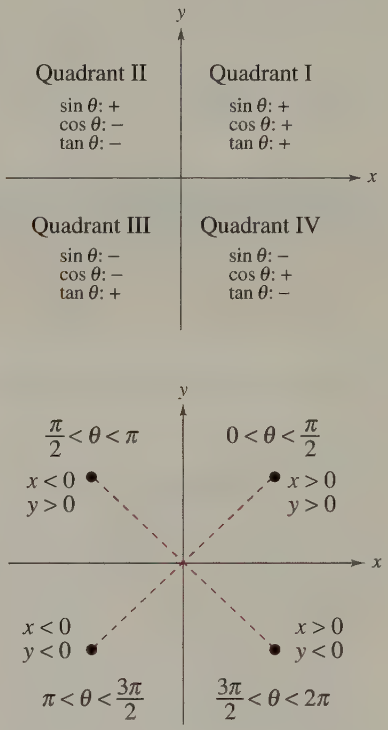


Figure 5.26

EXAMPLE 2 Evaluating Trigonometric Functions

Given $\tan\theta = -\frac{5}{4}$ and $\cos\theta > 0$, find $\sin\theta$ and $\sec\theta$.

Solution

Note that θ lies in Quadrant IV because that is the only quadrant in which the tangent is negative and the cosine is positive. Moreover, using

$$\tan\theta = \frac{y}{x} = -\frac{5}{4}$$

and the fact that y is negative in Quadrant IV, you can let $y = -5$ and $x = 4$. So, $r = \sqrt{16 + 25} = \sqrt{41}$, and you have

$$\sin\theta = \frac{y}{r} = \frac{-5}{\sqrt{41}}$$ Exact value

$$\approx -0.7809$$ Approximate value

and

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{41}}{4}$$ Exact value

$$\approx 1.6008.$$ Approximate value

EXAMPLE 3 Trigonometric Functions of Quadrant Angles

Evaluate the sine and cosine functions at the four quadrant angles 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$.

Solution

To begin, choose a point on the terminal side of each angle, as shown in Figure 5.27. For each of the four given points, $r = 1$, and you have the following.

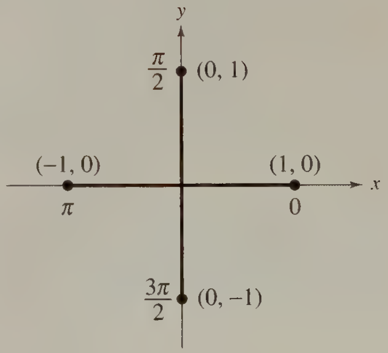


Figure 5.27

$\sin 0 = \frac{y}{r} = \frac{0}{1} = 0$ $\cos 0 = \frac{x}{r} = \frac{1}{1} = 1$ $(x, y) = (1, 0)$

$\sin\frac{\pi}{2} = \frac{y}{r} = \frac{1}{1} = 1$ $\cos\frac{\pi}{2} = \frac{x}{r} = \frac{0}{1} = 0$ $(x, y) = (0, 1)$

$\sin\pi = \frac{y}{r} = \frac{0}{1} = 0$ $\cos\pi = \frac{x}{r} = \frac{-1}{1} = -1$ $(x, y) = (-1, 0)$

$\sin\frac{3\pi}{2} = \frac{y}{r} = \frac{-1}{1} = -1$ $\cos\frac{3\pi}{2} = \frac{x}{r} = \frac{0}{1} = 0$ $(x, y) = (0, -1)$

Reference Angles

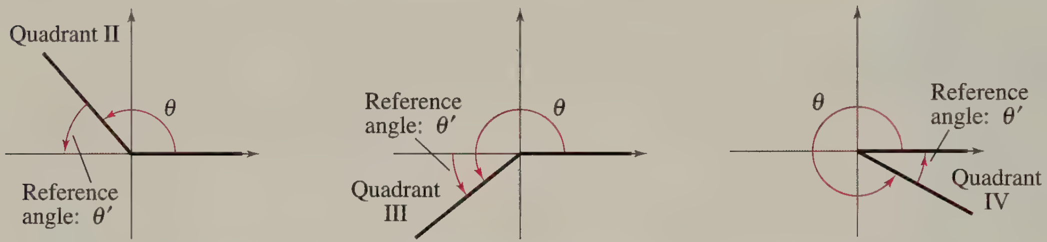
The values of the trigonometric functions of angles greater than 90° (or less than 0°) can be determined from their values at corresponding acute angles called **reference angles.**

A computer simulation to accompany this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Definition of Reference Angle

Let θ be an angle in standard position. Its **reference angle** is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

Figure 5.28 shows the reference angles for θ in Quadrants II, III, and IV.


Quadrant II
θ
Reference angle: θ'
Reference angle: θ'
θ
Quadrant III
θ
Reference angle: θ'
Quadrant IV

$\theta' = \pi - \theta$ (radians)
$\theta' = 180° - \theta$ (degrees)

$\theta' = \theta - \pi$ (radians)
$\theta' = \theta - 180°$ (degrees)

$\theta' = 2\pi - \theta$ (radians)
$\theta' = 360° - \theta$ (degrees)

Figure 5.28

EXAMPLE 4 Finding Reference Angles

Find the reference angle θ'.

a. $\theta = 300°$ **b.** $\theta = 2.3$ **c.** $\theta = -135°$

Solution

a. Because 300° lies in Quadrant IV, the angle it makes with the x-axis is

$$\theta' = 360° - 300° = 60°. \quad \text{Degrees}$$

b. Because 2.3 lies between $\pi/2 \approx 1.5708$ and $\pi \approx 3.1416$, it follows that it is in Quadrant II and its reference angle is

$$\theta' = \pi - 2.3 \approx 0.8416. \quad \text{Radians}$$

c. First, determine that $-135°$ is coterminal with 225°, which lies in Quadrant III. So, the reference angle is

$$\theta' = 225° - 180° = 45°. \quad \text{Degrees}$$

Figure 5.29 shows each angle θ and its reference angle θ'.

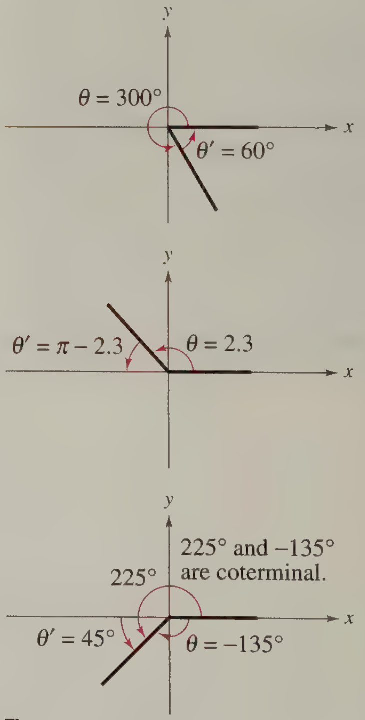


Figure 5.29

To see how a reference angle is used to evaluate a trigonometric function, consider the point (x, y) on the terminal side of θ as shown in Figure 5.30. By definition, you know that

$$\sin\theta = \frac{y}{r} \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

For the right triangle with acute angle θ' and sides of lengths $|x|$ and $|y|$, you have

$$\sin\theta' = \frac{\text{opp}}{\text{hyp}} = \frac{|y|}{r}$$

and

$$\tan\theta' = \frac{\text{opp}}{\text{adj}} = \frac{|y|}{|x|}.$$

So, it follows that $\sin\theta$ and $\sin\theta'$ are equal, *except possibly in sign*. The same is true for $\tan\theta$ and $\tan\theta'$, and for the other four trigonometric functions. In all cases, the sign of the function value can be determined by the quadrant in which θ lies.

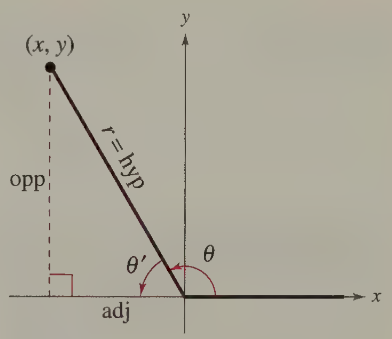


Figure 5.30

Evaluating Trigonometric Functions of Any Angle

To find the value of a trigonometric function of any angle θ:

1. Determine the function value for the associated reference angle θ'.
2. Depending on the quadrant in which θ lies, prefix the appropriate sign to the function value.

By using reference angles and the special angles discussed in the previous section, you can greatly extend the scope of *exact* trigonometric values. For instance, knowing the function values of 30° means that you know the function values of all angles for which 30° is a reference angle. For convenience, the following table gives the exact values of the trigonometric functions of special angles and quadrant angles.

Trigonometric Values of Common Angles

θ (degrees)	0°	30°	45°	60°	90°	180°	270°
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undef.	0	Undef.

Library of Functions

The trigonometric functions are important for modeling periodic behavior, such as business cycles, planetary orbits, pendulums, and light rays. These functions play a prominent role in calculus.

Consult the Library of Functions Summary inside the front cover for a description of the trigonometric functions.

STUDY T!P

Learning the table of values at the left is worth the effort. Doing so will increase both your efficiency and your confidence. Here is a pattern for the sine function that may help you remember the values.

θ	0°	30°	45°	60°	90°
$\sin\theta$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$

Reverse the order to get cosine values of the same angles.

EXAMPLE 5 Trigonometric Functions of Nonacute Angles

Evaluate the trigonometric functions.

a. $\cos \dfrac{4\pi}{3}$ **b.** $\tan(-210^\circ)$ **c.** $\csc \dfrac{11\pi}{4}$

Solution

a. Because $\theta = 4\pi/3$ lies in Quadrant III, the reference angle is $\theta' = (4\pi/3) - \pi = \pi/3$, as shown in Figure 5.31(a). Moreover, the cosine is negative in Quadrant III, so

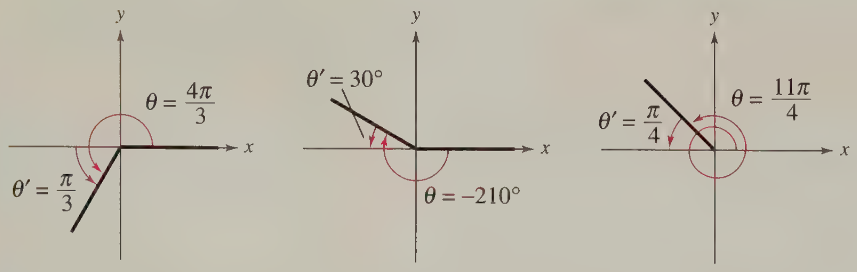
$$\begin{aligned}\cos \frac{4\pi}{3} &= (-)\cos \frac{\pi}{3} \\ &= -\frac{1}{2}.\end{aligned}$$

b. Because $-210^\circ + 360^\circ = 150^\circ$, it follows that -210° is coterminal with the second-quadrant angle 150°. Therefore, the reference angle is $\theta' = 180^\circ - 150^\circ = 30^\circ$, as shown in Figure 5.31(b). Finally, because the tangent is negative in Quadrant II, you have

$$\begin{aligned}\tan(-210^\circ) &= (-)\tan 30^\circ \\ &= -\frac{\sqrt{3}}{3}.\end{aligned}$$

c. Because $(11\pi/4) - 2\pi = 3\pi/4$, it follows that $11\pi/4$ is coterminal with the second-quadrant angle $3\pi/4$. Therefore, the reference angle is $\theta' = \pi - (3\pi/4) = \pi/4$, as shown in Figure 5.31(c). Because the cosecant is positive in Quadrant II, you have

$$\begin{aligned}\csc \frac{11\pi}{4} &= (+)\csc \frac{\pi}{4} \\ &= \frac{1}{\sin(\pi/4)} \\ &= \sqrt{2}.\end{aligned}$$



(a) (b) (c)

Figure 5.31

The fundamental trigonometric identities listed in the previous section (for an acute angle θ) are also valid when θ is any angle in the domain of the function.

EXAMPLE 6 Using Trigonometric Identities

Let θ be an angle in Quadrant II such that $\sin\theta = \frac{1}{3}$. Find (a) $\cos\theta$ and (b) $\tan\theta$ by using trigonometric identities.

Solution

a. Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, you obtain

$$\left(\frac{1}{3}\right)^2 + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{1}{9} = \frac{8}{9}.$$

Because $\cos\theta < 0$ in Quadrant II, you can use the negative root to obtain

$$\cos\theta = -\frac{\sqrt{8}}{\sqrt{9}} = -\frac{2\sqrt{2}}{3}.$$

b. Using the trigonometric identity $\tan\theta = \sin\theta/\cos\theta$, you obtain

$$\tan\theta = \frac{1/3}{-2\sqrt{2}/3}$$

$$= -\frac{1}{2\sqrt{2}}$$

$$= -\frac{\sqrt{2}}{4}.$$

You can use a calculator to evaluate trigonometric functions, as shown in the next example.

EXAMPLE 7 Using a Calculator

a. Use a calculator to evaluate cot 410° and $\sin(-7)$.

b. Use a calculator to solve $\tan\theta = 4.812,\ 0 \le \theta \le 2\pi$.

Solution

	Function	*Mode*	*Graphing Calculator Keystrokes*	*Display*
a.	cot 410°	Degree	$\boxed{(}\ \boxed{\text{TAN}}\ 410\ \boxed{)}\ \boxed{x^{-1}}\ \boxed{\text{ENTER}}$	0.8390996
	$\sin(-7)$	Radian	$\boxed{\text{SIN}}\ \boxed{(}\ \boxed{(-)}\ 7\ \boxed{)}\ \boxed{\text{ENTER}}$	-0.6569866

b. To solve the equation $\tan\theta = 4.812$, you can use the inverse tangent key, as follows.

Equation	*Mode*	*Graphing Calculator Keystrokes*	*Display*
$\tan\theta = 4.812$	Radian	$\boxed{\text{TAN}^{-1}}\ 4.812\ \boxed{\text{ENTER}}$	1.365898912

The angle $\theta \approx 1.366$ lies in Quadrant I. A second value of θ lies in Quadrant III (tangent is positive) and is

$$\theta = \pi + 1.366 \approx 4.507.$$

The *Interactive* CD-ROM and *Internet* versions of this text offer a built-in graphing calculator, which can be used with the Examples, Explorations, and Exercises.

Exploration

Set your graphing utility to *degree* mode and enter tan 90. What happens? Why? Now set your graphing utility to *radian* mode and enter $\tan(\pi/2)$. Explain the graphing utility's answer.

Trigonometric Functions of Real Numbers

To define a trigonometric function of a real number (rather than an angle), let t represent any real number. Then imagine that the real number line is wrapped around a *unit circle*, as shown in Figure 5.32. Note that positive numbers correspond to a counterclockwise wrapping, and negative numbers correspond to a clockwise wrapping.

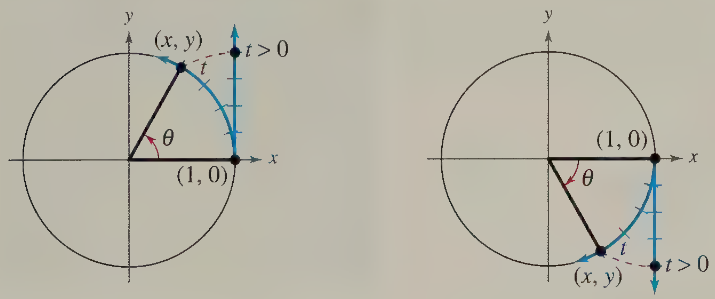


Figure 5.32

As the real line is wrapped around the unit circle, each real number t will correspond with a central angle θ. Moreover, because the circle has a radius of 1, the arc intercepted by the angle θ will have a length of t. The point is that *if θ is measured in radians*, then $t = \theta$. So, you can define $\sin t$ to be $\sin t = \sin(t \text{ radians})$. Similarly, $\cos t = \cos(t \text{ radians})$, $\tan t = \tan(t \text{ radians})$, and so on.

EXAMPLE 8 Evaluating Trigonometric Functions

Evaluate $f(t) = \sin t$ for (a) $t = 1$, (b) $t = 3\pi/2$, and (c) $t = 2\pi$.

Algebraic Solution

a. $f(1) = \sin 1$ Radian mode

≈ 0.841471

b. $f\left(\frac{3\pi}{2}\right) = \sin \frac{3\pi}{2}$ Common angle

$= -1$

c. $f(2\pi) = \sin 2\pi$ Reference angle: $\theta' = 0$

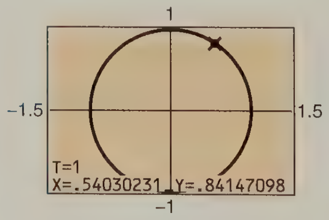
$= 0$

Graphical Solution

Use a graphing utility set in *parametric* and *radian* modes to graph $X_{1T} = \cos T$ and $Y_{1T} = \sin T$ using the following settings.

Tmin = 0, Tmax = 6.3, Tstep = 0.1
Xmin = −1.5, Xmax = 1.5, Xscl = 1
Ymin = −1, Ymax = 1, Yscl = 1

a. Use the *value* or *trace* feature to estimate that $\sin 1 \approx 0.841471$, as shown in Figure 5.33.



```
T=1
X=.54030231 Y=.84147098
```

Figure 5.33

b. Use the *value* or *trace* feature to estimate that $\sin \frac{3\pi}{2} \approx \sin 4.7 \approx -1$.

c. Use the *value* or *trace* feature to estimate that $\sin 2\pi \approx \sin 6.3 \approx 0$.

The *domain* of the sine and cosine functions is the set of all real numbers. To determine the *range* of these two functions, consider the unit circle shown in Figure 5.34. Because $r = 1$, it follows that $\sin t = y$ and $\cos t = x$. Moreover, because (x, y) is on the unit circle, you know that $-1 \le y \le 1$ and $-1 \le x \le 1$. So, the values of sine and cosine also range between -1 and 1.

$$-1 \le \quad y \quad \le 1 \qquad \text{and} \qquad -1 \le \quad x \quad \le 1$$

$$-1 \le \sin t \le 1 \qquad\qquad -1 \le \cos t \le 1$$

You add 2π to each value of t in the interval $[0, 2\pi]$, completing a second revolution around the unit circle, as shown in Figure 5.35. The values of $\sin(t + 2\pi)$ and $\cos(t + 2\pi)$ correspond to those of $\sin t$ and $\cos t$. Similar results can be obtained for repeated revolutions (positive or negative) on the unit circle. This leads to the general result

$$\sin(t + 2\pi n) = \sin t \qquad \text{and} \qquad \cos(t + 2\pi n) = \cos t$$

for any integer n and real number t. Functions that behave in such a repetitive (or cyclic) manner are called **periodic.**

Exploration

With your graphing utility in *radian* mode, select a number t and calculate $\cos t$ and $\cos(t + 2\pi)$. Repeat this experiment for other values of t and explain why the answers are always the same. Perform a similar experiment with $\tan t$ and $\tan(t + \pi)$.

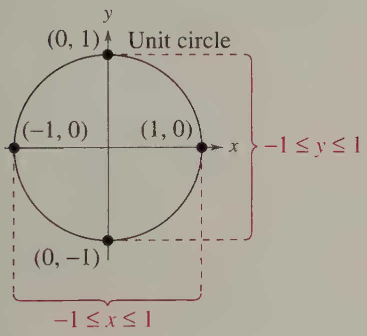


Figure 5.34

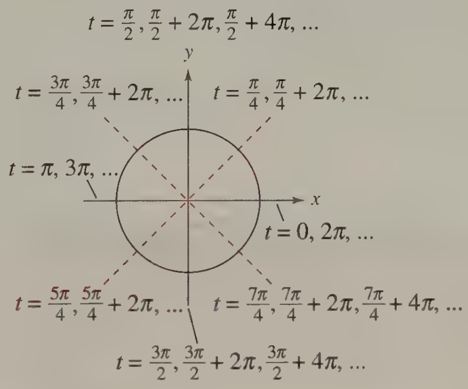

$t = \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 4\pi, \ldots$
y
$t = \frac{3\pi}{4}, \frac{3\pi}{4} + 2\pi, \ldots$
$t = \frac{\pi}{4}, \frac{\pi}{4} + 2\pi, \ldots$
$t = \pi, 3\pi, \ldots$
x
$t = 0, 2\pi, \ldots$
$t = \frac{5\pi}{4}, \frac{5\pi}{4} + 2\pi, \ldots$
$t = \frac{7\pi}{4}, \frac{7\pi}{4} + 2\pi, \frac{7\pi}{4} + 4\pi, \ldots$
$t = \frac{3\pi}{2}, \frac{3\pi}{2} + 2\pi, \frac{3\pi}{2} + 4\pi, \ldots$

Figure 5.35

Definition of Periodic Function

A function f is **periodic** if there exists a positive real number c such that

$$f(t + c) = f(t)$$

for all t in the domain of f. The least number c for which f is periodic is called the **period** of f.

From this definition it follows that the sine and cosine functions are periodic and have a period of 2π. The other four trigonometric functions are also periodic, and more will be said about this in Section 5.5.

Recall from Section 1.4 that a function f is *even* if $f(-t) = f(t)$ and is *odd* if $f(-t) = -f(t)$.

Even and Odd Trigonometric Functions

The cosine and secant functions are *even*.

$\cos(-t) = \cos t \qquad \sec(-t) = \sec t$

The sine, cosecant, tangent, and cotangent functions are *odd*.

$\sin(-t) = -\sin t \qquad \csc(-t) = -\csc t$

$\tan(-t) = -\tan t \qquad \cot(-t) = -\cot t$

Exploration

Select a number t and use your graphing utility to calculate $\cos t$ and $\cos(-t)$. Repeat this experiment for other values of t and explain why the answer is always the same. Perform a similar experiment to determine the relationship between $\sin t$ and $\sin(-t)$.

At this point, you have completed your introduction to basic trigonometry. You have measured angles in both degrees and radians. You have defined the six trigonometric functions from a right triangle perspective and as functions of real numbers. In your remaining work with trigonometry, you should continue to rely on both perspectives. For instance, in the next two sections on graphing techniques, it helps to think of the trigonometric functions as functions of real numbers. Later, in Section 5.7, you will look at applications involving angles and triangles.

For your convenience, a summary of basic trigonometry is included on the inside back cover of this text.

Writing About Math *Patterns in Trigonometric Functions*

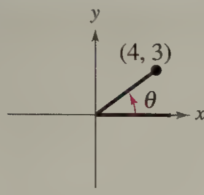
Complete the following table. Then identify and write a short paragraph describing any inherent patterns in the trigonometric functions. What can you conclude?

Function	Domain	Range	Even/Odd	Period	Zeros
Sine					
Cosine					
Tangent					
Cosecant					
Secant					
Cotangent					

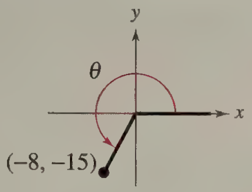
5.3 Exercises

In Exercises 1–4, determine the exact values of the six trigonometric functions of the angle θ.

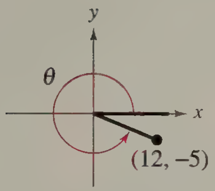
1. (a)



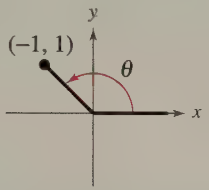
(b)



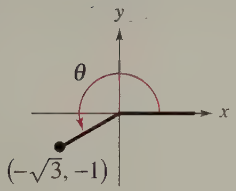
2. (a)



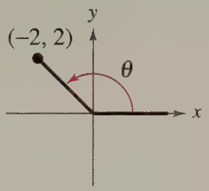
(b)



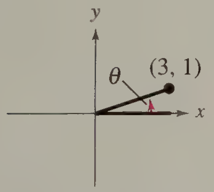
3. (a)



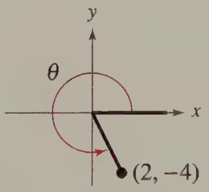
(b)



4. (a)



(b)



In Exercises 5–12, the given point is on the terminal side of an angle in standard position. Determine the exact values of the six trigonometric functions of the angle.

5. $(7, 24)$

6. $(8, 15)$

7. $(5, -12)$

8. $(-24, 10)$

9. $(-4, 10)$

10. $(-5, -6)$

11. $(-2, 9)$

12. $(6, -14)$

In Exercises 13–18, state the quadrant in which θ lies.

13. $\sin\theta < 0$ and $\cos\theta < 0$

14. $\sin\theta > 0$ and $\cos\theta > 0$

15. $\sin\theta > 0$ and $\tan\theta < 0$

16. $\sec\theta > 0$ and $\cot\theta < 0$

17. $\cot\theta > 0$ and $\cos\theta > 0$

18. $\tan\theta > 0$ and $\csc\theta < 0$

In Exercises 19–26, find the values of the six trigonometric functions of θ.

Function Value	*Constraint*
19. $\sin\theta = \frac{3}{5}$	θ lies in Quadrant II.
20. $\cos\theta = -\frac{4}{5}$	θ lies in Quadrant III.
21. $\tan\theta = -\frac{15}{8}$	$\sin\theta < 0$
22. $\csc\theta = 4$	$\cot\theta < 0$
23. $\sec\theta = -2$	$0 \le \theta \le \pi$
24. $\cot\theta$ is undefined.	$\dfrac{\pi}{2} \le \theta \le \dfrac{3\pi}{2}$
25. $\sin\theta = 0$	$\sec\theta = -1$
26. $\tan\theta$ is undefined.	$\csc\theta = -1$

In Exercises 27–30, the terminal side of θ lies on the given line in the specified quadrant. Find the values of the six trigonometric functions of θ.

Line	*Quadrant*
27. $y = -x$	Quadrant II
28. $y = \frac{1}{3}x$	Quadrant III
29. $y = 2x$	Quadrant III
30. $4x + 3y = 0$	Quadrant IV

In Exercises 31–38, evaluate the trigonometric function of the quadrant angle.

31. $\sec\pi$

32. $\tan\dfrac{\pi}{2}$

33. $\cot\dfrac{\pi}{2}$

34. $\csc\pi$

35. $\sec 0$

36. $\csc\dfrac{3\pi}{2}$

37. $\cot\pi$

38. $\csc\dfrac{\pi}{2}$

In Exercises 39–50, find the reference angle θ' and sketch θ and θ' in standard position.

39. $\theta = 208°$

40. $\theta = 322°$

41. $\theta = -245°$

42. $\theta = -145°$

43. $\theta = -292°$

44. $\theta = -95°$

45. $\theta = \dfrac{11\pi}{3}$

46. $\theta = \dfrac{17\pi}{6}$

47. $\theta = 3.5$

48. $\theta = 4.8$

49. $\theta = -3.68$

50. $\theta = -1.72$

In Exercises 51–64, evaluate the sine, cosine, and tangent of the angle without using a calculator.

51. 225° **52.** 300°

53. -750° **54.** -495°

55. -240° **56.** -330°

57. $\frac{5\pi}{3}$ **58.** $\frac{\pi}{4}$

59. $-\frac{\pi}{6}$ **60.** $-\frac{\pi}{2}$

61. $\frac{11\pi}{4}$ **62.** $\frac{10\pi}{3}$

63. $-\frac{7\pi}{6}$ **64.** $-\frac{20\pi}{3}$

In Exercises 65–78, use a calculator to evaluate the trigonometric function. Round your answer to four decimal places. (Be sure the calculator is set in the correct mode.)

65. $\sin 10^\circ$ **66.** $\sec 225^\circ$

67. $\tan 240^\circ$ **68.** $\csc 330^\circ$

69. $\cos(-110^\circ)$ **70.** $\cot(-220^\circ)$

71. $\sec(-280^\circ)$ **72.** $\sin(-195^\circ)$

73. $\sin 0.65$ **74.** $\sin(-0.65)$

75. $\tan \frac{\pi}{9}$ **76.** $\tan\left(-\frac{\pi}{9}\right)$

77. $\csc\left(-\frac{8\pi}{9}\right)$ **78.** $\cos\left(-\frac{15\pi}{14}\right)$

In Exercises 79–84, find two solutions of the equation. Give your answers in degrees ($0^\circ \leq \theta \leq 360^\circ$) and radians ($0 \leq \theta \leq 2\pi$). Do not use a calculator.

79. (a) $\sin \theta = \frac{1}{2}$ (b) $\sin \theta = -\frac{1}{2}$

80. (a) $\cos \theta = \frac{\sqrt{2}}{2}$ (b) $\cos \theta = -\frac{\sqrt{2}}{2}$

81. (a) $\csc \theta = \frac{2\sqrt{3}}{3}$ (b) $\cot \theta = -1$

82. (a) $\csc \theta = -\sqrt{2}$ (b) $\csc \theta = 2$

83. (a) $\sec \theta = -\frac{2\sqrt{3}}{3}$ (b) $\cos \theta = -\frac{1}{2}$

84. (a) $\cot \theta = -\sqrt{3}$ (b) $\sec \theta = \sqrt{2}$

In Exercises 85–96, use a calculator to approximate two values of θ ($0^\circ \leq \theta < 360^\circ$) that satisfy the equation. Round your answers to two decimal places.

85. $\sin \theta = 0.8191$ **86.** $\cos \theta = 0.8746$

87. $\tan \theta = 0.6524$ **88.** $\cot \theta = 0.7521$

89. $\sec \theta = -1.2241$ **90.** $\csc \theta = -1.0038$

91. $\sin \theta = -0.4793$ **92.** $\tan \theta = -2.1832$

93. $\cos \theta = 0.9848$ **94.** $\sin \theta = 0.0175$

95. $\tan \theta = 1.192$ **96.** $\cot \theta = 5.671$

In Exercises 97–102, find the indicated trigonometric value in the specified quadrant.

Function	*Quadrant*	*Trigonometric Value*
97. $\sin \theta = -\frac{3}{5}$	IV	$\cos \theta$
98. $\cot \theta = -3$	II	$\sin \theta$
99. $\tan \theta = \frac{3}{2}$	III	$\sec \theta$
100. $\csc \theta = -2$	IV	$\cot \theta$
101. $\cos \theta = \frac{5}{8}$	I	$\sec \theta$
102. $\sec \theta = -\frac{9}{4}$	III	$\tan \theta$

In Exercises 103–110, find the point (x, y) on the unit circle that corresponds to the real number t. Use the result to evaluate $\sin t$, $\cos t$, and $\tan t$.

103. $t = \frac{\pi}{4}$ **104.** $t = \frac{\pi}{3}$

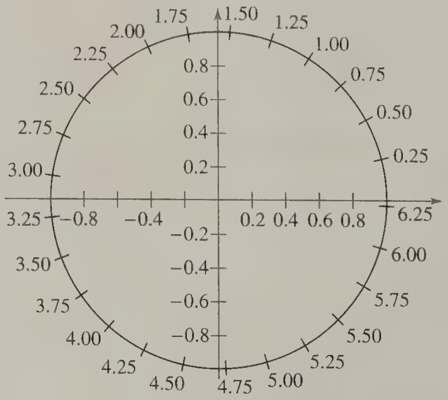
105. $t = \frac{5\pi}{6}$ **106.** $t = \frac{5\pi}{4}$

107. $t = \frac{4\pi}{3}$ **108.** $t = \frac{11\pi}{6}$

109. $t = \frac{3\pi}{2}$ **110.** $t = \pi$

Estimation **In Exercises 111 and 112, use the figure below and a straightedge to approximate the value of each trigonometric function. Check your approximation using a graphing utility.**

111. (a) $\sin 5$ (b) $\cos 2$ **112.** (a) $\sin 0.75$ (b) $\cos 2.5$



Estimation **In Exercises 113 and 114, use the figure in Exercises 111 and 112 to approximate the solution of the equation where $0 \le t < 2\pi$. Check your approximation using a graphing utility.**

113. (a) $\sin t = 0.25$ (b) $\cos t = -0.25$

114. (a) $\sin t = -0.75$ (b) $\cos t = 0.75$

115. ***Average Temperature*** The average daily temperature T (in degrees Fahrenheit) for a certain city is

$$T = 45 - 23\cos\left[\frac{2\pi}{365}(t - 32)\right]$$

where t is the time in days, with $t = 1$ corresponding to January 1. Find the average daily temperatures on the following days.

(a) January 1

(b) July 4 ($t = 185$)

(c) October 18 ($t = 291$)

116. ***Sales*** A company that sells seasonal products forecasts monthly sales over a 2-year period to be

$$S = 23.1 + 0.442t + 4.3\sin\frac{\pi t}{6}$$

where S is measured in thousands of units and t is the time (in months), with $t = 1$ representing January 2000. Estimate sales for the following months.

(a) February 2000 (b) February 2001

(c) September 2000 (d) September 2001

117. ***Harmonic Motion*** The displacement from equilibrium of an oscillating weight suspended by a spring is

$$y(t) = 2e^{-t}\cos 6t$$

where y is the displacement (in centimeters) and t is the time (in seconds).

(a) What was the initial displacement ($t = 0$)?

(b) Use a graphing utility to complete the table.

t	0.50	1.02	1.54	2.07	2.59
y					

(c) The approximate times when the weight is at its maximum distance from equilibrium are shown in the table in part (b). Explain why the magnitude of the maximum displacement is decreasing. What causes this decrease in maximum displacement in the physical system? What factor in the model measures this decrease?

(d) Find the first two times that the weight is at the equilibrium point ($y = 0$).

118. ***Harmonic Motion*** The displacement from equilibrium of an oscillating weight suspended by a spring and subject to the damping effect of friction is

$$y(t) = 2e^{-t}\cos 6t$$

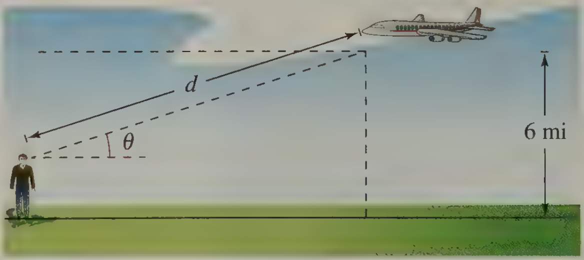
where y is the displacement (in centimeters) and t is the time (in seconds). Find the displacement when (a) $t = 0$, (b) $t = 1/4$, and (c) $t = 1/2$.

119. ***Electric Circuits*** The initial current and charge in an electric circuit are zero. The current when 100 volts is applied to the circuit is

$$I = 5e^{-2t}\sin t$$

where the resistance, inductance, and capacitance are respectively 80 ohms, 20 henrys, and 0.01 farad. Approximate the current (in amperes) $t = 0.7$ seconds after the voltage is applied.

120. ***Distance*** An airplane, flying at an altitude of 6 miles, is on a flight path that passes directly over an observer. If θ is the angle of elevation from the observer to the plane, find the distance from the observer to the plane when (a) $\theta = 30°$, (b) $\theta = 90°$, and (c) $\theta = 120°$.



Synthesis

True or False? **In Exercises 121–124, determine whether the statement is true or false. Justify your answer.**

121. $\sin 151° = \sin 29°$ **122.** $\tan 24° = \tan 156°$

123. $\csc\left(-\frac{7\pi}{6}\right) = \csc\left(-\frac{11\pi}{6}\right)$

124. $-\cot\left(\frac{3\pi}{4}\right) = \cot\left(-\frac{\pi}{4}\right)$

125. ***Exploration*** Use a graphing utility to complete the table to demonstrate the properties of even and odd functions. Identify each function as being either even or odd.

θ	$0°$	$20°$	$40°$	$60°$	$80°$
$\sec\theta$					
$\sec(-\theta)$					
$\tan\theta$					
$\tan(-\theta)$					

126. ***Conjecture***

(a) Use a graphing utility to complete the table.

θ	$0°$	$20°$	$40°$	$60°$	$80°$
$\sin\theta$					
$\sin(180° - \theta)$					

(b) Make a conjecture about the relationship between $\sin\theta$ and $\sin(180° - \theta)$.

127. ***Conjecture***

(a) Use a graphing utility to complete the table.

θ	0	0.3	0.6	0.9	1.2	1.5
$\cos\left(\frac{3\pi}{2} - \theta\right)$						
$-\sin\theta$						

(b) Make a conjecture about the relationship between $\cos\left(\frac{3\pi}{2} - \theta\right)$ and $-\sin\theta$.

128. ***Conjecture***

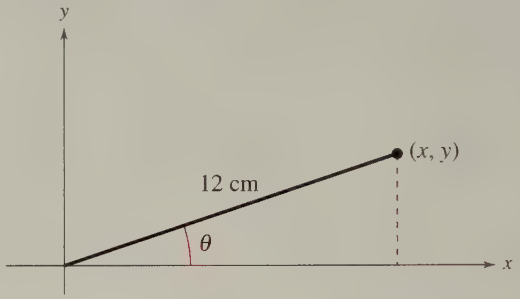
(a) Use a graphing utility to complete the table.

θ	0.1	0.4	0.7	1.0	1.3
$\tan\left(\theta - \frac{\pi}{2}\right)$					
$-\cot\theta$					

(b) Make a conjecture about the relationship between $\tan\left(\theta - \frac{\pi}{2}\right)$ and $-\cot\theta$.

129. ***Think About It*** Because $f(t) = \sin t$ is an odd function and $g(t) = \cos t$ is an even function, what can be said about the function $h(t) = f(t)g(t)$?

130. ***Writing*** Consider an angle in standard position with $r = 12$ centimeters, as shown in the figure. Write a short paragraph describing the change in the magnitudes of x, y, $\sin\theta$, $\cos\theta$, and $\tan\theta$ as θ increases continually from 0° to 90°.



Review

In Exercises 131–134, sketch the graph of the function. Identify any intercepts and asymptotes.

131. $y = 2^{x-1}$

132. $y = 3^{x+2}$

133. $y = \ln(x - 1)$

134. $y = \ln(x + 1)$

In Exercises 135–138, solve the equation. Round your answer to three decimal places.

135. $4^{3-x} = 726$

136. $\dfrac{4500}{4 + e^{2x}} = 50$

137. $\ln x = -6$

138. $\ln\sqrt{x + 10} = 1$

In Exercises 139–142, sketch a right triangle corresponding to the trigonometric function of the acute angle θ. Use the Pythagorean Theorem to determine the third side and then find the other five trigonometric functions of θ.

139. $\cot\theta = \dfrac{8}{3}$

140. $\sec\theta = 8$

141. $\tan\theta = \dfrac{7}{24}$

142. $\cos\theta = \dfrac{2}{9}$

5.4 Graphs of Sine and Cosine Functions

What You Should Learn:

- How to sketch the graphs of basic sine and cosine functions
- How to use amplitude and period to help sketch the graphs of sine and cosine functions
- How to sketch translations of graphs of sine and cosine functions
- How to use sine and cosine functions to model real-life data

Why You Should Learn It:

You can use sine and cosine functions to model real-life data. For instance, Exercise 83 on page 410 shows you how a sine function can be used to model the monthly precipitation for Seattle, Washington.

Jim Corwin/Tony Stone Images

Basic Sine and Cosine Curves

In this section you will study techniques for sketching the graphs of the sine and cosine functions. The graph of the sine function is a **sine curve.** In Figure 5.36, the black portion of the graph represents one period of the function and is called **one cycle** of the sine curve. The gray portion of the graph indicates that the basic sine wave repeats indefinitely to the right and left. The graph of the cosine function is shown in Figure 5.37. To produce these graphs with a graphing utility, make sure you have set the graphing utility to *radian* mode.

Recall from Section 5.3 that the domain of the sine and cosine functions is the set of all real numbers. Moreover, the range of each function is the interval $[-1, 1]$, and each function has a period of 2π. Do you see how this information is consistent with the basic graphs given in Figures 5.36 and 5.37?

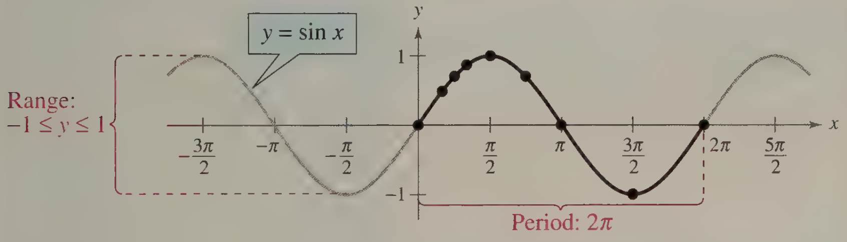


Figure 5.36

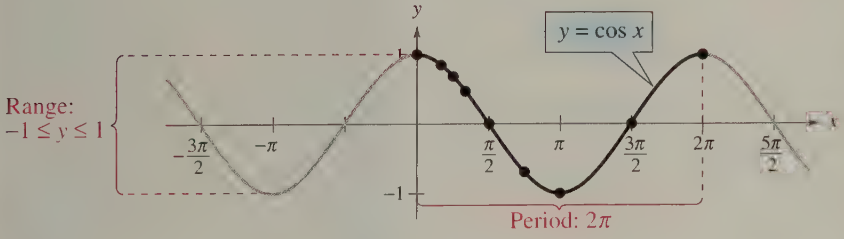


Figure 5.37

The table below lists key points on the graphs of $y = \sin x$ and $y = \cos x$.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	-1	0
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	0	1

Note from Figures 5.36 and 5.37 that the sine graph is symmetric with respect to the *origin,* whereas the cosine graph is symmetric with respect to the *y-axis.* These properties of symmetry follow from the fact that the sine function is odd whereas the cosine function is even.

To sketch the graphs of the basic sine and cosine functions by hand, it helps to note five *key points* in one period of each graph: the *intercepts*, *maximum points*, and *minimum points*. See Figure 5.38

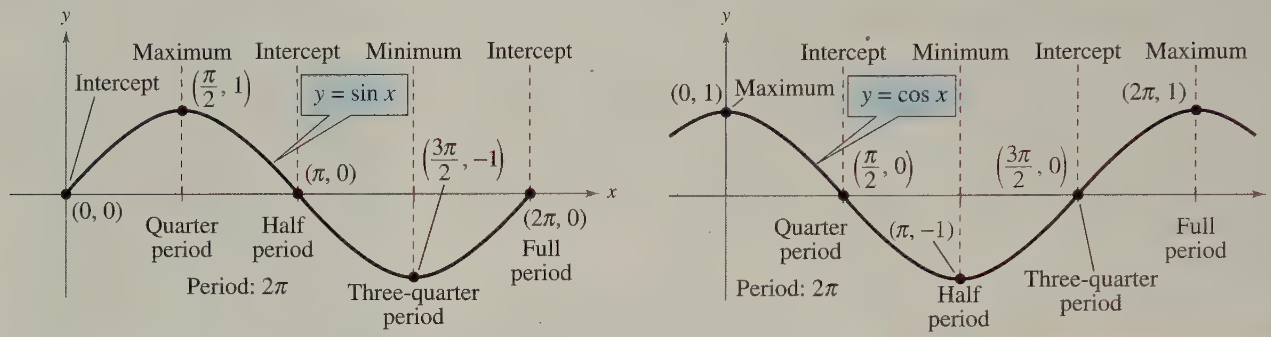


Figure 5.38

EXAMPLE 1 Using Key Points to Sketch a Sine Curve

Sketch the graph of $y = 2 \sin x$ on the interval $[-\pi, 4\pi]$.

Solution

Note that

$$y = 2 \sin x = 2(\sin x)$$

indicates that the y-values for the key points will have twice the magnitude of the graph of $y = \sin x$. Divide the period 2π into four equal parts to get the key points for $y = 2 \sin x$.

$$(0, 0), \quad \left(\frac{\pi}{2}, 2\right), \quad (\pi, 0), \quad \left(\frac{3\pi}{2}, -2\right), \quad \text{and} \quad (2\pi, 0)$$

By connecting these key points with a smooth curve and extending the curve in both directions over the interval $[-\pi, 4\pi]$, you obtain the graph shown in Figure 5.39. Use a graphing utility to confirm this graph. Be sure to set the graphing utility to *radian* mode.

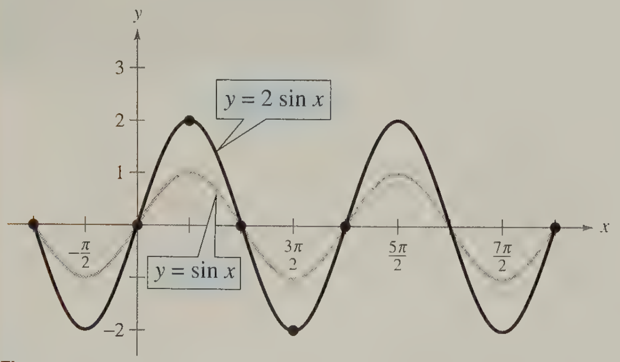
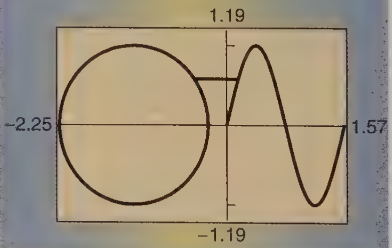


Figure 5.39

Exploration

Enter the program SINESHOW given on our website, *college.hmco.com*, into your graphing utility. This program simultaneously draws the unit circle and the corresponding points on the sine curve, as shown below. After the circle and sine curve are drawn, you can connect the points on the unit circle with their corresponding points on the sine curve by pressing ENTER. Discuss the relationship that is illustrated.



Amplitude and Period of Sine and Cosine Curves

In the rest of this section you will study the graphic effect of each of the constants a, b, c, and d in equations of the forms

$$y = d + a\sin(bx - c)$$

and

$$y = d + a\cos(bx - c).$$

A quick review of the transformations studied in Section 1.5 should help in this investigation.

The constant factor a in $y = a\sin x$ acts as a *scaling factor*—a *vertical stretch* or *vertical shrink* of the basic sine curve. If $|a| > 1$, the basic sine curve is stretched, and if $|a| < 1$, the basic sine curve is shrunk. The result is that the graph of $y = a\sin x$ ranges between $-a$ and a instead of between -1 and 1. The absolute value of a is the **amplitude** of the function $y = a\sin x$. The range of the function $y = a\sin x$ is $-a \le y \le a$.

Definition of Amplitude of Sine and Cosine Curves

The **amplitude** of $y = a\sin x$ and $y = a\cos x$ represents half the distance between the maximum and minimum values of the function and is given by

$$\text{Amplitude} = |a|.$$

EXAMPLE 2 Scaling: Vertical Shrinking and Stretching

On the same coordinate axes, sketch the graph of each function.

a. $y = \dfrac{1}{2}\cos x$ **b.** $y = 3\cos x$

Solution

a. Because the amplitude of $y = \frac{1}{2}\cos x$ is $\frac{1}{2}$, the maximum value is $\frac{1}{2}$ and the minimum value is $-\frac{1}{2}$. Divide one cycle, $0 \le x \le 2\pi$, into four equal parts to get the key points

$$\left(0, \frac{1}{2}\right), \quad \left(\frac{\pi}{2}, 0\right), \quad \left(\pi, -\frac{1}{2}\right), \quad \left(\frac{3\pi}{2}, 0\right), \quad \text{and} \quad \left(2\pi, \frac{1}{2}\right).$$

b. A similar analysis shows that the amplitude of $y = 3\cos x$ is 3, and the key points are

$$(0, 3), \quad \left(\frac{\pi}{2}, 0\right), \quad (\pi, -3), \quad \left(\frac{3\pi}{2}, 0\right), \quad \text{and} \quad (2\pi, 3).$$

The graphs of these two functions are shown in Figure 5.40. Use a graphing utility to confirm these graphs.

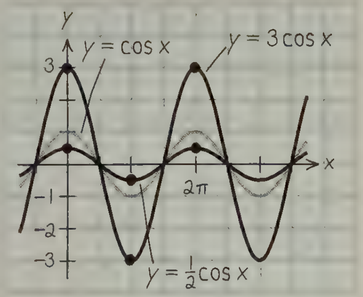


Figure 5.40

Exploration

Use a graphing utility to graph $y = a\sin x$, where $a = 0.5$, 1, 2, and 3. How does the value of a affect the shape of the graph? What happens to the graph if a is negative?

STUDY T!P

When using a graphing utility to graph trigonometric functions, pay special attention to the viewing window you use. For instance, try graphing $y = [\sin(10x)]/10$ in the standard viewing window in *radian* mode. What do you observe? Use the *zoom* feature to find a viewing window that displays a good view of the graph.

You know from Section 1.5 that the graph of $y = -f(x)$ is a *reflection* in the x-axis of the graph of $y = f(x)$. For instance, the graph of

$$y = -3 \cos x$$

is a reflection of the graph of

$$y = 3 \cos x,$$

as shown in Figure 5.41.

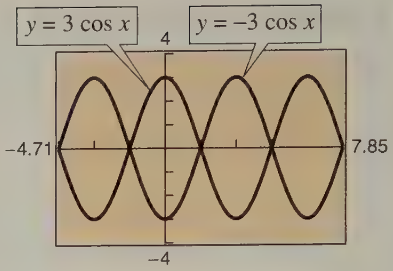


Figure 5.41

Because $y = a \sin x$ completes one cycle from $x = 0$ to $x = 2\pi$, it follows that $y = a \sin bx$ completes one cycle from $x = 0$ to $x = 2\pi/b$.

Period of Sine and Cosine Functions

Let b be a positive real number. The **period** of $y = a \sin bx$ and $y = a \cos bx$ is given by

$$\text{Period} = \frac{2\pi}{b}.$$

Exploration

Use a graphing utility to graph $y = \sin bx$, where $b = 0.5$, 1, and 2. How does the value of b affect the graph?

Note that if $0 < b < 1$, the period of $y = a \sin bx$ is greater than 2π and represents a *horizontal stretching* of the graph of $y = a \sin x$. Similarly, if $b > 1$, the period of $y = a \sin bx$ is less than 2π and represents a *horizontal shrinking* of the graph of $y = a \sin x$. If b is negative, the identities $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ are used to rewrite the function.

EXAMPLE 3 Scaling: Horizontal Stretching

Sketch the graph of $y = \sin \dfrac{x}{2}$.

Solution

The amplitude is 1. Moreover, because $b = \frac{1}{2}$, the period is

$$\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi.$$

Now, divide the period-interval $[0, 4\pi]$ into four equal parts with the values π, 2π, and 3π, to obtain the key points on the graph

$$(0, 0), \quad (\pi, 1), \quad (2\pi, 0), \quad (3\pi, -1), \quad \text{and} \quad (4\pi, 0).$$

The graph is shown in Figure 5.42. Use a graphing utility to confirm this graph.

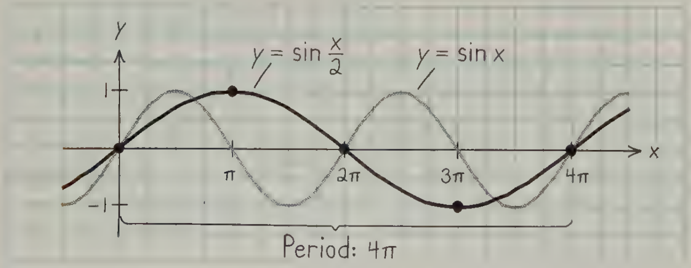


Figure 5.42

STUDY TIP

In general, to divide a period-interval into four equal parts, successively add "period/4," starting with the left endpoint of the interval. For instance, for the period-interval $[-\pi/6, \pi/2]$ of length $2\pi/3$, you would successively add

$$\frac{2\pi/3}{4} = \frac{\pi}{6}$$

to get $-\pi/6, 0, \pi/6, \pi/3$, and $\pi/2$.

Translations of Sine and Cosine Curves

The constant c in the general equations

$$y = a\sin(bx - c) \qquad \text{and} \qquad y = a\cos(bx - c)$$

creates *horizontal translations* (shifts) of the basic sine and cosine curves. Comparing $y = a\sin bx$ with $y = a\sin(bx - c)$, you find that the graph of $y = a\sin(bx - c)$ completes one cycle from $bx - c = 0$ to $bx - c = 2\pi$. By solving for x, you can find the interval for one cycle to be

$$\overbrace{\frac{c}{b}}^{\text{Left endpoint}} \le x \le \overbrace{\frac{c}{b} + \underbrace{\frac{2\pi}{b}}_{\text{Period}}}^{\text{Right endpoint}}.$$

This implies that the period of $y = a\sin(bx - c)$ is $2\pi/b$, and the graph of $y = a\sin bx$ is shifted by an amount c/b. The number c/b is the **phase shift.**

Exploration

Use a graphing utility to graph

$$y = \sin(x + c)$$

where $c = -\pi/4$, 0, and $\pi/4$. Use a viewing window in which $-1.6 \le x \le 6.3$ and $-2 \le y \le 2$. How does the value of c affect the graph?

Graphs of Sine and Cosine Functions

The graphs of $y = a\sin(bx - c)$ and $y = a\cos(bx - c)$ have the following characteristics. (Assume $b > 0$.)

Amplitude $= |a|$ Period $= 2\pi/b$

The left and right endpoints of a one-cycle interval can be determined by solving the equations $bx - c = 0$ and $bx - c = 2\pi$.

EXAMPLE 4 Horizontal Translation

Analyze the graph of $y = \frac{1}{2}\sin\left(x - \frac{\pi}{3}\right)$.

Algebraic Solution

The amplitude is $\frac{1}{2}$ and the period is 2π. By solving the equations

$$x - \frac{\pi}{3} = 0 \qquad \text{and} \qquad x - \frac{\pi}{3} = 2\pi$$

$$x = \frac{\pi}{3} \qquad\qquad x = \frac{7\pi}{3}$$

you see that the interval $[\pi/3, 7\pi/3]$ corresponds to one cycle of the graph. Dividing this interval into four equal parts produces the following key points.

$$\left(\frac{\pi}{3}, 0\right), \quad \left(\frac{5\pi}{6}, \frac{1}{2}\right), \quad \left(\frac{4\pi}{3}, 0\right), \quad \left(\frac{11\pi}{6}, -\frac{1}{2}\right), \quad \left(\frac{7\pi}{3}, 0\right)$$

Graphical Solution

Use a graphing utility set in *radian* mode to graph $y = (1/2)\sin(x - \pi/3)$, as shown in Figure 5.43. Use the *minimum*, *maximum*, and *zero* or *root* features of the graphing utility to approximate the key points (1.047, 0), (2.618, 0.5), (4.189, 0), (5.760, −0.5), and (7.330, 0).

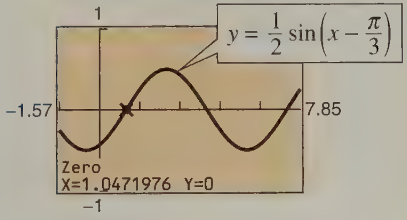


Figure 5.43

EXAMPLE 5 Horizontal Translation

Use a graphing utility to analyze the graph of $y = -3\cos(2\pi x + 4\pi)$.

Solution

The amplitude is 3 and the period is $2\pi/2\pi = 1$. By solving the equations

$$2\pi x + 4\pi = 0 \qquad \text{and} \qquad 2\pi x + 4\pi = 2\pi$$

$$2\pi x = -4\pi \qquad\qquad 2\pi x = -2\pi$$

$$x = -2 \qquad\qquad x = -1$$

you see that the interval $[-2, -1]$ corresponds to one cycle of the graph. Dividing this interval into four equal parts produces the key points

$$(-2, -3), \quad (-7/4, 0), \quad (-3/2, 3), \quad (-5/4, 0), \quad \text{and} \quad (-1, -3).$$

The graph is shown in Figure 5.44.

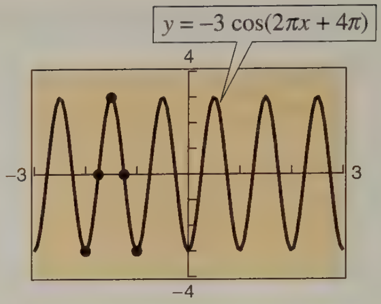


Figure 5.44

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

The final type of transformation is the *vertical translation* caused by the constant d in the equations

$$y = d + a\sin(bx - c) \qquad \text{and} \qquad y = d + a\cos(bx - c).$$

The shift is d units upward for $d > 0$ and d units downward for $d < 0$. In other words, the graph oscillates about the horizontal line $y = d$ instead of about the x-axis.

Exploration

Use a graphing utility to graph $y = d + \sin x$, where $d = -2$, 1, and 3. How does the value of d affect the graph?

EXAMPLE 6 Vertical Translation

Use a graphing utility to analyze the graph of $y = 2 + 3\cos 2x$.

Solution

The amplitude is 3 and the period is π. The key points over the interval $[0, \pi]$ are

$$(0, 5), \quad (\pi/4, 2), \quad (\pi/2, -1), \quad (3\pi/4, 2), \quad \text{and} \quad (\pi, 5).$$

The graph is shown in Figure 5.45. Compared with the graph of $f(x) = 3\cos 2x$, the graph of $y = 2 + 3\cos 2x$ is shifted upward two units.

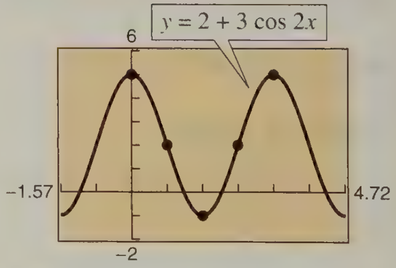


Figure 5.45

EXAMPLE 7 Finding an Equation for a Graph

Find the amplitude, period, and phase shift for the sine function whose graph is shown in Figure 5.46. Write an equation for this graph.

Solution

The amplitude for this sine curve is 2. The period is 2π, and there is a right phase shift of $\pi/2$. So, you can write

$$y = 2\sin\left(x - \frac{\pi}{2}\right).$$

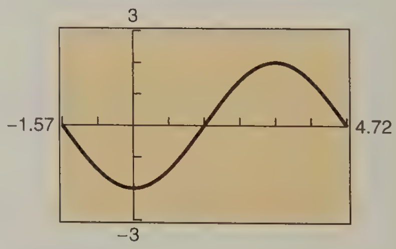


Figure 5.46

In Example 7, you can find a cosine function with the same graph. The amplitude for this cosine curve is 2. The period is 2π, and there is a right phase shift of π. So, you can write

$$y = 2\cos(x - \pi).$$

Mathematical Modeling

Sine and cosine functions can be used to model many real-life situations, including electric currents, musical tones, radio waves, tides, and weather patterns.

EXAMPLE 8 Finding a Trigonometric Model

Throughout the day, the depth of water at the end of a dock varies with the tides. The table shows the depths (in meters) at various times during the morning.

t (time)	Midnight	2 A.M.	4 A.M.	6 A.M.	8 A.M.	10 A.M.	Noon
y (depth)	2.55	3.80	4.40	3.80	2.55	1.80	2.27

a. Use a trigonometric function to model this data.

b. Find the depths at 9 A.M. and 3 P.M.

c. A boat needs at least 3 meters of water to moor at the dock. During what times in the afternoon can it safely dock?

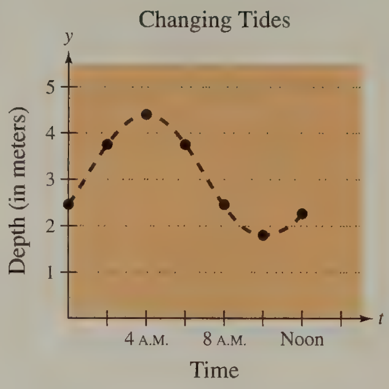


Figure 5.47

Solution

a. Begin by graphing the data, as shown in Figure 5.47. You can use either a sine or cosine model. Suppose you use a cosine model of the form

$$y = a\cos(bt - c) + d.$$

The difference between the maximum height and minimum height of the graph is twice the amplitude of the function. So, the amplitude is

$$a = \tfrac{1}{2}[(\text{maximum depth}) - (\text{minimum depth})] = \tfrac{1}{2}(4.4 - 1.8) = 1.3.$$

The cosine function completes one half of a cycle between when the maximum and minimum depths occur. So, the period is

$$p = 2[(\text{time of min. depth}) - (\text{time of max. depth})] = 2(10 - 4) = 12$$

which implies that $b = 2\pi/p \approx 0.524$. Because high tide occurs 4 hours after midnight, consider the left endpoint to be $c/b = 4$, so $c \approx 2.094$. Moreover, because the average depth is $\frac{1}{2}(4.4 + 1.8) = 3.1$, it follows that $d = 3.1$. So, you can model the depth with the function

$$y = 1.3\cos(0.524t - 2.094) + 3.1.$$

b. The depths at 9 A.M. and 3 P.M. are as follows:

$$y = 1.3\cos(0.524 \cdot 9 - 2.094) + 3.1 \approx 1.97 \text{ meters} \qquad \text{9 A.M.}$$

$$y = 1.3\cos(0.524 \cdot 15 - 2.094) + 3.1 \approx 4.23 \text{ meters.} \qquad \text{3 P.M.}$$

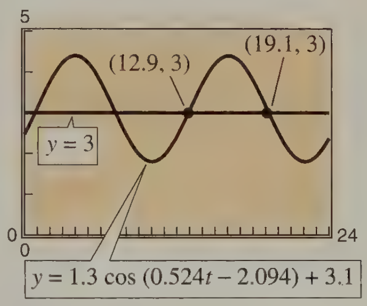


Figure 5.48

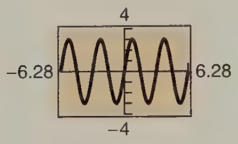
c. Using a graphing utility, graph the model with the line $y = 3$, as shown in Figure 5.48. From the graph, it follows that the depth is at least 3 meters between 12:54 P.M. ($t \approx 12.9$) and 7:06 P.M. ($t \approx 19.1$).

Note that a sine model for the data in Example 8 is $y = 1.3\sin(0.524t - 0.524) + 3.1$. Can you see how to find this?

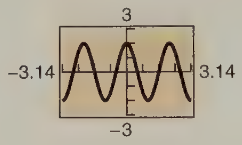
5.4 Exercises

In Exercises 1–14, find the period and amplitude.

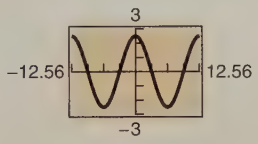
1. $y = 3 \sin 2x$



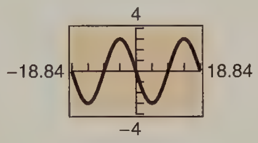
2. $y = 2 \cos 3x$



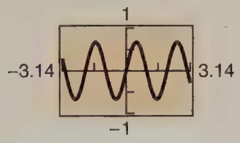
3. $y = \frac{5}{2} \cos \frac{x}{2}$



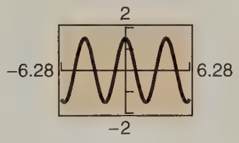
4. $y = -3 \sin \frac{x}{3}$



5. $y = \frac{2}{3} \sin \pi x$



6. $y = \frac{3}{2} \cos \frac{\pi x}{2}$



7. $y = -2 \sin x$

8. $y = -\cos \frac{2x}{5}$

9. $y = 3 \sin 6x$

10. $y = \frac{1}{3} \sin 10x$

11. $y = \frac{1}{4} \cos \frac{2x}{3}$

12. $y = \frac{5}{2} \cos \frac{x}{4}$

13. $y = 3 \sin 4\pi x$

14. $y = \frac{3}{4} \cos \frac{\pi x}{12}$

In Exercises 15–22, describe the relationship between the graphs of *f* and *g*.

15. $f(x) = \sin x$
$g(x) = \sin(x - \pi)$

16. $f(x) = \cos x$
$g(x) = \cos(x + \pi)$

17. $f(x) = \cos 2x$
$g(x) = -\cos 2x$

18. $f(x) = \sin 3x$
$g(x) = \sin(-3x)$

19. $f(x) = \cos x$
$g(x) = -5 \cos x$

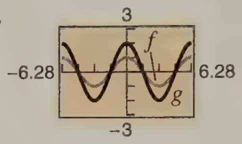
20. $f(x) = \sin x$
$g(x) = \sin 3x$

21. $f(x) = \sin x$
$g(x) = 4 + \sin x$

22. $f(x) = \cos 4x$
$g(x) = -6 + \cos 4x$

In Exercises 23–26, describe the relationship between the graphs of *f* and *g*.

23.



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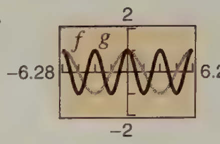

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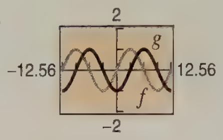
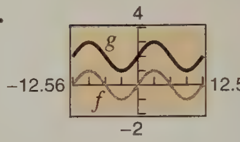


26.



In Exercises 27–34, sketch the graphs of the two functions in the same coordinate plane. (Include two full periods.)

27. $f(x) = -2 \sin x$
$g(x) = 4 \sin x$

28. $f(x) = \sin x$
$g(x) = \sin \frac{x}{3}$

29. $f(x) = \cos x$
$g(x) = 4 + \cos x$

30. $f(x) = 2 \cos 2x$
$g(x) = -\cos 4x$

31. $f(x) = -\frac{1}{2} \sin \frac{x}{2}$
$g(x) = 5 - \frac{1}{2} \sin \frac{x}{2}$

32. $f(x) = 4 \sin \pi x$
$g(x) = 4 \sin \pi x - 1$

33. $f(x) = 2 \cos x$
$g(x) = 2 \cos(x + \pi)$

34. $f(x) = -\cos x$
$g(x) = -\cos\left(x - \frac{\pi}{2}\right)$

Conjecture **In Exercises 35–38, use a graphing utility to graph *f* and *g* in the same viewing window. (Include two full periods.) Make a conjecture about the functions.**

35. $f(x) = \sin x$
$g(x) = \cos\left(x - \frac{\pi}{2}\right)$

36. $f(x) = \sin x$
$g(x) = -\cos\left(x + \frac{\pi}{2}\right)$

37. $f(x) = \cos x$
$g(x) = -\sin\left(x - \frac{\pi}{2}\right)$

38. $f(x) = \cos x$
$g(x) = -\cos(x - \pi)$

In Exercises 39–54, sketch the graph of the function by hand. Use a graphing utility to verify your sketch. (Include two full periods.)

39. $y = -2 \sin 4x$

40. $y = -3 \cos 2x$

41. $y = \cos 2\pi x$

42. $y = \dfrac{5}{2} \sin \dfrac{\pi x}{4}$

43. $y = -2 \sin \dfrac{2\pi x}{3}$

44. $y = -10 \cos \dfrac{\pi x}{6}$

45. $y = \sin\left(x - \dfrac{\pi}{4}\right)$

46. $y = \dfrac{1}{2} \sin(x - \pi)$

47. $y = 8 \cos(x + \pi)$

48. $y = 6 \cos\left(x + \dfrac{\pi}{6}\right)$

49. $y = \dfrac{1}{10} \cos 60\pi x$

50. $y = -4 + 5 \cos \dfrac{\pi t}{12}$

51. $y = 2 - 2 \sin \dfrac{2\pi x}{3}$

52. $y = 2 \cos x - 3$

53. $y = \dfrac{2}{3} \cos\left(\dfrac{x}{2} - \dfrac{\pi}{4}\right)$

54. $y = -3 \cos(6x + \pi)$

In Exercises 55–62, use a graphing utility to graph the function. (Include two full periods.) Identify the amplitude and period of the graph.

55. $y = -2 \sin(4x + \pi)$

56. $y = -4 \sin\left(\dfrac{2}{3}x - \dfrac{\pi}{3}\right)$

57. $y = \cos\left(2\pi x - \dfrac{\pi}{2}\right) + 1$

58. $y = 3 \cos\left(\dfrac{\pi x}{2} + \dfrac{\pi}{2}\right) - 3$

59. $y = 5 \sin(\pi - 2x) + 10$

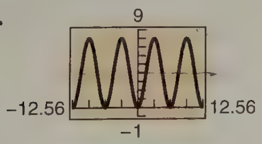
60. $y = 5 \cos(\pi - 2x) + 6$

61. $y = \frac{1}{100} \sin 120\pi t$

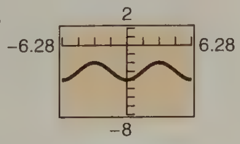
62. $y = -\frac{97}{100} \cos 50\pi t$

Graphical Reasoning **In Exercises 63–66, find** ***a*** **and** ***d*** **for the function** $f(x) = a \cos x + d$ **so that the graph of** f **matches the figure.**

63.



64.



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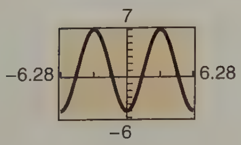

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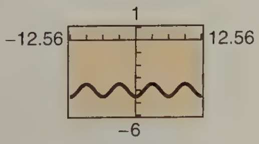
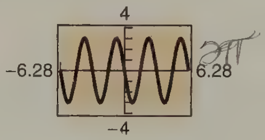


Graphical Reasoning **In Exercises 67–70, find** ***a, b,*** **and** ***c*** **for the function** $f(x) = a \sin(bx - c)$ **so that the graph of** f **matches the figure.**

67.



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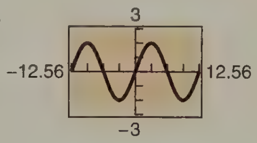
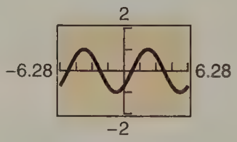
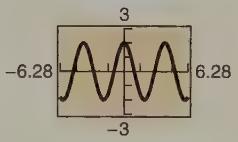


69.



70.



In Exercises 71–74, use a graphing utility to graph y_1 **and** y_2 **for all real numbers** x **in the interval** $[-2\pi, 2\pi]$**. Use the graphs to find the real numbers** x **such that** $y_1 = y_2$**.**

71. $y_1 = \sin x$
$y_2 = -\frac{1}{2}$

72. $y_1 = \cos x$
$y_2 = -1$

73. $y_1 = \cos x$
$y_2 = \dfrac{\sqrt{2}}{2}$

74. $y_1 = \sin x$
$y_2 = \dfrac{\sqrt{3}}{2}$

75. ***Respiratory Cycle*** For a person at rest, the velocity v (in liters per second) of air flow during a respiratory cycle is

$$v = 0.85 \sin \frac{\pi t}{3}$$

where t is the time (in seconds). (Inhalation occurs when $v > 0$, and exhalation occurs when $v < 0$.)

(a) Use a graphing utility to graph v.

(b) Find the time for one full respiratory cycle.

(c) Find the number of cycles per minute.

76. ***Respiratory Cycle*** The model in Exercise 75 is for a person at rest. How might the model change for a person who is exercising? Explain.

Sales **In Exercises 77 and 78, use a graphing utility to graph the sales function over 1 year, where** S **is the sales in thousands of units and** t **is the time in months, with** $t = 1$ **corresponding to January. Determine the months of maximum and minimum sales.**

77. $S = 22.3 - 3.4 \cos \dfrac{\pi t}{6}$

78. $S = 74.50 + 43.75 \sin \dfrac{\pi t}{6}$

79. ***Ferris wheel*** You are riding a Ferris wheel. Your height h (in feet) above the ground at any time t (in seconds) can be modeled by

$$h = 25\sin\frac{\pi}{15}(t - 75) + 30.$$

The ferris wheel turns for 135 seconds before it stops to let the first passengers off.

(a) Graph the model.

(b) What are your minimum and maximum heights above the ground?

80. ***Blood Pressure*** The pressure P (in millimeters of mercury) against the walls of the blood vessels of a certain person is modeled by

$$P = 100 - 20\cos\frac{8\pi}{3}t$$

where t is the time (in seconds). Graph the model. If one cycle is equivalent to one heartbeat, what is the person's pulse rate in heartbeats per minute?

81. ***Fuel Consumption*** The daily consumption C (in gallons) of diesel fuel on a farm is modeled by

$$C = 30.3 + 21.6\sin\left(\frac{2\pi t}{365} + 10.9\right)$$

where t is the time in days, with $t = 1$ corresponding to January 1.

(a) What is the period of the model? Is it what you expected? Explain.

(b) What is the average daily fuel consumption? Which term of the model did you use? Explain.

(c) Use a graphing utility to graph the model. Use the graph to approximate the time of the year when consumption exceeds 40 gallons per day.

82. ***Data Analysis*** The motion of an oscillating weight suspended by a spring was measured by a motion detector. The data was collected, and the approximate maximum displacements from equilibrium ($y = 2$) are labeled in the figure. The distance y from the motion detector is measured in centimeters and the time t is measured in seconds.

(a) Is y a function of t? Explain.

(b) Approximate the amplitude and period.

(c) Find a model for the data.

(d) Use a graphing utility to graph the model in part (c). Compare the result with the data in the figure.

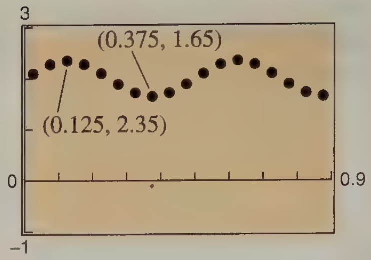


FIGURE FOR 82

83. ***Data Analysis*** The normal monthly precipitation p (in inches) for Seattle, Washington, for month t is shown in the table, with $t = 1$ corresponding to January. (Source: National Climatic Data Center)

t	1	2	3	4	5	6	7	8	9	10	11	12
p	5.4	4.0	3.8	2.5	1.8	1.6	0.9	1.2	1.9	3.3	5.7	6.0

(a) Find a trigonometric model for the normal monthly precipitation in Seattle.

(b) Use a graphing utility to graph the data and the model found in part (a). How well does the model fit the data?

84. ***Data Analysis*** The table shows the normal daily high temperatures for Honolulu H and Chicago C (in degrees Fahrenheit) for month t, with $t = 1$ corresponding to January. (Source: NOAA)

t	1	2	3	4	5	6	7	8	9	10	11	12
H	80.1	80.5	81.6	82.8	84.7	86.5	87.5	88.7	88.5	86.9	84.1	81.2
C	29.0	33.5	45.8	58.6	70.1	79.6	83.7	81.8	74.8	63.3	48.4	34.0

(a) A model for Honolulu is

$$H(t) = 84.40 + 4.28\sin\left(\frac{\pi t}{6} + 3.86\right).$$

Find a trigonometric model for Chicago.

(b) Use a graphing utility to graph the data points and the model for the temperatures in Honolulu. How well does the model fit the data?

(c) Use a graphing utility to graph the data points and the model for the temperatures in Chicago. How well does the model fit the data?

(d) Use the models to estimate the average annual temperature in each city. Which term of the models did you use? Explain.

(e) What are the periods of the two models? Are they what you expected? Explain.

(f) Which city has the greater variability in temperature throughout the year? Which factor of the models determines this variability? Explain.

Synthesis

True or False? **In Exercises 85 and 86, determine whether the statement is true or false. Justify your answer.**

85. The graph of $y = 6 - \frac{3}{4}\sin\frac{3x}{10}$ has a period of $\frac{20\pi}{3}$.

86. The graph of $y = -\cos x$ is a reflection of the graph of $y = \sin(x + \pi/2)$ in the x-axis.

87. ***Writing*** Use a graphing utility to graph the function $y = a\sin x$ for $a = \frac{1}{2}$, $a = \frac{3}{2}$, and $a = -3$. Write a paragraph describing the changes in the graph for the specified changes in a.

88. ***Writing*** Use a graphing utility to graph the function $y = d + \sin x$ for $d = 2$, $d = 3.5$, and $d = -2$. Write a paragraph describing the changes in the graph for the specified changes in d.

89. ***Writing*** Use a graphing utility to graph the function $y = \sin bx$ for $b = \frac{1}{2}$, $b = \frac{3}{2}$, and $b = 4$. Write a paragraph describing the changes in the graph for the specified changes in b.

90. ***Writing*** Use a graphing utility to graph the function $y = \sin(x - c)$ for $c = 1$, $c = 3$, and $c = -2$. Write a paragraph describing the changes in the graph for the specified changes in c.

91. ***Exploration*** In Section 5.3 it was shown that $f(x) = \cos x$ is an even function and $g(x) = \sin x$ is an odd function. Use a graphing utility to graph h and use the graph to determine whether h is even, odd, or neither.

(a) $h(x) = \cos^2 x$

(b) $h(x) = \sin^2 x$

(c) $h(x) = \sin x \cos x$

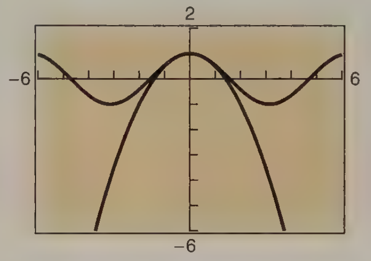
92. ***Conjecture*** If f is an even function and g is an odd function, use the results of Exercise 91 to make a conjecture about the following.

(a) $h(x) = [f(x)]^2$

(b) $h(x) = [g(x)]^2$

(c) $h(x) = f(x)g(x)$

93. ***Graphical Reasoning*** The figure shows the graphs of the functions $f(x) = 1 - \frac{1}{2}x^2$ and $g(x) = \cos x$. Identify the graphs and explain your reasoning.



Review

In Exercises 94–97, sketch the graph of the rational function and identify its asymptotes. Use a graphing utility to verify your results.

94. $f(x) = \frac{2}{11 + x}$

95. $f(x) = \frac{5x - 3}{x}$

96. $f(x) = \frac{10}{x^2 + x - 12}$

97. $f(x) = \frac{x^2 - 7x + 12}{x + 2}$

In Exercises 98–101, convert the angle measure from radians to degrees. Round your answer to three decimal places.

98. $\frac{13\pi}{2}$

99. $-\frac{\pi}{9}$

100. 8.5π

101. -0.48

5.5 Graphs of Other Trigonometric Functions

Graph of the Tangent Function

Recall that the tangent function is odd. That is, $\tan(-x) = -\tan x$. Consequently, the graph of $y = \tan x$ is symmetric with respect to the origin. You also know from the identity $\tan x = \sin x/\cos x$ that the tangent is undefined when $\cos x = 0$. Two such values are $x = \pm\pi/2 \approx \pm 1.5708$.

x	$-\frac{\pi}{2}$	-1.57	-1.5	-1	0	1	1.5	1.57	$\frac{\pi}{2}$
$\tan x$	Undef.	-1255.8	-14.1	-1.56	0	1.56	14.1	1255.8	Undef.

$\tan x$ approaches $-\infty$ as x approaches $-\pi/2$ from the right.

$\tan x$ approaches ∞ as x approaches $\pi/2$ from the left.

As indicated in the table, $\tan x$ increases without bound as x approaches $\pi/2$ from the left, and it decreases without bound as x approaches $-\pi/2$ from the right. So, the graph of $y = \tan x$ has *vertical asymptotes* at $x = \pi/2$ and $-\pi/2$, as shown in Figure 5.49. Moreover, because the period of the tangent function is π, vertical asymptotes also occur when $x = \pi/2 + n\pi$, where n is an integer. The domain of the tangent function is the set of all real numbers other than $x = \pi/2 + n\pi$, and the range is the set of all real numbers.

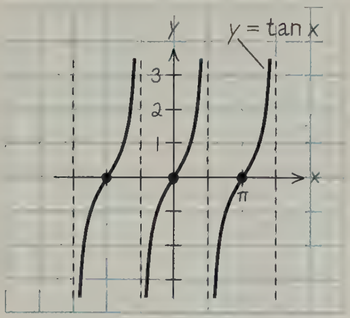


Figure 5.49

Period: π

Domain: all $x \neq \frac{\pi}{2} + n\pi$

Range: $(-\infty, \infty)$

Vertical asymptotes: $x = \frac{\pi}{2} + n\pi$

Sketching the graph of a function of the form $y = a\tan(bx - c)$ is similar to sketching the graph of $y = a\sin(bx - c)$ in that you locate key points that identify the intercepts and asymptotes. Two consecutive asymptotes can be found by solving the equations $bx - c = -\pi/2$ and $bx - c = \pi/2$. The midpoint between two consecutive asymptotes is an x-intercept of the graph. The period of the function $y = a\tan(bx - c)$ is the distance between two consecutive asymptotes. The amplitude of a tangent function is not defined. After plotting the asymptotes and the x-intercept, plot a few additional points between the two asymptotes and sketch one cycle. Finally, sketch one or two additional cycles to the left and right.

What You Should Learn:

- How to sketch the graphs of tangent functions
- How to sketch the graphs of cotangent functions
- How to sketch the graphs of secant and cosecant functions
- How to sketch the graphs of damped trigonometric functions

Why You Should Learn It:

You can use tangent, cotangent, secant, and cosecant functions to model real-life data. For instance, Exercise 70 on page 420 shows you how a tangent function can be used to model and analyze the distance between a television camera and a parade unit.

A. Ramey/PhotoEdit

EXAMPLE 1 Sketching the Graph of a Tangent Function

Sketch the graph of $y = \tan \frac{x}{2}$.

Solution

By solving the equations

$$\frac{x}{2} = -\frac{\pi}{2} \quad \text{and} \quad \frac{x}{2} = \frac{\pi}{2}$$

$$x = -\pi \qquad\qquad x = \pi$$

you can see that two consecutive asymptotes occur at $x = -\pi$ and $x = \pi$. Between these two asymptotes, plot a few points, including the x-intercept, as shown in the table. Three cycles of the graph are shown in Figure 5.50. Use a graphing utility to confirm this graph.

x	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
$\tan \frac{x}{2}$	Undef.	-1	0	1	Undef.

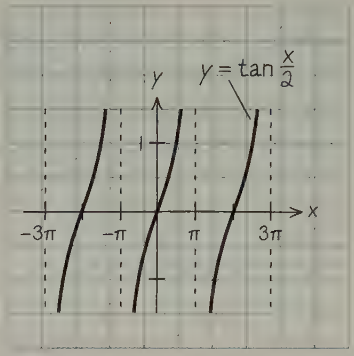


Figure 5.50

EXAMPLE 2 Sketching the Graph of a Tangent Function

Sketch the graph of $y = -3 \tan 2x$.

Algebraic Solution

By solving the equations $2x = -\pi/2$ and $2x = \pi/2$, you can see that two consecutive asymptotes occur at $x = -\pi/4$ and $x = \pi/4$. Between these two asymptotes, plot a few points, including the x-intercept, as shown in the table. Three complete cycles of the graph are shown in Figure 5.51.

x	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$
$-3 \tan 2x$	Undef.	3	0	-3	Undef.

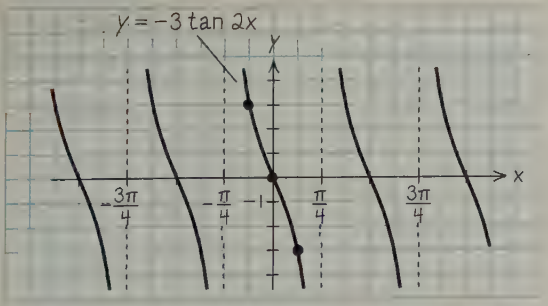


Figure 5.51

A computer animation of this example appears in the *Interactive* CD-ROM and Internet versions of this text.

Graphical Solution

Use a graphing utility set in *radian* mode and *dot* mode to graph $y = -3 \tan 2x$, as shown in Figure 5.52. Use the *zero* or *root* feature or the *zoom* and *trace* features of the graphing utility to approximate the key points as shown in the table.

x	-0.7854	-0.3927	0	0.3927	0.7854
$-3 \tan 2x$	Undef.	3	0	-3	Undef.

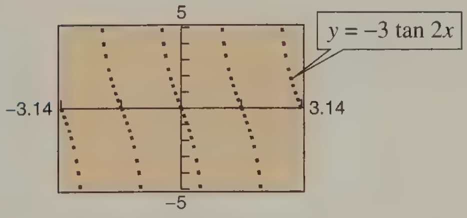


Figure 5.52

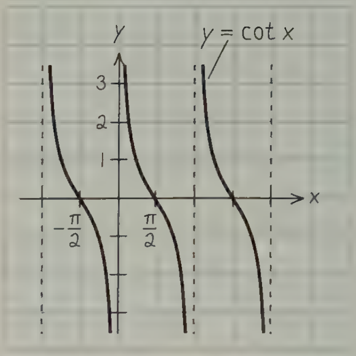
By comparing the graphs in Examples 1 and 2, you can see that the graph of $y = a\tan(bx - c)$ is increasing between consecutive vertical asymptotes if $a > 0$ and decreasing between consecutive vertical asymptotes if $a < 0$. In other words, the graph for $a < 0$ is a reflection in the x-axis of the graph for $a > 0$.

Graph of the Cotangent Function

The graph of the cotangent function is similar to the graph of the tangent function. It also has a period of π. However, from the identity

$$y = \cot x = \frac{\cos x}{\sin x}$$

you can see that the cotangent function has vertical asymptotes $x = n\pi$, where n is an integer, because $\sin x$ is zero at these x-values. The graph of the cotangent function is shown in Figure 5.53.



Period: π

Domain: all $x \neq n\pi$

Range: $(-\infty, \infty)$

Vertical asymptotes: $x = n\pi$

Figure 5.53

EXAMPLE 3 Sketching the Graph of a Cotangent Function

Sketch the graph of $y = 2\cot\frac{x}{3}$.

Solution

To locate two consecutive vertical asymptotes of the graph, solve the equations $x/3 = 0$ and $x/3 = \pi$ to see that two consecutive asymptotes occur at $x = 0$ and $x = 3\pi$.

Then, between these two asymptotes, plot a few points, including the x-intercept, as shown in the table. Three cycles of the graph are shown in Figure 5.54. Use a graphing utility to confirm this graph. [Enter the function as $y = 2/\tan(x/3)$.] Note that the period is 3π, the distance between consecutive asymptotes.

x	0	$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$\frac{9\pi}{4}$	3π
$2\cot\frac{x}{3}$	Undef.	2	0	-2	Undef.

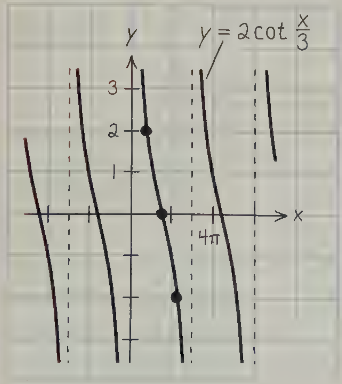


Figure 5.54

Graphs of the Reciprocal Functions

The graphs of the two remaining trigonometric functions can be obtained from the graphs of the sine and cosine functions using the reciprocal identities

$$\csc x = \frac{1}{\sin x} \qquad \text{and} \qquad \sec x = \frac{1}{\cos x}.$$

For instance, at a given value of x, the y-coordinate for $\sec x$ is the reciprocal of the y-coordinate for $\cos x$. Of course, when $\cos x = 0$, the reciprocal does not exist. Near such values of x, the behavior of the secant function is similar to that of the tangent function. In other words, the graphs of

$$\tan x = \frac{\sin x}{\cos x} \qquad \text{and} \qquad \sec x = \frac{1}{\cos x}$$

have vertical asymptotes at $x = \pi/2 + n\pi$, where n is an integer and the cosine is zero at these x-values. Similarly,

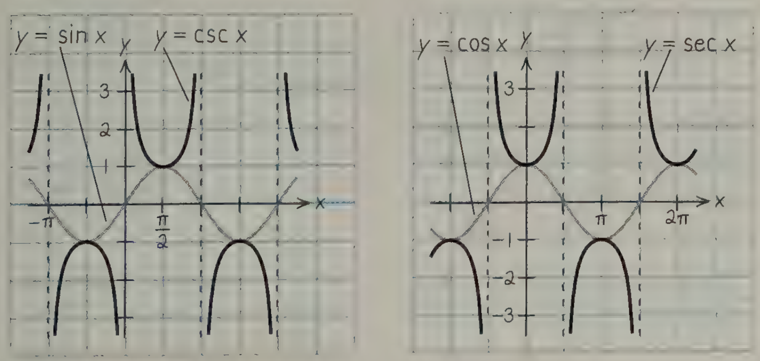
$$\cot x = \frac{\cos x}{\sin x} \qquad \text{and} \qquad \csc x = \frac{1}{\sin x}$$

have vertical asymptotes where $\sin x = 0$, that is, at $x = n\pi$.

To sketch the graph of a secant or cosecant function, you should first make a sketch of its reciprocal function. For instance, to sketch the graph of $y = \csc x$, first sketch the graph of $y = \sin x$. Then take reciprocals of the y-coordinates to obtain points on the graph of $y = \csc x$. You can use this procedure to obtain the graphs shown in Figure 5.55.

Exploration

Use a graphing utility to graph the functions $y_1 = \sin x$ and $y_2 = \csc x = 1/\sin x$ in the same viewing window. How are the graphs related? What happens to the graph of the cosecant function as x approaches the zeros of the sine function?



Period: 2π

Domain: all $x \neq n\pi$

Range: all y not in $(-1, 1)$

Vertical asymptotes: $x = n\pi$

Symmetry: origin

Period: 2π

Domain: all $x \neq \frac{\pi}{2} + n\pi$

Range: all y not in $(-1, 1)$

Vertical asymptotes: $x = \frac{\pi}{2} + n\pi$

Symmetry: y-axis

Figure 5.55

In comparing the graphs of the secant and cosecant functions with those of the sine and cosine functions, note that the "hills" and "valleys" are interchanged. For example, a hill (or maximum point) on the sine curve corresponds to a valley (a local minimum) on the cosecant curve. Similarly, a valley (or minimum point) on the sine curve corresponds to a hill (a local maximum) on the cosecant curve, as shown in Figure 5.56.

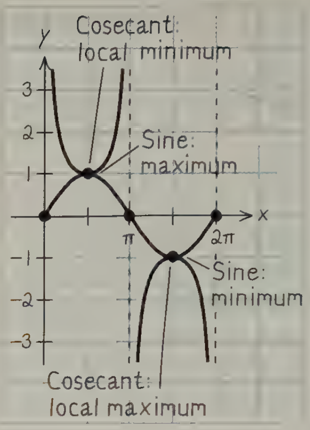


Figure 5.56

EXAMPLE 4 Comparing Trigonometric Graphs

Use a graphing utility to compare the graphs of

$$y = 2\sin\left(x + \frac{\pi}{4}\right) \quad \text{and} \quad y = 2\csc\left(x + \frac{\pi}{4}\right).$$

Solution

The two graphs are shown in Figure 5.57. Note how the hills and valleys of the graphs are related. For the function $y = 2\sin[x + (\pi/4)]$, the amplitude is 2 and the period is 2π. By solving the double inequality

$$0 < x + \frac{\pi}{4} < 2\pi \quad \Longrightarrow \quad -\frac{\pi}{4} < x < \frac{7\pi}{4}$$

you can see that one cycle of the sine function corresponds to the interval from $x = -\pi/4$ to $x = 7\pi/4$. The graph of this sine function is represented by the gray curve in Figure 5.57. Because the sine function is zero at the endpoints of this interval, the corresponding cosecant function

$$y = 2\csc\left(x + \frac{\pi}{4}\right) = 2\left(\frac{1}{\sin[x + (\pi/4)]}\right)$$

has vertical asymptotes at $x = -\pi/4, 3\pi/4, 7\pi/4$, etc. The graph of the cosecant function is represented by the black curve.

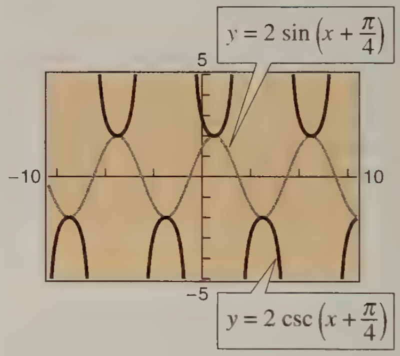


Figure 5.57

EXAMPLE 5 Comparing Trigonometric Graphs

Use a graphing utility to compare the graphs of

$$y = \cos 2x \quad \text{and} \quad y = \sec 2x.$$

Solution

Begin by graphing $y_1 = \cos 2x$ and $y_2 = \sec 2x = 1/\cos 2x$ in the same viewing window, as shown in Figure 5.58. Note that the x-intercepts of $y = \cos 2x$

$$\left(-\frac{\pi}{4}, 0\right), \quad \left(\frac{\pi}{4}, 0\right), \quad \left(\frac{3\pi}{4}, 0\right), \ldots$$

correspond to the vertical asymptotes

$$x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}, \quad x = \frac{3\pi}{4}, \ldots$$

of the graph of $y = \sec 2x$.

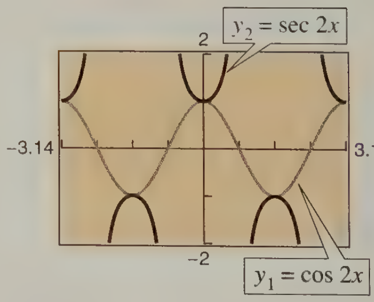


Figure 5.58

Damped Trigonometric Graphs

A *product* of two functions can be graphed using properties of the individual functions. For instance, consider the function

$$f(x) = x \sin x$$

as the product of the functions $y = x$ and $y = \sin x$. Using properties of absolute value and the fact that $|\sin x| \le 1$, you have $0 \le |x|\,|\sin x| \le |x|$. Consequently,

$$-|x| \le x \sin x \le |x|$$

which means that the graph of $f(x) = x \sin x$ lies between the lines $y = -x$ and $y = x$. Furthermore, because

$$f(x) = x \sin x = \pm x \qquad \text{at} \qquad x = \frac{\pi}{2} + n\pi$$

$$f(x) = x \sin x = 0 \qquad \text{at} \qquad x = n\pi$$

the graph of f touches the line $y = -x$ or the line $y = x$ at $x = \pi/2 + n\pi$ and has x-intercepts at $x = n\pi$. A sketch of f is shown in Figure 5.59. In the function $f(x) = x \sin x$, the factor x is called the **damping factor.**

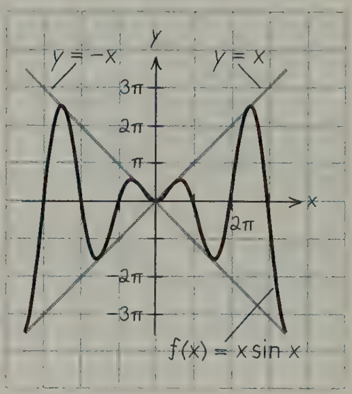


Figure 5.59

EXAMPLE 6 Analyzing a Damped Sine Curve

Analyze the graph of

$$f(x) = e^{-x} \sin 3x.$$

Solution

Consider $f(x)$ as the product of the two functions

$$y = e^{-x} \qquad \text{and} \qquad y = \sin 3x$$

each of which has the set of real numbers as its domain. For any real number x, you know that $e^{-x} \ge 0$ and $|\sin 3x| \le 1$. Therefore, $|e^{-x}|\,|\sin 3x| \le e^{-x}$, which means that

$$-e^{-x} \le e^{-x} \sin 3x \le e^{-x}.$$

Furthermore, because

$$f(x) = e^{-x} \sin 3x = \pm e^{-x} \qquad \text{at} \qquad x = \frac{\pi}{6} + \frac{n\pi}{3}$$

and

$$f(x) = e^{-x} \sin 3x = 0 \qquad \text{at} \qquad x = \frac{n\pi}{3}$$

the graph of f touches the curves $y = -e^{-x}$ and $y = e^{-x}$ at $x = \dfrac{\pi}{6} + \dfrac{n\pi}{3}$ and has intercepts at $x = \dfrac{n\pi}{3}$. The graph is shown in Figure 5.60.

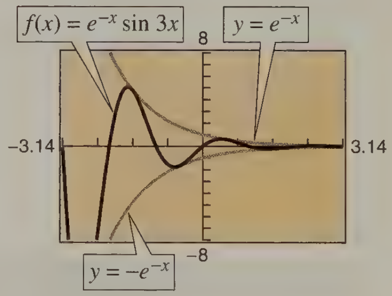
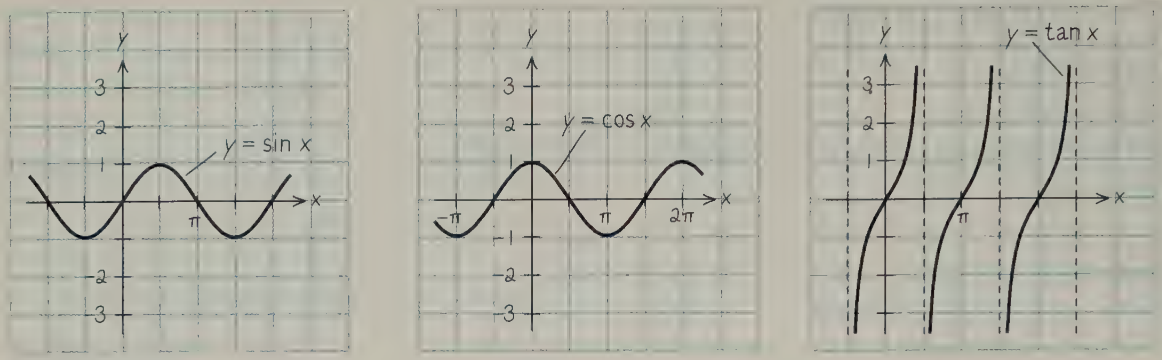


Figure 5.60

Figure 5.61 summarizes the graphs, domains, ranges, and periods of the six basic trigonometric functions.



Domain: all reals

Range: $[-1, 1]$

Period: 2π

Domain: all reals

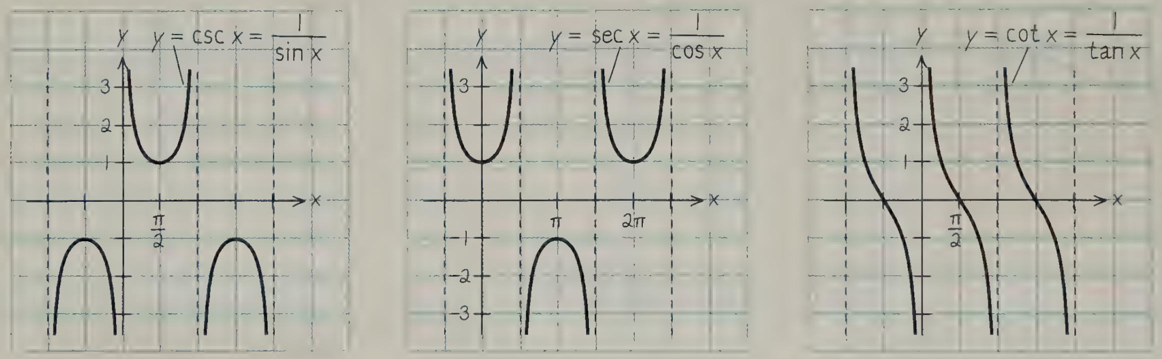
Range: $[-1, 1]$

Period: 2π

Domain: all $x \neq \frac{\pi}{2} + n\pi$

Range: $(-\infty, \infty)$

Period: π



Domain: all $x \neq n\pi$

Range: $(-\infty, -1]$ and $[1, \infty)$

Period: 2π

Domain: all $x \neq \frac{\pi}{2} + n\pi$

Range: $(-\infty, -1]$ and $[1, \infty)$

Period: 2π

Domain: all $x \neq n\pi$

Range: $(-\infty, \infty)$

Period: π

Figure 5.61

Writing About Math *Combining Trigonometric Functions*

Recall from Section 1.6 that functions can be combined arithmetically. This also applies to trigonometric functions. For each of the functions $h(x) = x + \sin x$ and $h(x) = \cos x - \sin 3x$, (a) identify two simpler functions f and g that comprise the combination, (b) use a table to show how to obtain the numerical values of $h(x)$ from the numerical values of $f(x)$ and $g(x)$, and (c) use graphing utility graphs of f and g to show how h may be formed.

Can you find functions

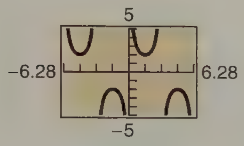
$$f(x) = d + a\sin(bx - c) \qquad \text{and} \qquad g(x) = d + a\cos(bx - c)$$

such that $f(x) + g(x) = 0$ for all x? If so, write a short paragraph explaining how you found the functions.

5.5 Exercises

In Exercises 1–8, match the function with its graph. State the period of the function. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

(a)



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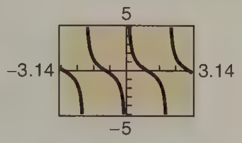
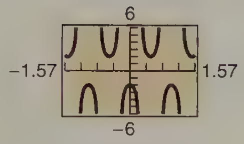
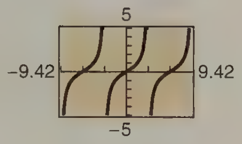


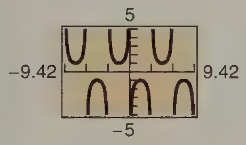
(c)



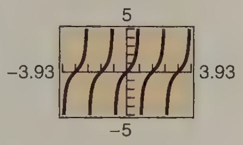
(d)



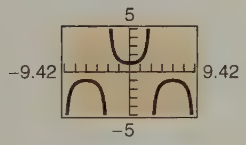
(e)



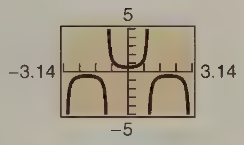
(f)



(g)



(h)



1. $y = \sec \dfrac{x}{2}$

2. $y = \tan \dfrac{x}{2}$

3. $y = \tan 2x$

4. $y = 2 \csc x$

5. $y = \cot \dfrac{\pi x}{2}$

6. $y = \dfrac{1}{2} \sec \dfrac{\pi x}{2}$

7. $y = -\csc x$

8. $y = -2 \sec 2\pi x$

In Exercises 9–30, sketch the graph of the function. (Include two full periods.) Use a graphing utility to verify your result.

9. $y = \frac{1}{3} \tan x$

10. $y = \frac{1}{4} \tan 2x$

11. $y = -2 \tan 2x$

12. $y = -3 \tan \pi x$

13. $y = -\frac{1}{2} \sec x$

14. $y = \frac{1}{4} \sec 2x$

15. $y = -\sec \pi x$

16. $y = 2 \sec 4x$

17. $y = \sec \pi x - 3$

18. $y = -2 \sec 4x + 2$

19. $y = \csc \dfrac{x}{2}$

20. $y = -\csc \dfrac{x}{3}$

21. $y = \dfrac{1}{2} \cot \dfrac{x}{2}$

22. $y = 3 \cot \dfrac{\pi x}{2}$

23. $y = \frac{1}{2} \sec 2x$

24. $y = -\frac{1}{2} \tan \pi x$

25. $y = 2 \tan \dfrac{\pi x}{4}$

26. $y = \sec(x + \pi)$

27. $y = \csc(\pi - x)$

28. $y = \sec(\pi - x)$

29. $y = 2 \cot\left(x - \dfrac{\pi}{2}\right)$

30. $y = \dfrac{1}{4} \csc\left(x + \dfrac{\pi}{4}\right)$

In Exercises 31–40, use a graphing utility to graph the function. (Include two full periods.) Describe your viewing window.

31. $y = \dfrac{1}{3} \tan \dfrac{x}{3}$

32. $y = -2 \tan 2\pi x$

33. $y = -2 \sec 4x$

34. $y = \dfrac{1}{4} \sec \pi x$

35. $y = \tan\left(x - \dfrac{\pi}{4}\right)$

36. $y = -\csc(4x - \pi)$

37. $y = \dfrac{1}{4} \cot\left(x + \dfrac{\pi}{2}\right)$

38. $y = 0.1 \tan\left(\dfrac{\pi x}{4} + \dfrac{\pi}{4}\right)$

39. $y = \dfrac{1}{2} \sec(2x - \pi)$

40. $y = \dfrac{1}{3} \sec\left(\dfrac{\pi x}{2} + \dfrac{\pi}{2}\right)$

In Exercises 41–44, use a graph to solve the equation in the interval $[-2\pi, 2\pi]$.

41. $\tan x = 1$

42. $\cot x = -\sqrt{3}$

43. $\sec x = -2$

44. $\csc x = \sqrt{2}$

In Exercises 45 and 46, use the graph of the function to determine whether the function is even, odd, or neither.

45. $f(x) = \sec x$

46. $f(x) = \tan x$

In Exercises 47–50, use a graphing utility to graph the two equations in the same viewing window. Use the graphs to lend evidence that the expressions are equivalent. Verify the results algebraically.

47. $y_1 = \sin x \csc x, \quad y_2 = 1$

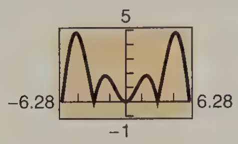
48. $y_1 = \sin x \sec x, \quad y_2 = \tan x$

49. $y_1 = \dfrac{\cos x}{\sin x}, \quad y_2 = \cot x$

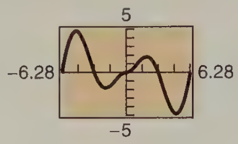
50. $y_1 = \sec^2 x - 1, \quad y_2 = \tan^2 x$

In Exercises 51–54, match the function with its graph. Describe the behavior of the function as x approaches zero. [The graphs are labeled (a), (b), (c), and (d).]

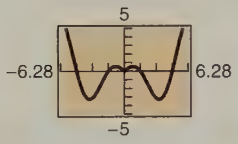
(a)



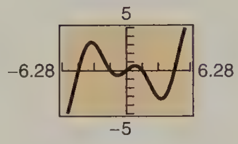
(b)



(c)



(d)



51. $f(x) = x \cos x$

52. $f(x) = |x \sin x|$

53. $g(x) = |x| \sin x$

54. $g(x) = |x| \cos x$

Conjecture **In Exercises 55–58, use a graphing utility to graph the functions f and g. Use the graphs to make a conjecture about the relationship between the functions.**

55. $f(x) = \sin x + \cos\left(x + \dfrac{\pi}{2}\right), \quad g(x) = 0$

56. $f(x) = \sin x - \cos\left(x + \dfrac{\pi}{2}\right), \quad g(x) = 2 \sin x$

57. $f(x) = \sin^2 x, \quad g(x) = \frac{1}{2}(1 - \cos 2x)$

58. $f(x) = \cos^2 \dfrac{\pi x}{2}, \quad g(x) = \dfrac{1}{2}(1 + \cos \pi x)$

In Exercises 59–62, use a graphing utility to graph the function and the damping factor of the function in the same viewing window. Describe the behavior of the function as x increases without bound.

59. $f(x) = e^{-x} \cos x$

60. $g(x) = e^{-x^2/2} \sin x$

61. $f(x) = 2^{-x/4} \cos \pi x$

62. $h(x) = 2^{-x^2/4} \sin x$

Exploration **In Exercises 63–68, use a graphing utility to graph the function. Describe the behavior of the function as x approaches zero.**

63. $f(x) = \dfrac{6}{x} + \cos x$

64. $f(x) = \sin x - \dfrac{4}{x}$

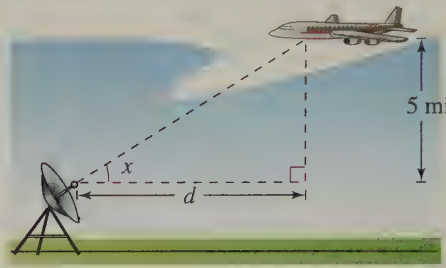
65. $f(x) = \dfrac{\sin x}{x}$

66. $f(x) = \dfrac{1 - \cos x}{x}$

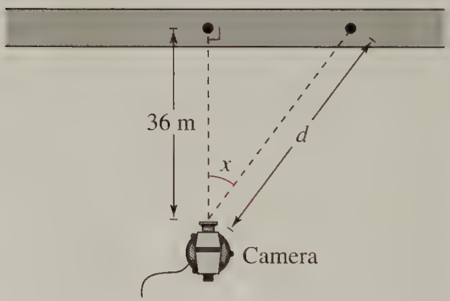
67. $f(x) = \dfrac{\tan x}{x}$

68. $f(x) = \dfrac{x}{\cot x}$

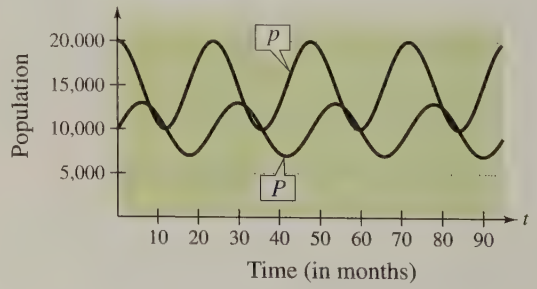
69. ***Distance*** A plane flying at an altitude of 5 miles over level ground will pass directly over a radar antenna. Let d be the ground distance from the antenna to the point directly under the plane and let x be the angle of elevation to the plane from the antenna. Write d as a function of x and graph the function over the interval $0 < x < \pi$.



70. ***Television Coverage*** A television camera is on a reviewing platform 36 meters from the street on which a parade will be passing from left to right. Express the distance d from the camera to a particular unit in the parade as a function of the angle x, and graph the function over the interval $-\pi/2 < x < \pi/2$. (Consider x as negative when a unit in the parade approaches from the left.)



71. ***Predatory-Prey Model*** Suppose the population of a certain predator at time t (in months) in a region is estimated to be $P = 10{,}000 + 3000 \sin(2\pi t/24)$ and the population of its primary food source (its prey) is estimated to be $p = 15{,}000 + 5000 \cos(2\pi t/24)$. Use the graph of the models to explain the oscillations in the size of each population.



72. ***Normal Temperatures*** The normal monthly high temperatures in degrees Fahrenheit for Erie, Pennsylvania, are approximated by

$$H(t) = 54.33 - 20.38 \cos \frac{\pi t}{6} - 15.69 \sin \frac{\pi t}{6}$$

and the normal monthly low temperatures are approximated by

$$L(t) = 39.36 - 15.70 \cos \frac{\pi t}{6} - 14.16 \sin \frac{\pi t}{6}$$

where t is the time (in months), with $t = 1$ corresponding to January. (Source: National Oceanic and Atmospheric Association)

(a) Use a graphing utility to graph each function. What is the period of each function?

(b) During what part of the year is the difference between the normal high and low temperatures greatest? When is it smallest?

(c) The sun is the farthest north in the sky around June 21, but the graph shows the warmest temperatures at a later date. Approximate the lag time of the temperatures relative to the position of the sun.

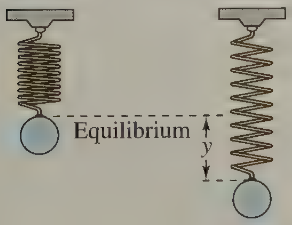
73. ***Harmonic Motion*** An object weighing W pounds is suspended from a ceiling by a steel spring. The weight is pulled downward (positive direction) from its equilibrium position and released. The resulting motion of the weight is described by the function

$$y = \frac{1}{2}e^{-t/4} \cos 4t, \quad t > 0$$

where y is the distance (in feet) and t is the time (in seconds).

(a) Use a graphing utility to graph the function.

(b) Describe the behavior of the displacement function for increasing values of time t.



74. ***Sales*** The projected monthly sales S (in thousands of units) of a seasonal product is modeled by

$$S = 74 + 3t + 40 \sin \frac{\pi t}{6}$$

where t is the time (in months), with $t = 1$ corresponding to January.

(a) Graph the sales function over 1 year. Use a graphing utility to verify your graph.

(b) Which month should have the greatest number of sales? Which month should have the least number of sales? Justify your answers.

75. ***Sales*** The projected monthly sales S (in thousands of units) of a seasonal product is modeled by

$$S = 52 + 5t - 28 \cos \frac{\pi t}{6}$$

where t is the time (in months), with $t = 1$ corresponding to January.

(a) Graph the sales function over 1 year. Use a graphing utility to verify your graph.

(b) Which month should have the greatest number of sales? Which month should have the least number of sales? Justify your answers.

76. ***Data Analysis*** The motion of an oscillating weight suspended by a spring was measured by a motion detector. The data was collected, and the approximate maximum (positive and negative) displacements from equilibrium are shown in the graph. The displacement y is measured in centimeters and the time t is measured in seconds.

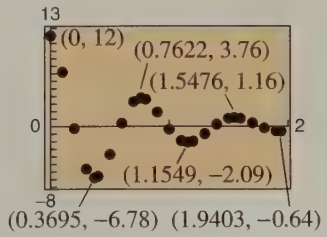
(a) Is y a function of t? Explain.

(b) Approximate the frequency of the oscillations.

(c) Fit a model of the form $y = ab^t \cos ct$ to the data. Use the result of part (b) to approximate c. Use the regression capabilities of a graphing utility to fit an exponential model to the positive maximum displacements of the weight.

(d) Rewrite the model in the form $y = ae^{kt} \cos ct$.

(e) Use a graphing utility to graph the model. Compare the result with the data in the graph.



77. ***Writing*** Write a short paragraph describing the specified change in the physical system of Exercise 76.

(a) A spring of less stiffness is used, and so the length of time in each oscillation is greater.

(b) The effect of friction is decreased.

78. ***Numerical and Graphical Reasoning*** A crossed belt connects a 10-centimeter pulley on an electric motor with a 20-centimeter pulley on a saw arbor. The electric motor runs at 1700 revolutions per minute.

(a) Determine the number of revolutions per minute of the saw.

(b) How does crossing the belt affect the saw in relation to the motor?

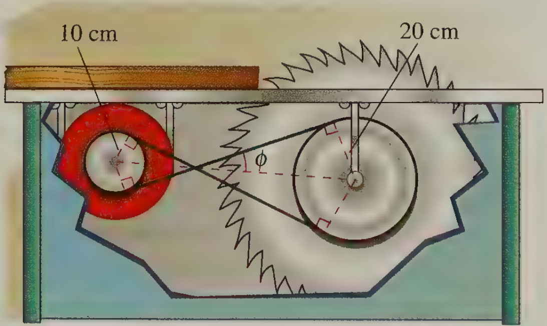
(c) Let L be the total length of the belt. Write L as a function of ϕ, where ϕ is measured in radians. What is the domain of the function? (*Hint:* Add the lengths of the straight sections of the belt and the length of belt around each pulley.)

(d) Use a graphing utility to complete the table.

ϕ	0.3	0.6	0.9	1.2	1.5
L					

(e) As ϕ increases, do the lengths of the straight sections of the belt change faster or slower than the lengths of the belt around each pulley?

(f) Use a graphing utility to graph the function over the appropriate domain.



Synthesis

True or False? **In Exercises 79–81, determine whether the statement is true or false. Justify your answer.**

79. The graph of $y = -\frac{1}{8}\tan\left(\frac{x}{2} + \pi\right)$ has an asymptote at $x = -3\pi$.

80. The graph of $y = -2\csc\left(x + \frac{\pi}{3}\right)$ has an asymptote at $x = -\frac{\pi}{3}$.

81. In the graph of $y = 2^x \sin x$, as x approaches $-\infty$, y approaches 0.

82. ***Writing*** Describe the behavior of $f(x) = \tan x$ as x approaches $\pi/2$ from the left and from the right.

83. ***Writing*** Describe the behavior of $f(x) = \csc x$ as x approaches π from the left and from the right.

84. ***Graphical Reasoning*** Consider the two functions $f(x) = 2\sin x$ and $g(x) = \frac{1}{2}\csc x$ on the interval $(0, \pi)$.

(a) Use a graphing utility to graph f and g in the same viewing window.

(b) Approximate the interval where $f > g$.

(c) Describe the behavior of each of the functions as x approaches π. How is the behavior of g related to the behavior of f as x approaches π?

85. ***Pattern Recognition***

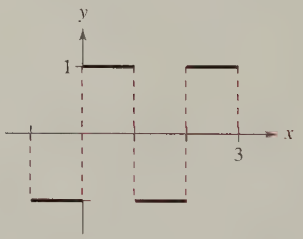
(a) Use a graphing utility to graph each function.

$$y_1 = \frac{4}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x\right)$$

$$y_2 = \frac{4}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x\right)$$

(b) Identify the pattern in part (a) and find a function y_3 that continues the pattern one more term. Use a graphing utility to graph y_3.

(c) The graphs of parts (a) and (b) approximate the periodic function in the figure. Find a function y_4 that is a better approximation.



Review

In Exercises 86–89, determine whether the function is one-to-one. If it is, find its inverse.

86. $f(x) = -10$

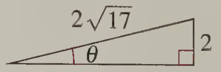
87. $f(x) = (x - 7)^2 + 3$

88. $f(x) = \sqrt{3x - 14}$

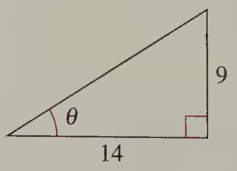
89. $f(x) = \sqrt[3]{x - 5}$

In Exercises 90 and 91, find the exact values of the six trigonometric functions of the angle θ.

90.



91.



5.6 Inverse Trigonometric Functions

What You Should Learn:

- How to evaluate inverse sine functions
- How to evaluate other inverse trigonometric functions
- How to evaluate compositions of trigonometric functions

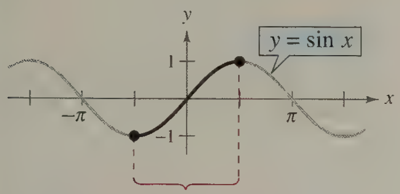
Why You Should Learn It:

Inverse trigonometric functions can be useful in exploring how two aspects of a real-life problem relate to each other. Exercise 70 on page 431 investigates the relationship between the length of rope from a winch to a boat and the angle of elevation between them.

Arlene Collins

Inverse Sine Function

Recall from Section 1.7 that for a function to have an inverse, it must pass the Horizontal Line Test. From Figure 5.62 it is obvious that $y = \sin x$ does not pass the test because different values of x yield the same y-value.



Sin x has an inverse on this interval.

Figure 5.62

However, if you restrict the domain to the interval $-\pi/2 \le x \le \pi/2$ (corresponding to the black portion of the graph in Figure 5.62), the following properties hold.

1. On the interval $[-\pi/2, \pi/2]$, the function $y = \sin x$ is increasing.
2. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ takes on its full range of values, $-1 \le \sin x \le 1$.
3. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ passes the Horizontal Line Test.

So, on the restricted domain $-\pi/2 \le x \le \pi/2$, $y = \sin x$ has a unique inverse called the **inverse sine function.** It is denoted by

$$y = \arcsin x \qquad \text{or} \qquad y = \sin^{-1} x.$$

The notation $\sin^{-1} x$ is consistent with the inverse function notation $f^{-1}(x)$. The arcsin x notation (read as "the arcsine of x") comes from the association of a central angle with its intercepted *arc length* on a unit circle. So, arcsin x means the angle (or arc) whose sine is x. Both notations, arcsin x and $\sin^{-1} x$, are commonly used in mathematics, so remember that $\sin^{-1} x$ denotes the *inverse* sine function rather than $1/\sin x$. The values of arcsin x lie in the interval $-\pi/2 \le \arcsin x \le \pi/2$. The graph of $y = \arcsin x$ is shown in Example 2.

Definition of Inverse Sine Function

The **inverse sine function** is defined by

$$y = \arcsin x \qquad \text{if and only if} \qquad \sin y = x$$

where $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$. The domain of $y = \arcsin x$ is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$.

When evaluating the inverse sine function, it helps to remember the phrase "the arcsine of x is the angle (or number) whose sine is x."

EXAMPLE 1 Evaluating the Inverse Sine Function

If possible, find the exact value.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\sin^{-1}\frac{\sqrt{3}}{2}$ **c.** $\sin^{-1} 2$

Solution

a. Because $\sin(-\pi/6) = -\frac{1}{2}$ for $-\pi/2 \le y \le \pi/2$, it follows that

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}. \qquad \text{Angle whose sine is } -\tfrac{1}{2}$$

b. Because $\sin(\pi/3) = \sqrt{3}/2$ for $-\pi/2 \le y \le \pi/2$, it follows that

$$\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}. \qquad \text{Angle whose sine is } \tfrac{\sqrt{3}}{2}$$

c. It is not possible to evaluate $y = \sin^{-1} x$ when $x = 2$ because there is no angle whose sine is 2. Remember that the domain of the inverse sine function is $[-1, 1]$.

EXAMPLE 2 Graphing the Arcsine Function

Sketch a graph of $y = \arcsin x$.

Solution

By definition, the equations

$$y = \arcsin x \qquad \text{and} \qquad \sin y = x$$

are equivalent for $-\pi/2 \le y \le \pi/2$. So, their graphs are the same. From the interval $[-\pi/2, \pi/2]$, you can assign values to y in the second equation to make a table of values.

y	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$x = \sin y$	-1	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	1

The resulting graph for

$$y = \arcsin x$$

is shown in Figure 5.63. Note that it is the reflection (in the line $y = x$) of the black portion of the graph in Figure 5.62. Use a graphing utility to confirm this graph. Be sure you see that Figure 5.63 shows the *entire* graph of the inverse sine function. Remember that the range of $y = \arcsin x$ is the closed interval $[-\pi/2, \pi/2]$.

STUDY T!P

As with the trigonometric functions, much of the work with the inverse trigonometric functions can be done by *exact* calculations rather than by calculator approximations. Exact calculations help to increase your understanding of the inverse functions by relating them to the triangle definitions of the trigonometric functions.

Library of Functions

The inverse trigonometric functions are obtained from the trigonometric functions in much the same way that the logarithmic function was developed from the exponential function. However, unlike the exponential function, the trigonometric functions are not one-to-one, and so it is necessary to restrict their domains to regions that pass the Horizontal Line Test. Consult the Library of Functions Summary inside the front cover for a description of the inverse trigonometric functions.

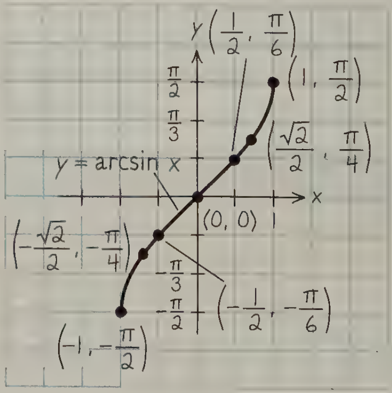


Figure 5.63

Other Inverse Trigonometric Functions

The cosine function is decreasing on the interval $0 \le x \le \pi$, as shown in Figure 5.64.

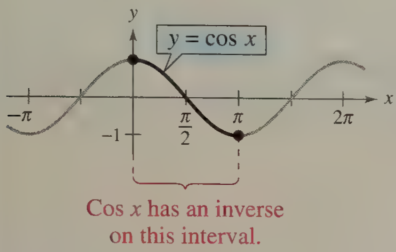


Figure 5.64

Consequently, on this interval the cosine function has an inverse function—the **inverse cosine function**—denoted by

$$y = \arccos x \qquad \text{or} \qquad y = \cos^{-1} x.$$

Because these equations are equivalent for $0 \le y \le \pi$, their values are the same, as shown in the table below.

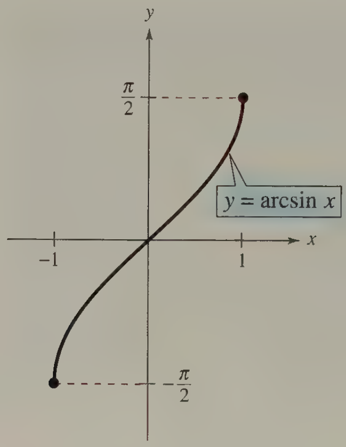
y	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$x = \cos y$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Similarly, you can define an **inverse tangent function** by restricting the domain of $y = \tan x$ to the interval $(-\pi/2, \pi/2)$. The following list summarizes the definitions of the three most common inverse trigonometric functions. The remaining three are discussed in Exercises 80–82 in this section.

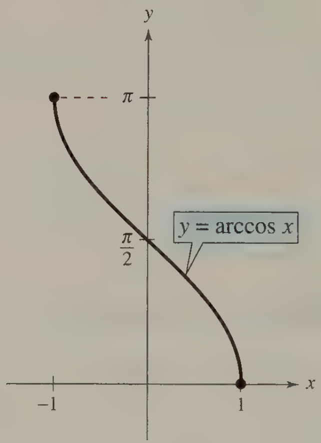
Definition of Inverse Trigonometric Functions

Function	*Domain*	*Range*
$y = \arcsin x$ if and only if $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$ if and only if $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ if and only if $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

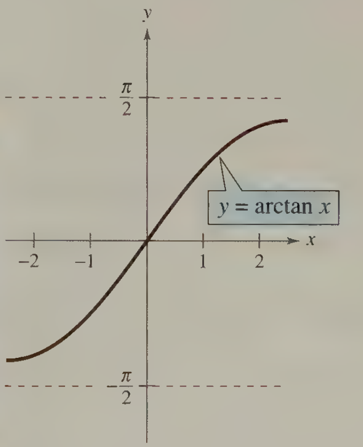
The graphs of these three inverse trigonometric functions are shown in Figure 5.65.



Domain: $[-1, 1]$; *Range:* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Domain: $[-1, 1]$; *Range:* $[0, \pi]$



Domain: $(-\infty, \infty)$; *Range:* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Figure 5.65

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 3 Evaluating Inverse Trigonometric Functions

Find the exact value.

a. $\arccos \frac{\sqrt{2}}{2}$ **b.** $\arccos(-1)$ **c.** $\arctan 0$ **d.** $\arctan(-1)$

Solution

a. Because $\cos(\pi/4) = \sqrt{2}/2$, and $\pi/4$ lies in $[0, \pi]$, it follows that

$$\arccos \frac{\sqrt{2}}{2} = \frac{\pi}{4}. \qquad \text{Angle whose cosine is } \frac{\sqrt{2}}{2}$$

b. Because $\cos \pi = -1$, and π lies in $[0, \pi]$, it follows that

$$\arccos(-1) = \pi. \qquad \text{Angle whose cosine is } -1$$

c. Because $\tan 0 = 0$, and 0 lies in $(-\pi/2, \pi/2)$, it follows that

$$\arctan 0 = 0. \qquad \text{Angle whose tangent is } 0$$

d. Because $\tan(-\pi/4) = -1$ and $-\pi/4$ lies in $(-\pi/2, \pi/2)$, it follows that

$$\arctan(-1) = -\frac{\pi}{4}. \qquad \text{Angle whose tangent is } -1$$

You can use the $\boxed{\text{SIN}^{-1}}$, $\boxed{\text{COS}^{-1}}$, and $\boxed{\text{TAN}^{-1}}$ keys on your calculator to approximate values of trigonometric functions.

STUDY TIP

Most graphing utilities do not have function keys for the inverse cosecant function, the inverse secant function, or the inverse cotangent function. To evaluate these functions, you can use the inverse sine, inverse cosine, and inverse tangent functions, respectively. For instance, to evaluate arcsec 3.4, enter the expression as arccos(1/3.4).

EXAMPLE 4 Calculators and Inverse Trigonometric Functions

Use a calculator to approximate the value (if possible).

a. $\arctan(-8.45)$ **b.** $\arcsin 0.2447$ **c.** $\arccos 2$

Solution

Function	*Mode*	*Graphing Calculator Keystrokes*
a. $\arctan(-8.45)$	Radian	$\boxed{\text{TAN}^{-1}}$ $\boxed{(}$ $\boxed{(-)}$ 8.45 $\boxed{)}$ $\boxed{\text{ENTER}}$

From the display, it follows that $\arctan(-8.45) \approx -1.453001$.

b. $\arcsin 0.2447$	Radian	$\boxed{\text{SIN}^{-1}}$ 0.2447 $\boxed{\text{ENTER}}$

From the display, it follows that $\arcsin 0.2447 \approx 0.2472103$.

c. $\arccos 2$	Radian	$\boxed{\text{COS}^{-1}}$ 2 $\boxed{\text{ENTER}}$

In real number mode, the calculator should display an error message because the domain of the inverse cosine function is $[-1, 1]$.

In Example 4, if you had set the calculator to *degree* mode, the display would have been in degrees rather than in radians. This convention is peculiar to calculators. By definition, the values of inverse trigonometric functions are always in *radians*.

Compositions of Functions

Recall from Section 1.7 that inverse functions possess the properties

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

The inverse trigonometric versions of these properties are given below.

Inverse Properties

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \qquad \text{and} \qquad \arcsin(\sin y) = y.$$

If $-1 \le x \le 1$ and $0 \le y \le \pi$, then

$$\cos(\arccos x) = x \qquad \text{and} \qquad \arccos(\cos y) = y.$$

If x is a real number and $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \qquad \text{and} \qquad \arctan(\tan y) = y.$$

Keep in mind that these inverse properties do not apply for arbitrary values of x and y. For instance,

$$\arcsin\left(\sin\frac{3\pi}{2}\right) = \arcsin(-1) = -\frac{\pi}{2} \neq \frac{3\pi}{2}.$$

In other words, the property

$$\arcsin(\sin y) = y$$

is not valid for values of y outside the interval $[-\pi/2, \pi/2]$.

Exploration

Use a graphing utility to graph $y = \arcsin(\sin x)$. What are the domain and range of this function? Explain why $\arcsin(\sin 4)$ does not equal 4.

Now graph $y = \sin(\arcsin x)$ and determine the domain and range. Explain why $\sin(\arcsin 4)$ is not defined.

EXAMPLE 5 Using Inverse Properties

If possible, find the exact value.

a. $\tan[\arctan(-5)]$ **b.** $\arcsin\left(\sin\frac{5\pi}{3}\right)$ **c.** $\cos(\cos^{-1}\pi)$

Solution

a. Because -5 lies in the domain of $\arctan x$, the inverse property applies, and you have

$$\tan[\arctan(-5)] = -5.$$

b. In this case, $5\pi/3$ does not lie within the range of the arcsine function, $-\pi/2 \le y \le \pi/2$. However, $5\pi/3$ is coterminal with

$$\frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$$

which does lie in the range of the arcsine function, and you have

$$\arcsin\left(\sin\frac{5\pi}{3}\right) = \arcsin\left[\sin\left(-\frac{\pi}{3}\right)\right] = -\frac{\pi}{3}.$$

c. The expression $\cos(\cos^{-1}\pi)$ is not defined because $\cos^{-1}\pi$ is not defined. Remember that the domain of the inverse cosine function is $[-1, 1]$.

Example 6 shows how to use right triangles to find exact values of compositions of inverse functions. Then, Example 7 shows how to use triangles to convert a trigonometric expression into an algebraic expression. This conversion technique is used frequently in calculus.

EXAMPLE 6 Evaluating Compositions of Functions

Find the exact value of (a) $\tan\left(\arccos\dfrac{2}{3}\right)$ and (b) $\cos\left[\arcsin\left(-\dfrac{3}{5}\right)\right]$.

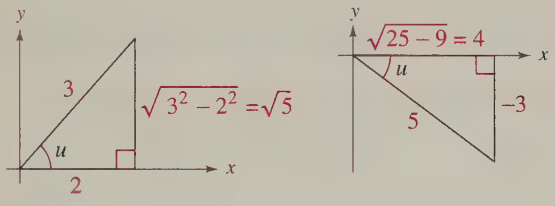
Algebraic Solution

a. If you let $u = \arccos\frac{2}{3}$, then $\cos u = \frac{2}{3}$. Because $\cos u$ is positive, u is a first-quadrant angle. You can sketch and label angle u as shown in Figure 5.66(a). Consequently,

$$\tan\left(\arccos\frac{2}{3}\right) = \tan u = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{5}}{2}.$$

b. If you let $u = \arcsin\left(-\frac{3}{5}\right)$, then $\sin u = -\frac{3}{5}$. Because $\sin u$ is negative, u is a fourth-quadrant angle. You can sketch and label angle u as shown in Figure 5.66(b). Consequently,

$$\cos\left[\arcsin\left(-\frac{3}{5}\right)\right] = \cos u = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5}.$$



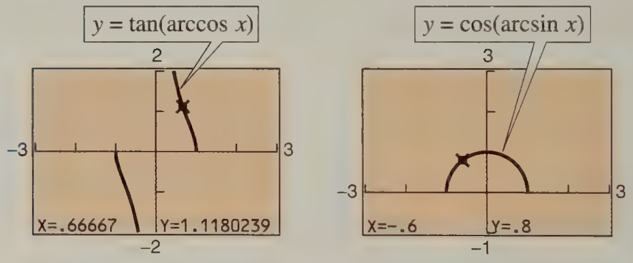
(a) (b)

Figure 5.66

Graphical Solution

a. Use a graphing utility set in *radian* mode to graph $y = \tan(\arccos x)$, as shown in Figure 5.67(a). Use the *value* feature or *zoom* and *trace* features of the graphing utility to find that the value of the composition of functions when $x = \frac{2}{3} \approx 0.66667$ is $y = 1.118 \approx \sqrt{5}/2$.

b. Use a graphing utility set in *radian* mode to graph $y = \cos(\arcsin x)$, as shown in Figure 5.67(b). Use the *value* feature or *zoom* and *trace* features of the graphing utility to find that the value of the composition of functions when $x = -\frac{3}{5} = -0.6$ is $y = 0.8 = \frac{4}{5}$.



(a) (b)

Figure 5.67

EXAMPLE 7 Some Problems from Calculus

Write each of the following as an algebraic expression in x.

a. $\sin(\arccos 3x)$, $0 \le x \le \dfrac{1}{3}$ **b.** $\cot(\arccos 3x)$, $0 \le x \le \dfrac{1}{3}$

Solution

If you let $u = \arccos 3x$, then $\cos u = 3x$. Because $\cos u = 3x/1 = \text{adj}/\text{hyp}$ you can sketch a right triangle with acute angle u, as shown in Figure 5.68. From this triangle, you can easily convert each expression to algebraic form.

a. $\sin(\arccos 3x) = \sin u = \dfrac{\text{opp}}{\text{hyp}} = \sqrt{1 - 9x^2}, \qquad 0 \le x < \dfrac{1}{3}$

b. $\cot(\arccos 3x) = \cot u = \dfrac{\text{adj}}{\text{opp}} = \dfrac{3x}{\sqrt{1 - 9x^2}}, \qquad 0 \le x < \dfrac{1}{3}$

A similar argument can be made here for x-values lying in the interval $\left[-\frac{1}{3}, 0\right]$.

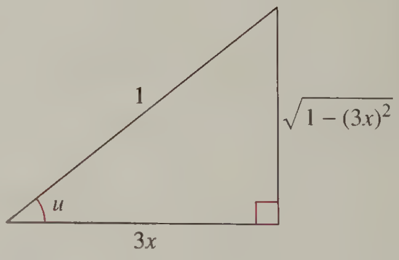


Figure 5.68

5.6 Exercises

1. ***Numerical and Graphical Analysis*** Consider the function $y = \arcsin x$.

(a) Use a graphing utility to complete the table.

x	-1	-0.8	-0.6	-0.4	-0.2
y					

x	0	0.2	0.4	0.6	0.8	1
y						

(b) Plot the points from the table in part (a) and graph the function. (Do not use a graphing utility.)

(c) Use a graphing utility to graph the inverse sine function and compare the result with your hand-drawn graph in part (b).

(d) Determine any intercepts and symmetry of the graph.

2. ***Numerical and Graphical Analysis*** Consider the function $y = \arccos x$.

(a) Use a graphing utility to complete the table.

x	-1	-0.8	-0.6	-0.4	-0.2
y					

x	0	0.2	0.4	0.6	0.8	1
y						

(b) Plot the points from the table in part (a) and graph the function. (Do not use a graphing utility.)

(c) Use a graphing utility to graph the inverse cosine function and compare the result with your hand-drawn graph in part (b).

(d) Determine any intercepts and symmetry of the graph.

3. ***Numerical and Graphical Analysis*** Consider the function $y = \arctan x$.

(a) Use a graphing utility to complete the table.

x	-10	-8	-6	-4	-2
y					

x	0	2	4	6	8	10
y						

(b) Plot the points from the table in part (a) and graph the function. (Do not use a graphing utility.)

(c) Use a graphing utility to graph the inverse tangent function and compare the result with your hand-drawn graph in part (b).

(d) Determine the horizontal asymptotes of the graph.

In Exercises 4–7, write each trigonometric expression in inverse function form, or vice versa. For example,

$\sin\frac{\pi}{2} = 1$ ⇨ $\arcsin 1 = \frac{\pi}{2}$.

4. $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ ⇨

5. $\tan\left(-\frac{\pi}{4}\right) = -1$ ⇨

6. $\arccos\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ ⇨

7. $\arcsin(-1) = -\frac{\pi}{2}$ ⇨

In Exercises 8–15, evaluate the expression using a calculator.

8. (a) $\arcsin\frac{1}{2}$ (b) $\arcsin 0$

9. (a) $\arccos\frac{1}{2}$ (b) $\arccos 0$

10. (a) $\arctan\frac{\sqrt{3}}{3}$ (b) $\arctan 1$

11. (a) $\arccos\left(-\frac{\sqrt{2}}{2}\right)$ (b) $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

12. (a) $\arctan(-\sqrt{3})$ (b) $\arctan\sqrt{3}$

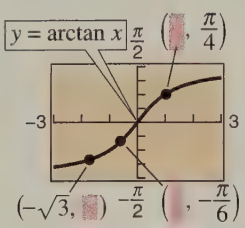
13. (a) $\arccos\left(-\frac{1}{2}\right)$ (b) $\arcsin\frac{\sqrt{2}}{2}$

14. (a) $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$ (b) $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

15. (a) $\arcsin(-1)$ (b) $\arccos 1$

In Exercises 16 and 17, determine the missing coordinates of the points on the graph of the function.

16.



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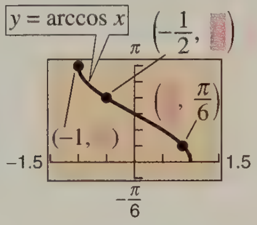


In Exercises 18–23, use a calculator to approximate the value of the expression. (Round your answer to two decimal places.)

18. (a) $\arccos 0.22$ (b) $\arcsin 0.45$

19. (a) $\arcsin(-0.75)$ (b) $\arccos(-0.7)$

20. (a) $\arctan(-6)$ (b) $\arctan 18$

21. (a) $\arcsin 0.41$ (b) $\arccos 0.36$

22. (a) $\arccos(-0.51)$ (b) $\arcsin(-0.125)$

23. (a) $\arctan 0.98$ (b) $\arctan 4.7$

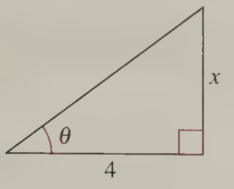
In Exercises 24 and 25, use a graphing utility to graph *f*, *g*, and $y = x$ in the same viewing window to verify geometrically that *g* is the inverse of *f*. (Be sure to properly restrict the domain of *f*.)

24. $f(x) = \tan x, \quad g(x) = \arctan x$

25. $f(x) = \sin x, \quad g(x) = \arcsin x$

In Exercises 26–29, use an inverse trigonometric function to write θ as a function of x.

26.



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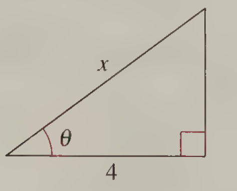

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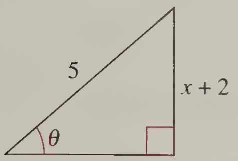

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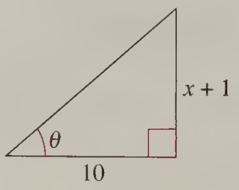


In Exercises 30–37, use the properties of inverse functions to evaluate the expression.

30. $\sin(\arcsin 0.7)$ **31.** $\tan(\arctan 35)$

32. $\cos[\arccos(-0.3)]$ **33.** $\sin[\arcsin(-0.1)]$

34. $\arcsin(\sin 3\pi)$ **35.** $\arccos\left(\cos \frac{7\pi}{2}\right)$

36. $\arctan\left(\tan \frac{11\pi}{6}\right)$ **37.** $\arcsin\left(\sin \frac{7\pi}{4}\right)$

In Exercises 38–45, find the exact value of the expression. Use a graphing utility to verify your result. (*Hint:* Make a sketch of a right triangle.)

38. $\sin\left(\arctan \frac{4}{3}\right)$ **39.** $\sec\left(\arcsin \frac{3}{5}\right)$

40. $\cos\left(\arcsin \frac{24}{25}\right)$ **41.** $\csc\left[\arctan\left(-\frac{12}{5}\right)\right]$

42. $\sec\left[\arctan\left(-\frac{3}{5}\right)\right]$ **43.** $\tan\left[\arcsin\left(-\frac{3}{8}\right)\right]$

44. $\sin\left[\arccos\left(-\frac{2}{3}\right)\right]$ **45.** $\cot\left(\arctan \frac{6}{11}\right)$

In Exercises 46–53, write an algebraic expression that is equivalent to the expression. (*Hint:* Sketch a right triangle, as demonstrated in Example 7.)

46. $\cot(\arctan x)$ **47.** $\sin(\arctan x)$

48. $\sin(\arccos x)$ **49.** $\sec[\arcsin(x - 1)]$

50. $\tan\left(\arccos \frac{x}{5}\right)$ **51.** $\cot\left(\arctan \frac{1}{x}\right)$

52. $\csc\left(\arctan \frac{x}{\sqrt{7}}\right)$ **53.** $\cos\left(\arcsin \frac{x - h}{r}\right)$

In Exercises 54 and 55, use a graphing utility to graph *f* and *g* in the same viewing window to verify that the two are equal. Explain why they are equal. Identify any asymptotes of the graphs.

54. $f(x) = \sin(\arctan 2x), \quad g(x) = \frac{2x}{\sqrt{1 + 4x^2}}$

55. $f(x) = \tan\left(\arccos \frac{x}{2}\right), \quad g(x) = \frac{\sqrt{4 - x^2}}{x}$

In Exercises 56–59, complete the equation.

56. $\arctan \frac{14}{x} = \arcsin(\quad), \quad x \neq 0$

57. $\arcsin \frac{\sqrt{36 - x^2}}{6} = \arccos(\quad), \quad 0 \leq x \leq 6$

58. $\arccos \dfrac{3}{\sqrt{x^2 - 2x + 10}} = \arcsin(\quad)$

59. $\arccos \dfrac{x - 2}{2} = \arctan(\quad), \quad |x - 2| < 2$

In Exercises 60–67, use a graphing utility to graph the function.

60. $y = 2 \arccos x$

61. $y = \arcsin \dfrac{x}{2}$

62. $f(x) = \arcsin(x - 2)$

63. $g(t) = \arccos(t + 2)$

64. $f(x) = \arctan 2x$

65. $f(x) = \pi + \arctan x$

66. $h(v) = \tan(\arccos v)$

67. $f(x) = \arccos \dfrac{x}{4}$

In Exercises 68 and 69, write the given function in terms of the sine function by using the identity

$$A \cos \omega t + B \sin \omega t = \sqrt{A^2 + B^2} \sin\left(\omega t + \arctan \frac{A}{B}\right).$$

Use a graphing utility to graph both forms of the function. What does the graph imply?

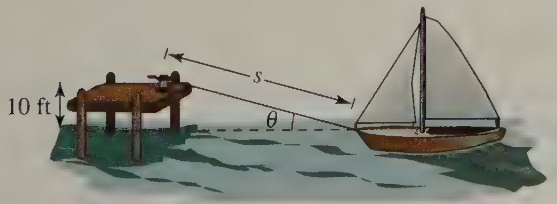
68. $f(t) = 3 \cos 2t + 3 \sin 2t$

69. $f(t) = 4 \cos \pi t + 3 \sin \pi t$

70. ***Docking a Boat*** A boat is pulled in by means of a winch located on a dock 10 feet above the deck of the boat. Let θ be the angle of elevation from the boat to the winch and let s be the length of the rope from the winch to the boat.

(a) Write θ as a function of s.

(b) Find θ when $s = 52$ feet and when $s = 26$ feet.



71. ***Photography*** A television camera at ground level is filming the lift-off of a space shuttle at a point 750 meters from the launch pad. Let θ be the angle of elevation to the shuttle and let s be the height of the shuttle.

(a) Write θ as a function of s.

(b) Find θ when $s = 400$ meters and when $s = 1600$ meters.

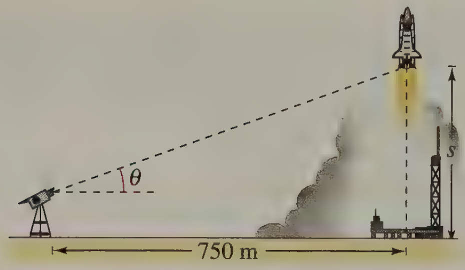


FIGURE FOR 71

72. ***Rock Salt*** Different types of granular substances naturally settle at different angles when stored in cone-shaped piles. This angle θ is called the *angle of repose*. When rock salt is stored in a cone-shaped pile 11 feet high, the diameter of the pile's base is about 34 feet. (Source: Bulk-Store Structures, Inc.)

(a) Find the angle of repose for rock salt.

(b) How tall is a pile of rock salt that has a base diameter of 40 feet?

73. ***Granular Angle of Repose*** When whole corn is stored in a cone-shaped pile 20 feet high, the diameter of the pile's base is about 82 feet.

(a) Find the angle of repose for whole corn.

(b) How tall is a pile of corn that has a base diameter of 100 feet?

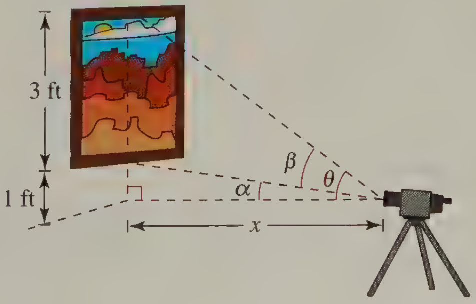
74. ***Photography*** A photographer is taking a picture of a 3-foot painting hung in an art gallery. The camera lens is 1 foot below the lower edge of the painting. The angle β subtended by the camera lens x feet from the painting is

$$\beta = \arctan \frac{3x}{x^2 + 4}, \quad x > 0.$$

(a) Use a graphing utility to graph β as a function of x.

(b) Move the cursor to approximate the distance from the picture when β is maximum.

(c) Identify the asymptote of the graph and discuss its meaning in the context of the problem.



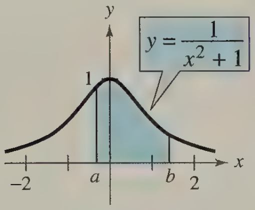
75. ***Area*** In calculus, it is shown that the area of the region bounded by the graphs of $y = 0$, $y = 1/(x^2 + 1)$, $x = a$, and $x = b$ is

Area = $\arctan b - \arctan a$.

Find the areas for the following values of a and b.

(a) $a = 0, b = 1$ (b) $a = -1, b = 1$

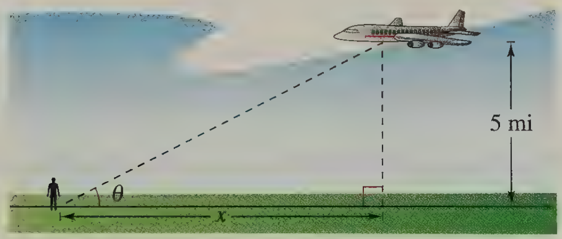
(c) $a = 0, b = 3$ (d) $a = -1, b = 3$



76. ***Angle of Elevation*** An airplane flies at an altitude of 5 miles toward a point directly over an observer. Consider θ and x as shown in the figure.

(a) Write θ as a function of x.

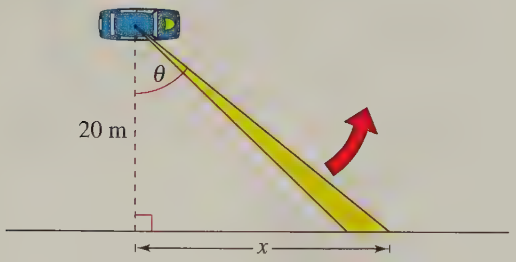
(b) Find θ when $x = 10$ miles and $x = 3$ miles.



77. ***Security Patrol*** A security car with its spotlight on is parked 20 meters from a long warehouse. Consider θ and x as shown in the figure.

(a) Write θ as a function of x.

(b) Find θ when $x = 5$ meters and when $x = 12$ meters.



Synthesis

True or False? **In Exercises 78 and 79, determine whether the statement is true or false. Justify your answer.**

78. $\sin \frac{5\pi}{6} = \frac{1}{2} \Longrightarrow \arcsin \frac{1}{2} = \frac{5\pi}{6}$

79. $\arctan x = \frac{\arcsin x}{\arccos x}$

80. Define the inverse cotangent function by restricting the domain of the cotangent function to the interval $(0, \pi)$, and sketch the inverse function's graph.

81. Define the inverse secant function by restricting the domain of the secant function to the intervals $[0, \pi/2)$ and $(\pi/2, \pi]$, and sketch the inverse function's graph.

82. Define the inverse cosecant function by restricting the domain of the cosecant function to the intervals $[-\pi/2, 0)$ and $(0, \pi/2]$, and sketch the inverse function's graph.

83. Use the results of Exercises 80–82 to evaluate the following without using a calculator.

(a) $\operatorname{arcsec} \sqrt{2}$ (b) $\operatorname{arcsec} 1$

(c) $\operatorname{arccot}\left(-\sqrt{3}\right)$ (d) $\operatorname{arccsc} 2$

In Exercises 84–89, prove the identity.

84. $\arcsin(-x) = -\arcsin x$

85. $\arctan(-x) = -\arctan x$

86. $\arccos(-x) = \pi - \arccos x$

87. $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}, \quad x > 0$

88. $\arcsin x + \arccos x = \frac{\pi}{2}$

89. $\arcsin x = \arctan \frac{x}{\sqrt{1 - x^2}}$

Review

In Exercises 90–93, evaluate the sine, cosine, and tangent of the angle without using a calculator.

90. $840°$ **91.** $-585°$

92. $\frac{17\pi}{3}$ **93.** $-\frac{19\pi}{4}$

In Exercises 94–97, sketch the graph of the function. (Include two full periods.) Use a graphing utility to verify your result.

94. $y = \frac{1}{2}\sin(x + \pi)$ **95.** $y = 2 - \cos(x + \pi)$

96. $y = \frac{1}{2}\tan\left(x + \frac{\pi}{2}\right)$ **97.** $y = 4 \cot \frac{1}{2}x$

5.7 Applications and Models

Applications Involving Right Triangles

In this section, the three angles of a right triangle are denoted by the letters A, B, and C (where C is the right angle), and the lengths of the sides opposite these angles by the letters a, b, and c (where c is the hypotenuse).

What You Should Learn:

- How to solve real-life problems involving right triangles
- How to solve real-life problems involving directional bearings
- How to solve real-life problems involving harmonic motion

Why You Should Learn It:

You can use trigonometric functions to model and solve real-life problems. For instance, Exercise 60 on page 443 shows you how a trigonometric function can be used to model the harmonic motion of a buoy.

Mary Kate Denny/PhotoEdit

EXAMPLE 1 Solving a Right Triangle

Solve the right triangle shown in Figure 5.69.

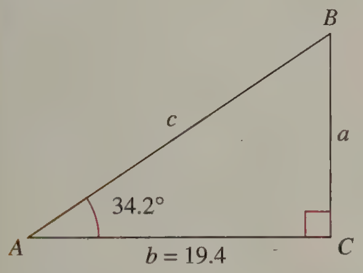


Figure 5.69

Solution

Because $C = 90°$, it follows that $A + B = 90°$ and $B = 90° - 34.2° = 55.8°$. To solve for a, use the fact that

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{a}{b} \quad \Rightarrow \quad a = b \tan A.$$

So, $a = 19.4 \tan 34.2° \approx 13.18$. Similarly, to solve for c, use the fact that

$$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{b}{c} \quad \Rightarrow \quad c = \frac{b}{\cos A}.$$

So, $c = \dfrac{19.4}{\cos 34.2°} \approx 23.46$.

Recall from Section 5.2 that the term *angle of elevation* denotes the angle from the horizontal upward to an object and that the term *angle of depression* denotes the angle from the horizontal downward to an object. An angle of elevation and an angle of depression are shown in Figure 5.70.

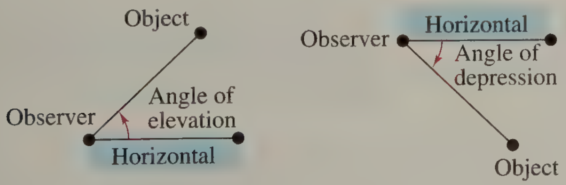


Figure 5.70

EXAMPLE 2 Finding a Side of a Right Triangle

A safety regulation states that the maximum angle of elevation for a rescue ladder is 72°. If a fire department's longest ladder is 110 feet, what is the maximum safe rescue height?

Solution

A sketch is shown in Figure 5.71. From the equation $\sin A = a/c$, it follows that

$$a = c \sin A = 110 \sin 72^\circ \approx 104.6.$$

So, the maximum safe rescue height is about 104.6 feet above the height of the fire truck.

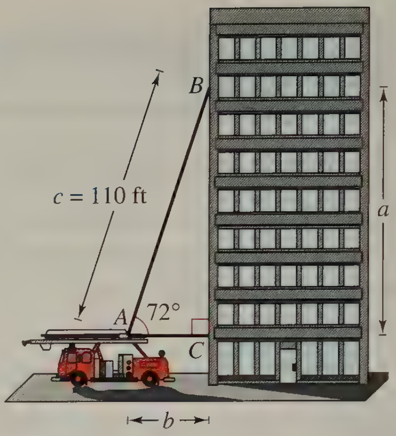


Figure 5.71

EXAMPLE 3 Finding a Side of a Right Triangle

At a point 200 feet from the base of a building, the angle of elevation to the *bottom* of a smokestack is 35°, and the angle of elevation to the *top* is 53°, as shown in Figure 5.72. Find the height s of the smokestack alone.

Solution

Note from Figure 5.72 that this problem involves two right triangles. In the smaller right triangle, use the fact that $\tan 35^\circ = a/200$ to conclude that the height of the building is

$$a = 200 \tan 35^\circ.$$

Now, in the larger right triangle, use the equation

$$\tan 53^\circ = \frac{a + s}{200}$$

to conclude that $s = 200 \tan 53^\circ - a$. So, the height of the smokestack is

$$s = 200 \tan 53^\circ - a = 200 \tan 53^\circ - 200 \tan 35^\circ \approx 125.4 \text{ feet}.$$

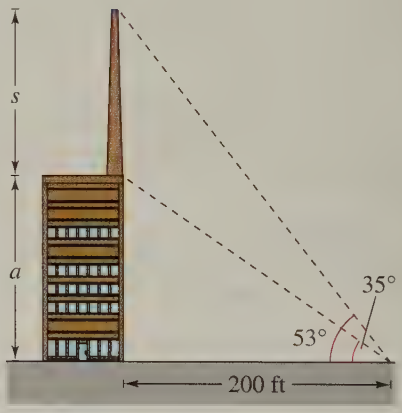


Figure 5.72

EXAMPLE 4 Finding an Angle of Depression

A swimming pool is 20 meters long and 12 meters wide. The bottom of the pool has a constant slant so that the water depth is 1.3 meters at the shallow end and 4 meters at the deep end, as shown in Figure 5.73. Find the angle of depression of the bottom of the pool.

Solution

Using the tangent function, you see that

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{2.7}{20} = 0.135.$$

So, the angle of depression is

$$A = \arctan 0.135 \approx 0.13419 \text{ radian} \approx 7.69^\circ.$$

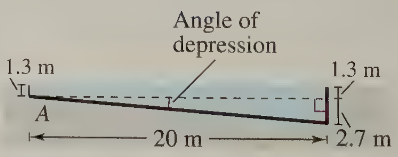
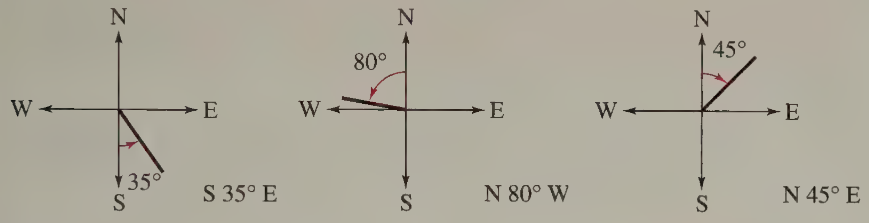


Figure 5.73

Trigonometry and Bearings

In surveying and navigation, directions are generally given in terms of **bearings.** A bearing measures the acute angle a path or line of sight makes with a fixed north–south line, as shown in Figure 5.74. For instance, the bearing of S 35° E in Figure 5.74(a) means 35 degrees east of south.



(a) (b) (c)

Figure 5.74

EXAMPLE 5 Finding Directions in Terms of Bearings

A ship leaves port at noon and heads due west at 20 knots (nautical miles per hour). At 2 P.M. the ship changes course to N 54° W, as shown in Figure 5.75. Find the ship's bearing and distance from the port of departure at 3 P.M.

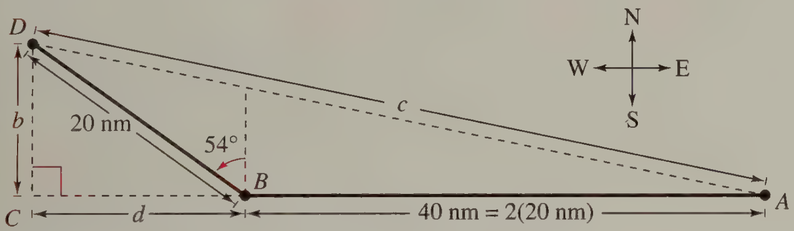


Figure 5.75

Solution

In triangle BCD, you have $B = 90° - 54° = 36°$. The two sides of this triangle can be determined to be

$$b = 20 \sin 36° \qquad \text{and} \qquad d = 20 \cos 36°.$$

In triangle ACD, you can find angle A as follows.

$$\tan A = \frac{b}{d + 40} = \frac{20 \sin 36°}{20 \cos 36° + 40} \approx 0.2092494$$

$$A \approx \arctan 0.2092494 \approx 0.2062732 \text{ radian} \approx 11.82°$$

The angle with the north–south line is $90° - 11.82° = 78.18°$. Therefore, the bearing of the ship is

N 78.18° W. Bearing

Finally, from triangle ACD, you have $\sin A = b/c$, which yields

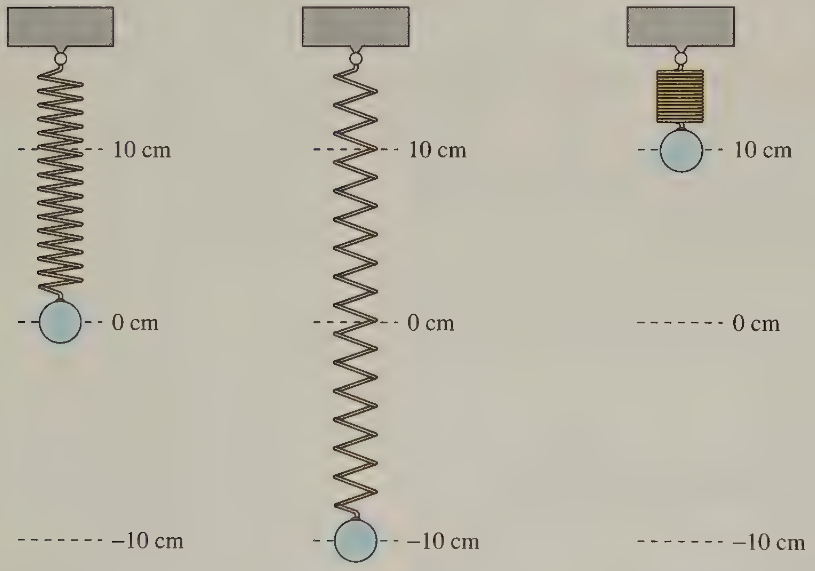
$$c = \frac{b}{\sin A} = \frac{20 \sin 36°}{\sin 11.82°} \approx 57.4 \text{ nautical miles.}$$

Distance from port

Harmonic Motion

The periodic nature of the trigonometric functions is useful for describing the motion of a point on an object that vibrates, oscillates, rotates, or is moved by wave motion.

For example, consider a ball that is bobbing up and down on the end of a spring,* as shown in Figure 5.76. Suppose that 10 centimeters is the maximum distance the ball moves vertically upward or downward from its equilibrium (at-rest) position. Suppose further that the time it takes for the ball to move from its maximum displacement above zero to its maximum displacement below zero and back again is $t = 4$ seconds. Assuming the ideal conditions of perfect elasticity and no friction or air resistance, the ball would continue to move up and down in a uniform and regular manner.



Equilibrium

Maximum negative displacement

Maximum positive displacement

Figure 5.76

From this spring you can conclude that the period (time for one complete cycle) of the motion is

Period = 4 seconds

and that its amplitude (maximum displacement from equilibrium) is

Amplitude = 10 centimeters.

Motion of this nature can be described by a sine or cosine function, and is called **simple harmonic motion.**

Definition of Simple Harmonic Motion

A point that moves on a coordinate line is said to be in **simple harmonic motion** if its distance d from the origin at time t is given by either

$$d = a \sin \omega t \qquad \text{or} \qquad d = a \cos \omega t$$

where a and ω are real numbers such that $\omega > 0$. The motion has amplitude $|a|$, period $2\pi/\omega$, and frequency $\omega/2\pi$.

EXAMPLE 6 Simple Harmonic Motion

Write the equation for the simple harmonic motion of the ball illustrated in Figure 5.76, where the period is four seconds. What is the frequency of this motion?

Solution

Because the spring is at equilibrium ($d = 0$) when $t = 0$, you use the equation

$$d = a \sin \omega t.$$

Moreover, because the maximum displacement from zero is 10 and the period is 4, you have

$$\text{Amplitude} = |a| = 10$$

$$\text{Period} = \frac{2\pi}{\omega} = 4 \quad \Longrightarrow \quad \omega = \frac{\pi}{2}.$$

Consequently, the equation of motion is

$$d = 10 \sin \frac{\pi}{2} t.$$

Note that the choice of $a = 10$ or $a = -10$ depends on whether the ball initially moves up or down. The frequency is

$$\begin{aligned} \text{Frequency} &= \frac{\omega}{2\pi} \\ &= \frac{\pi/2}{2\pi} \\ &= \frac{1}{4} \quad \text{cycle per second.} \end{aligned}$$

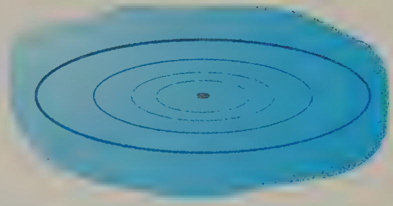


Figure 5.77

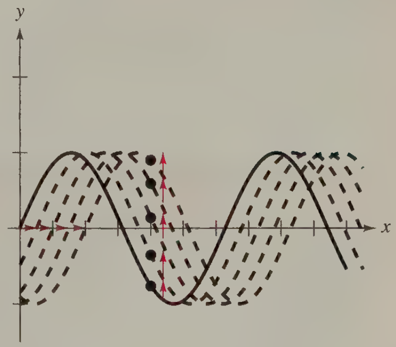


Figure 5.78

One illustration of the relationship between sine waves and harmonic motion is the wave motion that results when a stone is dropped into a calm pool of water. The waves move outward in roughly the shape of sine (or cosine) waves, as shown in Figure 5.77. As an example, suppose you are fishing and your fishing bob is attached so that it does not move horizontally. As the waves move outward from the dropped stone, your fishing bob will move up and down in simple harmonic motion, as shown in Figure 5.78.

EXAMPLE 7 Simple Harmonic Motion

Given the equation for simple harmonic motion

$$d = 6\cos\frac{3\pi}{4}t$$

find (a) the maximum displacement, (b) the frequency, (c) the value of d when $t = 4$, and (d) the least positive value of t for which $d = 0$.

Algebraic Solution

The given equation has the form $d = a\cos\omega t$, with $a = 6$ and $\omega = 3\pi/4$.

a. The maximum displacement (from the point of equilibrium) is given by the amplitude. So, the maximum displacement is 6.

b. Frequency $= \dfrac{\omega}{2\pi} = \dfrac{3\pi/4}{2\pi} = \dfrac{3}{8}$ cycle per unit of time

c. $d = 6\cos\left[\dfrac{3\pi}{4}(4)\right] = 6\cos 3\pi = 6(-1) = -6$

d. To find the least positive value of t for which $d = 0$, solve the equation

$$d = 6\cos\frac{3\pi}{4}t = 0$$

to obtain

$$\frac{3\pi}{4}t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \quad \Longrightarrow \quad t = \frac{2}{3}, 2, \frac{10}{3}, \ldots.$$

So, the least positive value of t is $t = \frac{2}{3}$.

Graphical Solution

Use a graphing utility set in *radian* mode to graph

$$y = 6\cos\frac{3\pi}{4}x.$$

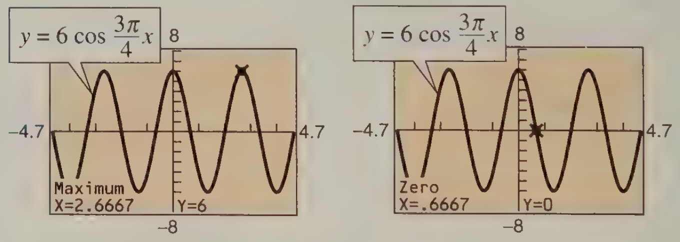
a. Use the *maximum* feature of the graphing utility to estimate that the maximum displacement from the point of equilibrium $y = 0$ is 6, as shown in Figure 5.79(a).

b. The period is the time for the graph to complete one cycle, which is $x \approx 2.667$, as shown in Figure 5.79(a). You can estimate the frequency as follows.

$$\text{Frequency} \approx \frac{1}{2.667} \approx 0.375 \text{ cycles per unit of time}$$

c. Use the *value* or *trace* feature to estimate that the value of y when $x = 4$ is $y = -6$.

d. Use the *zero* or *root* feature to estimate that the least positive value of x for which $y = 0$ is $x \approx 0.6667$, as shown in Figure 5.79(b).



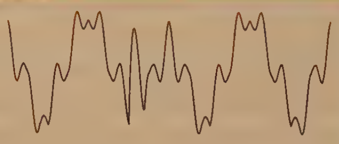
(a) (b)

Figure 5.79

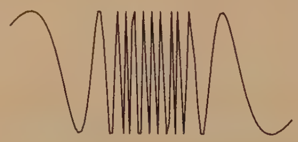
Writing About Math *Radio Waves*

Many different physical phenomena can be characterized by wave motion. These include electromagnetic waves such as radio waves, television waves, and microwaves. Radio waves transmit sound in two different ways. For an AM station, the *amplitude* of the wave is modified to carry sound. The letters AM stand for *amplitude modulation*. An FM radio signal has its *frequency* modified in order to carry sound, hence the term *frequency modulation*. Of the two graphs at the left, one shows an AM wave and the other shows an FM wave. Which is which? Write a short paragraph explaining your reasoning.

(a)



(b)



5.7 Exercises

In Exercises 1–10, solve the right triangle shown in the figure below. (Round your answers to two decimal places.)

1. $A = 25^\circ,\ b = 10$

2. $B = 56^\circ,\ c = 15$

3. $B = 71^\circ,\ b = 24$

4. $A = 7.4^\circ,\ a = 40.5$

5. $a = 6,\ b = 16$

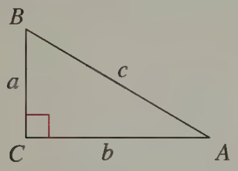
6. $a = 25,\ c = 35$

7. $b = 18,\ c = 42$

8. $b = 1.72,\ c = 8.35$

9. $A = 12^\circ\, 15',\ c = 430.5$

10. $B = 65^\circ\, 12',\ a = 14.2$



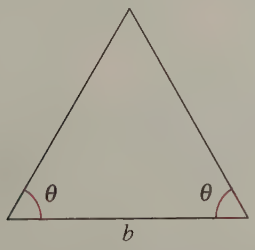
In Exercises 11–14, find the altitude of the isosceles triangle shown in the figure below. (Round your answers to two decimal places.)

11. $\theta = 52^\circ,\ b = 4$ inches

12. $\theta = 18^\circ,\ b = 10$ meters

13. $\theta = 41.6^\circ,\ b = 14.2$ feet

14. $\theta = 72.94^\circ,\ b = 3.36$ centimeters



15. ***Length of a Shadow*** A shadow of length L is created by a 60-foot silo when the sun is θ° above the horizon.

(a) Write L as a function of θ.

(b) Use a graphing utility to complete the table.

θ	10°	20°	30°	40°	50°
L					

(c) The angle measure increases in equal increments in the table. Does the length of the shadow change in equal increments? Explain.

16. ***Length of a Shadow*** A shadow of length L is created by an 850-foot building when the sun is θ° above the horizon.

(a) Write L as a function of θ.

(b) Use a graphing utility to complete the table.

θ	10°	20°	30°	40°	50°
L					

(c) The angle measure increases in equal increments in the table. Does the length of the shadow change in equal increments? Explain.

17. ***Height*** A ladder of length 20 feet leans against the side of a house. The angle of elevation of the ladder is θ°.

(a) Write the height h of the ladder as a function of θ.

(b) Use a graphing utility to complete the table.

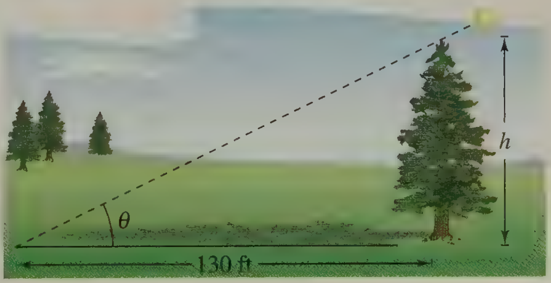
θ	60°	65°	70°	75°	80°
h					

18. ***Height*** The length of a shadow of a tree is 130 feet when the angle of elevation of the sun is θ°.

(a) Write the height h of the tree as a function of θ.

(b) Use a graphing utility to complete the table.

θ	10°	15°	20°	25°	30°
h					



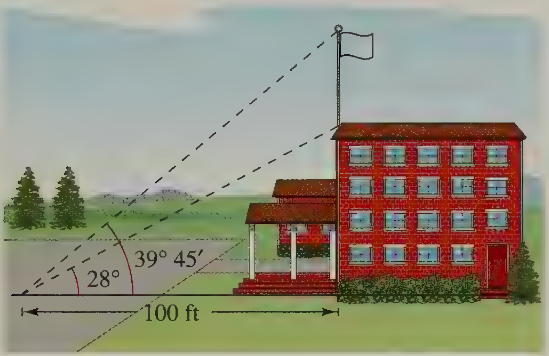
19. ***Height*** From a point 50 feet in front of a church, the angles of elevation to the base of the steeple and the top of the steeple are 35° and 47° 40′, respectively.

(a) Draw right triangles that give a visual representation of the problem. Label the known quantities and the unknown height of the steeple.

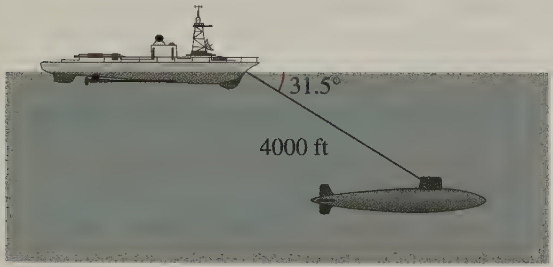
(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) Find the height of the steeple.

20. ***Height*** From a point 100 feet in front of a public library, the angles of elevation to the base of the flagpole and the top of the flagpole are 28° and 39° 45′, respectively. The flagpole is mounted on the front of the library's roof. Find the height of the pole.



21. ***Depth of a Submarine*** The sonar of a navy cruiser detects a nuclear submarine that is 4000 feet from the cruiser. The angle between the water level and the submarine is 31.5°. How deep is the submarine?



22. ***Height of a Kite*** A 100-foot line is attached to a kite. When the kite has pulled the line taut, the angle of elevation to the kite is approximately 50°. Approximate the height of the kite.

23. ***Angle of Elevation*** An engineer erects a 75-foot vertical cellular-phone tower. Find the angle of elevation to the top of the tower from a point on level ground 95 feet from its base.

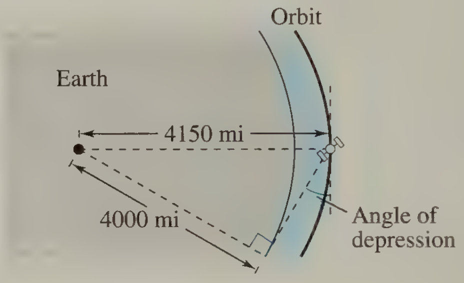
24. ***Angle of Elevation*** The height of an outdoor basketball backboard is $12\frac{1}{2}$ feet, and the backboard casts a shadow $17\frac{1}{3}$ feet long.

(a) Draw a right triangle that represents the problem. Label the known and unknown quantities.

(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) Find the angle of elevation of the sun.

25. ***Angle of Depression*** A television satellite is traveling in a circular orbit 150 miles above the surface of the earth. Find the angle of depression from the television satellite to the horizon. Assume that the radius of the earth is 4000 miles.

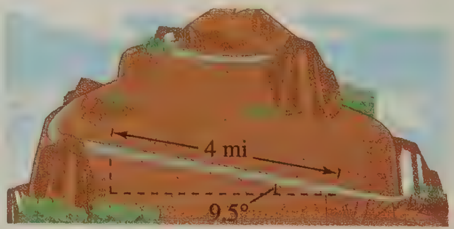


26. ***Angle of Depression*** Find the angle of depression from the top of a lighthouse 250 feet above water level to the water line of a ship $2\frac{1}{2}$ miles offshore.

27. ***Airplane Ascent*** When an airplane leaves the runway, its angle of climb is 18° and its speed is 275 feet per second. Find the plane's altitude after 1 minute.

28. ***Airplane Ascent*** How long will it take the plane in Exercise 27 to climb to an altitude of 10,000 feet? 16,000 feet?

29. ***Mountain Descent*** A sign on the roadway at the top of a mountain indicates that for the next 4 miles the grade is 9.5°. Find the change in elevation for a car descending the mountain.

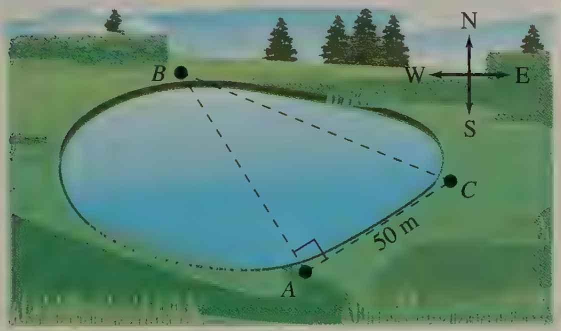


30. ***Mountain Descent*** A road sign at the top of a mountain indicates that for the next 4 miles the grade is 12%. Find the angle of the grade and the change in elevation for a car descending the mountain.

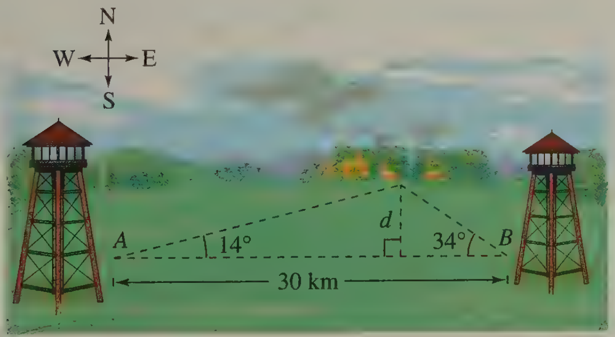
31. ***Navigation*** An airplane flying at 550 miles per hour has a bearing of N 58° E. After flying 1.5 hours, how far north and how far east has the plane traveled from its point of departure?

32. ***Navigation*** A ship leaves port at noon and has a bearing of S 29° W. If the ship sails at 20 knots, how many nautical miles south and how many nautical miles west will the ship have traveled by 6:00 P.M.?

33. ***Surveying*** A surveyor wishes to find the distance across a swamp. The bearing from A to B is N 32° W. The surveyor walks 50 meters from A, and at the point C the bearing to B is N 68° W. Find (a) the bearing from A to C and (b) the distance from A to B.



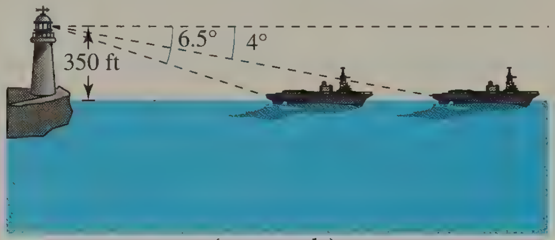
34. ***Location of a Fire*** Two fire towers are 30 kilometers apart, tower A being due west of tower B. A fire is spotted from the towers, and the bearings from A and B are E 14° N and W 34° N, respectively. Find the distance d of the fire from the line segment AB.



35. ***Navigation*** A ship is 50 miles east and 35 miles south of port. If the captain wants to sail directly to port, what bearing should be taken?

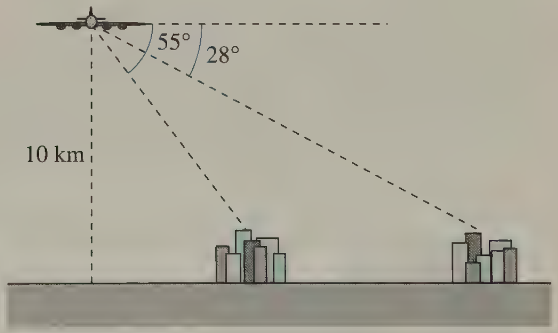
36. ***Navigation*** A plane is 160 miles north and 85 miles east of an airport. If the pilot wants to fly directly to the airport, what bearing should be taken?

37. ***Distance Between Ships*** An observer in a lighthouse 350 feet above sea level observes two ships directly offshore. The angles of depression to the ships are 4° and 6.5°. How far apart are the ships?



(not to scale)

38. ***Distance Between Towns*** A passenger in an airplane flying at an altitude of 10 kilometers sees two towns directly to the left of the plane. The angles of depression to the towns are 28° and 55°. How far apart are the towns?



39. ***Altitude of a Plane*** A plane is observed approaching your home and you assume its speed is 550 miles per hour. If the angle of elevation of the plane is 16° at one time and 57° 1 minute later, approximate the altitude of the plane.

40. ***Height of a Mountain*** While traveling across flat land, you notice a mountain directly in front of you. The angle of elevation to the peak is 2.5°. After you drive 18 miles closer to the mountain, the angle of elevation is 10°. Approximate the height of the mountain.

Geometry **In Exercises 41 and 42, find the angle α between two nonvertical lines L_1 and L_2. The angle α satisfies the equation**

$$\tan \alpha = \left| \frac{m_2 - m_1}{1 + m_2 m_1} \right|$$

where m_1 and m_2 are the slopes of L_1 and L_2, respectively. (Assume $m_1 m_2 \neq -1$.)

41. $3x - 2y = 5$
$x + y = 1$

42. $2x + y = 8$
$x - 5y = -4$

43. ***Geometry*** Determine the angle between the diagonal of a cube and the diagonal of its base, as shown in the figure.

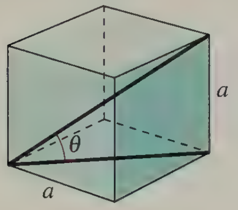


FIGURE FOR 43

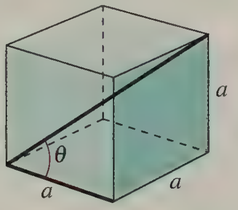
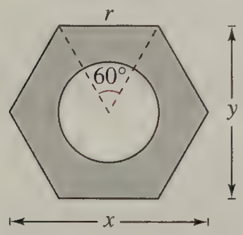


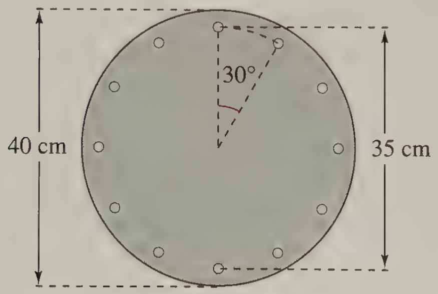
FIGURE FOR 44

44. ***Geometry*** Determine the angle between the diagonal of a cube and its edge, as shown in the figure.

45. ***Wrench Size*** Express the distance y across the flat sides of a hexagonal nut as a function of r, as shown in the figure.



46. ***Bolt Circle*** The figure shows a circular sheet of diameter 40 centimeters. The sheet contains 12 equally spaced bolt holes. Determine the straight-line distance between the centers of the bolt holes.

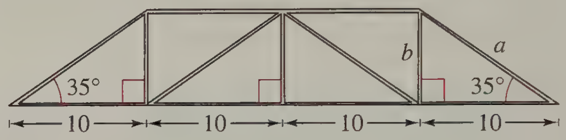


47. ***Geometry*** A regular pentagon (a pentagon with congruent sides and angles) is inscribed in a circle of radius 24 inches. Find the length of the sides of the pentagon.

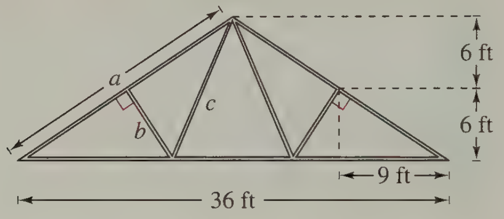
48. ***Geometry*** A regular hexagon (a hexagon with congruent sides and angles) is inscribed in a circle of radius 24 inches. Find the length of the sides of the hexagon.

Trusses **In Exercises 49 and 50, find the lengths of all the unknown members of the truss.**

49.



50.



Harmonic Motion **In Exercises 51–54, for the simple harmonic motion described by the trigonometric function, find (a) the maximum displacement, (b) the frequency, and (c) the least positive value of t for which $d = 0$. Use a graphing utility to verify your results.**

51. $d = 4 \cos 8\pi t$

52. $d = \frac{1}{2} \cos 20\pi t$

53. $d = \frac{1}{16} \sin 140\pi t$

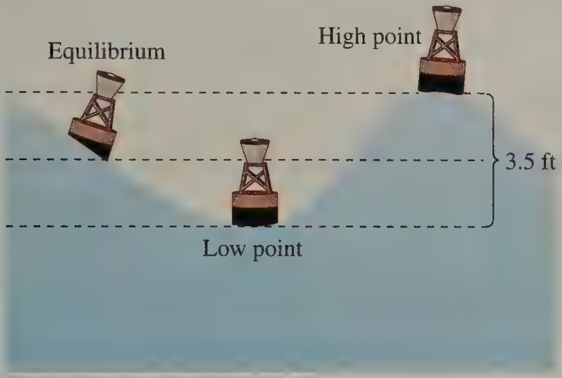
54. $d = \frac{1}{64} \sin 792\pi t$

Harmonic Motion **In Exercises 55–58, find a model for simple harmonic motion satisfying the specified conditions.**

	Displacement ($t = 0$)	*Amplitude*	*Period*
55.	0	8 cm	2 sec
56.	0	3 m	6 sec
57.	3 in.	3 in.	1.8 sec
58.	2 ft	2 ft	10 sec

59. ***Tuning Fork*** A point on the end of a tuning fork moves in the simple harmonic motion described by $d = a \sin \omega t$. Find ω given that the tuning fork for middle C has a frequency of 264 vibrations per second.

60. ***Wave Motion*** A buoy oscillates in simple harmonic motion as waves go past. At a given time it is noted that the buoy moves a total of 3.5 feet from its low point to its high point, and that it returns to its high point every 10 seconds. Write an equation that describes the motion of the buoy if it is at its high point at $t = 0$.



61. ***Springs*** A weight stretches a spring 1.5 inches. The weight is pushed 3 inches above the point of equilibrium and released. Its motion is modeled by

$$y = \frac{1}{4}\cos 16t, \quad t > 0$$

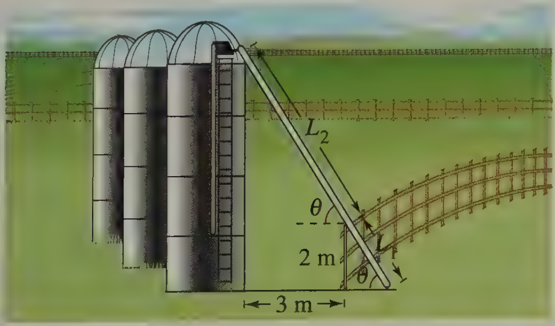
where y is in feet and t is in seconds.

(a) Use a graphing utility to graph the function.

(b) What is the period of the oscillations?

(c) Determine the first time the weight passes the point of equilibrium ($y = 0$).

62. ***Numerical and Graphical Analysis*** A 2-meter-high fence is 3 meters from the side of a grain storage bin. A grain elevator must reach from ground level outside the fence to the storage bin. The objective is to determine the shortest elevator meeting the constraints.



(a) Complete four rows of the table.

θ	L_1	L_2	$L_1 + L_2$
0.1	$\frac{2}{\sin 0.1}$	$\frac{3}{\cos 0.1}$	23.0
0.2	$\frac{2}{\sin 0.2}$	$\frac{3}{\cos 0.2}$	13.1

(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the minimum length.

(c) Write the length L as a function of θ.

(d) Use a graphing utility to graph the function. Use the graph to estimate the minimum length. How does your estimate compare with that of part (b)?

63. ***Data Analysis*** The times S of sunset (Greenwich Mean Time) at 40° north latitude on the 15th of each month are: 1(16:59), 2(17:35), 3(18:06), 4(18:38), 5(19:08), 6(19:30), 7(19:28), 8(18:57), 9(18:09), 10(17:21), 11(16:44), 12(16:36). The month is represented by t, with $t = 1$ corresponding to January. A model (where minutes have been converted to the decimal part of an hour) for this data is

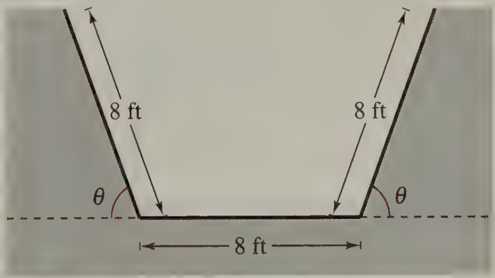
$$S(t) = 18.09 + 1.41\sin\left(\frac{\pi t}{6} + 4.60\right).$$

(a) Use a graphing utility to graph the data points and the model in the same viewing window.

(b) What is the period of the model? Is it what you expected? Explain.

(c) What is the amplitude of the function? What does it represent in the model? Explain.

64. ***Numerical and Graphical Analysis*** The cross sections of an irrigation canal are isosceles trapezoids where the length of three of the sides is 8 feet. The objective is to find the angle θ that maximizes the area of the cross sections.



Base 1	*Base 2*	*Altitude*	*Area*
8	$8 + 16 \cos 10°$	$8 \sin 10°$	22.1
8	$8 + 16 \cos 20°$	$8 \sin 20°$	42.5

(a) Complete six rows of the table.

(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the maximum cross-sectional area.

(c) Write the area A as a function of θ.

(d) Use a graphing utility to graph the function. Use the graph to estimate the maximum cross-sectional area. How does your estimate compare with that of part (b)?

65. ***Data Analysis*** The table shows the average sales S (in millions of dollars) of an outerwear manufacturer for each month t, where $t = 1$ represents January.

t	1	2	3	4	5	6
S	13.46	11.15	8.00	4.85	2.54	1.70
t	7	8	9	10	11	12
S	2.54	4.85	8.00	11.15	13.46	14.30

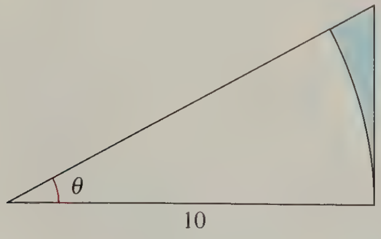
(a) Create a scatter plot of the data.

(b) Find a trigonometric model that fits the data. Graph the model on your scatter plot. How well does the model fit?

(c) What is the period of the model? Do you think it is reasonable given the context? Explain your reasoning.

(d) Interpret the meaning of the model's amplitude in the context of the problem.

66. ***Numerical and Graphical Analysis*** Consider the shaded region outside the sector of the circle of radius 10 meters and inside the right triangle.



(a) Write the area A of the region as a function of θ. Determine the domain of the function.

(b) Use a graphing utility to complete the table.

θ	0	0.3	0.6	0.9	1.2	1.5
A						

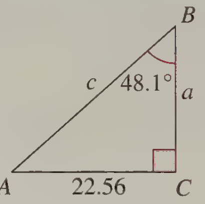
(c) Use a graphing utility to graph the function over the appropriate domain.

(d) What does the area approach as θ approaches $\pi/2$?

Synthesis

True or False? **In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.**

67. In the right triangle shown below, $a = \dfrac{22.56}{\tan 41.9°}$.



68. N 24° E means 24 degrees north of east.

Review

In Exercises 69–72, perform the operation and simplify.

69. $\dfrac{x^2}{x+6} - \dfrac{1}{2x+1}$

70. $\dfrac{x-10}{2(x-5)} \div \dfrac{x+10}{2(x-5)}$

71. $\dfrac{3x^2 - 13x + 4}{x^2 + 8x + 12} \cdot \dfrac{x^3 + 3x^2 - 18x}{12x^2 - 4x}$

72. $\dfrac{4}{x-3} + \dfrac{8}{x} - \dfrac{x+1}{x^2 - 2x - 3}$

In Exercises 73–76, solve the equation. (Round your answer to three decimal places.)

73. $e^{2x} = 54$

74. $\dfrac{300}{1 + e^{-x}} = 100$

75. $\ln(x^2 + 1) = 3.2$

76. $\log_8 x + \log_8(x - 1) = \frac{1}{3}$

In Exercises 77–80, use a calculator to approximate the value of the expression. Round your answer to two decimal places.

77. arccos 0.13

78. arcsin 0.96

79. arcsin(−0.11)

80. arctan(−12)

5 Chapter Summary

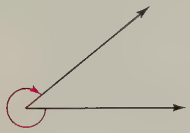
What did you learn?

Section 5.1	**Review Exercises**
☐ How to describe angles	1, 2
☐ How to use degree measure	3–18
☐ How to use radian measure	19–26
☐ How to convert between degree and radian measure	27–34
☐ How to use angles to model and solve real-life problems	35–40
Section 5.2	
☐ How to evaluate trigonometric functions of acute angles	41–44
☐ How to use the fundamental trigonometric identities	45, 46
☐ How to use a calculator to evaluate trigonometric functions	47–50
☐ How to use trigonometric functions to model and solve real-life problems	51, 52
Section 5.3	
☐ How to evaluate trigonometric functions of any angle	53–72
☐ How to use reference angles to evaluate trigonometric functions	73–84
☐ How to evaluate trigonometric functions of real numbers	85–88
Section 5.4	
☐ How to sketch the graphs of basic sine and cosine functions	89–92
☐ How to use amplitude and period to help sketch the graphs of sine and cosine functions	93–102
☐ How to sketch translations of graphs of sine and cosine functions	103–112
☐ How to use sine and cosine functions to model real-life data	113, 114
Section 5.5	
☐ How to sketch the graphs of tangent functions	115–122
☐ How to sketch the graphs of cotangent functions	123–126
☐ How to sketch the graphs of secant and cosecant functions	127–136
☐ How to sketch the graphs of damped trigonometric functions	137–140
Section 5.6	
☐ How to evaluate inverse sine functions	141, 142
☐ How to evaluate other inverse trigonometric functions	143–150
☐ How to evaluate compositions of trigonometric functions	151–154
Section 5.7	
☐ How to solve real-life problems involving right triangles	155–158
☐ How to solve real-life problems involving directional bearings	159, 160
☐ How to solve real-life problems involving harmonic motion	161

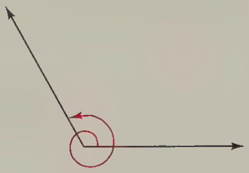
5 Review Exercises

5.1 **In Exercises 1 and 2, determine whether the angle is positive or negative.**

1.



2.



In Exercises 3–6, (a) sketch the angle in standard position, (b) determine the quadrant in which the angle lies, and (c) list one positive and one negative coterminal angle.

3. 40°
4. 190°
5. -110°
6. -405°

In Exercises 7–10, find the complement (if possible) and supplement of each angle.

7. 8°
8. 94°
9. 171°
10. 49°

In Exercises 11–14, convert the measure to decimal degree form. Round your answer to two decimal places.

11. $135^\circ\, 16'\, 65''$
12. $-234^\circ\, 40''$
13. $5^\circ\, 22'\, 53''$
14. $280^\circ\, 8'\, 50''$

In Exercises 15–18, convert the measure to D°M′S″ form.

15. 135.29°
16. 25.8°
17. -85.36°
18. -327.93°

In Exercises 19–22, (a) sketch the angle in standard position, (b) determine the quadrant in which the angle lies, and (c) list one positive and one negative coterminal angle.

19. $\frac{\pi}{16}$
20. $\frac{40\pi}{47}$
21. $-\frac{9\pi}{15}$
22. $-\frac{11\pi}{3}$

In Exercises 23–26, find the complement (if possible) and supplement of each angle.

23. $\frac{\pi}{8}$
24. $\frac{13\pi}{18}$
25. $\frac{3\pi}{10}$
26. $\frac{2\pi}{21}$

In Exercises 27–30, convert the measure from radians to degrees. Round your answer to two decimal places.

27. $\frac{5\pi}{7}$
28. $-\frac{3\pi}{5}$
29. -3.5
30. 1.55

In Exercises 31–34, convert the measure from degrees to radians. Round your answer to four decimal places.

31. 480°
32. -16.5°
33. $-33^\circ\, 45'$
34. $84^\circ\, 15'$

35. Find the radian measure of the central angle of a circle with a radius of 12 feet that intercepts an arc of length 25 feet.

36. Find the radian measure of the central angle of a circle with a radius of 60 inches that intercepts an arc of length 245 inches.

37. Find the length of the arc on a circle with a radius of 20 meters intercepted by a central angle of 138°.

38. Find the radius of a circle with a central angle of $\frac{\pi}{3}$ that intercepts an arc of length 8.5 kilometers.

39. ***Difference in Latitudes*** Assuming that the earth is a sphere of radius 6378 kilometers, what is the difference in latitude of two cities, one of which is 600 kilometers due north of the other?

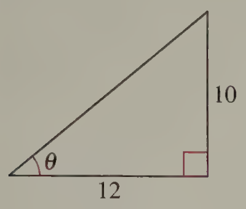
40. ***Angular Speed*** A car is moving at a rate of 28 miles per hour, and the diameter of its wheels is about $2\frac{1}{3}$ feet.

(a) Find the rotational speed of the wheels in revolutions per minute.

(b) Find the angular speed of the wheels in radians per minute.

5.2 **In Exercises 41–44, find the exact values of the six trigonometric functions of the angle θ given in the figure.**

41.



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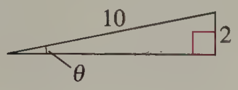
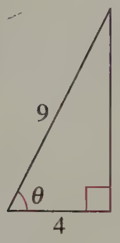
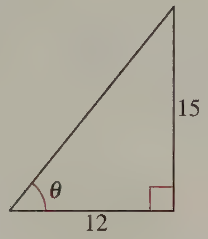


43.



44.



In Exercises 45 and 46, use trigonometric identities to transform one side of the equation into the other.

45. $\csc\theta\tan\theta = \sec\theta$

46. $\dfrac{\cot\theta + \tan\theta}{\cot\theta} = \sec^2\theta$

In Exercises 47–50, use a calculator to evaluate each function. Round your answer to four decimal places.

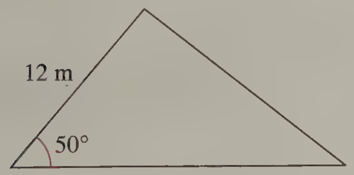
47. (a) $\cos 84°$ (b) $\sin 6°$

48. (a) $\csc 52° 12'$ (b) $\sec 54° 7'$

49. (a) $\cos\dfrac{\pi}{4}$ (b) $\sec\dfrac{\pi}{4}$

50. (a) $\tan\left(\dfrac{3\pi}{20}\right)$ (b) $\cot\left(\dfrac{3\pi}{20}\right)$

51. ***Altitude of Elevation*** Find the altitude of the triangle shown.



52. ***Width of a River*** An engineer is trying to determine the width of a river. From point P, the engineer walks downstream 125 feet and sights to point Q. From this sighting, it is determined that $\theta = 62°$. How wide is the river?

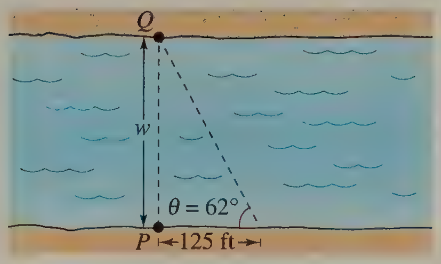


FIGURE FOR 52

5.3 **In Exercises 53–58, find the six trigonometric functions of the angle θ (in standard position) whose terminal side passes through the given point.**

53. $(12, 16)$ **54.** $(-4, -6)$

55. $(-7, 2)$ **56.** $(4, -8)$

57. $\left(\frac{2}{3}, \frac{5}{2}\right)$ **58.** $(x, 4x),\quad x > 0$

In Exercises 59–62, find the remaining five trigonometric functions of θ satisfying the given conditions.

59. $\sec\theta = \frac{6}{5},\quad \tan\theta < 0$

60. $\tan\theta = -\frac{12}{5},\quad \sin\theta > 0$

61. $\sin\theta = \frac{3}{8},\quad \cos\theta < 0$

62. $\cos\theta = -\frac{2}{5},\quad \sin\theta > 0$

In Exercises 63–68, evaluate the trigonometric function without using a calculator.

63. $\tan\dfrac{\pi}{3}$ **64.** $\sec\dfrac{\pi}{4}$

65. $\sin\left(\dfrac{-5\pi}{3}\right)$ **66.** $\cot\left(-\dfrac{5\pi}{6}\right)$

67. $\cos 495°$ **68.** $\csc 270°$

In Exercises 69–72, use a calculator to evaluate the trigonometric function. Round to two decimal places.

69. $\tan 33°$ **70.** $\csc 105°$

71. $\sec\dfrac{12\pi}{5}$ **72.** $\sin\left(-\dfrac{\pi}{9}\right)$

In Exercises 73–76, find the reference angle for the given angle.

73. $264°$ **74.** $635°$

75. $-\dfrac{6\pi}{5}$ **76.** $\dfrac{17\pi}{3}$

In Exercises 77–84, evaluate the sine, cosine, and tangent of the angle without using a calculator.

77. 240°

78. 315°

79. −210°

80. −315°

81. $-\frac{9\pi}{4}$

82. $-\frac{11\pi}{6}$

83. $-\frac{\pi}{2}$

84. $-\frac{\pi}{3}$

In Exercises 85–88, find the point (x, y) on the unit circle that corresponds to the given real number t. Use the result to evaluate $\sin t$, $\cos t$, and $\tan t$.

85. $\frac{2\pi}{3}$

86. $\frac{7\pi}{4}$

87. $\frac{7\pi}{6}$

88. $\frac{3\pi}{4}$

5.4 **In Exercises 89–92, sketch the graph of the function.**

89. $f(x) = 3 \sin x$

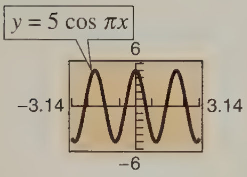
90. $f(x) = 2 \cos x$

91. $f(x) = \frac{7}{2} \sin x$

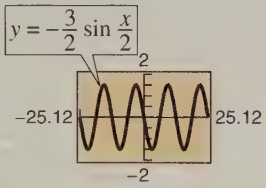
92. $f(x) = \frac{1}{4} \cos x$

In Exercises 93–96, find the period and amplitude.

93.



94.



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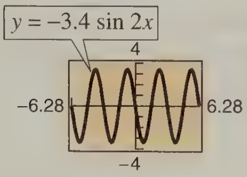

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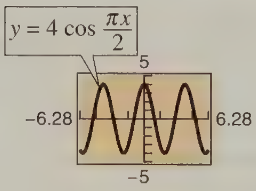


In Exercises 97–108, sketch the graph of the function.

97. $f(x) = 3 \cos 2\pi x$

98. $f(x) = -2 \sin \pi x$

99. $f(x) = 5 \sin \frac{2x}{5}$

100. $f(x) = 8 \cos\left(-\frac{x}{4}\right)$

101. $f(x) = -\frac{5}{2} \cos\left(\frac{x}{4}\right)$

102. $f(x) = -\frac{1}{2} \sin \frac{\pi x}{4}$

103. $f(x) = \frac{5}{2} \sin(x - \pi)$

104. $f(x) = 3 \cos(x + \pi)$

105. $f(x) = 4 \sin\left(x - \frac{\pi}{2}\right)$

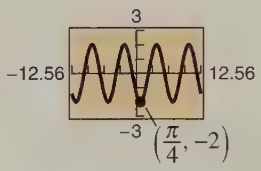
106. $f(x) = \frac{1}{2} \sin(\pi x) - 3$

107. $f(x) = -3 \cos\left(\frac{x}{2} - \frac{\pi}{4}\right)$

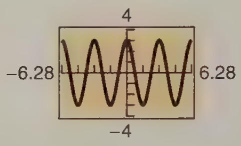
108. $f(x) = 4 - 2 \cos(4x + \pi)$

In Exercises 109–112, find a, b, and c for the function $f(x) = a \cos(bx - c)$ so that the graph of f matches the graph shown.

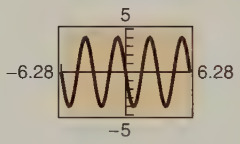
109.



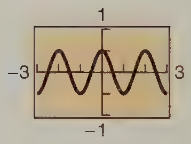
110.



111.



112.



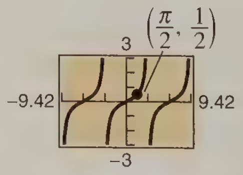
Sales **In Exercises 113 and 114, use a graphing utility to graph the sales function over one year, where S is the sales (in thousands of units) and t is the time (in months), with $t = 1$ corresponding to January. Determine the months of maximum and minimum sales.**

113. $S = 48.4 - 6.1 \cos \frac{\pi t}{6}$

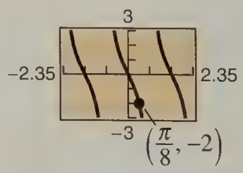
114. $S = 56.25 + 9.50 \sin \frac{\pi t}{6}$

5.5 **In Exercises 115 and 116, find a and b for $f(x) = a \tan bx$ so that the graph of f matches the graph shown.**

115.



116.



In Exercises 117–136, sketch the graph of the function.

117. $f(x) = -\tan \dfrac{\pi x}{4}$ **118.** $f(x) = 4 \tan \pi x$

119. $f(x) = -\dfrac{1}{4} \tan \dfrac{\pi x}{2}$ **120.** $f(x) = \tan\left(x + \dfrac{\pi}{4}\right)$

121. $f(x) = \dfrac{1}{4} \tan\left(x - \dfrac{\pi}{2}\right)$ **122.** $f(x) = 2 + 2 \tan \dfrac{x}{3}$

123. $f(x) = 3 \cot \dfrac{x}{2}$ **124.** $f(x) = \dfrac{1}{2} \cot \dfrac{\pi x}{2}$

125. $f(x) = \frac{1}{2} \cot\left(x - \dfrac{\pi}{2}\right)$

126. $f(x) = 4 \cot\left(x + \dfrac{\pi}{4}\right)$

127. $f(x) = \frac{1}{4} \sec x$ **128.** $f(x) = \frac{1}{2} \csc x$

129. $f(x) = \frac{1}{4} \csc 2x$ **130.** $f(x) = \frac{1}{2} \sec 2\pi x$

131. $f(x) = \sec\left(x - \dfrac{\pi}{4}\right)$

132. $f(x) = \dfrac{1}{2} \csc(2x + \pi)$

133. $f(x) = 2 \sec(x - \pi)$

134. $f(x) = -2 \csc(x - \pi)$

135. $f(x) = \csc\left(3x - \dfrac{\pi}{2}\right)$

136. $f(x) = 3 \csc\left(2x + \dfrac{\pi}{4}\right)$

In Exercises 137–140, sketch the graph of the function and the damping factor of the function by hand. Use a graphing utility to verify your graph.

137. $f(x) = e^x \sin 2x$ **138.** $f(x) = 2x \cos x$

139. $f(x) = x \sin \pi x$

140. $f(x) = 2.5e^{-x/4} \sin 2\pi x$

5.6 **In Exercises 141–144, evaluate the expression without using a calculator.**

141. (a) $\arcsin 1$ (b) $\arcsin 4$

142. (a) $\arcsin\left(-\dfrac{\sqrt{2}}{2}\right)$ (b) $\arcsin \dfrac{\sqrt{3}}{2}$

143. (a) $\arccos \dfrac{\sqrt{2}}{2}$ (b) $\arccos\left(-\dfrac{\sqrt{3}}{2}\right)$

144. (a) $\arctan(-\sqrt{3})$ (b) $\arctan(-1)$

In Exercises 145–148, use a calculator to approximate the value of the expression. (Round your answer to three decimal places.)

145. (a) $\arccos 0.42$ (b) $\arcsin 0.63$

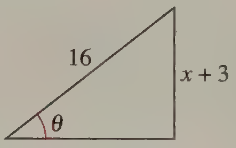
146. (a) $\arcsin(-0.94)$ (b) $\arccos(-0.12)$

147. (a) $\arctan(-12)$ (b) $\arctan 21$

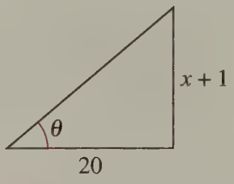
148. (a) $\arctan 0.81$ (b) $\arctan 6.4$

In Exercises 149 and 150, use an inverse trigonometric function to write θ as a function of x.

149.



150.

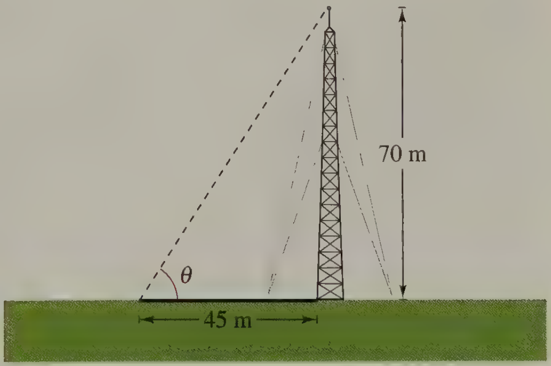


In Exercises 151–154, write an algebraic expression for the given expression.

151. $\sec[\arcsin(x - 1)]$ **152.** $\tan\left(\arccos \dfrac{x}{2}\right)$

153. $\sin\left(\arccos \dfrac{x^2}{4 - x^2}\right)$ **154.** $\csc(\arcsin 10x)$

5.7 **155.** ***Angle of Elevation*** The height of a radio transmission tower is 70 meters, and it casts a shadow of length 45 meters. Find the angle of elevation of the sun.



156. ***Shuttle Height*** An observer 2.5 miles from the launch pad of a space shuttle launch measures the angle of elevation to the base of the shuttle to be 25° soon after lift-off. How high is the shuttle at that instant? (Assume the shuttle is still moving vertically.)

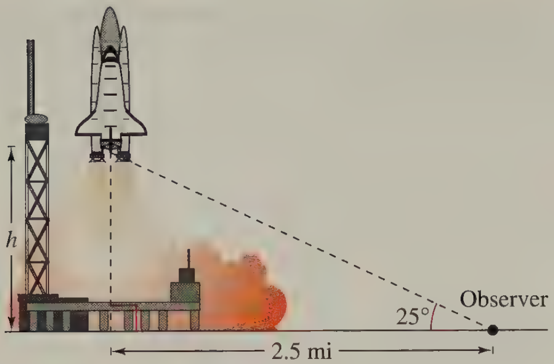
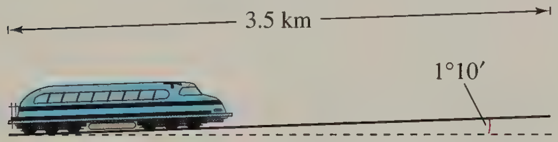
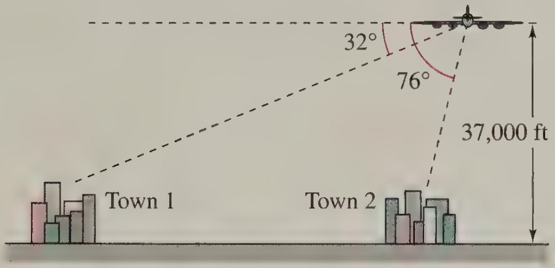


FIGURE FOR 156

157. ***Railroad Grade*** A train travels 3.5 kilometers on a straight track with a grade of 1° 10′. What is the vertical rise of the train in that distance?



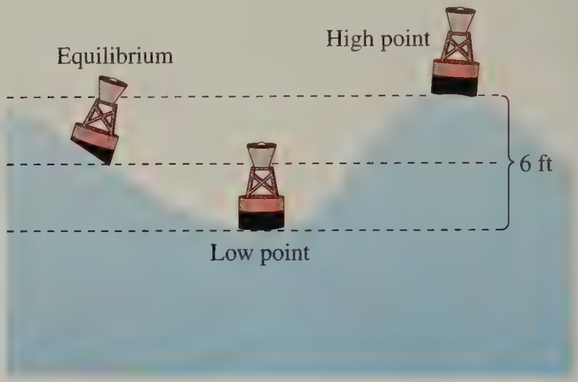
158. ***Distance Between Towns*** A passenger in an airplane flying at an altitude of 37,000 feet sees two towns directly to the left of the airplane. The angles of depression to the towns are 32° and 76°. How far apart are the towns?



159. ***Navigation*** An airplane flying at 490 miles per hour has a bearing of N 46° W. After flying 1.8 hours, how far north and how far west has the plane traveled from its point of departure?

160. ***Distance*** From city A to city B, a plane flies 650 miles at a bearing of N 48° E. From city B to city C, the plane flies 810 miles at a bearing of S 65° E. Find the distance from A to C and the bearing from A to C.

161. ***Wave Motion*** A buoy oscillates in simple harmonic motion as waves go past. At a given time it is noted that the buoy moves a total of 6 feet from its low point to its high point, and that it returns to its high point every 15 seconds. Write an equation that describes the motion of the buoy if it is at its high point at $t = 0$.



Synthesis

True or False? **In Exercises 162–165, determine whether the statement is true or false. Justify your answer.**

162. $\dfrac{\sin 60°}{\sin 30°} = \sin 2°$

163. $\tan[(0.8)^2] = \tan^2(0.8)$

164. $y = \sin \theta$ is not a function because $\sin 30° = \sin 150°$.

165. $\tan \dfrac{3\pi}{4} = -1$ ➡ $\arctan(-1) = \dfrac{3\pi}{4}$

166. In your own words, explain the meaning of (a) an angle in standard position, (b) a negative angle, (c) coterminal angles, and (d) an obtuse angle.

167. A fan motor turns at a given angular speed. How does the speed of the tips of the blades change if a fan of greater diameter is installed on the motor? Explain.

Chapter Project *Fitting a Model to Data*

In this project, you will find and use models relating to the carbon dioxide level of the earth's atmosphere.

Since 1958, the Mauna Loa Climate Observatory in Hawaii has been collecting data on the carbon dioxide level of the earth's atmosphere. The table shows the average monthly readings for January of each year from 1974 through 1999. The readings measure the carbon dioxide concentration in parts per million. (Source: National Oceanic Atmospheric Administration, Climate Monitoring and Diagnostic Laboratory, CarbonCycle-Greenhouse Gases)

1974	1975	1976	1977	1978	1979	1980	1981	1982	1983	1984	1985	1986
329.7	331.1	332.0	333.8	335.4	336.7	338.7	340.1	341.2	342.8	344.5	345.9	347.2

1987	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999
349.0	351.5	352.9	354.2	355.5	356.3	357.0	358.6	360.6	362.4	363.5	366.5	368.6

a. Enter the data in the table into a graphing utility. (Let $t = 0$ represent 1970.) Draw a scatter plot for the data. Does the data appear to be modeled best with a linear, quadratic, or exponential model?

b. Find the linear, quadratic, or exponential model that you think best fits the data.

Questions for Further Exploration

1. The data in the table represents the carbon dioxide levels for January of each year. Throughout each year, the level oscillated as follows.

- In April, the average reading was about 2.5 parts per million higher than the average reading given by the model in part (b) above.
- In July, the average reading was the same as the average reading given by the model in part (b) above.
- In October, the average reading was about 2.5 parts per million lower than the average reading given by the model in part (b) above.

Use a sine function to rewrite the model found in part (b) above so that the model incorporates the described oscillations.

2. Use a graphing utility to graph the revised model.

3. Make a careful sketch of the model for 1 year. What physical factors on the earth would contribute to the oscillation in the carbon dioxide level during the year?

4. Is the model you found periodic? Explain your reasoning.

5. Use the model to estimate the level of carbon dioxide in the earth's atmosphere in the following years.

(a) 2000

(b) 2010

(c) 2020

6. What significance does the trend in the carbon dioxide level in the earth's atmosphere have?

5 Chapter Test

Take this test as you would take a test in class. After you are done, check your work against the answers in the back of the book.

The *Interactive* CD-ROM and *Internet* versions of this text provide answers to the Chapter Tests and Cumulative Tests. They also offer Chapter Pre-Tests (that test key skills and concepts covered in previous chapters) and Chapter Post-Tests, both of which have randomly generated exercises with diagnostic capabilities.

1. Consider the angle of magnitude $5\pi/4$ radians.

(a) Sketch the angle in standard position.

(b) Determine two coterminal angles (one positive and one negative).

(c) Convert the angle to degree measure.

2. A truck is moving at a rate of 90 kilometers per hour, and the diameter of its wheels is 1 meter. Find the angular speed of the wheels in radians per minute.

3. Find the exact values of the six trigonometric functions of the angle θ shown in the figure.

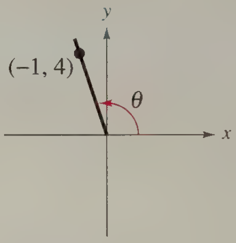


FIGURE FOR 3

4. Given that $\tan\theta = \frac{11}{6}$, find the other five trigonometric functions of θ.

5. Determine the reference angle θ' of the angle $\theta = 290°$ and sketch θ and θ' in standard position.

6. Determine the quadrant in which θ lies if $\sec\theta < 0$ and $\tan\theta > 0$.

7. Find two values of θ in degrees $(0 \le \theta < 360°)$ if $\cos\theta = -\sqrt{3}/2$. (Do not use a calculator.)

8. Use a calculator to approximate two values of θ in radians $(0 \le \theta < 2\pi)$ if $\csc\theta = 1.030$. Round the result to two decimal places.

9. Find the five remaining trigonometric functions of θ, given that $\sec\theta = \frac{12}{10}$ and $\tan\theta < 0$.

In Exercises 10–15, graph the function through two full periods without using a graphing utility.

10. $g(x) = -2\sin\left(x - \frac{\pi}{4}\right)$ **11.** $f(\alpha) = \frac{1}{2}\tan 2\alpha$

12. $f(x) = \frac{1}{2}\sec(x - \pi)$ **13.** $f(x) = 2\cos(\pi - 2x) + 3$

14. $f(x) = 2\csc\left(x + \dfrac{\pi}{2}\right)$ **15.** $f(x) = \dfrac{1}{4}\cot\left(x - \dfrac{\pi}{2}\right)$

In Exercises 16 and 17, use a graphing utility to graph the function. If the function is periodic, find its period.

16. $y = \sin 2\pi x + 2\cos\pi x$ **17.** $y = 6e^{-0.12t}\cos(0.25t),\ \ 0 \le t \le 32$

18. Find a, b, and c for the function $f(x) = a\sin(bx + c)$ so that the graph of f matches the graph at the right.

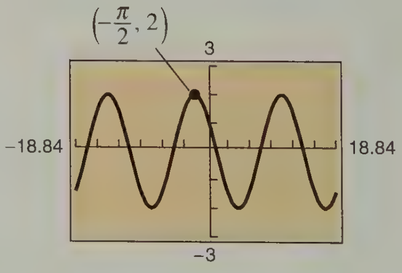


FIGURE FOR 18

19. Find the exact value of $\tan\left(\arccos\frac{2}{3}\right)$ without using a calculator.

In Exercises 20–22, sketch the graph of the function

20. $f(x) = 2\arcsin\left(\frac{1}{2}x\right)$ **21.** $f(x) = 2\arccos x$ **22.** $f(x) = \arctan\dfrac{x}{2}$

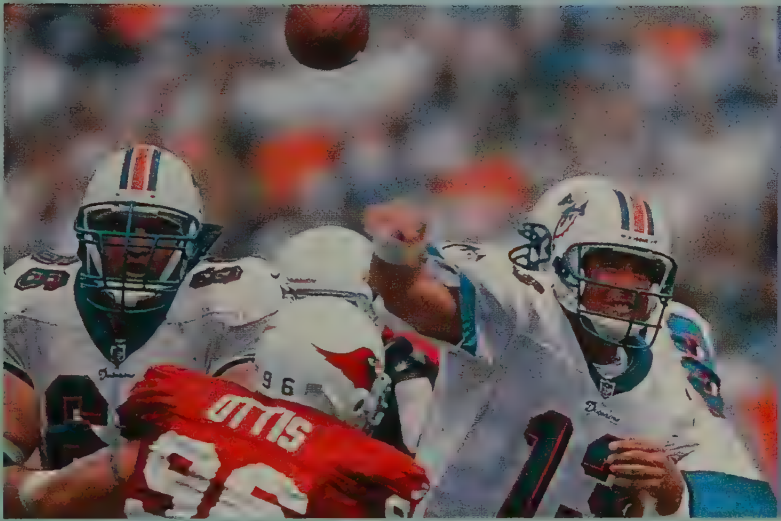
23. A plane is 160 miles north and 110 miles east of an airport. If the pilot wants to fly directly to the airport, what bearing should he take?

24. From a 100-foot roof of a condominium on the coast, a tourist sights a cruise

Analytic Trigonometry

6

6.1 Using Fundamental Identities
6.2 Verifying Trigonometric Identities
6.3 Solving Trigonometric Equations
6.4 Sum and Difference Formulas
6.5 Multiple-Angle and Product-Sum Formulas



Andy Lyons/Allsport

A football thrown by a quarterback follows a parabolic path. The horizontal distance the football travels depends not only on the speed of the throw, but on the angle at which the ball is thrown.

The Big Picture

In this chapter you will learn how to

- ☐ use fundamental trigonometric identities to evaluate trigonometric functions and simplify trigonometric expressions.
- ☐ verify trigonometric identities.
- ☐ use standard algebraic techniques and inverse trigonometric functions to solve trigonometric equations.
- ☐ use sum and difference formulas, multiple-angle formulas, power-reducing formulas, half-angle formulas, and product-sum formulas to rewrite and evaluate trigonometric functions.

Important Vocabulary

As you encounter each new vocabulary term in this chapter, add the term and its definition to your notebook glossary.

- reduction formulas (p. 484)
- double-angle formulas (p. 489)
- power-reducing formulas (p. 491)
- half-angle formulas (p. 492)
- product-to-sum formulas (p. 493)
- sum-to-product formulas (p. 494)

Additional Resources Text-specific additional resources are available to help you do well in this course. See page xvi for details.

6.1 Using Fundamental Identities

Introduction

In Chapter 5, you studied the basic definitions, properties, graphs, and applications of the individual trigonometric functions. In this chapter, you will learn how to use the fundamental identities to

1. evaluate trigonometric functions,
2. simplify trigonometric expressions,
3. develop additional trigonometric identities, and
4. solve trigonometric equations.

What You Should Learn:

- How to recognize and write fundamental trigonometric identities
- How to use fundamental trigonometric identities to evaluate trigonometric functions, simplify trigonometric expressions, and rewrite trigonometric expressions

Why You Should Learn It:

Fundamental trigonometric identities can be used to simplify trigonometric expressions. For instance, Exercise 109 on page 461 shows you how trigonometric identities can be used to simplify an expression for the rate of change of a function, a concept used in calculus.

Fundamental Trigonometric Identities

Reciprocal Identities

$$\sin u = \frac{1}{\csc u} \qquad \cos u = \frac{1}{\sec u} \qquad \tan u = \frac{1}{\cot u}$$

$$\csc u = \frac{1}{\sin u} \qquad \sec u = \frac{1}{\cos u} \qquad \cot u = \frac{1}{\tan u}$$

Quotient Identities

$$\tan u = \frac{\sin u}{\cos u} \qquad \cot u = \frac{\cos u}{\sin u}$$

Pythagorean Identities

$$\sin^2 u + \cos^2 u = 1 \qquad 1 + \tan^2 u = \sec^2 u \qquad 1 + \cot^2 u = \csc^2 u$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u \qquad \cos\left(\frac{\pi}{2} - u\right) = \sin u$$

$$\tan\left(\frac{\pi}{2} - u\right) = \cot u \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u$$

$$\sec\left(\frac{\pi}{2} - u\right) = \csc u \qquad \csc\left(\frac{\pi}{2} - u\right) = \sec u$$

Even/Odd Identities

$$\sin(-u) = -\sin u \qquad \sec(-u) = \sec u \qquad \tan(-u) = -\tan u$$

$$\csc(-u) = -\csc u \qquad \cos(-u) = \cos u \qquad \cot(-u) = -\cot u$$

Pythagorean identities are sometimes used in radical form such as

$$\sin u = \pm\sqrt{1 - \cos^2 u} \quad \text{or} \quad \tan u = \pm\sqrt{\sec^2 u - 1}$$

where the sign depends on the choice of u.

Using the Fundamental Identities

One common use of trigonometric identities is to use given values of trigonometric functions to evaluate other trigonometric functions.

The *Interactive* CD-ROM and *Internet* versions of this text show every example with its solution; clicking on the *Try It!* button brings up similar problems. Guided Examples and Integrated Examples show step-by-step solutions to additional examples. Integrated Examples are related to several concepts in the section.

EXAMPLE 1 Using Identities to Evaluate a Function

Use the values $\sec u = -\frac{3}{2}$ and $\tan u > 0$ to find the values of all six trigonometric functions.

Solution

Using a reciprocal identity, you have

$$\cos u = \frac{1}{\sec u} = \frac{1}{-3/2} = -\frac{2}{3}.$$

Using a Pythagorean identity, you have

$$\begin{aligned} \sin^2 u &= 1 - \cos^2 u && \text{Pythagorean identity} \\ &= 1 - \left(-\frac{2}{3}\right)^2 && \text{Substitute } -\tfrac{2}{3} \text{ for } \cos u. \\ &= 1 - \frac{4}{9} && \text{Simplify.} \\ &= \frac{5}{9}. && \text{Simplify.} \end{aligned}$$

Because $\sec u < 0$ and $\tan u > 0$, it follows that u lies in Quadrant III. Moreover, because $\sin u$ is negative when u is in Quadrant III, you can choose the negative root and obtain $\sin u = -\sqrt{5}/3$. Now, knowing the values of the sine and cosine, you can find the values of all six trigonometric functions.

$$\sin u = -\frac{\sqrt{5}}{3} \qquad \csc u = \frac{1}{\sin u} = -\frac{3}{\sqrt{5}}$$

$$\cos u = -\frac{2}{3} \qquad \sec u = \frac{1}{\cos u} = -\frac{3}{2}$$

$$\tan u = \frac{\sin u}{\cos u} = \frac{-\sqrt{5}/3}{-2/3} = \frac{\sqrt{5}}{2} \qquad \cot u = \frac{1}{\tan u} = \frac{2}{\sqrt{5}}$$

EXAMPLE 2 Simplifying a Trigonometric Expression

Simplify $\sin x \cos^2 x - \sin x$.

Solution

First factor out a common monomial factor and then use a fundamental identity.

$$\begin{aligned} \sin x \cos^2 x - \sin x &= \sin x(\cos^2 x - 1) && \text{Factor out monomial factor.} \\ &= -\sin x(1 - \cos^2 x) && \text{Distributive Property} \\ &= -\sin x(\sin^2 x) && \text{Pythagorean identity} \\ &= -\sin^3 x && \text{Multiply.} \end{aligned}$$

STUDY TIP

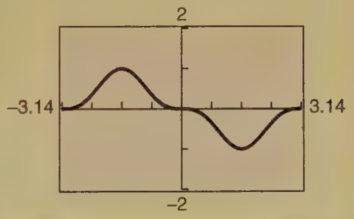
You can use a graphing utility to check the result of Example 2. To do this, graph

$$y_1 = \sin x \cos^2 x - \sin x$$

and

$$y_2 = -\sin^3 x$$

in the same viewing window, as shown below. The two graphs *appear* to coincide, so the expressions *appear* to be equivalent. Remember that in order to be certain that two expressions are equivalent, you need to show their equivalence algebraically, as in Example 2.



EXAMPLE 3 Verifying Trigonometric Identities

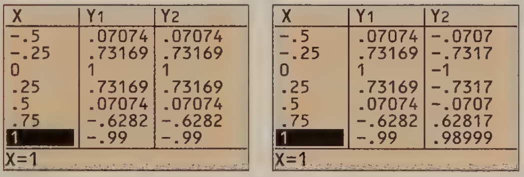
Determine which of the following appears to be an identity.

a. $\cos 3x \overset{?}{=} 4\cos^3 x - 3\cos x$ **b.** $\cos 3x \overset{?}{=} \sin\left(3x - \frac{\pi}{2}\right)$

Numerical Solution

a. Use the *table* feature of a graphing utility set in *radian* mode to create a table that shows the values of $y_1 = \cos 3x$ and $y_2 = 4\cos^3 x - 3\cos x$ for different values of x, as shown in Figure 6.1(a). The values of y_1 and y_2 appear to be identical, so $\cos 3x = 4\cos^3 x - 3\cos x$ appears to be an identity.

b. Create a table that shows the values of $y_1 = \cos 3x$ and $y_2 = \sin(3x - \pi/2)$ for different values of x, as shown in Figure 6.1(b). The values of y_1 and y_2 are not identical, so $\cos 3x = \sin(3x - \pi/2)$ is not an identity.



X	Y1	Y2
-.5	.07074	.07074
-.25	.73169	.73169
0	1	1
.25	.73169	.73169
.5	.07074	.07074
.75	-.6282	-.6282
1	-.99	-.99

X=1

X	Y1	Y2
-.5	.07074	-.0707
-.25	.73169	-.7317
0	1	-1
.25	.73169	-.7317
.5	.07074	-.0707
.75	-.6282	.62817
1	-.99	.98999

X=1

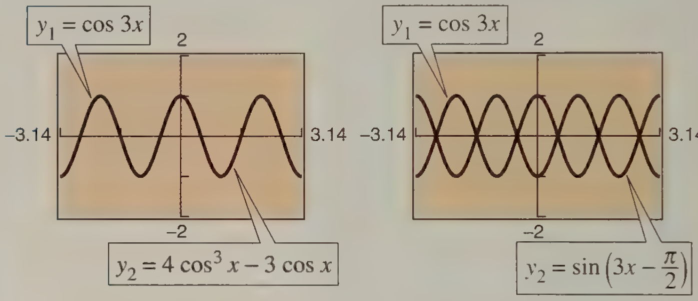
(a) (b)

Figure 6.1

Graphical Solution

a. Use a graphing utility set in *radian* mode to graph $y_1 = \cos 3x$ and $y_2 = 4\cos^3 x - 3\cos x$ in the same viewing window, as shown in Figure 6.2(a). Because the graphs appear to coincide, $\cos 3x = 4\cos^3 x - 3\cos x$ appears to be an identity.

b. Graph $y_1 = \cos 3x$ and $y_2 = \sin(3x - \pi/2)$ in the same viewing window, as shown in Figure 6.2(b). Because the graphs do not coincide, $\cos 3x = \sin(3x - \pi/2)$ is not an identity.



(a) (b)

Figure 6.2

EXAMPLE 4 Verifying a Trigonometric Identity

Verify the identity $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \csc\theta$.

Algebraic Solution

$$
\begin{aligned}
\frac{\sin\theta}{1+\cos\theta} + \frac{\cos\theta}{\sin\theta} &= \frac{(\sin\theta)(\sin\theta) + (\cos\theta)(1+\cos\theta)}{(1+\cos\theta)(\sin\theta)} \\
&= \frac{\sin^2\theta + \cos^2\theta + \cos\theta}{(1+\cos\theta)(\sin\theta)} && \text{Multiply.} \\
&= \frac{\cancel{1+\cos\theta}}{\cancel{(1+\cos\theta)}(\sin\theta)} && \text{Pythagorean identity} \\
&= \csc\theta && \text{Divide out common factor and use reciprocal identity.}
\end{aligned}
$$

Notice how the identity in Example 4 is verified. You start with the left side of the equation (the more complicated side) and use the fundamental trigonometric identities to simplify it until you obtain the right side.

Graphical Solution

Use a graphing utility set in *radian* and *dot* modes to graph y_1 and y_2 in the same viewing window, as shown in Figure 6.3. Because the graphs appear to coincide, this equation appears to be an identity.

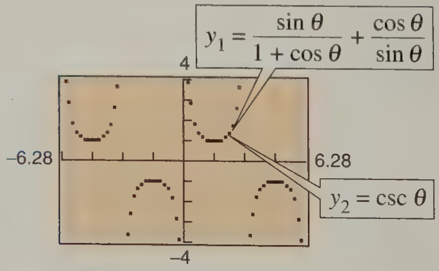


Figure 6.3

When factoring trigonometric expressions, it is helpful to find a polynomial form that fits the expression, as shown in Example 5.

EXAMPLE 5 Factoring Trigonometric Expressions

Factor each expression.

a. $\sec^2 \theta - 1$

b. $4 \tan^2 \theta + \tan \theta - 3$

Solution

a. Here the expression has the polynomial form $u^2 - v^2$ (the difference of two squares), which factors as

$$\sec^2 \theta - 1 = (\sec \theta - 1)(\sec \theta + 1).$$

b. This expression has the polynomial form, $ax^2 + bx + c$, and it factors as

$$4 \tan^2 \theta + \tan \theta - 3 = (4 \tan \theta - 3)(\tan \theta + 1).$$

On occasion, factoring or simplifying can best be done by first rewriting the expression in terms of just *one* trigonometric function or in terms of *sine* or *cosine alone*. These strategies are illustrated in Examples 6 and 7.

EXAMPLE 6 Factoring a Trigonometric Expression

Factor $\csc^2 x - \cot x - 3$.

Solution

You can use the identity $\csc^2 x = 1 + \cot^2 x$ to rewrite the expression in terms of the cotangent alone.

$$\begin{aligned}\csc^2 x - \cot x - 3 &= (1 + \cot^2 x) - \cot x - 3 && \text{Pythagorean identity}\\ &= \cot^2 x - \cot x - 2 && \text{Combine like terms.}\\ &= (\cot x - 2)(\cot x + 1) && \text{Factor.}\end{aligned}$$

EXAMPLE 7 Simplifying a Trigonometric Expression

Simplify $\sin t + \cot t \cos t$.

Solution

Begin by rewriting the expression in terms of sine and cosine.

$$\begin{aligned}\sin t + \cot t \cos t &= \sin t + \left(\frac{\cos t}{\sin t}\right)\cos t && \text{Quotient identity}\\ &= \frac{\sin^2 t + \cos^2 t}{\sin t} && \text{Add fractions.}\\ &= \frac{1}{\sin t} && \text{Pythagorean identity}\\ &= \csc t && \text{Reciprocal identity}\end{aligned}$$

The last two examples in this section involve techniques for rewriting expressions into forms that are used in calculus.

EXAMPLE 8 Rewriting a Trigonometric Expression

Rewrite $\dfrac{1}{1 + \sin x}$ so that it is *not* in fractional form.

Solution

From the Pythagorean identity $\cos^2 x = 1 - \sin^2 x = (1 - \sin x)(1 + \sin x)$, you can see that by multiplying both the numerator and the denominator by $(1 - \sin x)$ you will produce a monomial denominator.

$$\begin{aligned}
\frac{1}{1+\sin x} &= \frac{1}{1+\sin x} \cdot \frac{1-\sin x}{1-\sin x} && \text{Multiply numerator and denominator by } (1-\sin x).\\
&= \frac{1-\sin x}{1-\sin^2 x} && \text{Multiply.}\\
&= \frac{1-\sin x}{\cos^2 x} && \text{Pythagorean identity}\\
&= \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} && \text{Write as separate fractions.}\\
&= \frac{1}{\cos^2 x} - \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} && \text{Write as separate fractions.}\\
&= \sec^2 x - \tan x \sec x && \text{Reciprocal and quotient identities}
\end{aligned}$$

EXAMPLE 9 Trigonometric Substitution

Use the substitution $x = 2 \tan \theta$, $0 < \theta < \pi/2$, to express $\sqrt{4 + x^2}$ as a trigonometric function of θ.

Solution

Begin by letting $x = 2 \tan \theta$. Then you can obtain

$$\begin{aligned}
\sqrt{4+x^2} &= \sqrt{4 + (2\tan\theta)^2} && \text{Substitute } 2\tan\theta \text{ for } x.\\
&= \sqrt{4(1+\tan^2\theta)} && \text{Distributive Property}\\
&= \sqrt{4\sec^2\theta} && \text{Pythagorean identity}\\
&= 2\sec\theta. && \sec\theta > 0 \text{ for } 0 < \theta < \tfrac{\pi}{2}
\end{aligned}$$

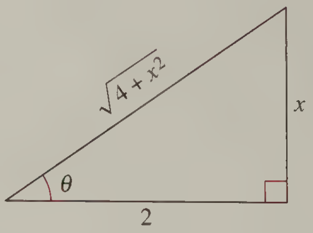


Figure 6.4

Figure 6.4 shows the right triangle illustration of this substitution. For $0 < \theta < \pi/2$, you have

$$\text{opposite} = x,\ \text{adjacent} = 2,\ \text{and hypotenuse} = \sqrt{4 + x^2}.$$

Try using these expressions to obtain the result shown above.

6.1 Exercises

In Exercises 1–14, use the given values to evaluate (if possible) the remaining trigonometric functions.

1. $\sin x = \frac{\sqrt{3}}{2}, \quad \cos x = \frac{1}{2}$

2. $\csc \theta = \frac{5}{3}, \quad \tan \theta = \frac{3}{4}$

3. $\sec \theta = \sqrt{2}, \quad \sin \theta = -\frac{\sqrt{2}}{2}$

4. $\tan x = \frac{\sqrt{3}}{3}, \quad \cos x = -\frac{\sqrt{3}}{2}$

5. $\tan x = \frac{7}{24}, \quad \sec x = -\frac{25}{24}$

6. $\cot \phi = -5, \quad \sin \phi = \frac{\sqrt{26}}{26}$

7. $\sec \phi = -1, \quad \sin \phi = 0$

8. $\cos\left(\frac{\pi}{2} - x\right) = \frac{3}{5}, \quad \cos x = \frac{4}{5}$

9. $\sin(-x) = -\frac{2}{3}, \quad \tan x = -\frac{2\sqrt{5}}{5}$

10. $\csc x = 5, \quad \cos x > 0$

11. $\tan \theta = 4, \quad \sin \theta < 0$

12. $\sec \theta = -3, \quad \tan \theta < 0$

13. $\sin \theta = -1, \quad \cot \theta = 0$

14. $\tan \theta$ is undefined, $\quad \sin \theta > 0$

In Exercises 15–18, fill in the blanks. (*Note:* $x \to c^+$ indicates that x approaches c from the right, and $x \to c^-$ indicates that x approaches c from the left.)

15. As $x \to \frac{\pi}{2}^-$, $\sin x \to$ ______ and $\csc x \to$ ______ .

16. As $x \to 0^+$, $\cos x \to$ ______ and $\sec x \to$ ______ .

17. As $x \to \frac{\pi}{2}^-$, $\tan x \to$ ______ and $\cot x \to$ ______ .

18. As $x \to \pi^+$, $\sin x \to$ ______ and $\csc x \to$ ______ .

In Exercises 19–24, match the trigonometric expression with one of the following.

(a) $\mathbf{-1}$ **(b)** $\mathbf{\cos x}$ **(c)** $\mathbf{\cot x}$

(d) $\mathbf{1}$ **(e)** $\mathbf{-\tan x}$ **(f)** $\mathbf{\sin x}$

19. $\csc x \sin x$

20. $\tan x \cos x$

21. $\tan^2 x - \sec^2 x$

22. $(1 - \sin^2 x)(\sec x)$

23. $\frac{\sin(-x)}{\cos(-x)}$

24. $\frac{\sin[(\pi/2) - x]}{\cos[(\pi/2) - x]}$

In Exercises 25–30, match the trigonometric expression with one of the following.

(a) $\mathbf{\csc x}$ **(b)** $\mathbf{\cot x}$ **(c)** $\mathbf{\cos^2 x}$

(d) $\mathbf{\sin x \tan x}$ **(e)** $\mathbf{\sec^2 x}$ **(f)** $\mathbf{\sec^2 x + \tan^2 x}$

25. $\cos x \csc x$

26. $\sin^2 x(\csc^2 x - 1)$

27. $\sec^4 x - \tan^4 x$

28. $\cot x \sec x$

29. $\frac{\sec^2 x - 1}{\sin^2 x}$

30. $\frac{\cos^2[(\pi/2) - x]}{\cos x}$

In Exercises 31–44, use the fundamental identities to simplify the expression. Use the *table* feature of a graphing utility to check your result numerically.

31. $\cot x \sin x$

32. $\cos \beta \tan \beta$

33. $\sin \phi(\csc \phi - \sin \phi)$

34. $\sec^2 x(1 - \sin^2 x)$

35. $\frac{\cot x}{\csc x}$

36. $\frac{\sec \theta}{\csc \theta}$

37. $\sec \alpha \cdot \frac{\sin \alpha}{\tan \alpha}$

38. $\frac{1}{\tan^2 x + 1}$

39. $\frac{\sin(-x)}{\cos x}$

40. $\frac{\tan^2 \theta}{\sec^2 \theta}$

41. $\sin\left(\frac{\pi}{2} - x\right)\csc x$

42. $\cot\left(\frac{\pi}{2} - x\right)\cos x$

43. $\frac{\cos^2 y}{1 - \sin y}$

44. $\cos t(1 + \tan^2 t)$

In Exercises 45–52, verify the identity algebraically. Then use a graphing utility to check your result graphically.

45. $\csc \theta \tan \theta = \sec \theta$

46. $\cos \theta \sec \theta - \cos^2 \theta = \sin^2 \theta$

47. $\frac{\csc \theta}{\sec \theta} + \frac{\cos \theta}{\sin \theta} = 2 \cot \theta$

48. $\frac{\sec^2 \theta - \tan^2 \theta + \tan \theta}{\sec \theta} = \cos \theta + \sin \theta$

The *Interactive* CD-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered Section and Review Exercises. They also provide Tutorial Exercises, which link to Guided Examples for additional help.

49. $1 - \dfrac{\sin^2 \theta}{1 - \cos \theta} = -\cos \theta$

50. $\dfrac{\tan \theta}{1 + \sec \theta} + \dfrac{1 + \sec \theta}{\tan \theta} = 2 \csc \theta$

51. $\dfrac{\cot(-\theta)}{\csc \theta} = -\cos \theta$ **52.** $\dfrac{\csc\left(\dfrac{\pi}{2} - \theta\right)}{\tan(-\theta)} = -\csc \theta$

In Exercises 53–62, verify the identity algebraically. Then use the *table* feature of a graphing utility to check your result numerically.

53. $\sin \theta + \cos \theta \cot \theta = \csc \theta$

54. $(\sec \theta - \tan \theta)(\csc \theta + 1) = \cot \theta$

55. $\dfrac{\cos \theta}{1 - \sin \theta} = \sec \theta + \tan \theta$

56. $\dfrac{1 + \csc \theta}{\cot \theta + \cos \theta} = \sec \theta$

57. $\dfrac{\sin \theta}{\csc \theta} + \dfrac{\cos \theta}{\sec \theta} = 1$

58. $\dfrac{1 + \csc \theta}{\sec \theta} - \cot \theta = \cos \theta$

59. $\dfrac{1 + \cos \theta}{\sin \theta} + \dfrac{\sin \theta}{1 + \cos \theta} = 2 \csc \theta$

60. $\dfrac{\sin \theta + \cos \theta}{\sin \theta} - \dfrac{\cos \theta - \sin \theta}{\cos \theta} = \sec \theta \csc \theta$

61. $\ln|\csc \theta| = -\ln|\sin \theta|$

62. $\ln|1 + \cos \theta| + \ln|1 - \cos \theta| = 2 \ln|\sin \theta|$

In Exercises 63–70, factor the expression and use the fundamental identities to simplify. Use a graphing utility to check your result graphically.

63. $\cot^2 x - \cot^2 x \cos^2 x$

64. $\sec^2 x \tan^2 x + \sec^2 x$

65. $\sin^2 x \sec^2 x - \sin^2 x$

66. $\dfrac{\csc^2 x - 1}{\csc x - 1}$

67. $\tan^4 x + 2 \tan^2 x + 1$

68. $1 - 2 \sin^2 x + \sin^4 x$

69. $\sin^4 x - \cos^4 x$

70. $\sec^3 x - \sec^2 x - \sec x + 1$

In Exercises 71–74, perform the multiplication and use the fundamental identities to simplify.

71. $(\sin x + \cos x)^2$

72. $(\cot x + \csc x)(\cot x - \csc x)$

73. $(\sec x + 1)(\sec x - 1)$

74. $(3 - 3 \sin x)(3 + 3 \sin x)$

In Exercises 75–78, perform the addition or subtraction and use the fundamental identities to simplify.

75. $\dfrac{1}{1 + \cos x} + \dfrac{1}{1 - \cos x}$

76. $\dfrac{1}{\sec x + 1} - \dfrac{1}{\sec x - 1}$

77. $\dfrac{\cos x}{1 + \sin x} + \dfrac{1 + \sin x}{\cos x}$

78. $\tan x - \dfrac{\sec^2 x}{\tan x}$

In Exercises 79–82, rewrite the expression so that it is *not* in fractional form.

79. $\dfrac{\sin^2 y}{1 - \cos y}$ **80.** $\dfrac{5}{\tan x + \sec x}$

81 $\dfrac{3}{\sec x - \tan x}$ **82.** $\dfrac{\tan^2 x}{\csc x + 1}$

Numerical and Graphical Analysis **In Exercises 83–86, use a graphing utility to complete the table and graph the functions. Make a conjecture about y_1 and y_2.**

x	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y_1							
y_2							

83. $y_1 = \cos\left(\dfrac{\pi}{2} - x\right), \quad y_2 = \sin x$

84. $y_1 = \cos x + \sin x \tan x, \quad y_2 = \sec x$

85. $y_1 = \dfrac{\cos x}{1 - \sin x}, \quad y_2 = \dfrac{1 + \sin x}{\cos x}$

86. $y_1 = \sec^4 x - \sec^2 x, \quad y_2 = \tan^2 x + \tan^4 x$

In Exercises 87–90, use a graphing utility to determine which of the six trigonometric functions is equal to the expression.

87. $\cos x \cot x + \sin x$

88. $\sin x(\cot x + \tan x)$

89. $\sec x - \dfrac{\cos x}{1 + \sin x}$

90. $\dfrac{1}{2}\left(\dfrac{1 + \sin \theta}{\cos \theta} + \dfrac{\cos \theta}{1 + \sin \theta}\right)$

In Exercises 91–96, use the trigonometric substitution to write the algebraic expression as a trigonometric function of θ, where $0 < \theta < \pi/2$.

91. $\sqrt{25 - x^2}$, $x = 5 \sin \theta$

92. $\sqrt{16 - 4x^2}$, $x = 2 \sin \theta$

93. $\sqrt{x^2 - 9}$, $x = 3 \sec \theta$

94. $\sqrt{x^2 - 4}$, $x = 2 \sec \theta$

95. $\sqrt{x^2 + 25}$, $x = 5 \tan \theta$

96. $\sqrt{x^2 + 100}$, $x = 10 \tan \theta$

In Exercises 97–100, use a graphing utility to solve the equation for θ, where $0 \le \theta < 2\pi$.

97. $\sin \theta = \sqrt{1 - \cos^2 \theta}$

98. $\cos \theta = -\sqrt{1 - \sin^2 \theta}$

99. $\sec \theta = \sqrt{1 + \tan^2 \theta}$

100. $\tan \theta = \sqrt{\sec^2 \theta - 1}$

In Exercises 101–104, rewrite the expression as a single logarithm and simplify the result.

101. $\ln|\cos \theta| - \ln|\sin \theta|$

102. $\ln|\csc \theta| + \ln|\tan \theta|$

103. $\ln(1 + \sin x) - \ln|\sec x|$

104. $\ln|\cot t| + \ln(1 + \tan^2 t)$

In Exercises 105–108, use the *table* feature of a graphing utility to demonstrate the identity for the given values of θ.

105. $\csc^2 \theta - \cot^2 \theta = 1$, (a) $\theta = 132°$, (b) $\theta = \dfrac{2\pi}{7}$

106. $\tan^2 \theta + 1 = \sec^2 \theta$, (a) $\theta = 346°$, (b) $\theta = 3.1$

107. $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin \theta$, (a) $\theta = 80°$, (b) $\theta = 0.8$

108. $\sin(-\theta) = -\sin \theta$, (a) $\theta = 250°$, (b) $\theta = \frac{1}{2}$

109. ***Rate of Change*** The rate of change of the function $f(x) = -\csc x - \sin x$ is given by the expression $\csc x \cot x - \cos x$. Show that this expression can also be written as $\cos x \cot^2 x$.

Synthesis

True or False? **In Exercises 110–113, determine whether the statement is true or false. Justify your answer.**

110. $\dfrac{\sin k\theta}{\cos k\theta} = \tan \theta$, k is a constant.

111. $\dfrac{1}{5 \cos \theta} = 5 \sec \theta$

112. $\sin \theta \csc \theta = 1$

113. $\sin \theta \csc \phi = 1$

114. Express each of the other trigonometric functions of θ in terms of $\sin \theta$.

115. Express each of the other trigonometric functions of θ in terms of $\cos \theta$.

Review

In Exercises 116–121, find the reference angle θ' and sketch θ and θ' in standard position.

116. $\theta = 254°$

117. $\theta = 341°$

118. $\theta = -178°$

119. $\theta = -212°$

120. $\theta = \dfrac{13\pi}{15}$

121. $\theta = \dfrac{35\pi}{6}$

In Exercises 122–125, sketch the graph of the function.

122. $f(x) = \dfrac{1}{2} \sin \pi x$

123. $f(x) = -2 \tan \dfrac{\pi x}{2}$

124. $f(x) = \dfrac{1}{2} \cot\left(x + \dfrac{\pi}{4}\right)$

125. $f(x) = \dfrac{3}{2} \cos(x - \pi) + 3$

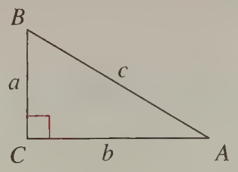
In Exercises 126–129, solve the right triangle shown in the figure. (Round your answers to two decimal places.)

126. $B = 80°$, $a = 16$

127. $A = 28°$, $c = 20$

128. $a = 14$, $b = 8$

129. $b = 6.2$, $c = 12.54$



6.2 Verifying Trigonometric Identities

Introduction

In this section, you will study techniques for verifying trigonometric identities. In the next section, you will study techniques for solving trigonometric equations. The key to both verifying identities *and* solving equations is the ability to use the fundamental identities and the rules of algebra to rewrite trigonometric expressions.

Remember that a *conditional equation* is an equation that is true for only some of the values in its domain. For example, the conditional equation

$\sin x = 0$ Conditional equation

is true only for $x = n\pi$, where n is an integer. When you find these values, you are *solving* the equation.

On the other hand, an equation that is true for all real values in the domain of the variable is an *identity*. For example, the familiar equation

$\sin^2 x = 1 - \cos^2 x$ Identity

is true for all real numbers x. So, it is an identity.

Verifying Trigonometric Identities

Although there are similarities, proving that a trigonometric equation is an identity is quite different from solving an equation. There is no well-defined set of rules to follow in verifying trigonometric identities, and the process is best learned by practice.

Guidelines for Verifying Trigonometric Identities

1. Work with one side of the equation at a time. It is often better to work with the more complicated side first.
2. Look for opportunities to factor an expression, add fractions, square a binomial, or create a monomial denominator.
3. Look for opportunities to use the fundamental identities. Note which functions are in the final expression you want. Sines and cosines pair up well, as do secants and tangents, and cosecants and cotangents.
4. If the preceding guidelines do not help, try converting all terms to sines and cosines.
5. Try something! Even making an attempt that leads to a dead end gives insight.

What You Should Learn:

- How to verify trigonometric identities

Why You Should Learn It:

You can use trigonometric identities to rewrite trigonometric expressions. For instance, Exercise 69 on page 469, shows you how trigonometric identities can be used to solve a problem about the coefficient of friction for an object on an inclined plane.

Nancy Dudley/Stock Boston

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 1 Verifying a Trigonometric Identity

Verify the identity $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$.

Solution

Because the left side is more complicated, start with it.

$$\begin{aligned}
\frac{\sec^2\theta - 1}{\sec^2\theta} &= \frac{(\tan^2\theta + 1) - 1}{\sec^2\theta} && \text{Pythagorean identity}\\
&= \frac{\tan^2\theta}{\sec^2\theta} && \text{Simplify.}\\
&= \tan^2\theta(\cos^2\theta) && \text{Reciprocal identity}\\
&= \frac{\sin^2\theta}{\cancel{\cos^2\theta}}\cancel{(\cos^2\theta)} && \text{Quotient identity}\\
&= \sin^2\theta && \text{Simplify.}
\end{aligned}$$

The *Interactive* CD-ROM and *Internet* versions of this text offer a built-in graphing calculator, which can be used with the Examples, Explorations, and Exercises.

Here is another way to verify the identity in Example 1.

$$\begin{aligned}
\frac{\sec^2\theta - 1}{\sec^2\theta} &= \frac{\sec^2\theta}{\sec^2\theta} - \frac{1}{\sec^2\theta}\\
&= 1 - \cos^2\theta\\
&= \sin^2\theta
\end{aligned}$$

There can be more than one way to verify an identity. Your method may differ from that used by your instructor or fellow students. This is a good chance to be creative and establish your own style, but try to be as efficient as possible.

EXAMPLE 2 Combining Fractions Before Using Identities

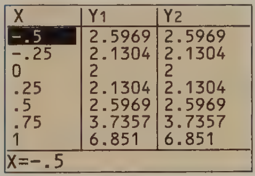
Verify the identity $\dfrac{1}{1 - \sin\alpha} + \dfrac{1}{1 + \sin\alpha} = 2\sec^2\alpha$.

Algebraic Solution

$$\begin{aligned}
\frac{1}{1 - \sin\alpha} + \frac{1}{1 + \sin\alpha} &= \frac{1 + \sin\alpha + 1 - \sin\alpha}{(1 - \sin\alpha)(1 + \sin\alpha)} && \text{Add fractions.}\\
&= \frac{2}{1 - \sin^2\alpha} && \text{Simplify.}\\
&= \frac{2}{\cos^2\alpha} && \text{Pythagorean identity}\\
&= 2\sec^2\alpha && \text{Reciprocal identity}
\end{aligned}$$

Numerical Solution

Use the *table* feature of a graphing utility set in *radian* mode to create a table that shows the values of $y_1 = 1/(1 - \sin x) + 1/(1 + \sin x)$ and $y_2 = 2/\cos^2 x$ for different values of x, as shown in Figure 6.5. From the table, you can see that the values appear to be identical, so $1/(1 - \sin x) + 1/(1 + \sin x) = 2\sec^2$ appears to be an identity.



```
X      Y1      Y2
-.5    2.5969  2.5969
-.25   2.1304  2.1304
0      2       2
.25    2.1304  2.1304
.5     2.5969  2.5969
.75    3.7357  3.7357
1      6.851   6.851
X=-.5
```

Figure 6.5

EXAMPLE 3 Verifying a Trigonometric Identity

Verify the identity $(\tan^2 x + 1)(\cos^2 x - 1) = -\tan^2 x$.

Solution

By applying identities before multiplying, you obtain the following.

$$\begin{aligned}(\tan^2 x + 1)(\cos^2 x - 1) &= (\sec^2 x)(-\sin^2 x) && \text{Pythagorean identities}\\ &= -\frac{\sin^2 x}{\cos^2 x} && \text{Reciprocal identity}\\ &= -\left(\frac{\sin x}{\cos x}\right)^2 && \text{Rule of exponents}\\ &= -\tan^2 x && \text{Quotient identity}\end{aligned}$$

You can use a graphing utility to verify your results by graphing both sides of the original identity in the same viewing window, as shown in Figure 6.6. The graphs appear to coincide.

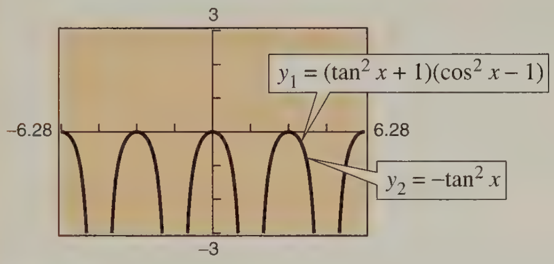


Figure 6.6

STUDY TIP

Although a graphing utility can be useful in helping to verify an identity, you must use algebraic techniques to produce a valid proof. For example, graph the two functions

$$y_1 = \sin 50x$$

$$y_2 = \sin 2x$$

in a trigonometric viewing window. Although their graphs seem identical, $\sin 50x \neq \sin 2x$.

EXAMPLE 4 Converting to Sines and Cosines

Verify the identity $\tan x + \cot x = \sec x \csc x$.

Solution

In this case there appear to be no fractions to add, no products to find, and no opportunity to use one of the Pythagorean identities. So, try converting the left side into sines and cosines to see what happens.

$$\begin{aligned}\tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} && \text{Quotient identities}\\ &= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} && \text{Add fractions.}\\ &= \frac{1}{\cos x \sin x} && \text{Pythagorean identity}\\ &= \frac{1}{\cos x} \cdot \frac{1}{\sin x} && \text{Product of fractions}\\ &= \sec x \csc x && \text{Reciprocal identities}\end{aligned}$$

Recall from algebra that *rationalizing the denominator* is, on occasion, a powerful simplification technique. A related form of this technique works for simplifying trigonometric expressions as well.

EXAMPLE 5 Verifying a Trigonometric Identity

Verify the identity $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$.

Algebraic Solution

Work with the *right* side because you can create a monomial denominator by multiplying the numerator and denominator by $(1 + \sin x)$.

$$\begin{aligned}
\frac{\cos x}{1 - \sin x} &= \frac{\cos x}{1 - \sin x}\left(\frac{1 + \sin x}{1 + \sin x}\right) && \text{Multiply numerator and denominator by } (1 + \sin x). \\
&= \frac{\cos x + \cos x \sin x}{1 - \sin^2 x} && \text{Multiply.} \\
&= \frac{\cos x + \cos x \sin x}{\cos^2 x} && \text{Pythagorean identity} \\
&= \frac{\cos x}{\cos^2 x} + \frac{\cos x \sin x}{\cos^2 x} && \text{Separate fractions.} \\
&= \frac{1}{\cos x} + \frac{\sin x}{\cos x} && \text{Simplify.} \\
&= \sec x + \tan x && \text{Identities}
\end{aligned}$$

Graphical Solution

Use a graphing utility set in *radian* and *dot* modes to graph $y_1 = \sec x + \tan x = 1/\cos x + \tan x$ and $y_2 = \cos x/(1 - \sin x)$ in the same viewing window, as shown in Figure 6.7. Because the graphs appear to coincide, $\sec x + \tan x = \cos x/(1 - \sin x)$ appears to be an identity.

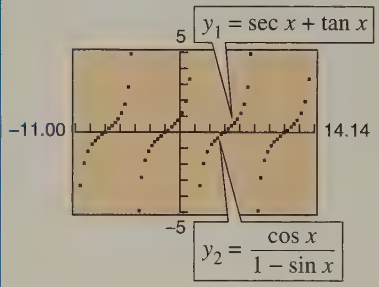


Figure 6.7

In Examples 1 through 5, you have been verifying trigonometric identities by working with one side of the equation and converting to the form given on the other side. On occasion it is practical to work with each side *separately* to obtain one common form equivalent to both sides. This is illustrated in Example 6.

EXAMPLE 6 Working with Each Side Separately

Verify the identity $\dfrac{\cot^2 \theta}{1 + \csc \theta} = \dfrac{1 - \sin \theta}{\sin \theta}$.

Solution

Working with the left side, you have

$$\begin{aligned}
\frac{\cot^2 \theta}{1 + \csc \theta} &= \frac{\csc^2 \theta - 1}{1 + \csc \theta} && \text{Pythagorean identity} \\
&= \frac{(\csc \theta - 1)\cancel{(\csc \theta + 1)}}{\cancel{1 + \csc \theta}} && \text{Factor.} \\
&= \csc \theta - 1. && \text{Simplify.}
\end{aligned}$$

Now, simplifying the right side, you have

$$\begin{aligned}
\frac{1 - \sin \theta}{\sin \theta} &= \frac{1}{\sin \theta} - \frac{\sin \theta}{\sin \theta} && \text{Separate fractions.} \\
&= \csc \theta - 1. && \text{Reciprocal identity}
\end{aligned}$$

The identity is verified because both sides are equal to $\csc \theta - 1$.

In Example 7, powers of trigonometric functions are rewritten as more complicated sums of products of trigonometric functions. This is a common procedure used in calculus.

EXAMPLE 7 Three Examples from Calculus

Verify each identity.

a. $\tan^4 x = \tan^2 x \sec^2 x - \tan^2 x$

b. $\sin^3 x \cos^4 x = (\cos^4 x - \cos^6 x)\sin x$

c. $\sec^4 x \tan^2 x = (\tan^2 x + \tan^4 x)\sec^2 x$

Solution

a.
$$\begin{aligned} \tan^4 x &= (\tan^2 x)(\tan^2 x) && \text{Write as separate factors.} \\ &= \tan^2 x(\sec^2 x - 1) && \text{Pythagorean identity} \\ &= \tan^2 x \sec^2 x - \tan^2 x && \text{Multiply.} \end{aligned}$$

b.
$$\begin{aligned} \sin^3 x \cos^4 x &= \sin^2 x \cos^4 x \sin x && \text{Write as separate factors.} \\ &= (1 - \cos^2 x)\cos^4 x \sin x && \text{Pythagorean identity} \\ &= (\cos^4 x - \cos^6 x)\sin x && \text{Multiply.} \end{aligned}$$

c.
$$\begin{aligned} \sec^4 x \tan^2 x &= \sec^2 x \tan^2 x \sec^2 x && \text{Write as separate facors.} \\ &= (1 + \tan^2 x)\tan^2 x \sec^2 x && \text{Pythagorean identity} \\ &= (\tan^2 x + \tan^4 x)\sec^2 x && \text{Multiply.} \end{aligned}$$

Writing About Math *Error Analysis*

Suppose you are tutoring a student in trigonometry. One of the homework problems your student encounters asks whether the following statement is an identity.

$$\tan^2 x \sin^2 x \overset{?}{=} \frac{5}{6}\tan^2 x$$

Your student does not attempt to verify the equivalence algebraically, but mistakenly uses only a graphical approach. Using window settings of $-3\pi \le x \le 3\pi$ with an x-scale of $\pi/2$, and $-20 \le y \le 20$ with a y-scale of 1, your student graphs both sides of the expression on a graphing utility and concludes that the statement is an identity.

Write a short paragraph explaining what is wrong with your student's reasoning.

6.2 Exercises

In Exercises 1–10, verify the identity.

1. $\sin t \csc t = 1$

2. $\tan y \cot y = 1$

3. $\dfrac{\csc^2 x}{\cot x} = \csc x \sec x$

4. $\cot^2 y(\sec^2 y - 1) = 1$

5. $\cos^2 \beta - \sin^2 \beta = 1 - 2\sin^2 \beta$

6. $\cos^2 \beta - \sin^2 \beta = 2\cos^2 \beta - 1$

7. $\tan^2 \theta + 6 = \sec^2 \theta + 5$

8. $2 - \csc^2 z = 1 - \cot^2 z$

9. $\cos x + \sin x \tan x = \sec x$

10. $\dfrac{\cot^3 t}{\csc t} = \cos t(\csc^2 t - 1)$

Numerical, Graphical, and Algebraic Analysis **In Exercises 11–18, use a graphing utility to complete the table and graph the functions. Use both as evidence that $y_1 = y_2$. Then verify the identity algebraically.**

x	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y_1							
y_2							

11. $y_1 = \dfrac{1}{\sec x \tan x}, \quad y_2 = \csc x - \sin x$

12. $y_1 = \dfrac{\csc x - 1}{1 - \sin x}, \quad y_2 = \csc x$

13. $y_1 = \csc x - \sin x, \quad y_2 = \cos x \cot x$

14. $y_1 = \sec x - \cos x, \quad y_2 = \sin x \tan x$

15. $y_1 = \sin x + \cos x \cot x, \quad y_2 = \csc x$

16. $y_1 = \dfrac{\sec x + \tan x}{\sec x - \tan x}, \quad y_2 = (\sec x + \tan x)^2$

17. $y_1 = \dfrac{1}{\tan x} + \dfrac{1}{\cot x}, \quad y_2 = \tan x + \cot x$

18. $y_1 = \dfrac{1}{\cos x} - \dfrac{1}{\sec x}, \quad y_2 = \sec x - \cos x$

In Exercises 19 and 20, find the error.

19.
$$\begin{aligned}
(1 + \tan x)[1 + \cot(-x)] &= (1 + \tan x)(1 + \cot x)\\
&= 1 + \cot x + \tan x + \tan x \cot x\\
&= 1 + \cot x + \tan x + 1\\
&= 2 + \cot x + \tan x
\end{aligned}$$

20.
$$\begin{aligned}
\frac{1 + \sec(-\theta)}{\sin(-\theta) + \tan(-\theta)} &= \frac{1 - \sec\theta}{\sin\theta - \tan\theta}\\
&= \frac{1 - \sec\theta}{(\sin\theta)\left[1 - \left(\dfrac{1}{\cos\theta}\right)\right]}\\
&= \frac{1 - \sec\theta}{\sin\theta(1 - \sec\theta)}\\
&= \frac{1}{\sin\theta}\\
&= \csc\theta
\end{aligned}$$

In Exercises 21 and 22, fill in the missing step(s).

21.
$$\begin{aligned}
\sec^4 x - 2\sec^2 x + 1 &= (\sec^2 x - 1)^2\\
&= \\
&= \tan^4 x
\end{aligned}$$

22.
$$\begin{aligned}
\frac{\tan x - \cot x}{\tan x + \cot x} &= \frac{\dfrac{\sin x}{\cos x} - \dfrac{\cos x}{\sin x}}{\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}}\\
&= \\
&= \frac{\sin^2 x - \cos^2 x}{1}\\
&= \sin^2 x - \cos^2 x\\
&= \\
&= 1 - 2\cos^2 x
\end{aligned}$$

In Exercises 23–32, verify the identity.

23. $\sin^{1/2} x \cos x - \sin^{5/2} x \cos x = \cos^3 x\sqrt{\sin x}$

24. $\sec^6 x(\sec x \tan x) - \sec^4 x(\sec x \tan x) = \sec^5 x \tan^3 x$

25. $\tan\left(\frac{\pi}{2} - x\right)\sec x = \csc x$

26. $\dfrac{\sin[(\pi/2) - x]}{\cos[(\pi/2) - x]} = \cot x$

27. $\dfrac{\sec(-x)}{\csc(-x)} = -\tan x$

28. $(1 + \sin y)[1 + \sin(-y)] = \cos^2 y$

29. $\dfrac{\cos(-\theta)}{1 + \sin(-\theta)} = \sec\theta + \tan\theta$

30. $\dfrac{1 + \csc(-\theta)}{\cos(-\theta) + \cot(-\theta)} = \sec\theta$

31. $\dfrac{\sin x\cos y + \cos x\sin y}{\cos x\cos y - \sin x\sin y} = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$

32. $\dfrac{\tan x + \tan y}{1 - \tan x\tan y} = \dfrac{\cot x + \cot y}{\cot x\cot y - 1}$

In Exercises 33–40, verify the identity algebraically, and use the *table* feature of a graphing utility to confirm it numerically.

33. $\dfrac{\tan x + \cot y}{\tan x\cot y} = \tan y + \cot x$

34. $\dfrac{\cos x - \cos y}{\sin x + \sin y} + \dfrac{\sin x - \sin y}{\cos x + \cos y} = 0$

35. $\sqrt{\dfrac{1 + \sin\theta}{1 - \sin\theta}} = \dfrac{1 + \sin\theta}{|\cos\theta|}$

36. $\sqrt{\dfrac{1 - \cos\theta}{1 + \cos\theta}} = \dfrac{1 - \cos\theta}{|\sin\theta|}$

37. $\cos^2 x + \cos^2\left(\frac{\pi}{2} - x\right) = 1$

38. $\sec^2 y - \cot^2\left(\frac{\pi}{2} - y\right) = 1$

39. $\sec x\sin\left(\frac{\pi}{2} - x\right) = 1$

40. $\csc^2\left(\frac{\pi}{2} - x\right) - 1 = \tan^2 x$

In Exercises 41–52, verify the identity algebraically, and use a graphing utility to confirm it graphically.

41. $2\sec^2 x - 2\sec^2 x\sin^2 x - \sin^2 x - \cos^2 x = 1$

42. $\csc x(\csc x - \sin x) + \dfrac{\sin x - \cos x}{\sin x} + \cot x = \csc^2 x$

43. $2 + \cos^2 x - 3\cos^4 x = \sin^2 x(2 + 3\cos^2 x)$

44. $4\tan^4 x + \tan^2 x - 3 = \sec^2 x(4\tan^2 x - 3)$

45. $\csc^4 x - 2\csc^2 x + 1 = \cot^4 x$

46. $\sin x(1 - 2\cos^2 x + \cos^4 x) = \sin^5 x$

47. $\sec^4\theta - \tan^4\theta = 1 + 2\tan^2\theta$

48. $\csc^4\theta - \cot^4\theta = 2\csc^2\theta - 1$

49. $\dfrac{\sin\beta}{1 - \cos\beta} = \dfrac{1 + \cos\beta}{\sin\beta}$

50. $\dfrac{\cot\alpha}{\csc\alpha - 1} = \dfrac{\csc\alpha + 1}{\cot\alpha}$

51. $\dfrac{\tan^3\alpha - 1}{\tan\alpha - 1} = \tan^2\alpha + \tan\alpha + 1$

52. $\dfrac{\sin^3\beta + \cos^3\beta}{\sin\beta + \cos\beta} = 1 - \sin\beta\cos\beta$

Conjecture **In Exercises 53–56, use a graphing utility to graph the trigonometric function. Use the graph to make a conjecture about a simplification of the expression. Verify the resulting identity algebraically.**

53. $y = \dfrac{1}{\cot x + 1} + \dfrac{1}{\tan x + 1}$

54. $y = \dfrac{\cos x}{1 - \tan x} + \dfrac{\sin x\cos x}{\sin x - \cos x}$

55. $y = \dfrac{1}{\sin x} - \dfrac{\cos^2 x}{\sin x}$

56. $y = \sin t + \dfrac{\cot^2 t}{\csc t}$

In Exercises 57–60, use the properties of logarithms and trigonometric identities to verify the identity.

57. $\ln|\cot\theta| = \ln|\cos\theta| - \ln|\sin\theta|$

58. $\ln|\sec\theta| = -\ln|\cos\theta|$

59. $-\ln(1 + \cos\theta) = \ln(1 - \cos\theta) - 2\ln|\sin\theta|$

60. $-\ln|\csc\theta + \cot\theta| = \ln|\csc\theta - \cot\theta|$

In Exercises 61–64, use the cofunction identities to evaluate the expression without using a calculator.

61. $\sin^2 25° + \sin^2 65°$

62. $\cos^2 14° + \cos^2 76°$

63. $\cos^2 20° + \cos^2 52° + \cos^2 38° + \cos^2 70°$

64. $\sin^2 12° + \sin^2 40° + \sin^2 50° + \sin^2 78°$

In Exercises 65–68, the powers of trigonometric functions are rewritten to be useful in calculus. Verify the identity.

65. $\tan^5 x = \tan^3 x \sec^2 x - \tan^3 x$

66. $\tan^3 x \sec^3 x = \sec^4 x(\sec x \tan x) - \sec^2 x(\sec x \tan x)$

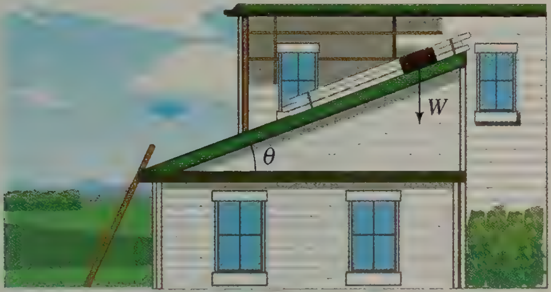
67. $\cos^3 x \sin^2 x = (\sin^2 x - \sin^4 x)\cos x$

68. $\sin^4 x + \cos^4 x = 1 - 2\cos^2 x + 2\cos^4 x$

69. ***Friction*** The forces acting on an object weighing W units on an inclined plane positioned at an angle of θ with the horizontal are modeled by

$$\mu W \cos\theta = W \sin\theta$$

where μ is the coefficient of friction. Solve the equation for μ and simplify the result.



70. ***Shadow of a Sundial*** The length s of a shadow cast by a vertical *gnomon* (column or shaft on a sundial) of height h when the angle of the sun above the horizon is θ can be modeled by the equation

$$s = \frac{h\sin(90^\circ - \theta)}{\sin\theta}.$$

Show that the equation is equivalent to $s = h\cot\theta$.

71. ***Rate of Change*** The rate of change of the function $f(x) = \sin x + \csc x$ is given by $\cos x - \csc x \cot x$. Show that the expression for the rate of change can also be given by $-\cos x \cot^2 x$.

Synthesis

True or False? **In Exercises 72–75, determine whether the statement is true or false. Justify your answer.**

72. There can be more than one way to verify a trigonometric identity.

73. Of the six trigonometric functions, two are even.

74. $\ln(\cos x) = \cos(\ln x)$

75. $\sin(x^2) = \sin^2(x)$

Think About It **In Exercises 76–79, explain why the equation is *not* an identity and find one value of the variable for which the equation is not true.**

76. $\sqrt{\tan^2 x} = \tan x$

77. $\sin\theta = \sqrt{1 - \cos^2\theta}$

78. $\tan\theta = \sqrt{\sec^2\theta - 1}$

79. $\sqrt{\sin^2 x + \cos^2 x} = \sin x + \cos x$

80. Verify that for all integers n

$$\cos\left[\frac{(2n + 1)\pi}{2}\right] = 0.$$

81. Verify that for all integers n

$$\sin\left[\frac{(12n + 1)\pi}{6}\right] = \frac{1}{2}.$$

Review

In Exercises 82–85, find a polynomial function with integer coefficients that has the given zeros. (There are many correct answers.)

82. $1, 8i, -8i$

83. $i, -i, 4i, -4i$

84. $4, 6 + i, 6 - i$

85. $0, 0, 2, 1 - i$

In Exercises 86–89, sketch the graph of the function by hand.

86. $f(x) = 2^x + 3$

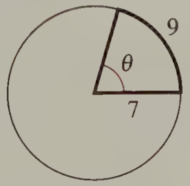
87. $f(x) = -2^{x-3}$

88. $f(x) = \left(\frac{3}{2}\right)^{x+1}$

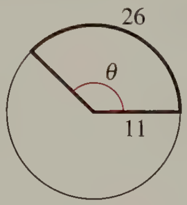
89. $f(x) = 5^{-x} - 2$

In Exercises 90 and 91, find the measure of the angle in radians.

90.



91.



In Exercises 92–95, state the quadrant in which θ lies.

92. $\csc\theta > 0$ and $\tan\theta < 0$

93. $\cot\theta > 0$ and $\cos\theta < 0$

94. $\sec\theta > 0$ and $\sin\theta < 0$

95. $\cot\theta > 0$ and $\sec\theta < 0$

6.3 Solving Trigonometric Equations

What You Should Learn:

- How to use standard algebraic techniques to solve trigonometric equations
- How to solve trigonometric equations of quadratic type
- How to solve trigonometric equations involving multiple angles
- How to use inverse trigonometric functions to solve trigonometric equations

Why You Should Learn It:

You can use trigonometric equations to solve a variety of real-life problems. For instance, Exercise 75 on page 480 shows you how solving a trigonometric equation can help answer questions about the position of the sun in Cheyenne, Wyoming.

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Introduction

To solve a trigonometric equation, use standard algebraic techniques such as collecting like terms and factoring. Your preliminary goal is to isolate the trigonometric function involved in the equation.

EXAMPLE 1 Solving a Trigonometric Equation

$$2\sin x - 1 = 0 \qquad \text{Original equation}$$

$$2\sin x = 1 \qquad \text{Add 1 to each side.}$$

$$\sin x = \tfrac{1}{2} \qquad \text{Divide each side by 2.}$$

To solve for x, note that the equation $\sin x = \frac{1}{2}$ has solutions $x = \pi/6$ and $x = 5\pi/6$ in the interval $[0, 2\pi)$. Moreover, because $\sin x$ has a period of 2π, there are infinitely many other solutions, which can be written as

$$x = \frac{\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + 2n\pi \qquad \text{General solution}$$

where n is an integer, as shown in Figure 6.8.

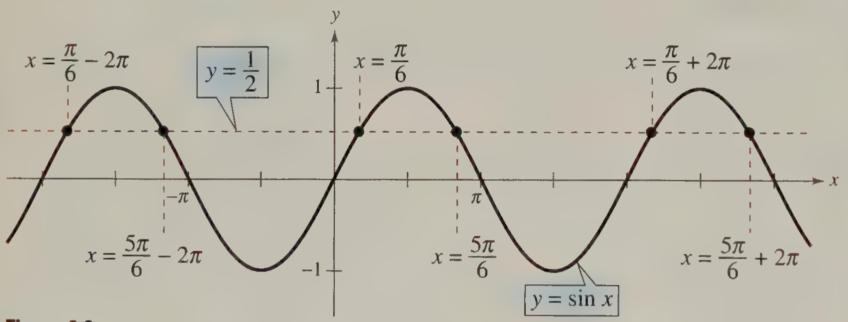


Figure 6.8

Another way to see that the equation $\sin x = \frac{1}{2}$ has infinitely many solutions is indicated in Figure 6.9. Any angles that are coterminal with $\pi/6$ or $5\pi/6$ are also solutions of the equation.

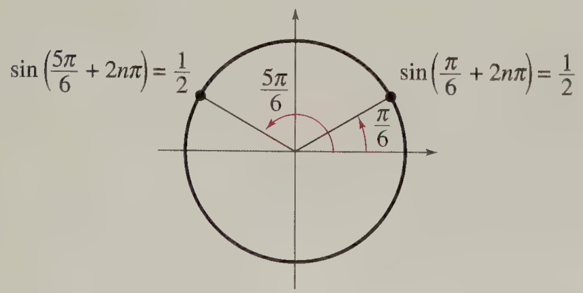

$\sin\left(\frac{5\pi}{6} + 2n\pi\right) = \frac{1}{2}$
$\frac{5\pi}{6}$
$\sin\left(\frac{\pi}{6} + 2n\pi\right) = \frac{1}{2}$
$\frac{\pi}{6}$

Figure 6.9

EXAMPLE 2 Collecting Like Terms

Find all solutions of $\sin x + \sqrt{2} = -\sin x$ in the interval $[0, 2\pi)$.

Algebraic Solution

Rewrite the equation so that $\sin x$ is isolated on one side of the equation.

$$\sin x + \sqrt{2} = -\sin x$$ Write original equation.

$$\sin x + \sin x = -\sqrt{2}$$ Add $\sin x$ and subtract $\sqrt{2}$ from each side.

$$2 \sin x = -\sqrt{2}$$ Combine like terms.

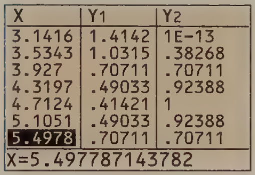
$$\sin x = -\frac{\sqrt{2}}{2}$$ Divide each side by 2.

The solutions in the interval $[0, 2\pi)$ are

$$x = \frac{5\pi}{4} \quad \text{and} \quad x = \frac{7\pi}{4}.$$

Numerical Solution

Use the *table* feature of a graphing utility set in *radian* mode to create a table that shows the values of $y_1 = \sin x + \sqrt{2}$ and $y_2 = -\sin x$ for different values of x. Your table should go from $x = 0$ to $x = 2\pi$ using increments in the table of $\pi/8$, as shown in Figure 6.10. From the table, you can see that the values of y_1 and y_2 appear to be identical when $x \approx 3.927 \approx 5\pi/4$ and $x \approx 5.4978 \approx 7\pi/4$. These values are the approximate solutions of $\sin x + \sqrt{2} = -\sin x$.



```
X      | Y1     | Y2
3.1416 | 1.4142 | 1E-13
3.5343 | 1.0315 | .38268
3.927  | .70711 | .70711
4.3197 | .49033 | .92388
4.7124 | .41421 | 1
5.1051 | .49033 | .92388
5.4978 | .70711 | .70711
X=5.497787143782
```

Figure 6.10

EXAMPLE 3 Extracting Square Roots

Solve $3 \tan^2 x - 1 = 0$.

Solution

Rewrite the equation so that $\tan x$ is isolated on one side of the equation.

$$3 \tan^2 x - 1 = 0$$ Write original equation.

$$3 \tan^2 x = 1$$ Add 1 to each side.

$$\tan^2 x = \frac{1}{3}$$ Divide each side by 3.

$$\tan x = \pm\frac{1}{\sqrt{3}}$$ Extract square roots.

Because $\tan x$ has a period of π, first find all solutions in the interval $[0, \pi)$. These are $x = \pi/6$ and $x = 5\pi/6$. Finally, add $n\pi$ to each of these solutions to get the general form

$$x = \frac{\pi}{6} + n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + n\pi$$ General solution

where n is an integer. The graph of $y = 3 \tan^2 x - 1$, shown in Figure 6.11, confirms this result.

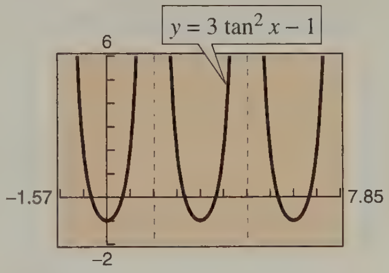


Figure 6.11

The equations in Examples 1, 2, and 3 involved only one trigonometric function. When two or more functions occur in the same equation, collect all terms on one side and try to separate the functions by factoring or by using appropriate identities. This may produce factors that yield no solutions, as illustrated in Example 4.

EXAMPLE 4 Factoring

Solve $\cot x \cos^2 x = 2 \cot x$.

Solution

$$\cot x \cos^2 x = 2 \cot x \qquad \text{Write original equation.}$$

$$\cot x \cos^2 x - 2 \cot x = 0 \qquad \text{Subtract } 2 \cot x \text{ from each side.}$$

$$\cot x(\cos^2 x - 2) = 0 \qquad \text{Factor.}$$

By setting each of these factors equal to zero, you obtain the following.

$$\cot x = 0 \qquad \text{and} \qquad \cos^2 x - 2 = 0$$

$$x = \frac{\pi}{2} \qquad\qquad \cos^2 x = 2$$

$$\cos x = \pm\sqrt{2}$$

The equation $\cot x = 0$ has the solution $x = \pi/2$. No solution is obtained from $\cos x = \pm\sqrt{2}$ because $\pm\sqrt{2}$ are outside the range of the cosine function. Therefore, the general form of the solution is obtained by adding multiples of π to $x = \pi/2$, to get

$$x = \frac{\pi}{2} + n\pi \qquad \text{General solution}$$

where n is an integer. The graph of $y = \cot x \cos^2 x - 2 \cot x$, shown in Figure 6.12, confirms this result.

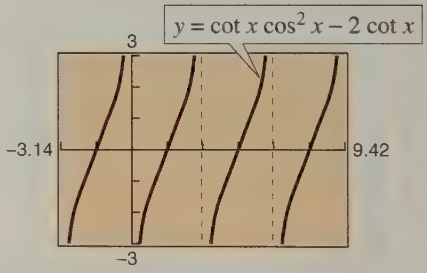


Figure 6.12

In Example 4, don't make the mistake of dividing each side of the equation by $\cot x$. Doing this would lose the solutions. Can you see why?

Equations of Quadratic Type

Many trigonometric equations are of quadratic type. Here are a few examples.

Quadratic in sin x	*Quadratic in sec x*
$2 \sin^2 x - \sin x - 1 = 0$	$\sec^2 x - 3 \sec x - 2 = 0$
$2(\sin x)^2 - \sin x - 1 = 0$	$(\sec x)^2 - 3 \sec x - 2 = 0$

To solve equations of this type, factor the quadratic or, if factoring is not possible, use the Quadratic Formula.

EXAMPLE 5 Factoring an Equation of Quadratic Type

Find all solutions of $2\sin^2 x - \sin x - 1 = 0$ in the interval $[0, 2\pi)$.

Algebraic Solution

Treating the equation as a quadratic in $\sin x$ and factoring produces the following.

$$2\sin^2 x - \sin x - 1 = 0 \qquad \text{Write original equation.}$$

$$(2\sin x + 1)(\sin x - 1) = 0 \qquad \text{Factor.}$$

Setting each factor equal to zero, you obtain the following solutions in the interval $[0, 2\pi)$.

$$2\sin x + 1 = 0 \qquad \text{and} \qquad \sin x - 1 = 0$$

$$\sin x = -\frac{1}{2} \qquad \qquad \sin x = 1$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6} \qquad \qquad x = \frac{\pi}{2}$$

So, the solutions in the interval $[0, 2\pi)$ are $\pi/2$, $7\pi/6$, and $11\pi/6$.

Graphical Solution

Use a graphing utility set in *radian* mode to graph $y = 2\sin^2 x - \sin x - 1$ for $0 \le x < 2\pi$, as shown in Figure 6.13. Use the *zero* or *root* feature or the *zoom* and *trace* features to approximate the x-intercepts to be

$$x \approx 1.571 \approx \frac{\pi}{2}, \quad x \approx 3.665 \approx \frac{7\pi}{6}, \quad x \approx 5.760 \approx \frac{11\pi}{6}.$$

These values are the approximate solutions of $2\sin^2 x - \sin x - 1 = 0$.

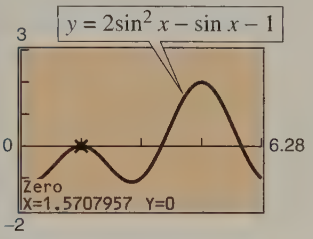


Figure 6.13

In Example 5, the general solution would be

$$x = \frac{\pi}{2} + 2n\pi, \qquad x = \frac{7\pi}{6} + 2n\pi, \qquad x = \frac{11\pi}{6} + 2n\pi \qquad \text{General solution}$$

where n is an integer.

When working with an equation of quadratic type, be sure that the equation involves a *single* trigonometric function, as shown in the next example.

EXAMPLE 6 Rewriting with a Single Trigonometric Function

Solve $2\sin^2 x + 3\cos x - 3 = 0$.

Solution

Begin by rewriting the equation so that it has only cosine functions.

$$2\sin^2 x + 3\cos x - 3 = 0 \qquad \text{Write original equation.}$$

$$2(1 - \cos^2 x) + 3\cos x - 3 = 0 \qquad \text{Pythagorean identity}$$

$$2\cos^2 x - 3\cos x + 1 = 0 \qquad \text{Combine like terms and multiply each side by } -1.$$

$$(2\cos x - 1)(\cos x - 1) = 0 \qquad \text{Factor.}$$

By setting each factor equal to zero, you can find the solutions in the interval $[0, 2\pi)$ to be $x = 0$, $x = \pi/3$, and $x = 5\pi/3$. The general solution is therefore

$$x = 2n\pi, \qquad x = \frac{\pi}{3} + 2n\pi, \qquad x = \frac{5\pi}{3} + 2n\pi \qquad \text{General solution}$$

where n is an integer. The graph of $y = 2\sin^2 x + 3\cos x - 3$, shown in Figure 6.14, confirms this result.

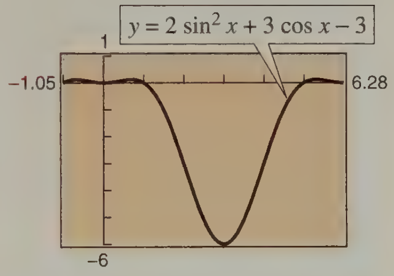


Figure 6.14

Sometimes you must square both sides of an equation to obtain a quadratic. This can introduce extraneous solutions.

EXAMPLE 7 Squaring and Converting to Quadratic Type

Find all solutions of $\cos x + 1 = \sin x$ in the interval $[0, 2\pi)$.

Solution

It is not clear how to rewrite this equation in terms of a single trigonometric function. See what happens when you square both sides of the equation.

$$\cos x + 1 = \sin x \qquad \text{Write original equation.}$$

$$\cos^2 x + 2\cos x + 1 = \sin^2 x \qquad \text{Square each side.}$$

$$\cos^2 x + 2\cos x + 1 = 1 - \cos^2 x \qquad \text{Pythagorean identity}$$

$$2\cos^2 x + 2\cos x = 0 \qquad \text{Combine like terms.}$$

$$2\cos x(\cos x + 1) = 0 \qquad \text{Factor.}$$

Setting each factor equal to zero produces the following.

$$2\cos x = 0 \qquad \text{and} \qquad \cos x + 1 = 0$$

$$\cos x = 0 \qquad\qquad \cos x = -1$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad\qquad x = \pi$$

Because you squared the original equation, check for extraneous solutions. Of the three possible solutions, $x = 3\pi/2$ is extraneous. (Try checking this.) So, in the interval $[0, 2\pi)$, the only solutions are $x = \pi/2$ and $x = \pi$. The graph of $y = \cos x + 1 - \sin x$, shown in Figure 6.15, confirms this result.

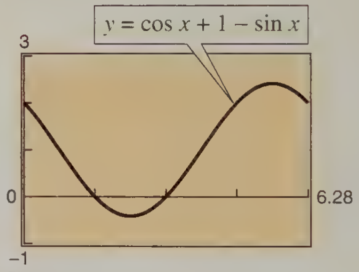


Figure 6.15

In Example 7, the general solution is

$$x = \frac{\pi}{2} + 2n\pi,\ x = \pi + 2n\pi \qquad \text{General solution}$$

where n is an integer.

Exploration

Use a graphing utility to confirm the solutions found in Example 7 in two different ways.

1. Graph both sides of the equation and find the x-coordinates of the points at which the graphs intersect.

   Left side: $y = \cos x + 1$ *Right side:* $y = \sin x$

2. Graph the equation $y = \cos x + 1 - \sin x$ and find the x-intercepts of the graph.

Do both methods produce the same x-values? Which method do you prefer? Why?

Functions Involving Multiple Angles

EXAMPLE 8 Functions of Multiple Angles

Find all solutions of $2\cos 3t - 1 = 0$.

Solution

$$2\cos 3t - 1 = 0 \qquad \text{Write original equation.}$$

$$2\cos 3t = 1 \qquad \text{Add 1 to each side.}$$

$$\cos 3t = \frac{1}{2} \qquad \text{Divide each side by 2.}$$

In the interval $[0, 2\pi)$, you know that $3t = \pi/3$ and $3t = 5\pi/3$ are the only solutions. So in general, you have

$$3t = \frac{\pi}{3} + 2n\pi \quad \text{and} \quad 3t = \frac{5\pi}{3} + 2n\pi.$$

Dividing this result by 3, you obtain the general solution

$$t = \frac{\pi}{9} + \frac{2n\pi}{3} \quad \text{and} \quad t = \frac{5\pi}{9} + \frac{2n\pi}{3} \qquad \text{General solution}$$

where n is an integer. This is confirmed graphically in Figure 6.16.

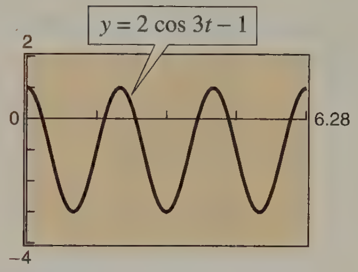


Figure 6.16

EXAMPLE 9 Functions of Multiple Angles

Find all solutions of $3\tan\left(\frac{x}{2}\right) + 3 = 0$.

Solution

$$3\tan\frac{x}{2} + 3 = 0 \qquad \text{Write original equation.}$$

$$3\tan\frac{x}{2} = -3 \qquad \text{Subtract 3 from each side.}$$

$$\tan\frac{x}{2} = -1 \qquad \text{Divide each side by 3.}$$

In the interval $[0, \pi)$, you know that $x/2 = 3\pi/4$ is the only solution. So in general, you have

$$\frac{x}{2} = \frac{3\pi}{4} + n\pi.$$

Multiplying this result by 2, you obtain the general solution

$$x = \frac{3\pi}{2} + 2n\pi \qquad \text{General solution}$$

where n is an integer. This is confirmed graphically in Figure 6.17.

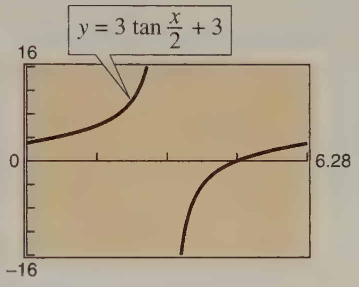


Figure 6.17

Using Inverse Functions

EXAMPLE 10 Using Inverse Functions

Find all solutions of $\sec^2 x - 2\tan x = 4$.

Solution

$$\begin{aligned} \sec^2 x - 2\tan x &= 4 && \text{Write original equation.} \\ 1 + \tan^2 x - 2\tan x - 4 &= 0 && \text{Pythagorean identity} \\ \tan^2 x - 2\tan x - 3 &= 0 && \text{Combine like terms.} \\ (\tan x - 3)(\tan x + 1) &= 0 && \text{Factor.} \end{aligned}$$

Setting each factor equal to zero, you obtain two solutions in the interval $(-\pi/2, \pi/2)$. [Recall that the range of the inverse tangent function is $(-\pi/2, \pi/2)$.]

$$\begin{aligned} \tan x &= 3, & \qquad \tan x &= -1 \\ x &= \arctan 3 & x &= \arctan(-1) \\ & & &= -\frac{\pi}{4} \end{aligned}$$

Finally, by adding multiples of π, you obtain the general solution

$$x = \arctan 3 + n\pi \quad \text{and} \quad x = -\frac{\pi}{4} + n\pi \qquad \text{General solution}$$

where n is an integer. This is confirmed graphically in Figure 6.18.

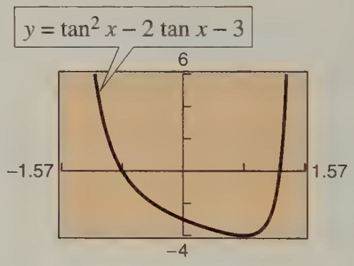


Figure 6.18

With some trigonometric equations, there is no reasonable way to find the solutions algebraically. In such cases, you can still use a graphing utility to approximate the solutions.

EXAMPLE 11 Approximating Solutions

Approximate the solutions of $x = 2\sin x$.

Solution

The graph of $y = x - 2\sin x$ is shown in Figure 6.19. From the graph, there appear to be three solutions. Using the *zero* or *root* feature or the *zoom* and *trace* features of a graphing utility, you can see that one solution is $x = 0$, and you can approximate the positive solution to be $x \approx 1.8955$. Finally, by symmetry, you can approximate the negative solution to be $x \approx -1.8955$.

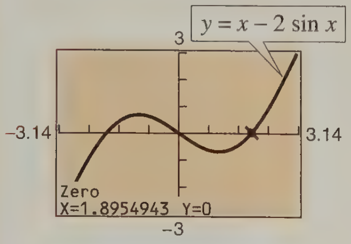


Figure 6.19

EXAMPLE 12 Surface Area of a Honeycomb

The surface area of a honeycomb is given by the equation

$$S = 6hs + \frac{3}{2}s^2\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right), \qquad 0 < \theta \le 90^\circ$$

where $h = 2.4$ inches, $s = 0.75$ inch, and θ is the angle indicated in Figure 6.20.

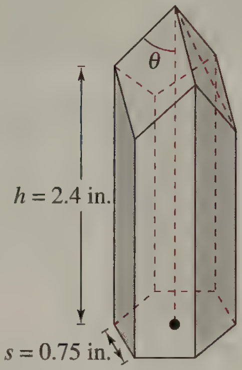


Figure 6.20

a. What value of θ gives a surface area of 12 square inches?

b. What value of θ gives the minimum surface area?

Solution

a. Let $h = 2.4$, $s = 0.75$, and $S = 12$.

$$S = 6hs + \frac{3}{2}s^2\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right)$$

$$12 = 6(2.4)(0.75) + \frac{3}{2}(0.75)^2\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right)$$

$$12 = 10.8 + 0.84375\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right)$$

$$0 = 0.84375\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right) - 1.2$$

Using a graphing utility set to *degree* mode, you can graph the function

$$y = 0.84375\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right) - 1.2$$

as shown in Figure 6.21. Using the *zero* or *root* feature or the *zoom* and *trace* features you can determine that $\theta \approx 59.9^\circ$ and $\theta \approx 49.9^\circ$.

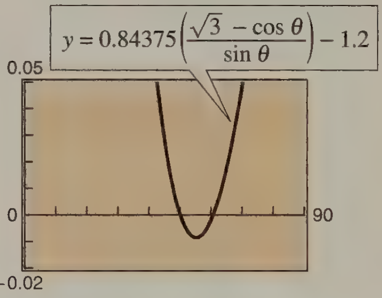


Figure 6.21

b. From part (a), let $h = 2.4$, $s = 0.75$, and $S = 12$ to obtain

$$S = 10.8 + 0.84375\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right).$$

The graph of this function is shown in Figure 6.22. Use the *minimum* feature or the *zoom* and *trace* features of a graphing utility to approximate the minimum point on the graph, which occurs at $\theta \approx 54.7^\circ$. By using calculus, it can be shown that

$$\theta = \arccos\left(\frac{1}{\sqrt{3}}\right) \approx 54.7356^\circ$$

is the exact minimum value.

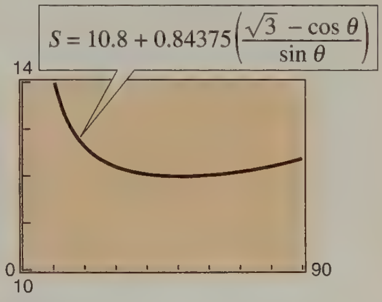


Figure 6.22

6.3 Exercises

In Exercises 1–6, verify that the x-values are solutions of the equation.

1. $2\cos x - 1 = 0$

(a) $x = \frac{\pi}{3}$ (b) $x = \frac{5\pi}{3}$

2. $\csc x - 2 = 0$

(a) $x = \frac{\pi}{6}$ (b) $x = \frac{5\pi}{6}$

3. $3\tan^2 2x - 1 = 0$

(a) $x = \frac{\pi}{12}$ (b) $x = \frac{5\pi}{12}$

4. $4\cos^2 2x - 2 = 0$

(a) $x = \frac{\pi}{8}$ (b) $x = \frac{7\pi}{8}$

5. $2\cos^2 x + 3\cos x + 1 = 0$

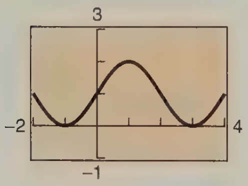
(a) $x = \frac{4\pi}{3}$ (b) $x = \pi$

6. $\sec^4 x - 3\sec^2 x - 4 = 0$

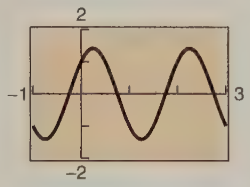
(a) $x = \frac{2\pi}{3}$ (b) $x = \frac{5\pi}{3}$

In Exercises 7–10, find the x-intercepts of the graph. Use a graphing utility to check your solutions.

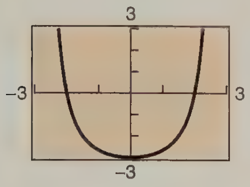
7. $y = \sin\frac{\pi x}{2} + 1$



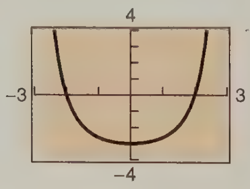
8. $y = \sin \pi x + \cos \pi x$



9. $y = \tan^2\left(\frac{\pi x}{6}\right) - 3$



10. $y = \sec^4\left(\frac{\pi x}{8}\right) - 4$



In Exercises 11–24, solve the equation for $0 \le x < 2\pi$.

11. $2\cos x + 1 = 0$

12. $\sqrt{2}\sin x + 1 = 0$

13. $\sqrt{3}\sec x - 2 = 0$

14. $\cot x + 1 = 0$

15. $3\csc^2 x - 4 = 0$

16. $\csc^2 x - 2 = 0$

17. $2\sin^2 2x = 1$

18. $\tan^2 3x = 3$

19. $4\cos^2 x - 3 = 0$

20. $\cos x(\cos x - 1) = 0$

21. $\sin^2 x = 3\cos^2 x$

22. $\tan 3x(\tan x - 1) = 0$

23. $(3\tan^2 x - 1)(\tan^2 x - 3) = 0$

24. $\cos 2x(2\cos x + 1) = 0$

In Exercises 25–42, find all solutions of the equation in the interval $[0, 2\pi)$ algebraically. Use the *table* feature of a graphing utility to confirm your answers numerically.

25. $\cos^3 x = \cos x$

26. $\tan^2 x - 1 = 0$

27. $3\tan^3 x = \tan x$

28. $2\sin^2 x = 2 + \cos x$

29. $\sec^2 x - \sec x = 2$

30. $\sec x \csc x = 2\csc x$

31. $2\sin x + \csc x = 0$

32. $\sin 2x = -\frac{\sqrt{3}}{2}$

33. $\csc x + \cot x = 1$

34. $\tan 3x = 1$

35. $\cos\frac{x}{2} = \frac{\sqrt{2}}{2}$

36. $\sec 4x = 2$

37. $\frac{1 + \cos x}{1 - \cos x} = 0$

38. $2\sin^2 x + 3\sin x + 1 = 0$

39. $2\sec^2 x + \tan^2 x - 3 = 0$

40. $\cos x + \sin x \tan x = 2$

41. $\sec^2 x + \tan x = 3$

42. $\csc^2 x - 4\cot x = -2$

In Exercises 43–48, use a graphing utility to approximate the solutions of the equation in the interval $[0, 2\pi)$ by setting the equation equal to 0, graphing the new equation, and using the *zero* or *root* feature to approximate the x-intercepts of the graph.

43. $9\cos x + 2 = 3$

44. $4\sin x = \cos x - 2$

45. $4\sin^2 x = 2\cos x + 1$

46. $\csc^2 x = 3\csc x + 4$

47. $4\sin^3 x + 2\sin^2 x = 2\sin x + 1$

48. $\frac{1 + \sin x}{\cos x} + \frac{\cos x}{1 + \sin x} = 4$

In Exercises 49–56, use a graphing utility to approximate the solutions of the equation in the interval $[0, 2\pi)$.

49. $\dfrac{\cos x \cot x}{1 - \sin x} = 3$ **50.** $2 \cos x - \sin x = 0$

51. $x \cos x - 1 = 0$ **52.** $2x \sin x - 2 = 0$

53. $\sec^2 x + 0.5 \tan x = 1$

54. $\csc^2 x + 0.5 \cot x = 5$

55. $12 \sin^2 x - 13 \sin x + 3 = 0$

56. $3 \tan^2 x + 4 \tan x - 4 = 0$

In Exercises 57–60, approximate the solutions (to three decimal places) of the equation in the given interval. Use a graphing utility to check your result.

57. $3 \tan^2 x + 5 \tan x - 4 = 0, \quad \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

58. $\cos^2 x - 2 \cos x - 1 = 0, \quad [0, \pi]$

59. $4 \cos^2 x - 2 \sin x + 1 = 0, \quad \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

60. $2 \sec^2 x + \tan x - 6 = 0, \quad \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

In Exercises 61–64, consider the function in the interval $[0, 2\pi)$. (a) Use a graphing utility to complete the table. Then make a conjecture about any unit intervals containing a zero of the function. Give a reason for your answer. (b) Use a graphing utility to graph the function and determine any unit intervals containing a zero of the function. Do the results agree with those of part (a)? Explain. (c) Use a graphing utility to approximate the zero of the function.

x	0	1	2	3	4	5	6
$f(x)$							

61. $f(x) = \cot \dfrac{x}{2} - x$ **62.** $f(x) = x \sec x - 1$

63. $f(x) = \sin^2 x + 2 \sin x - 1$

64. $f(x) = 4 \cos^2 x - 4 \cos x - 1$

In Exercises 65–68, (a) use a graphing utility to graph the function and approximate the maximum and minimum points on the graph in the interval $[0, 2\pi]$, and (b) solve the trigonometric equation and verify that the x-coordinates of the maximum and minimum points of f are among its solutions (the trigonometric equation is found using calculus).

	Function	*Trigonometric Equation*
65.	$f(x) = \sin x + \cos x$	$\cos x - \sin x = 0$
66.	$f(x) = 3 \sin x - 5 \cos x$	$3 \cos x + 5 \sin x = 0$
67.	$f(x) = x \sin 2x$	$2x \cos 2x + \sin 2x = 0$
68.	$f(x) = 2 \sin x + \cos 2x$	$2 \cos x - 4 \sin x \cos x = 0$

Fixed Point **In Exercises 69 and 70, find the smallest positive fixed point of the function f. [A fixed point of a function f is a real number c such that $f(c) = c$.]**

69. $f(x) = \tan \dfrac{\pi x}{4}$ **70.** $f(x) = \cos x$

71. ***Graphical Reasoning*** Consider the function

$$f(x) = \cos \frac{1}{x}$$

and its graph shown below.

(a) What is the domain of the function?

(b) Identify any symmetry or asymptotes of the graph.

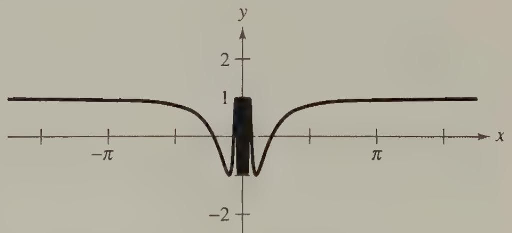
(c) Describe the behavior of the function as $x \to 0$.

(d) How many solutions does the equation

$$\cos \frac{1}{x} = 0$$

have in the interval $[-1, 1]$? Find the solutions.

(e) Does the equation $\cos(1/x) = 0$ have a greatest solution? If so, approximate the solution. If not, explain.



72. ***Graphical Reasoning*** Consider the function

$f(x) = \dfrac{\sin x}{x}$ and its graph on the following page.

(a) What is the domain of the function?

(b) Identify any symmetry or asymptotes of the graph.

(c) Describe the behavior of the function as $x \to 0$.

(d) How many solutions does the equation

$\dfrac{\sin x}{x} = 0$ have in the interval $[-8, 8]$?

Find the solutions.

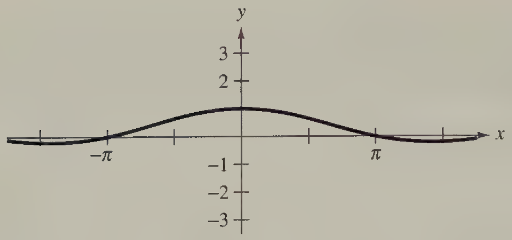
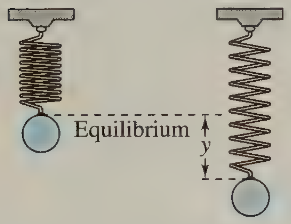


FIGURE FOR 72

73. ***Harmonic Motion*** A weight is oscillating on the end of a spring. The position of the weight relative to the point of equilibrium is

$$y = \frac{1}{12}(\cos 8t - 3\sin 8t)$$

where y is the displacement (in meters) and t is the time (in seconds). Find the times when the weight is at the point of equilibrium ($y = 0$) for $0 \le t \le 1$.



74. ***Sales*** The monthly sales (in thousands of units) of a seasonal product are approximated by

$$S = 74.50 + 43.75 \sin \frac{\pi t}{6}$$

where t is the time (in months), with $t = 1$ corresponding to January. Determine the months when sales exceed 100,000 units.

75. ***Position of the Sun*** Cheyenne, Wyoming, has a latitude of 41° N. At this latitude, the position of the sun at sunrise can be modeled by

$$D = 31 \sin\left(\frac{2\pi}{365}t - 1.4\right)$$

where t is the time (in days) and $t = 1$ represents January 1. In this model, D represents the number of degrees north or south of due east that the sun rises. Use a graphing calculator to determine the days that the sun is more than 20° north of due east at sunrise.

76. ***Projectile Motion*** A batted baseball leaves the bat at an angle of θ with the horizontal and an initial velocity of $v_0 = 100$ feet per second. The ball is caught by an outfielder 300 feet from home plate. Find θ if the range r of a projectile is

$$r = \frac{1}{32}v_0^2 \sin 2\theta.$$

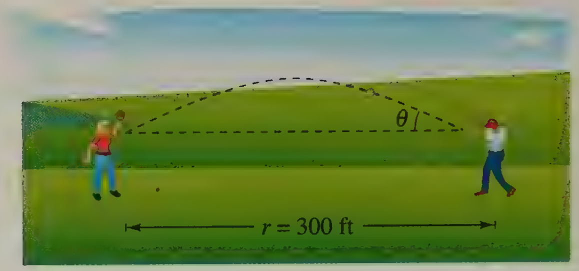
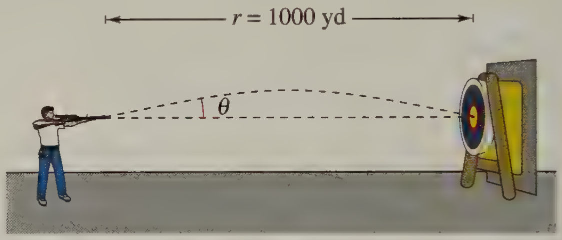


FIGURE FOR 76

77. ***Projectile Motion*** A sharpshooter intends to hit a target 1000 yards away with a gun that has a muzzle velocity of 1200 feet per second. Neglecting air resistance, determine the minimum angle of elevation of the gun if the range is $r = \frac{1}{32}v_0^2 \sin 2\theta$.

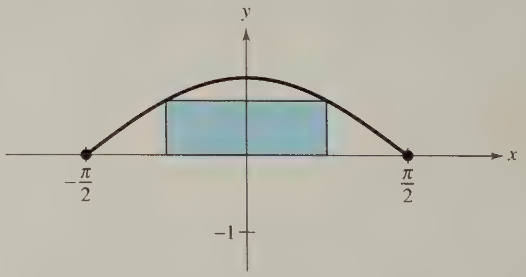


78. ***Area*** The area of a rectangle inscribed in one arch of the graph of $y = \cos x$ is

$$A = 2x\cos x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

(a) Use a graphing utility to graph the area function, and approximate the area of the largest inscribed rectangle.

(b) Determine the values of x for which $A \ge 1$.



79. ***Damped Harmonic Motion*** The displacement from equilibrium of a weight oscillating on the end of a spring is $y = 1.56e^{-0.22t}\cos 4.9t$, where y is the displacement (in feet) and t is the time (in seconds). Use a graphing utility to graph the displacement function for $0 \le t \le 10$. Find the time beyond which the displacement does not exceed 1 foot from equilibrium.

80. ***Data Analysis*** The table gives the unemployment rate r for the years 1989 through 1996 in the United States. The time t is measured in years, with $t = 0$ corresponding to 1990. (Source: U.S. Bureau of Labor Statistics)

t	-1	0	1	2
r	5.3	5.6	6.8	7.5

t	3	4	5	6
r	6.9	6.1	5.6	5.4

(a) Create a scatter plot of the data.

(b) Which of the following models best represents the data? Explain your reasoning.

(i) $r = 1.10 \cos(t + 0.93) + 6.35$

(ii) $r = 1.02 \sin(0.98t + 5.12) + 6.26$

(iii) $r = 1.04 \sin[0.98(t + 5.81)] + 6.34$

(iv) $r = 1.20 \sin[0.95(t + 5.48)] + 6.20$

(c) What term in the model gives the average unemployment rate? What is the rate?

(d) Economists study the lengths of business cycles, such as cycles in the unemployment rate. Based on this short span of time, use the model to give the length of this cycle.

(e) Use the model to estimate the next time the unemployment rate will be 5.5% or less.

81. ***Quadratic Approximation*** Consider the function

$f(x) = 3 \sin(0.6x - 2)$.

(a) Approximate the zero of the function in the interval $[0, 6]$.

(b) A quadratic approximation agreeing with f at $x = 5$ is

$g(x) = -0.45x^2 + 5.52x - 13.70$.

Use a graphing utility to graph f and g in the same viewing window. Describe the result.

(c) Use the Quadratic Formula to find the zeros of g. Compare the zero in the interval $[0, 6]$ with the result of part (a).

Synthesis

True or False? **In Exercises 82 and 83, determine whether the statement is true or false. Justify your answer.**

82. All trigonometric equations have either an infinite number of solutions or no solution.

83. The solutions of any trigonometric equation can always be found from its solutions in the interval $[0, 2\pi)$.

84. ***Writing*** Describe the difference between verifying an identity and solving an equation.

Review

In Exercises 85–88, convert the angle measure from degrees to radians. Round your result to three decimal places.

85. 124°

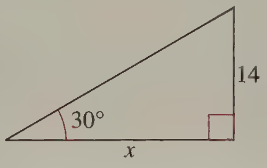
86. 486°

87. −0.41°

88. −210.55°

In Exercises 89–92, solve for x.

89.



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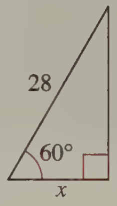

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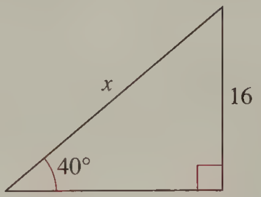
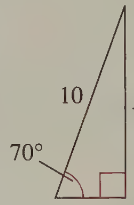


92.



In Exercises 93–96, sketch the graph of the function.

93. $f(x) = \frac{1}{4} \sin\left(x - \frac{\pi}{2}\right)$

94. $f(x) = -6 \cos \frac{\pi x}{4}$

95. $f(x) = \frac{1}{2} \cot\left(x - \frac{\pi}{4}\right)$

96. $f(x) = \frac{1}{2} \sec(2x + \pi)$

6.4 Sum and Difference Formulas

Using Sum and Difference Formulas

In this and the following section, you will study the uses of several trigonometric identities and formulas. For proofs of these formulas see Appendix A.

Sum and Difference Formulas

$$\sin(u+v) = \sin u \cos v + \cos u \sin v \qquad \tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$

$$\cos(u+v) = \cos u \cos v - \sin u \sin v \qquad \tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

Exploration

Use a graphing utility to graph $y = \cos(x+2)$ and $y = \cos x + \cos 2$ in the same viewing window. What can you conclude about the graphs? Is it true that $\cos(x+2) = \cos x + \cos 2$?

Use a graphing utility to graph $y = \sin(x+4)$ and $y = \sin x + \sin 4$ in the same viewing window. What can you conclude about the graphs? Is it true that $\sin(x+4) = \sin x + \sin 4$?

Examples 1 and 2 show how sum and difference formulas can be used to find exact values of trigonometric functions involving sums or differences of special angles.

EXAMPLE 1 Evaluating a Trigonometric Function

Find the exact value of cos 75°.

Solution

To find the exact value of cos 75°, use the fact that $75° = 30° + 45°$. Consequently, the formula for $\cos(u+v)$ yields

$$\begin{aligned}\cos 75° &= \cos(30° + 45°)\\ &= \cos 30° \cos 45° - \sin 30° \sin 45°\\ &= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}.\end{aligned}$$

Try checking this result on your calculator. You will find that $\cos 75° \approx 0.259$.

What You Should Learn:

- How to use sum and difference formulas to evaluate trigonometric functions
- How to use sum and difference formulas to verify identities and solve trigonometric equations

Why You Should Learn It:

You can use sum and difference formulas to rewrite trigonometric expressions. For instance, Exercise 65 on page 487 shows how to use sum and difference formulas to rewrite a trigonometric expression in a form that helps you find the equation of a standing wave.

Richard Megna/Fundemental Photographs

EXAMPLE 2 Evaluating a Trigonometric Function

Find the exact value of $\sin \dfrac{\pi}{12}$.

Solution

Using the fact that $\pi/12 = \pi/3 - \pi/4$ together with the formula for $\sin(u - v)$, you obtain

$$\begin{aligned}\sin \frac{\pi}{12} &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4} \\ &= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}.\end{aligned}$$

The Granger Collection, New York

Hipparchus, considered the most eminent of Greek astronomers, was born about 160 B.C. in Nicea. He is credited with the invention of trigonometry. He also derived the sum and difference formulas for $\sin(A \pm B)$ and $\cos(A \pm B)$.

EXAMPLE 3 Evaluating a Trigonometric Expression

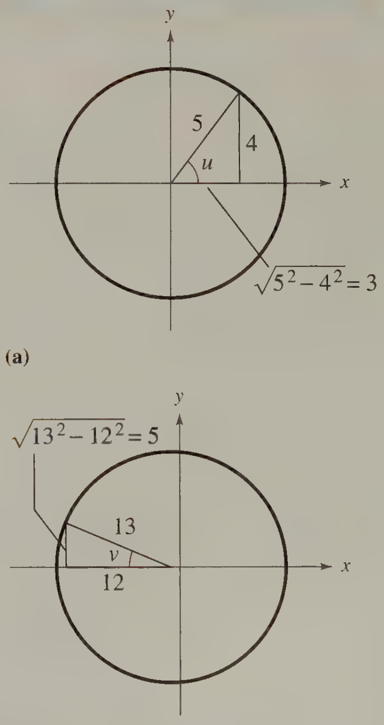
Find the exact value of $\sin(u + v)$ given that

$$\sin u = \frac{4}{5}, \text{ where } 0 < u < \frac{\pi}{2} \quad \text{and} \quad \cos v = -\frac{12}{13}, \text{ where } \frac{\pi}{2} < v < \pi.$$

Solution

Because $\sin u = 4/5$ and u is in Quadrant I, $\cos u = 3/5$, as shown in Figure 6.23(a). Because $\cos v = -12/13$ and v is in Quadrant II, $\sin v = 5/13$, as shown in Figure 6.23(b). You can find $\sin(u + v)$ as follows.

$$\begin{aligned}\sin(u + v) &= \sin u \cos v + \cos u \sin v \\ &= \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) = -\frac{48}{65} + \frac{15}{65} = -\frac{33}{65}\end{aligned}$$



(a)

(b)

Figure 6.23

EXAMPLE 4 An Application of a Sum Formula

Evaluate $\cos(\arctan 1 + \arccos x)$.

Solution

This expression fits the formula for $\cos(u + v)$. Angles $u = \arctan 1$ and $v = \arccos x$ are shown in Figure 6.24.

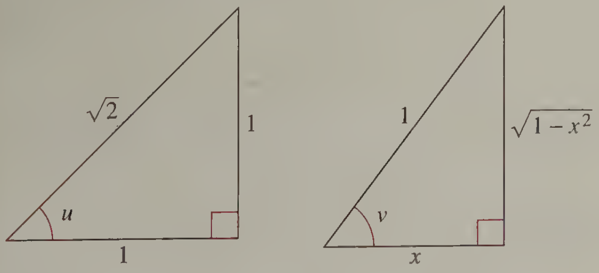


Figure 6.24

$$\begin{aligned}\cos(u + v) &= \cos(\arctan 1)\cos(\arccos x) - \sin(\arctan 1)\sin(\arccos x) \\ &= \frac{1}{\sqrt{2}} \cdot x - \frac{1}{\sqrt{2}} \cdot \sqrt{1 - x^2} = \frac{x - \sqrt{1 - x^2}}{\sqrt{2}}.\end{aligned}$$

EXAMPLE 5 Proving a Cofunction Identity

Prove the cofunction identity $\cos\left(\frac{\pi}{2} - x\right) = \sin x$.

Solution

Using the formula for $\cos(u - v)$, you have

$$\begin{aligned}\cos\left(\frac{\pi}{2} - x\right) &= \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x\\ &= (0)(\cos x) + (1)(\sin x)\\ &= \sin x.\end{aligned}$$

Sum and difference formulas can be used to derive **reduction formulas** involving expressions such as

$$\sin\left(\theta + \frac{n\pi}{2}\right) \quad \text{and} \quad \cos\left(\theta + \frac{n\pi}{2}\right)$$

where n is an integer, as shown in the following example.

EXAMPLE 6 Deriving Reduction Formulas

Simplify each expression.

a. $\cos\left(\theta - \frac{3\pi}{2}\right)$ **b.** $\tan(\theta + 3\pi)$

Solution

a. Using the formula for $\cos(u - v)$, you have

$$\begin{aligned}\cos\left(\theta - \frac{3\pi}{2}\right) &= \cos\theta\cos\frac{3\pi}{2} + \sin\theta\sin\frac{3\pi}{2}\\ &= (\cos\theta)(0) + (\sin\theta)(-1)\\ &= -\sin\theta.\end{aligned}$$

b. Using the formula for $\tan(u + v)$, you have

$$\begin{aligned}\tan(\theta + 3\pi) &= \frac{\tan\theta + \tan 3\pi}{1 - \tan\theta\tan 3\pi}\\ &= \frac{\tan\theta + 0}{1 - (\tan\theta)(0)}\\ &= \tan\theta.\end{aligned}$$

Note that the period of $\tan\theta$ is π, so the period of $\tan(\theta + 3\pi)$ is the same as the period of $\tan\theta$.

The next example was taken from calculus. It is used to derive the formula for the derivative of the sine function.

EXAMPLE 7 An Application from Calculus

Verify that

$$\frac{\sin(x+h)-\sin x}{h} = (\cos x)\left(\frac{\sin h}{h}\right) - (\sin x)\left(\frac{1-\cos h}{h}\right)$$

where $h \neq 0$.

Solution

Using the formula for $\sin(u+v)$, you have

$$\begin{aligned}\frac{\sin(x+h)-\sin x}{h} &= \frac{\sin x\cos h + \cos x\sin h - \sin x}{h}\\ &= \frac{\cos x\sin h - \sin x(1-\cos h)}{h}\\ &= (\cos x)\left(\frac{\sin h}{h}\right) - (\sin x)\left(\frac{1-\cos h}{h}\right).\end{aligned}$$

EXAMPLE 8 Solving a Trigonometric Equation

Find all solutions of $\sin\left(x+\frac{\pi}{4}\right) + \sin\left(x-\frac{\pi}{4}\right) = -1$ in the interval $[0, 2\pi)$.

Algebraic Solution

Using sum and difference formulas, rewrite the equation as

$$\begin{aligned}\sin x\cos\frac{\pi}{4} + \cos x\sin\frac{\pi}{4} + \sin x\cos\frac{\pi}{4} - \cos x\sin\frac{\pi}{4} &= -1\\ 2\sin x\cos\frac{\pi}{4} &= -1\\ 2(\sin x)\left(\frac{\sqrt{2}}{2}\right) &= -1\\ \sin x &= -\frac{1}{\sqrt{2}}\\ \sin x &= -\frac{\sqrt{2}}{2}.\end{aligned}$$

Therefore, the only solutions in the interval $[0, 2\pi)$ are

$$x = \frac{5\pi}{4} \quad \text{and} \quad x = \frac{7\pi}{4}.$$

Graphical Solution

Use a graphing utility set in *radian* mode to graph $y = \sin\left(x+\frac{\pi}{4}\right) + \sin\left(x-\frac{\pi}{4}\right) + 1$, as shown in Figure 6.25. Use the *zero* or *root* feature or the *zoom* and *trace* features to approximate the x-intercepts in the interval $[0, 2\pi)$ to be

$$x \approx 3.927 \approx \frac{5\pi}{4} \text{ and } x \approx 5.498 \approx \frac{7\pi}{4}.$$

These values are the approximate solutions in the interval $[0, 2\pi)$ of

$$\sin\left(x+\frac{\pi}{4}\right) + \sin\left(x-\frac{\pi}{4}\right) = -1.$$

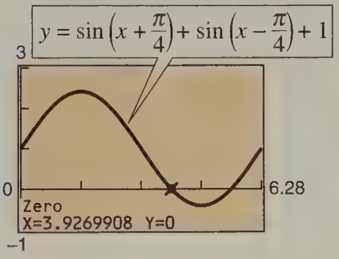


Figure 6.25

6.4 Exercises

In Exercises 1–8, find the exact value of each expression.

1. (a) $\cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right)$ (b) $\cos\frac{\pi}{6} + \cos\frac{\pi}{3}$

2. (a) $\sin\left(\frac{2\pi}{3} + \frac{3\pi}{4}\right)$ (b) $\sin\frac{2\pi}{3} + \sin\frac{3\pi}{4}$

3. (a) $\sin\left(\frac{7\pi}{6} - \frac{\pi}{3}\right)$ (b) $\sin\frac{7\pi}{6} - \sin\frac{\pi}{3}$

4. (a) $\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right)$ (b) $\cos\frac{5\pi}{4} - \cos\frac{\pi}{6}$

5. (a) $\cos(0° + 135°)$ (b) $\cos 0° + \cos 135°$

6. (a) $\cos(240° - 0°)$ (b) $\cos 240° - \cos 0°$

7. (a) $\sin(315° - 60°)$ (b) $\sin 315° - \sin 60°$

8. (a) $\sin(390° + 120°)$ (b) $\sin 390° + \sin 120°$

In Exercises 9–18, use the sum and difference formulas to find the exact values of the sine, cosine, and tangent of the angle.

9. $75° = 30° + 45°$

10. $15° = 45° - 30°$

11. $105° = 60° + 45°$

12. $165° = 135° + 30°$

13. $195° = 225° - 30°$

14. $285° = 330° - 45°$

15. $\frac{11\pi}{12} = \frac{3\pi}{4} + \frac{\pi}{6}$

16. $\frac{17\pi}{12} = \frac{7\pi}{6} + \frac{\pi}{4}$

17. $-\frac{\pi}{12} = \frac{\pi}{6} - \frac{\pi}{4}$

18. $-\frac{19\pi}{12} = \frac{2\pi}{3} - \frac{9\pi}{4}$

In Exercises 19–28, use the sum and difference formulas to write the expression as the sine, cosine, or tangent of an angle.

19. $\cos 40° \cos 15° - \sin 40° \sin 15°$

20. $\sin 110° \cos 80° + \cos 110° \sin 80°$

21. $\sin 340° \cos 50° - \cos 340° \sin 50°$

22. $\cos 20° \cos 30° + \sin 20° \sin 30°$

23. $\dfrac{\tan 325° - \tan 86°}{1 + \tan 325° \tan 86°}$

24. $\dfrac{\tan 140° - \tan 60°}{1 + \tan 140° \tan 60°}$

25. $\sin 3 \cos 1.2 - \cos 3 \sin 1.2$

26. $\cos 0.88 \cos 0.34 + \sin 0.88 \sin 0.34$

27. $\cos\frac{\pi}{7}\cos\frac{\pi}{5} - \sin\frac{\pi}{7}\sin\frac{\pi}{5}$

28. $\sin\frac{2\pi}{9}\cos\frac{\pi}{10} + \cos\frac{2\pi}{9}\sin\frac{\pi}{10}$

Numerical, Graphical, and Algebraic Analysis **In Exercises 29–34, use a graphing utility to complete the table and graph the two functions. Use both table and graph as evidence that $y_1 = y_2$. Then verify the identity algebraically.**

x	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y_1							
y_2							

29. $y_1 = \sin\left(\frac{\pi}{2} + x\right), \quad y_2 = \cos x$

30. $y_1 = \sin(3\pi - x), \quad y_2 = \sin x$

31. $y_1 = \sin\left(\frac{\pi}{6} + x\right), \quad y_2 = \frac{1}{2}\left(\cos x + \sqrt{3}\sin x\right)$

32. $y_1 = \cos\left(\frac{5\pi}{4} - x\right), \quad y_2 = -\frac{\sqrt{2}}{2}(\cos x + \sin x)$

33. $y_1 = \cos(x + \pi)\cos(x - \pi), \quad y_2 = \cos^2 x$

34. $y_1 = \sin(x + \pi)\sin(x - \pi), \quad y_2 = \sin^2 x$

In Exercises 35–38, find the exact value of the trigonometric function given that

$\sin u = 5/13$, where $0 < u < \pi/2$

$\cos v = -3/5$, where $\pi/2 < v < \pi$.

35. $\sin(u + v)$

36. $\cos(v - u)$

37. $\cos(u + v)$

38. $\sin(u - v)$

In Exercises 39–42, find the exact value of the trigonometric function given that

$\sin u = 7/25$, where $\pi/2 < u < \pi$

$\cos v = 4/5$, where $3\pi/2 < v < 2\pi$.

39. $\cos(u + v)$

40. $\sin(u + v)$

41. $\sin(v - u)$

42. $\cos(u - v)$

In Exercises 43–50, verify the identity.

43. $\cos(\pi - \theta) + \sin\left(\frac{\pi}{2} + \theta\right) = 0$

44. $\sin(\theta + \pi) + \cos\left(\theta - \frac{\pi}{2}\right) = 0$

45. $\tan(x + \pi) - \tan(\pi - x) = 2 \tan x$

46. $\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1 - \tan \theta}{1 + \tan \theta}$

47. $\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$

48. $\cos(x + y) + \cos(x - y) = 2 \cos x \cos y$

49. $\cos(x + y) \cos(x - y) = \cos^2 x - \sin^2 y$

50. $\sin(x + y) \sin(x - y) = \sin^2 x - \sin^2 y$

In Exercises 51–54, write the trigonometric expression as an algebraic expression.

51. $\sin(\arcsin x + \arccos x)$

52. $\cos(\arccos x - \arcsin x)$

53. $\sin(\arctan 2x - \arccos x)$

54. $\cos(\arcsin x - \arctan 2x)$

In Exercises 55–60, find all solutions of the equation in the interval $[0, 2\pi)$. Use a graphing utility to verify your results.

55. $\sin\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{3}\right) = 1$

56. $\sin\left(x + \frac{\pi}{6}\right) - \sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$

57. $\cos\left(x + \frac{\pi}{4}\right) - \cos\left(x - \frac{\pi}{4}\right) = 1$

58. $\cos\left(x + \frac{\pi}{6}\right) - \cos\left(x - \frac{\pi}{6}\right) = 1$

59. $\tan(x + \pi) + 2 \sin(x + \pi) = 0$

60. $2 \sin\left(x + \frac{\pi}{2}\right) + 3 \tan(\pi - x) = 0$

In Exercises 61–64, use a graphing utility to approximate all solutions of the equation in the interval $[0, 2\pi)$.

61. $\cos\left(x + \frac{\pi}{4}\right) + \cos\left(x - \frac{\pi}{4}\right) = 1$

62. $\sin\left(x + \frac{\pi}{2}\right) - \cos\left(x + \frac{3\pi}{2}\right) = 0$

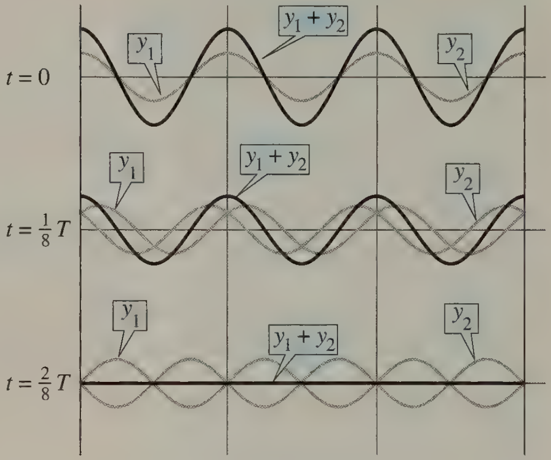
63. $\tan(x + \pi) - \cos\left(x + \frac{\pi}{2}\right) = 0$

64. $\tan(\pi - x) + 2 \cos\left(x + \frac{3\pi}{2}\right) = 0$

65. ***Standing Waves*** The equation of a standing wave is obtained by adding the displacements of two waves traveling in opposite directions. Assume that each of the waves has amplitude A, period T, and wavelength λ. If the models for these waves are y_1 and y_2 given below, show that

$$y_1 + y_2 = 2A \cos \frac{2\pi t}{T} \cos \frac{2\pi x}{\lambda}.$$

$$y_1 = A \cos 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right), \quad y_2 = A \cos 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$



66. ***Harmonic Motion*** A weight is attached to a spring suspended vertically from a ceiling. When a driving force is applied to the system, the weight moves vertically from its equilibrium position, and this motion is modeled by

$$y = \tfrac{1}{3} \sin 2t + \tfrac{1}{4} \cos 2t$$

where y is the distance from equilibrium (in feet) and t is the time (in seconds).

(a) Use a graphing utility to graph the model.

(b) Write the model in the form

$$y = \sqrt{a^2 + b^2} \sin(Bt + C).$$

Use a graphing utility to verify your result.

(c) Find the amplitude of the oscillations of the weight.

(d) Find the frequency of the oscillations of the weight.

Synthesis

True or False? **In Exercises 67–70, determine whether the statement is true or false. Justify your answer.**

67. $\sin(u \pm v) = \sin u \pm \sin v$

68. $\cos(u \pm v) = \cos u \pm \cos v$

69. It is not possible to find the exact value of sin 75°.

70. $\sin\left(x - \frac{11\pi}{2}\right) = \cos x$

In Exercises 71–74, verify the identity.

71. $\cos(n\pi + \theta) = (-1)^n \cos \theta$, n is an integer.

72. $\sin(n\pi + \theta) = (-1)^n \sin \theta$, n is an integer.

73. $a \sin B\theta + b \cos B\theta = \sqrt{a^2 + b^2} \sin(B\theta + C)$ where $C = \arctan(b/a)$ and $a > 0$.

74. $a \sin B\theta + b \cos B\theta = \sqrt{a^2 + b^2} \cos(B\theta - C)$ where $C = \arctan(a/b)$ and $b > 0$.

In Exercises 75–78, use the formulas given in Exercises 73 and 74 to write the expression in the following forms. Use a graphing utility to verify your results.

(a) $\sqrt{a^2 + b^2}\sin(B\theta + C)$

(b) $\sqrt{a^2 + b^2}\cos(B\theta - C)$

75. $\sin \theta + \cos \theta$

76. $3 \sin 2\theta + 4 \cos 2\theta$

77. $12 \sin 3\theta + 5 \cos 3\theta$

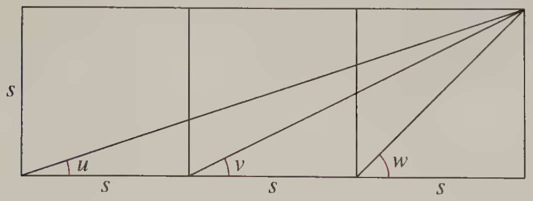
78. $\sin 2\theta - \cos 2\theta$

In Exercises 79 and 80, use the formulas given in Exercises 73 and 74 to write the trigonometric expression in the form $a \sin B\theta + b \cos B\theta$.

79. $2 \sin\left(\theta + \frac{\pi}{2}\right)$

80. $5 \cos\left(\theta + \frac{\pi}{4}\right)$

81. *Conjecture* Three squares of side s are placed side by side. Make a conjecture about the relationship between the sum $u + v$ and w. Prove your conjecture by using the identity for the tangent of the sum of two angles.



Conjecture **In Exercises 82 and 83, use a graphing utility to graph the trigonometric function. Use the graph to make a conjecture about a simplification of the expression. Verify the resulting identity algebraically.**

82. $g(x) = \cos(\pi + x)$

83. $h(\theta) = \tan(\pi + \theta)$

84. Verify the identity used in calculus.

$$\frac{\cos(x+h) - \cos x}{h} = \frac{\cos x(\cos h - 1)}{h} - \frac{\sin x \sin h}{h}$$

85. *Exploration* Let $x = \pi/6$ in the identity in Exercise 84 and define the functions f and g as follows.

$$f(h) = \frac{\cos(\pi/6 + h) - \cos(\pi/6)}{h}$$

$$g(h) = \cos\frac{\pi}{6}\left(\frac{\cos h - 1}{h}\right) - \sin\frac{\pi}{6}\left(\frac{\sin h}{h}\right)$$

(a) What are the domains of these functions?

(b) Use a graphing utility to complete the table.

h	0.01	0.02	0.05	0.1	0.2	0.5
$f(h)$						
$g(h)$						

(c) Use a graphing utility to graph the functions.

(d) Use the table and graph to make a conjecture about the values of the functions as $h \to 0$.

86. Use the sum formulas for sine and cosine to derive the formula

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}.$$

Review

In Exercises 87–90, find the x- and y-intercepts of the graph of the equation. Use a graphing utility to verify your results.

87. $y = -\frac{1}{2}(x - 10) + 14$

88. $y = x^2 - 3x - 40$

89. $y = |2x - 9| - 5$

90. $y = 2x\sqrt{x + 7}$

In Exercises 91–94, evaluate the expression without using a calculator.

91. $\arccos\left(\frac{\sqrt{3}}{2}\right)$

92. $\arctan(-\sqrt{3})$

93. $\arcsin 1$

94. $\arctan 0$

6.5 Multiple-Angle and Product-Sum Formulas

Multiple-Angle Formulas

In this section you will study four other categories of trigonometric identities.

1. The first category involves functions of multiple angles such as $\sin ku$ and $\cos ku$.
2. The second category involves squares of trigonometric functions such as $\sin^2 u$.
3. The third category involves functions of half-angles such as $\sin(u/2)$.
4. The fourth category involves products of trigonometric functions such as $\sin u \cos v$.

The most commonly used multiple-angle formulas are the **double-angle formulas.** They are used often, so you should learn them. (Proofs of the double-angle formulas are given in Appendix A.)

Double-Angle Formulas

$$\sin 2u = 2 \sin u \cos u \qquad \tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

$$\begin{aligned} \cos 2u &= \cos^2 u - \sin^2 u \\ &= 2\cos^2 u - 1 \\ &= 1 - 2\sin^2 u \end{aligned}$$

Note that $\sin 2u \neq 2 \sin u$, $\cos 2u \neq 2 \cos u$, and $\tan 2u \neq 2 \tan u$.

EXAMPLE 1 Solving a Multiple-Angle Equation

Find all solutions of $2 \cos x + \sin 2x = 0$.

Solution

Begin by rewriting the equation so that it involves functions of x (rather than $2x$). Then factor and solve as usual.

$$2 \cos x + \sin 2x = 0 \qquad \text{Write original equation.}$$

$$2 \cos x + 2 \sin x \cos x = 0 \qquad \text{Double-angle formula}$$

$$2 \cos x(1 + \sin x) = 0 \qquad \text{Factor.}$$

$$\cos x = 0, \quad 1 + \sin x = 0 \qquad \text{Set factors equal to zero.}$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = \frac{3\pi}{2} \qquad \text{Solutions in } [0, 2\pi)$$

Therefore, the general solution is

$$x = \frac{\pi}{2} + 2n\pi \quad \text{and} \quad x = \frac{3\pi}{2} + 2n\pi \qquad \text{General solution}$$

where n is an integer. Try verifying these solutions graphically.

What You Should Learn:

- How to use multiple-angle formulas to rewrite and evaluate trigonometric functions
- How to use power-reducing formulas to rewrite and evaluate trigonometric functions
- How to use half-angle formulas to rewrite and evaluate trigonometric functions
- How to use product-sum formulas to rewrite and evaluate trigonometric functions

Why You Should Learn It:

You can use a variety of trigonometric formulas to rewrite trigonometric functions in more convenient forms. For instance, Exercise 105 on page 498 shows you how to use a half-angle formula to determine the apex angle of a sound wave cone from the speed of an airplane.

NASA-Liaison Agency

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 2 Using Double-Angle Formulas in Sketching Graphs

Analyze the graph of $y = 4\cos^2 x - 2$ over the interval $[0, 2\pi]$.

Solution

Using a double-angle formula, you can rewrite the given function as

$$\begin{aligned} y &= 4\cos^2 x - 2 \\ &= 2(2\cos^2 x - 1) \\ &= 2\cos 2x. \end{aligned}$$

Using the techniques discussed in Section 5.4, you can recognize that the graph of this function has an amplitude of 2 and a period of π. The key points in the interval $[0, \pi]$ are as follows.

Maximum	*Intercept*	*Minimum*	*Intercept*	*Maximum*
$(0, 2)$	$\left(\frac{\pi}{4}, 0\right)$	$\left(\frac{\pi}{2}, -2\right)$	$\left(\frac{3\pi}{4}, 0\right)$	$(\pi, 2)$

Two cycles of the graph are shown in Figure 6.26.

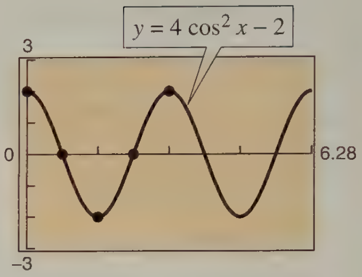


Figure 6.26

EXAMPLE 3 Evaluating Functions Involving Double Angles

Use the following to find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

$$\cos\theta = \frac{5}{13}, \qquad \frac{3\pi}{2} < \theta < 2\pi$$

Solution

From Figure 6.27, you can see that

$$\sin\theta = \frac{y}{r} = -\frac{12}{13}.$$

Consequently, you can write

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{12}{13}\right)\left(\frac{5}{13}\right) = -\frac{120}{169}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{25}{169}\right) - 1 = -\frac{119}{169}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{120}{119}.$$

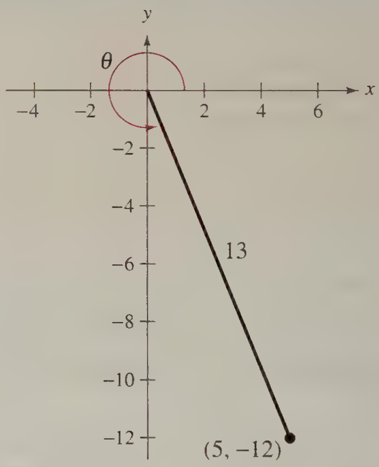


Figure 6.27

The double-angle formulas are not restricted to angles 2θ and θ. Other *double* combinations, such as 4θ and 2θ or 6θ and 3θ, are also valid. Here are two examples.

$$\sin 4\theta = 2\sin 2\theta\cos 2\theta \qquad \text{and} \qquad \cos 6\theta = \cos^2 3\theta - \sin^2 3\theta$$

By using double-angle formulas together with the sum formulas derived in the previous section, you can form other multiple-angle formulas.

STUDY T!P

Notice that you cannot solve Example 3 by simply evaluating $\theta = \cos^{-1}(5/13)$ and then calculating $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$. Do you see why?

EXAMPLE 4 Deriving a Triple-Angle Formula

Express $\sin 3x$ in terms of $\sin x$.

Solution

$$\begin{aligned}
\sin 3x &= \sin(2x + x) \\
&= \sin 2x \cos x + \cos 2x \sin x && \text{Sum formula} \\
&= 2 \sin x \cos x \cos x + (1 - 2 \sin^2 x)\sin x && \text{Double-angle formula} \\
&= 2 \sin x \cos^2 x + \sin x - 2 \sin^3 x && \text{Multiply.} \\
&= 2 \sin x(1 - \sin^2 x) + \sin x - 2 \sin^3 x && \text{Pythagorean identity} \\
&= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x && \text{Multiply.} \\
&= 3 \sin x - 4 \sin^3 x && \text{Simplify.}
\end{aligned}$$

Power-Reducing Formulas

The double-angle formulas can be used to obtain the following **power-reducing formulas.** (Proofs of the power-reducing formulas are given in Appendix A.)

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Power-Reducing Formulas

$$\sin^2 u = \frac{1 - \cos 2u}{2} \qquad \cos^2 u = \frac{1 + \cos 2u}{2} \qquad \tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

Example 5 shows a typical power reduction that is used in calculus.

EXAMPLE 5 Reducing a Power

Rewrite $\sin^4 x$ as a sum of first powers of the cosines of multiple angles.

Solution

Note the repeated use of power-reducing formulas.

$$\begin{aligned}
\sin^4 x &= (\sin^2 x)^2 && \text{Property of exponents} \\
&= \left(\frac{1 - \cos 2x}{2}\right)^2 && \text{Power-reducing formula} \\
&= \frac{1}{4}(1 - 2 \cos 2x + \cos^2 2x) && \text{Expand binomial.} \\
&= \frac{1}{4}\left(1 - 2 \cos 2x + \frac{1 + \cos 4x}{2}\right) && \text{Power-reducing formula} \\
&= \frac{1}{4} - \frac{1}{2} \cos 2x + \frac{1}{8} + \frac{1}{8} \cos 4x && \text{Distributive Property} \\
&= \frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x && \text{Simplify.} \\
&= \frac{1}{8}(3 - 4 \cos 2x + \cos 4x) && \text{Factor.}
\end{aligned}$$

Half-Angle Formulas

You can derive some useful alternative forms of the power-reducing formulas by replacing u with $u/2$. The results are called **half-angle formulas.**

Half-Angle Formulas

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The signs of $\sin(u/2)$ and $\cos(u/2)$ depend on the quadrant in which $u/2$ lies.

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 6 Using a Half-Angle Formula

Find the exact value of sin 105°.

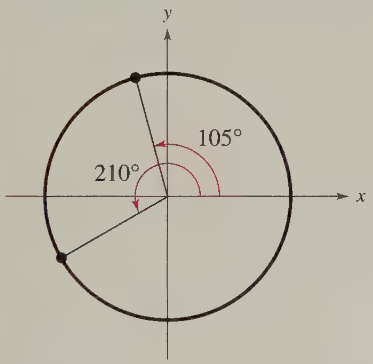


Figure 6.28

Solution

Begin by noting that 105° is half of 210°. Then, using the half-angle formula for $\sin(u/2)$ and the fact that 105° lies in Quadrant II (see Figure 6.28), you have

$$\begin{aligned}\sin 105^\circ &= \sqrt{\frac{1-\cos 210^\circ}{2}}\\ &= \sqrt{\frac{1-(-\cos 30^\circ)}{2}}\\ &= \sqrt{\frac{1+\left(\sqrt{3}/2\right)}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}.\end{aligned}$$

The positive square root is chosen because $\sin\theta$ is positive in Quadrant II.

Use your calculator to verify the result obtained in Example 6. That is, evaluate sin 105° and $\left(\sqrt{2+\sqrt{3}}\right)/2$.

EXAMPLE 7 Solving a Trigonometric Equation

Find all solutions of

$$2 - \sin^2 x = 2\cos^2 \frac{x}{2}$$

in the interval $[0, 2\pi)$.

Algebraic Solution

$$2 - \sin^2 x = 2\cos^2 \frac{x}{2}$$ Write original equation.

$$2 - \sin^2 x = 2\left(\pm\sqrt{\frac{1 + \cos x}{2}}\right)^2$$ Half-angle formula

$$2 - \sin^2 x = 2\left(\frac{1 + \cos x}{2}\right)$$ Simplify.

$$2 - \sin^2 x = 1 + \cos x$$ Simplify.

$$2 - (1 - \cos^2 x) = 1 + \cos x$$ Pythagorean identity

$$\cos^2 x - \cos x = 0$$ Simplify.

$$\cos x(\cos x - 1) = 0$$ Factor.

By setting the factors $\cos x$ and $(\cos x - 1)$ equal to zero, you find that the solutions in the interval $[0, 2\pi)$ are

$$x = \frac{\pi}{2}, \quad x = \frac{3\pi}{2}, \quad \text{and} \quad x = 0.$$

Graphical Solution

Use a graphing utility set in *radian* mode to graph $y = 2 - \sin^2 x - 2\cos^2 (x/2)$, as shown in Figure 6.29. Use the *zero* or *root* feature or the *zoom* and *trace* features to approximate the x-intercepts in the interval $[0, 2\pi)$ to be

$$x = 0, \quad x \approx 1.5708 \approx \frac{\pi}{2}, \quad \text{and} \quad x \approx 4.7124 \approx \frac{3\pi}{2}.$$

These values are the approximate solutions in the interval $[0, 2\pi)$ of $2 - \sin^2 x = \cos^2 \frac{x}{2}$.

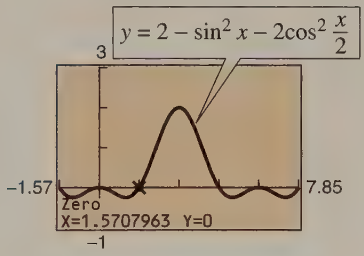


Figure 6.29

Product-to-Sum Formulas

Each of the following **product-to-sum formulas** is easily verified using the sum and difference formulas discussed in the preceding section.

Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$

$$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$$

EXAMPLE 8 Writing Products as Sums

Rewrite the product as a sum or difference.

$\cos 5x \sin 4x$

Solution

$$\begin{aligned}\cos 5x \sin 4x &= \frac{1}{2}[\sin(5x + 4x) - \sin(5x - 4x)]\\ &= \frac{1}{2}\sin 9x - \frac{1}{2}\sin x\end{aligned}$$

Occasionally, it is useful to reverse the procedure and write a sum of trigonometric functions as a product. This can be accomplished with the following **sum-to-product formulas.** (A proof of the first formula is given in Appendix A.)

Sum-to-Product Formulas

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

EXAMPLE 9 Using a Sum-to-Product Formula

Find the exact value of

$\cos 195° + \cos 105°$.

Solution

Using the appropriate sum-to-product formula, you obtain

$$\begin{aligned}\cos 195° + \cos 105° &= 2\cos\left(\frac{195° + 105°}{2}\right)\cos\left(\frac{195° - 105°}{2}\right)\\ &= 2\cos 150° \cos 45°\\ &= 2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= -\frac{\sqrt{6}}{2}.\end{aligned}$$

EXAMPLE 10 Solving a Trigonometric Equation

Find all solutions of $\sin 5x + \sin 3x = 0$ in the interval $[0, 2\pi)$.

Algebraic Solution

$$\sin 5x + \sin 3x = 0 \qquad \text{Write original equation.}$$

$$2\sin\left(\frac{5x + 3x}{2}\right)\cos\left(\frac{5x - 3x}{2}\right) = 0 \qquad \text{Sum-to-product formula}$$

$$2\sin 4x \cos x = 0 \qquad \text{Simplify.}$$

By setting the factor $\sin 4x$ equal to zero, you can find that the solutions in the interval $[0, 2\pi)$ are

$$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}.$$

Moreover, the equation $\cos x = 0$ yields no additional solutions.

Graphical Solution

Use a graphing utility set in *radian* mode to graph $y = \sin 5x + \sin 3x$, as shown in Figure 6.30. Use the *zero* or *root* feature or the *zoom* and *trace* features to approximate the x-intercepts in the interval $[0, 2\pi)$ to be

$$x \approx 0,\ x \approx 0.7854 \approx \frac{\pi}{4},\ x \approx 1.5708 \approx \frac{\pi}{2},$$

$$x \approx 2.3562 \approx \frac{3\pi}{4},\ x \approx 3.1416 \approx \pi,\ x \approx 3.9270 \approx \frac{5\pi}{4},$$

$$x \approx 4.7124 \approx \frac{3\pi}{2},\ x \approx 5.4978 \approx \frac{7\pi}{4}.$$

These values are the approximate solutions in the interval $[0, 2\pi)$ of $\sin 5x + \sin 3x = 0$.

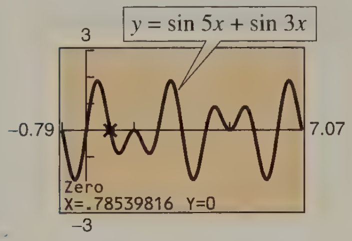


Figure 6.30

Note in Example 10 that the general solution would be $x = \frac{n\pi}{4}$, where n is an integer.

EXAMPLE 11 Verifying a Trigonometric Identity

Verify the identity $\dfrac{\sin t + \sin 3t}{\cos t + \cos 3t} = \tan 2t$.

Solution

Using appropriate sum-to-product formulas, you have

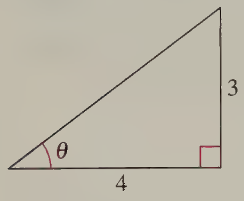
$$\frac{\sin t + \sin 3t}{\cos t + \cos 3t} = \frac{2\sin 2t\cos(-t)}{2\cos 2t\cos(-t)} = \frac{\sin 2t}{\cos 2t} = \tan 2t.$$

Writing About Math *Deriving an Area Formula*

Describe how you can use a double-angle formula or a half-angle formula to derive a formula for the area of an isosceles triangle. Use a labeled sketch to illustrate your derivation. Then write two examples that show how your formula can be used.

6.5 Exercises

In Exercises 1–8, use the figure to find the exact value of the trigonometric function.



1. $\sin \theta$

2. $\tan \theta$

3. $\cos 2\theta$

4. $\sin 2\theta$

5. $\tan 2\theta$

6. $\sec 2\theta$

7. $\csc 2\theta$

8. $\cot 2\theta$

In Exercises 9–16, use a graphing utility to approximate the equation's solutions in the interval $[0, 2\pi)$. If possible, find the exact solutions algebraically.

9. $\sin 2x - \sin x = 0$

10. $\sin 2x + \cos x = 0$

11. $8 \sin x \cos x = 1$

12. $\sin 2x \sin x = \cos x$

13. $\cos 2x + \cos x = 0$

14. $\tan 2x - \cot x = 0$

15. $\sin 4x = -2 \sin 2x$

16. $(\sin 2x + \cos 2x)^2 = 1$

In Exercises 17–20, use a double-angle formula to rewrite the expression. Use a graphing utility to graph both expressions to verify that both forms are the same.

17. $8 \sin x \cos x$

18. $6 \sin x \cos x + 4$

19. $5 - 10 \sin^2 x$

20. $(\cos x + \sin x)(\cos x - \sin x)$

In Exercises 21–24, find the exact values of $\sin 2u$, $\cos 2u$, and $\tan 2u$ using the double-angle formulas.

21. $\sin u = \frac{3}{5}, \quad 0 < u < \pi/2$

22. $\cos u = -\frac{2}{7}, \quad \pi/2 < u < \pi$

23. $\tan u = \frac{1}{2}, \quad \pi < u < 3\pi/2$

24. $\cot u = -6, \quad 3\pi/2 < u < 2\pi$

In Exercises 25–30, rewrite the expression in terms of the first power of the cosine. Use a graphing utility to graph both expressions and verify that both forms are the same.

25. $\cos^4 x$

26. $\sin^4 x$

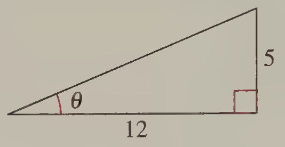
27. $\sin^2 x \cos^2 x$

28. $\cos^6 x$

29. $\sin^2 x \cos^4 x$

30. $\sin^4 x \cos^2 x$

In Exercises 31–38, use the figure to find the exact value of the trigonometric function.



31. $\cos \frac{\theta}{2}$

32. $\sin \frac{\theta}{2}$

33. $\tan \frac{\theta}{2}$

34. $\sec \frac{\theta}{2}$

35. $\csc \frac{\theta}{2}$

36. $\cot \frac{\theta}{2}$

37. $2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$

38. $2 \cos \frac{\theta}{2} \tan \frac{\theta}{2}$

In Exercises 39–46, use the half-angle formulas to determine the exact values of the sine, cosine, and tangent of the angle.

39. 15°

40. 165°

41. 112° 30′

42. 157° 30′

43. $\frac{\pi}{8}$

44. $\frac{\pi}{12}$

45. $\frac{3\pi}{8}$

46. $\frac{7\pi}{12}$

In Exercises 47–50, find the exact values of $\sin(u/2)$, $\cos(u/2)$ and $\tan(u/2)$ using the half-angle formulas.

47. $\sin u = \frac{5}{13}, \quad \pi/2 < u < \pi$

48. $\cos u = \frac{7}{25}, \quad 0 < u < \pi/2$

49. $\tan u = -\frac{8}{5}, \quad 3\pi/2 < u < 2\pi$

50. $\cot u = 7, \quad \pi < u < 3\pi/2$

In Exercises 51–54, use the half-angle formulas to simplify the expression.

51. $\sqrt{\frac{1 - \cos 6x}{2}}$

52. $\sqrt{\frac{1 + \cos 4x}{2}}$

53. $-\sqrt{\frac{1 - \cos 8x}{1 + \cos 8x}}$

54. $-\sqrt{\frac{1 - \cos(x - 1)}{2}}$

In Exercises 55–58, find the exact zeros of the function in the interval $[0, 2\pi)$. Use a graphing utility to graph the function and verify your answers.

55. $f(x) = \sin \frac{x}{2} - \cos x$

56. $h(x) = \sin \frac{x}{2} + \cos x - 1$

57. $h(x) = \cos \frac{x}{2} - \sin x$

58. $g(x) = \tan \frac{x}{2} - \sin x$

In Exercises 59–64, use the product-to-sum formulas to write the product as a sum or difference.

59. $6 \sin \frac{\pi}{3} \cos \frac{\pi}{3}$

60. $4 \sin \frac{\pi}{3} \cos \frac{5\pi}{6}$

61. $\sin 5\theta \cos 3\theta$

62. $5 \sin 3\alpha \sin 4\alpha$

63. $5 \cos(-5\beta) \cos 3\beta$

64. $\cos 2\theta \cos 4\theta$

In Exercises 65–72, use the sum-to-product formulas to write the sum or difference as a product.

65. $\sin 60° + \sin 30°$

66. $\cos 120° + \cos 30°$

67. $\sin 5\theta - \sin \theta$

68. $\sin x + \sin 7x$

69. $\sin(\alpha + \beta) - \sin(\alpha - \beta)$

70. $\cos(\phi + 2\pi) + \cos \phi$

71. $\cos\left(\theta + \frac{\pi}{2}\right) - \cos\left(\theta - \frac{\pi}{2}\right)$

72. $\sin\left(x + \frac{\pi}{2}\right) + \sin\left(x - \frac{\pi}{2}\right)$

In Exercises 73–76, find the exact zeros of the function in the interval $[0, 2\pi)$. Use a graphing utility to graph the function and verify your answers.

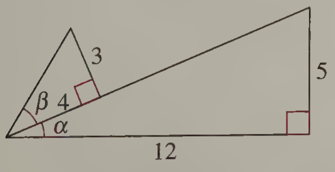
73. $g(x) = \sin 6x + \sin 2x$

74. $h(x) = \cos 2x - \cos 6x$

75. $f(x) = \dfrac{\cos 2x}{\sin 3x - \sin x} - 1$

76. $f(x) = \sin^2 3x - \sin^2 x$

In Exercises 77–80, use the figure to find the exact value of the trigonometric function in two ways.



77. $\sin^2 \alpha$

78. $\cos^2 \alpha$

79. $\sin \alpha \cos \beta$

80. $\cos \alpha \sin \beta$

In Exercises 81–92, verify the identity algebraically. Use a graphing utility to confirm the identity graphically.

81. $\csc 2\theta = \dfrac{\csc \theta}{2 \cos \theta}$

82. $\sec 2\theta = \dfrac{\sec^2 \theta}{2 - \sec^2 \theta}$

83. $\cos^2 2\alpha - \sin^2 2\alpha = \cos 4\alpha$

84. $\cos^4 x - \sin^4 x = \cos 2x$

85. $(\sin x + \cos x)^2 = 1 + \sin 2x$

86. $1 + \cos 10y = 2 \cos^2 5y$

87. $\sec \frac{u}{2} = \pm \sqrt{\dfrac{2 \tan u}{\tan u + \sin u}}$

88. $\tan \frac{u}{2} = \csc u - \cot u$

89. $\cos 3\beta = \cos^3 \beta - 3 \sin^2 \beta \cos \beta$

90. $\sin 4\beta = 4 \sin \beta \cos \beta(1 - 2 \sin^2 \beta)$

91. $\dfrac{\cos 4x - \cos 2x}{2 \sin 3x} = -\sin x$

92. $\dfrac{\cos 3x - \cos x}{\sin 3x - \sin x} = -\tan 2x$

In Exercises 93 and 94, graph the function by using the power-reducing formulas.

93. $f(x) = \sin^2 x$

94. $f(x) = \cos^2 x$

In Exercises 95–98, (a) use a graphing utility to graph the function and approximate the maximum and minimum points on the graph in the interval $[0, 2\pi]$, and (b) solve the trigonometric equation and verify that the x-coordinates of the maximum and minimum points of f are among its solutions (calculus is required to find the trigonometric equation).

	Function	*Trigonometric Equation*
95.	$f(x) = 4 \sin \frac{x}{2} + \cos x$	$2 \cos \frac{x}{2} - \sin x = 0$
96.	$f(x) = \cos 2x - 2 \sin x$	$-2 \cos x(2 \sin x + 1) = 0$
97.	$f(x) = 2 \cos \frac{x}{2} + \sin 2x$	$2 \cos 2x - \sin \frac{x}{2} = 0$
98.	$f(x) = 2 \sin \frac{x}{2} - 5 \cos\left(2x - \frac{\pi}{4}\right)$	$10 \sin\left(2x - \frac{\pi}{4}\right) + \cos \frac{x}{2} = 0$

In Exercises 99–102, write the trigonometric expression as an algebraic expression.

99. $\sin(2 \arcsin x)$ **100.** $\cos(2 \arccos x)$

101. $\cos(2 \arcsin x)$ **102.** $\sin(2 \arctan x)$

103. ***Projectile Motion*** The range of a projectile fired at an angle θ with the horizontal and with an initial velocity of v_0 feet per second is

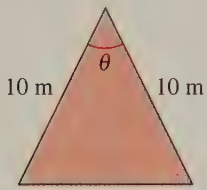
$$r = \tfrac{1}{32} v_0 \sin 2\theta$$

where r is measured in feet. Determine the expression for the range in terms of θ.

104. ***Area*** The length of each of the two equal sides of an isosceles triangle is 10 meters. The angle between the equal sides is θ.

(a) Express the area of the triangle as a function of $\theta/2$.

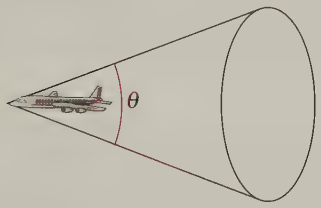
(b) Express the area of the triangle as a function of θ and determine the value of θ such that the area is a maximum.



105. ***Mach Number*** The mach number M of an airplane is the ratio of its speed to the speed of sound. When an airplane travels faster than the speed of sound, the sound waves form a cone behind the airplane. The mach number is related to the apex angle θ of the cone by

$$\sin \frac{\theta}{2} = \frac{1}{M}.$$

Find the angle θ that corresponds to a mach number of 4.5.



106. ***Railroad Track*** When two railroad tracks merge, the overlapping portions of the tracks are in the shape of a circular arcs. The radius of the arcs r (in feet) and the angle θ are related by

$$\frac{x}{2} = 2r \sin^2 \frac{\theta}{2}.$$

Write a formula for x in terms of $\cos \theta$.

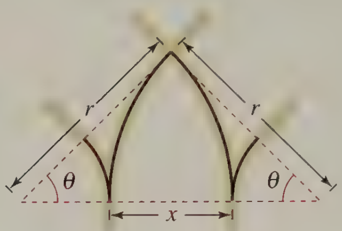


FIGURE FOR 106

Synthesis

True or False? **In Exercises 107 and 108, determine whether the statement is true or false. Justify your answer.**

107. $\sin \dfrac{x}{2} = -\sqrt{\dfrac{1 - \cos x}{2}}, \qquad \pi \le x \le 2\pi.$

108. The graph of $y = 4 - 8 \sin^2 x$ has a maximum at $(\pi, 4)$.

109. ***Conjecture*** Consider the function

$$f(x) = 2 \sin x \left[2 \cos^2\left(\frac{x}{2}\right) - 1\right].$$

(a) Use a graphing utility to graph the function.

(b) Make a conjecture about the function that is an identity with f.

(c) Verify your conjecture algebraically.

Review

In Exercises 110 and 111, find (if possible) the complement and supplement of each angle.

110. (a) 55° (b) 162°

111. (a) $\dfrac{\pi}{18}$ (b) $\dfrac{9\pi}{20}$

In Exercises 112–115, sketch the graph of the function. Use a graphing utility to verify your graph.

112. $f(x) = \dfrac{3}{2} \cos 2x$ **113.** $f(x) = \dfrac{5}{2} \sin \dfrac{1}{2}x$

114. $f(x) = \dfrac{1}{2} \tan 2\pi x$ **115.** $f(x) = \dfrac{1}{4} \sec \dfrac{\pi x}{2}$

6 Chapter Summary

What did you learn?

	Review Exercises
Section 6.1	
☐ How to recognize and write the fundamental trigonometric identities	1–10
☐ How to use fundamental trigonometric identities to evaluate trigonometric functions, simplify trigonometric expressions, and rewrite trigonometric expressions	11–24
Section 6.2	
☐ How to verify trigonometric identities	25–36
Section 6.3	
☐ How to use standard algebraic techniques to solve trigonometric equations	37–50
☐ How to solve trigonometric equations of quadratic type	51–54
☐ How to solve trigonometric equations involving multiple angles	55–60
☐ How to use inverse trigonometric functions to solve trigonometric equations	61–64
Section 6.4	
☐ How to use sum and difference formulas to evaluate trigonometric functions	65–78
☐ How to use sum and difference formulas to verify identities and solve trigonometric equations	79–86
Section 6.5	
☐ How to use multiple-angle formulas to rewrite and evaluate trigonometric formulas	87–95
☐ How to use power-reducing formulas to rewrite and evaluate trigonometric formulas	96–99
☐ How to use half-angle formulas to rewrite and evaluate trigonometric functions	100–108
☐ How to use product-sum formulas to rewrite and evaluate trigonometric functions	109–117

6 Review Exercises

6.1 **In Exercises 1–10, name the trigonometric function equivalent to the expression.**

1. $\dfrac{1}{\cos x}$

2. $\dfrac{1}{\sin x}$

3. $\dfrac{1}{\sec x}$

4. $\dfrac{1}{\tan x}$

5. $\dfrac{\cos x}{\sin x}$

6. $\sqrt{1 + \tan^2 x}$

7. $\csc\left(\dfrac{\pi}{2} - x\right)$

8. $\cot\left(\dfrac{\pi}{2} - x\right)$

9. $\sec(-x)$

10. $\tan(-x)$

In Exercises 11–14, use the given values and trigonometric identities to evaluate (if possible) the other trigonometric functions of the angle.

11. $\sin x = \dfrac{3}{5},\quad \cos x = \dfrac{4}{5}$

12. $\tan \theta = \dfrac{3}{2},\quad \sec \theta = \dfrac{\sqrt{13}}{2}$

13. $\sin\left(\dfrac{\pi}{2} - x\right) = \dfrac{1}{\sqrt{2}},\quad \sin x = -\dfrac{1}{\sqrt{2}}$

14. $\csc\left(\dfrac{\pi}{2} - \theta\right) = 3,\quad \sin \theta = \dfrac{2\sqrt{2}}{3}$

In Exercises 15–22, use the fundamental identities to simplify the trigonometric expression. Use the *table* feature of a graphing utility to check your result numerically.

15. $\dfrac{1}{\cot^2 x + 1}$

16. $\dfrac{\sec^2 x - 1}{\sec x - 1}$

17. $\dfrac{\sin^2 \alpha - \cos^2 \alpha}{\sin^2 \alpha - \sin \alpha \cos \alpha}$

18. $\dfrac{\sin^3 \beta + \cos^3 \beta}{\sin \beta + \cos \beta}$

19. $\tan^2 \theta(\csc^2 \theta - 1)$

20. $\csc^2 x(1 - \cos^2 x)$

21. $\tan\left(\dfrac{\pi}{2} - x\right) \sec x$

22. $\dfrac{\sin(-x) \cot x}{\sin\left(\dfrac{\pi}{2} - x\right)}$

23. ***Rate of Change*** The rate of change of the function $f(x) = 2\sqrt{\sin x}$ is given by the expression $\sin^{-1/2} x \cos x$. Show that this expression can also be written as $\cot x\sqrt{\sin x}$.

24. ***Rate of Change*** The rate of change of the function $f(x) = \csc x - \cot x$ is the expression $\csc^2 x - \csc x \cot x$. Show that this expression can also be written as

$$\frac{(1 - \cos x)}{\sin^2 x}.$$

6.2 **In Exercises 25–36, verify the identity.**

25. $\cos x(\tan^2 x + 1) = \sec x$

26. $\sec^2 x \cot x - \cot x = \tan x$

27. $\sin^3 \theta + \sin \theta \cos^2 \theta = \sin \theta$

28. $\cot^2 x - \cos^2 x = \cot^2 x \cos^2 x$

29. $\sin^5 x \cos^2 x = (\cos^2 x - 2\cos^4 x + \cos^6 x) \sin x$

30. $\cos^3 x \sin^2 x = (\sin^2 x - \sin^4 x) \cos x$

31. $\sqrt{\dfrac{1 - \sin \theta}{1 + \sin \theta}} = \dfrac{1 - \sin \theta}{|\cos \theta|}$

32. $\sqrt{1 - \cos x} = \dfrac{|\sin x|}{\sqrt{1 + \cos x}}$

33. $\dfrac{\csc(-x)}{\sec(-x)} = -\cot x$

34. $\dfrac{1 + \sec(-x)}{\sin(-x) + \tan(-x)} = -\csc x$

35. $\sin^2 x + \sin^2\left(\dfrac{\pi}{2} - x\right) = 1$

36. $\csc x \sin\left(\dfrac{\pi}{2} - x\right) = \cot x$

6.3 **In Exercises 37 and 38, verify that the x-values are solutions of the equation.**

37. $2\cos^2 4x - 1 = 0$

(a) $x = \dfrac{\pi}{16}$ (b) $x = \dfrac{3\pi}{16}$

38. $2\sin^2 x - \sin x - 1 = 0$

(a) $x = \dfrac{\pi}{2}$ (b) $x = \dfrac{7\pi}{6}$

In Exercises 39–50, solve the equation for $0 \le x < 2\pi$.

39. $2 \sin x - 1 = 0$

40. $\tan x + 1 = 0$

41. $\sin x = \sqrt{3} - \sin x$

42. $4 \cos x = 1 + 2 \cos x$

43. $3\sqrt{3} \tan x = 3$

44. $\frac{1}{2} \sec x - 1 = 0$

45. $3 \csc^2 x = 4$

46. $4 \tan^2 x - 1 = \tan^2 x$

47. $4 \cos^2 x - 3 = 0$

48. $\sin x(\sin x + 1) = 0$

49. $\sin x - \tan x = 0$

50. $\csc x - 2 \cot x = 0$

In Exercises 51–60, find all solutions of the equation in the interval $[0, 2\pi)$. Use a graphing utility to verify your answers.

51. $2 \cos^2 x - \cos x = 1$

52. $2 \sin^2 x - 3 \sin x = -1$

53. $\cos^2 x + \sin x = 1$

54. $\sin^2 x + 2 \cos x = 2$

55. $2 \sin 2x - \sqrt{2} = 0$

56. $\sqrt{3} \tan 3x = 0$

57. $\cos 4x(\cos x - 1) = 0$

58. $3 \csc^2 5x = -4$

59. $\cos 4x - 7 \cos 2x = 8$

60. $\sin 4x - \sin 2x = 0$

In Exercises 61–64, use the inverse functions where necessary to find all solutions of the equation in the interval $[0, 2\pi)$. Use a graphing utility to verify your answers.

61. $\sin^2 x - 2 \sin x = 0$

62. $2 \cos^2 x + 3 \cos x = 0$

63. $\tan^2 \theta + \tan \theta - 12 = 0$

64. $\sec^2 x + 6 \tan x + 4 = 0$

6.4 **In Exercises 65–68, find the exact value of the sine, cosine, and tangent of the angle by using the sum or difference formula.**

65. $285° = 225° + 60°$

66. $345° = 300° + 45°$

67. $\dfrac{5\pi}{12} = \dfrac{2\pi}{3} - \dfrac{\pi}{4}$

68. $\dfrac{19\pi}{12} = \dfrac{11\pi}{6} - \dfrac{\pi}{4}$

In Exercises 69–72, write the expression as the sine, cosine, or tangent of an angle.

69. $\sin 140° \cos 50° + \cos 140° \sin 50°$

70. $\cos 25° \cos 15° - \sin 25° \sin 15°$

71. $\dfrac{\tan 25° + \tan 10°}{1 - \tan 25° \tan 10°}$

72. $\dfrac{\tan 68° - \tan 115°}{1 + \tan 68° \tan 115°}$

In Exercises 73–78, find the exact value of the trigonometric function given that $\sin u = \frac{3}{4}$, $\cos v = -\frac{5}{13}$, and u and v are in Quadrant II.

73. $\sin(u + v)$

74. $\tan(u + v)$

75. $\cos(u - v)$

76. $\sin(u - v)$

77. $\cos(u + v)$

78. $\tan(u - v)$

In Exercises 79–84, verify the identity.

79. $\cos\left(x + \dfrac{\pi}{2}\right) = -\sin x$

80. $\sin\left(x - \dfrac{3\pi}{2}\right) = \cos x$

81. $\cot\left(\dfrac{\pi}{2} - x\right) = \tan x$

82. $\sin(\pi - x) = \sin x$

83. $\cos 3x = 4 \cos^3 x - 3 \cos x$

84. $\dfrac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta} = \tan \alpha + \tan \beta$

In Exercises 85 and 86, find all solutions of the equation in the interval $[0, 2\pi)$.

85. $\sin\left(x + \dfrac{\pi}{2}\right) - \sin\left(x - \dfrac{\pi}{2}\right) = \sqrt{3}$

86. $\cos\left(x + \dfrac{3\pi}{4}\right) - \cos\left(x - \dfrac{3\pi}{4}\right) = 0$

6.5 **In Exercises 87–92, use double-angle formulas to verify the identity algebraically and use a graphing utility to confirm it graphically.**

87. $6 \sin x \cos x = 3 \sin 2x$

88. $4 \sin x \cos x + 2 = 2 \sin 2x + 2$

89. $1 - 4 \sin^2 x \cos^2 x = \cos^2 2x$

90. $\sin 4x = 8 \cos^3 x \sin x - 4 \cos x \sin x$

91. $\dfrac{\sin 2\alpha}{\cos^2 \alpha - \sin^2 \alpha} = \tan 2\alpha$

92. $\tan^2 x = \dfrac{1 - \cos 2x}{1 + \cos 2x}$

In Exercises 93 and 94, find the exact values of $\sin 2u$, $\cos 2u$, and $\tan 2u$ using the double-angle formulas.

93. $\sin u = -\dfrac{5}{7}, \quad \pi < u < \dfrac{3\pi}{2}$

94. $\cos u = -\dfrac{2}{\sqrt{5}}, \quad \dfrac{\pi}{2} < u < \pi$

95. *Projectile Motion* A baseball leaves the hand of the first baseman at an angle of θ with the horizontal and an initial velocity of $v_0 = 80$ feet per second. The ball is caught by the second baseman 100 feet away. Find θ if the range r of a projectile is

$$r = \frac{1}{32} v_0^2 \sin 2\theta.$$

In Exercises 96–99, use the power-reducing formulas to rewrite the expression in terms of the first power of the cosine of the angle.

96. $\sin^6 x$

97. $\cos^4 x \sin^4 x$

98. $\cos^4 2x$

99. $\sin^4 2x$

In Exercises 100–103, use the half-angle formulas to determine the exact value of the sine, cosine, and tangent of the angle.

100. $105°$

101. $67° 30'$

102. $\dfrac{7\pi}{8}$

103. $\dfrac{11\pi}{12}$

In Exercises 104 and 105, use the half-angle formulas to simplify the expression.

104. $-\sqrt{\dfrac{1 + \cos 10x}{2}}$

105. $\dfrac{\sin 6x}{1 + \cos 6x}$

In Exercises 106 and 107, use the half-angle formulas to verify the identity.

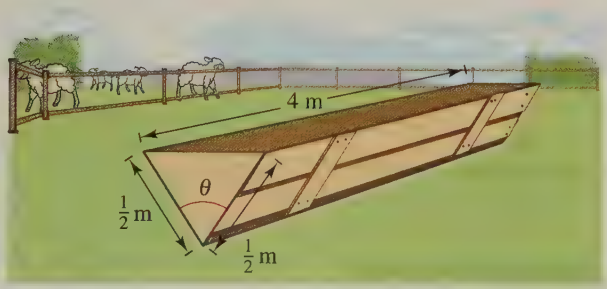
106. $\sqrt{\dfrac{1 - \cos^2 x}{1 + \cos x}} = \sqrt{2}\left|\sin \dfrac{x}{2}\right|$

107. $\dfrac{\sec x - 1}{\tan x} = \tan \dfrac{x}{2}$

108. ***Volume*** A trough for feeding cattle is 4 meters long and its cross sections are isosceles triangles with the two equal sides being $\frac{1}{2}$ meter. The angle between the equal sides is θ.

(a) Express the trough's volume as a function of $\theta/2$.

(b) Express the volume of the trough as a function of θ and determine the value of θ such that the volume is maximum.



In Exercises 109–112, write the trigonometric expression as a product.

109. $\cos 3\theta + \cos 2\theta$

110. $\sin 5\theta - \sin 3\theta$

111. $\cos \dfrac{3\pi}{4} - \cos \dfrac{\pi}{4}$

112. $\sin\left(x + \dfrac{\pi}{4}\right) - \sin\left(x - \dfrac{\pi}{4}\right)$

In Exercises 113–116, write the trigonometric expression as a sum or difference.

113. $\sin 3\alpha \sin 2\alpha$

114. $3 \sin 2x \sin 3x$

115. $6 \sin \dfrac{\pi}{4} \cos \dfrac{\pi}{4}$

116. $\cos \dfrac{x}{2} \cos \dfrac{x}{4}$

117. ***Harmonic Motion*** A weight is attached to a spring suspended vertically from a ceiling. When a driving force is applied to the system, the weight moves vertically from its equilibrium position, and this motion is described by the model

$$y = 1.5 \sin 8t - 0.5 \cos 8t$$

where y is the distance from equilibrium measured in feet and t is the time in seconds.

(a) Write the model in the form $y = \sqrt{a^2 + b^2} \sin(Bt + C)$.

(b) Use a graphing utility to graph the model.

(c) Find the amplitude of the weight's oscillations.

(d) Find the frequency of the weight's oscillations.

Synthesis

True or False? **In Exercises 118–121, determine whether the statement is true or false. Justify your answer.**

118. If $\dfrac{\pi}{2} < \theta < \pi$, then $\cos \dfrac{\theta}{2} < 0$.

119. $\sin(x + y) = \sin x + \sin y$

120. $4 \sin(-x) \cos(-x) = -2 \sin 2x$

121. $4 \sin 45° \cos 15° = 1 + \sqrt{3}$

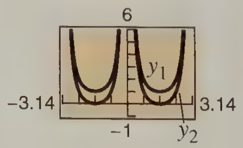
122. List the reciprocal identities, quotient identities, and Pythagorean identities from memory.

123. Is $\cos \theta = \sqrt{1 - \sin^2 \theta}$ an identity? Explain.

In Exercises 124 and 125, use the graphs of y_1 and y_2 to determine how to change y_2 to a new function y_3 so that $y_1 = y_3$.

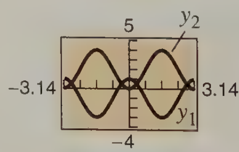
124. $y_1 = \sec^2\left(\dfrac{\pi}{2} - x\right)$

$y_2 = \cot^2 x$



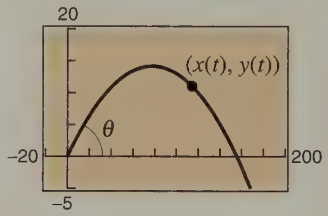
125. $y_1 = \dfrac{\cos 3x}{\cos x}$

$y_2 = (2 \sin x)^2$



Chapter Project *Projectile Motion*

Parametric equations are equations that express two variables in terms of a third variable, called the parameter. In this project, you will use parametric equations to model the path of a projectile. These parametric equations express position x and position y in terms of time t. For any time t, the horizontal position $x(t)$ and vertical position $y(t)$ of a projectile (ignoring air resistance) launched at ground level is given by the parameter equations



$$x(t) = (v_0 \cos \theta)t$$

$$y(t) = (v_0 \sin \theta)t - 16t^2.$$

In these equations, θ is the angle with the horizontal and v_0 is the initial velocity in feet per second, as indicated in the figure at the right.

Set your graphing utility to parametric and degree modes, and use the viewing window

$$0 \le t \le 5$$

$$-20 \le x \le 200$$

$$-5 \le y \le 20.$$

Letting $v_0 = 88$ feet per second and $\theta = 20°$, graph the parametric equations

$$x(t) = (88 \cos 20°)t$$

$$y(t) = (88 \sin 20°)t - 16t^2.$$

Use the *zoom* and *trace* features to (a) find the maximum height attained by the projectile, (b) find the time at which the maximum height occurs, (c) determine the length of time that the projectile is in the air, and (d) determine the range of the projectile.

Questions for Further Exploration

1. Verify algebraically the range of the projectile by solving the equation $(88 \sin 20°)t - 16t^2 = 0$ for t and then evaluating $x(t)$ at this t-value.
2. Use a graphing utility to find the maximum height and range of the projectile when $\theta = 30°$ and $v_0 = 132$ feet per second.
3. Let $v_0 = 60$ feet per second. Find the maximum range for the angles $\theta = 20°, 30°, 40°, 50°$, and $60°$. In general, what angle should you use to produce the maximum range?
4. What is the relationship between the time the projectile reaches its maximum height and the time it takes for the projectile to return to the ground? Explain.
5. Eliminate t from the parametric equations

$$x(t) = (v_0 \cos \theta)t \quad \text{and} \quad y(t) = (v_0 \sin \theta)t - 16t^2$$

by solving for t in the first equation and substituting this value into the equation for y. Show in this case that the height of the projectile is given by the equation

$$y = (\tan \theta)x - \frac{16 \sec^2 \theta}{v_0^2}x^2.$$

Use this equation to find the angle θ corresponding to a maximum range of 200 feet and initial velocity of $v_0 = 80$ feet per second.

6 Chapter Test

Take this test as you would take a test in class. After you are done, check your work against the answers given in the back of the book.

The *Interactive* CD-ROM and *Internet* versions of this text provide answers to the Chapter Tests and Cumulative Tests. They also offer Chapter Pre-Tests (that test key skills and concepts covered in previous chapters) and Chapter Post-Tests, both of which have randomly generated exercises with diagnostic capabilities.

1. If $\tan\theta = \frac{6}{5}$ and $\cos\theta < 0$, use the fundamental identities to evaluate the other five trigonometric functions of θ.

2. Use the fundamental identities to simplify $\csc^2\beta(1 - \cos^2\beta)$.

3. Factor and simplify $\dfrac{\sec^4 x - \tan^4 x}{\sec^2 x + \tan^2 x}$.

4. Add and simplify $\dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}$.

5. Determine the values of θ, $0 \le \theta < 2\pi$, for which $\tan\theta = -\sqrt{\sec^2\theta - 1}$ is true.

6. Use a graphing utility to graph the functions $y_1 = \cos x + \sin x \tan x$ and $y_2 = \sec x$. Make a conjecture about y_1 and y_2. Verify the result algebraically.

In Exercises 7–12, verify the identity.

7. $\sin\theta\sec\theta = \tan\theta$

8. $\sec^2 x\tan^2 x + \sec^2 x = \sec^4 x$

9. $\dfrac{\csc\alpha + \sec\alpha}{\sin\alpha + \cos\alpha} = \cot\alpha + \tan\alpha$

10. $\cos\left(x + \dfrac{\pi}{2}\right) = -\sin x$

11. $\sin(n\pi + \theta) = (-1)^n\sin\theta$, where n is an integer.

12. $(\sin x + \cos x)^2 = 1 + \sin 2x$

13. Find the exact values of the sine, cosine, and tangent of the angle $-\dfrac{7\pi}{12}$.

14. Use the fundamental identities to simplify the expression $\dfrac{\sin^4 x}{\tan^2 x}$.

15. Write $3\sin 2\theta\sin 6\theta$ as a sum or difference.

16. Write $\cos 5\theta + \cos 3\theta$ as a product.

In Exercises 17–20, find all solutions of the equation in the interval $[0, 2\pi)$.

17. $\tan^2 x + \tan x = 0$

18. $\sin 2\alpha - \cos\alpha = 0$

19. $4\cos^2 x - 3 = 0$

20. $\csc^2 x - \csc x - 2 = 0$

21. Use a graphing utility to approximate the solutions of the equation $5\cos x - x = 0$ in the interval $[0, 2\pi)$ accurate to three decimal places.

22. Use the figure at the right to find the exact values of $\sin 2u$ and $\tan 2u$.

23. Explain why the equation $\cos^2 x + \cos x - 6 = 0$ has no solution.

24. Find the exact value of tan 105°.

25. The index of refraction n of a transparent material is the ratio of the speed of light in a vacuum to the speed of light in the material. For the glass triangular prism in the figure at the right, $n = 1.5$ and $\alpha = 60°$. Find the angle θ for the glass prism if

$$n = \frac{\sin(\theta/2 + \alpha/2)}{\sin(\theta/2)}.$$

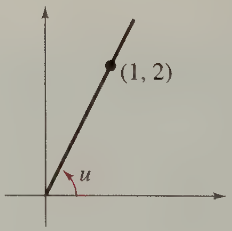


FIGURE FOR 22

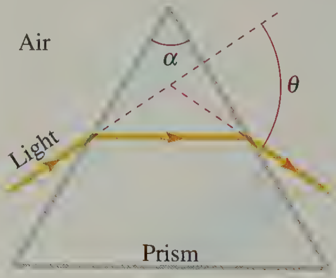


FIGURE FOR 25

Systems of Equations and Inequalities

8

- 8.1 Solving Systems of Equations
- 8.2 Systems of Linear Equations in Two Variables
- 8.3 Multivariable Linear Systems
- 8.4 Systems of Inequalities
- 8.5 Linear Programming

Ron Watts/CORBIS

In 1996, 57 million newspapers were printed daily in the United States. With a population of over 265 million, there were about 215 newspapers per 1000 people. (Source: U.S. Bureau of the Census and Editor & Publisher, Co.)

The Big Picture

In this chapter you will learn how to

- ☐ solve systems of equations by substitution, by elimination, by Gaussian elimination, and graphically.
- ☐ recognize linear systems in row-echelon form and to use back substitution to solve the system.
- ☐ solve nonsquare systems of equations.
- ☐ sketch the graphs of inequalities in two variables and solve systems of inequalities.
- ☐ solve linear programming problems.
- ☐ use systems of equations and inequalities to model and solve real-life problems.

Important Vocabulary

As you encounter each new vocabulary term in this chapter, add the term and its definition to your notebook glossary.

- systems of equations (p. 566)
- solution of a system of equations (p. 566)
- method of substitution (p. 566)
- point of intersection (p. 567)
- break-even point (p. 571)
- method of elimination (p. 577)
- equivalent systems (p. 578)
- consistent system (p. 579)
- inconsistent system (p. 579)
- row-echelon form (p. 587)
- ordered triple (p. 587)
- Gaussian elimination (p. 588)
- nonsquare system of equations (p. 591)
- three-dimensional coordinate system (p. 592)
- graph of an equation in three variables (p. 592)
- partial fraction (p. 593)
- partial fraction decomposition (p. 593)
- basic equation (p. 594)
- solution of an inequality (p. 603)
- graph of an inequality (p. 603)
- linear inequalities (p. 604)
- solution of a system of inequalities (p. 605)
- optimization (p. 613)
- linear programming (p. 613)
- objective function (p. 613)
- constraints (p. 613)
- feasible solutions (p. 613)

Additional Resources Text-specific additional resources are available to help you do well in this course. See page xvi for details.

8.1 Solving Systems of Equations

The Method of Substitution

Up to this point in the text, most problems have involved either a function of one variable or a single equation in two variables. However, many problems in science, business, and engineering involve two or more equations in two or more variables. To solve such problems, you need to find solutions of **systems of equations.** Here is an example of a system of two equations in two unknowns, x and y.

$$\begin{cases} 2x + y = 5 & \text{Equation 1} \\ 3x - 2y = 4 & \text{Equation 2} \end{cases}$$

A **solution** of this system is an ordered pair that satisfies each equation in the system. Finding the set of all such solutions is called **solving the system of equations.** For instance, the ordered pair (2, 1) is a solution of this system. To check this, you can substitute 2 for x and 1 for y in *each* equation.

Check (2, 1) *in Equation 1:*

$$2x + y = 5 \qquad \text{Write Equation 1.}$$

$$2(2) + 1 \overset{?}{=} 5 \qquad \text{Substitute 2 for } x \text{ and 1 for } y.$$

$$4 + 1 = 5 \qquad \text{Solution checks in Equation 1. } \checkmark$$

Check (2, 1) *in Equation 2:*

$$3x - 2y = 4 \qquad \text{Write Equation 2.}$$

$$3(2) - 2(1) \overset{?}{=} 4 \qquad \text{Substitute 2 for } x \text{ and 1 for } y.$$

$$6 - 2 = 4 \qquad \text{Solution checks in Equation 2. } \checkmark$$

In this chapter you will study four ways to solve equations, beginning with the **method of substitution.**

Method	*Section*	*Type of System*
1. Substitution	8.1	Linear or nonlinear, 2 variables
2. Graphical	8.1	Linear or nonlinear, 2 variables
3. Elimination	8.2	Linear, 2 variables
4. Gaussian Elimination	8.3	Linear, 3 or more variables

The Method of Substitution

1. Solve one of the equations for one variable in terms of the other.
2. Substitute the expression found in Step 1 into the other equation to obtain an equation in one variable.
3. Solve the equation obtained in Step 2.
4. Back-substitute the solution in Step 3 into the expression obtained in Step 1 to find the value of the other variable.
5. Check that the solution satisfies each of the original equations.

What You Should Learn:

- How to use the method of substitution to solve systems of equations in two variables
- How to solve systems of equations graphically
- How to use systems of equations to model and solve real-life problems

Why You Should Learn It:

You can use systems of equations in situations where the variables must satisfy two or more conditions. For instance, Exercise 73 on page 575 shows how to use a system of equations to compare two models for estimating the number of board feet in a 16-foot log.

Bruce Hands/Tony Stone Images

In the algebraic solution of Example 1, you use the method of substitution to solve the system of equations. In the graphical solution, note that the solution of the system corresponds to the **point of intersection** of the graphs.

EXAMPLE 1 Solving a System of Equations

Solve the system of equations.

$$\begin{cases} x + y = 4 & \text{Equation 1} \\ x - y = 2 & \text{Equation 2} \end{cases}$$

Algebraic Solution

Begin by solving for y in Equation 1.

$$y = 4 - x \qquad \text{Solve for } y \text{ in Equation 1.}$$

Next, substitute this expression for y into Equation 2 and solve the resulting single-variable equation for x.

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ x - (4 - x) &= 2 && \text{Substitute } 4 - x \text{ for } y. \\ x - 4 + x &= 2 && \text{Simplify.} \\ 2x &= 6 && \text{Combine like terms.} \\ x &= 3 && \text{Divide each side by 2.} \end{aligned}$$

Finally, you can solve for y by *back-substituting* $x = 3$ into the equation $y = 4 - x$, to obtain

$$\begin{aligned} y &= 4 - x && \text{Write Equation 1.} \\ y &= 4 - 3 && \text{Substitute 3 for } x. \\ y &= 1. && \text{Solve for } y. \end{aligned}$$

The solution is the ordered pair $(3, 1)$. Check this as follows.

Check $(3, 1)$ *in Equation 1:*

$$\begin{aligned} x + y &= 4 && \text{Write Equation 1.} \\ 3 + 1 &\overset{?}{=} 4 && \text{Substitute for } x \text{ and } y. \\ 4 &= 4 && \text{Solution checks in Equation 1. } \checkmark \end{aligned}$$

Check $(3, 1)$ *in Equation 2:*

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ 3 - 1 &\overset{?}{=} 2 && \text{Substitute for } x \text{ and } y. \\ 2 &= 2 && \text{Solution checks in Equation 2. } \checkmark \end{aligned}$$

Graphical Solution

Begin by solving both equations for y. Then use a graphing utility to graph the equations

$$y_1 = 4 - x \quad \text{and} \quad y_2 = x - 2$$

in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features of the graphing utility to approximate the point of intersection of the graphs. In Figure 8.1, the point of intersection is $(3, 1)$.

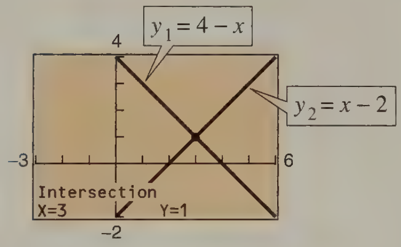


Figure 8.1

Check that $(3, 1)$ is the exact solution as follows.

Check $(3, 1)$ *in Equation 1:*

$$\begin{aligned} x + y &= 4 && \text{Write Equation 1.} \\ 3 + 1 &\overset{?}{=} 4 && \text{Substitute for } x \text{ and } y. \\ 4 &= 4 && \text{Solution checks in Equation 1. } \checkmark \end{aligned}$$

Check $(3, 1)$ *in Equation 2:*

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ 3 - 1 &\overset{?}{=} 2 && \text{Substitute for } x \text{ and } y. \\ 2 &= 2 && \text{Solution checks in Equation 2. } \checkmark \end{aligned}$$

In the algebraic solution of Example 1, note that the term *back-substitution* implies that you work *backward.* First you solve for one of the variables, and then you substitute that value *back* into one of the equations in the system to find the value of the other variable.

EXAMPLE 2 Solving a System by Substitution

A total of $12,000 is invested in two funds paying 9% and 11% simple interest. The yearly interest is $1180. How much is invested at each rate?

Solution

Verbal Model: 9% fund + 11% fund = Total investment

9% interest + 11% interest = Total interest

Labels: Amount in 9% fund $= x$ (dollars)
Interest for 9% fund $= 0.09x$ (dollars)
Amount in 11% fund $= y$ (dollars)
Interest for 11% fund $= 0.11y$ (dollars)
Total investment = $12,000 (dollars)
Total interest = $1180 (dollars)

System:

$$\begin{cases} x + y = 12{,}000 & \text{Equation 1} \\ 0.09x + 0.11y = 1{,}180 & \text{Equation 2} \end{cases}$$

To begin, it is convenient to multiply both sides of Equation 2 by 100 to obtain $9x + 11y = 118{,}000$. This eliminates the need to work with decimals.

$$9x + 11y = 118{,}000 \qquad \text{Revised Equation 2}$$

To solve this system, you can solve for x in Equation 1.

$$x = 12{,}000 - y \qquad \text{Revised Equation 1}$$

Next, substitute this expression for x into Revised Equation 2 and solve the resulting equation for y.

$$\begin{aligned} 9x + 11y &= 118{,}000 && \text{Write Revised Equation 2.} \\ 9(12{,}000 - y) + 11y &= 118{,}000 && \text{Substitute } 12{,}000 - y \text{ for } x. \\ 108{,}000 - 9y + 11y &= 118{,}000 && \text{Distributive Property} \\ 2y &= 10{,}000 && \text{Combine like terms.} \\ y &= 5000 && \text{Divide each side by 2.} \end{aligned}$$

Finally, back-substitute the value $y = 5000$ to solve for x.

$$\begin{aligned} x &= 12{,}000 - y && \text{Write Revised Equation 1.} \\ x &= 12{,}000 - 5000 && \text{Substitute 5000 for } y. \\ x &= 7000 && \text{Simplify.} \end{aligned}$$

The solution is (7000, 5000). This means that $7000 is invested at 9% and $5000 is invested at 11% to yield yearly interest of $1180. Check this in the original system.

The equations in Examples 1 and 2 are linear. Substitution can also be used to solve systems in which one or both of the equations are nonlinear.

The *Interactive* CD-ROM and *Internet* versions of this text offer a built-in graphing calculator, which can be used with the Examples, Explorations, and Exercises.

STUDY T!P

One way to check the answers you obtain in this section is to use a graphing utility. For instance, enter the two equations in Example 2

$$y_1 = 12{,}000 - x$$

$$y_2 = \frac{1180 - 0.09x}{0.11}$$

and find an appropriate viewing window that shows where the lines intersect. Then use the *intersect* feature or the *zoom* and *trace* features to find their point of intersection.

EXAMPLE 3 Substitution: Two-Solution Case

Solve the system of equations.

$$\begin{cases} x^2 + 4x - y = 7 & \qquad \text{Equation 1} \\ 2x - y = -1 & \qquad \text{Equation 2} \end{cases}$$

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Algebraic Solution

Begin by solving for y in Equation 2 to obtain $y = 2x + 1$. Next, substitute this expression for y into Equation 1 and solve for x.

$$\begin{aligned} x^2 + 4x - y &= 7 && \text{Write Equation 1.} \\ x^2 + 4x - (2x + 1) &= 7 && \text{Substitute } 2x + 1 \text{ for } y. \\ x^2 + 2x - 8 &= 0 && \text{Write in general form.} \\ (x + 4)(x - 2) &= 0 && \text{Factor.} \\ x + 4 &= 0 \Rightarrow x = -4 && \text{Set 1st factor equal to 0.} \\ x - 2 &= 0 \Rightarrow x = 2 && \text{Set 2nd factor equal to 0.} \end{aligned}$$

Back-substituting these values of x to solve for the corresponding values of y produces the solutions $(-4, -7)$ and $(2, 5)$. Check these in the original system.

Graphical Solution

To graph each equation, first solve both equations for y. Then use a graphing utility to graph the equations in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features to approximate the points of intersection of the graphs. In Figure 8.2, the points of intersection are $(-4, -7)$ and $(2, 5)$. Check that $(-4, -7)$ and $(2, 5)$ are the exact solutions by substituting *both* ordered pairs into *both* equations.

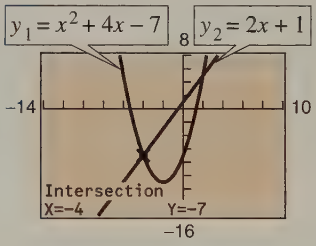


Figure 8.2

EXAMPLE 4 Substitution: No-Solution Case

Solve the system of equations.

$$\begin{cases} -x + y = 4 & \qquad \text{Equation 1} \\ x^2 + y = 3 & \qquad \text{Equation 2} \end{cases}$$

Solution

Begin by solving for y in Equation 1 to obtain $y = x + 4$. Next, substitute this expression for y into Equation 2 and solve for x.

$$\begin{aligned} x^2 + y &= 3 && \text{Write Equation 2.} \\ x^2 + (x + 4) &= 3 && \text{Substitute } x + 4 \text{ for } y. \\ x^2 + x + 1 &= 0 && \text{Simplify.} \\ x &= \frac{-1 \pm \sqrt{1 - 4}}{2} && \text{Quadratic Formula} \end{aligned}$$

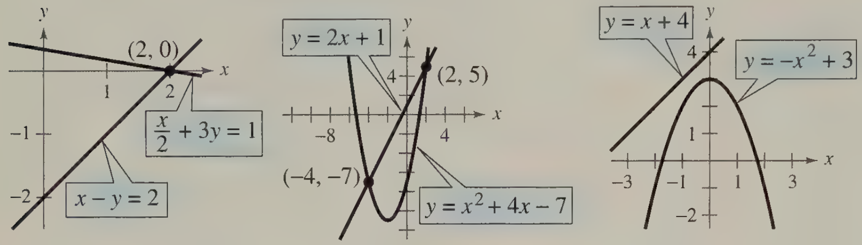
Because this yields two complex values, the equation $x^2 + x + 1 = 0$ has no *real* solution. So, the original system of equations has no *real* solution.

Try graphing the system of equations in Example 4. Do the graphs of the equations intersect? Why or why not?

STUDY TIP

When using substitution, solve for the variable that is not raised to a power in either equation. For instance, in Example 4 it would not be practical to solve for x in Equation 2. Can you see why?

From Examples 2, 3, and 4, you can see that a system of two equations in two unknowns can have exactly one solution, more than one solution, or no solution. For instance, in Figure 8.3(a), the two equations graph as two lines with a *single point* of intersection. The two equations in Example 3 graph as a parabola and a line with *two points* of intersection, as shown in Figure 8.3(b). The two equations in Example 4 graph as a line and a parabola that happen to have *no points* of intersection, as shown in Figure 8.3(c).



(a) One Intersection Point **(b) Two Intersection Points** **(c) No Intersection Points**

Figure 8.3

Example 5 shows the value of a graphical approach to solving systems of equations in two variables. Notice what would happen if you tried only the substitution method in Example 5. You would obtain the equation $x + \ln x = 1$. It would be difficult to solve this equation for x using standard algebraic techniques. In such cases, a graphical approach to solving a system of equations is more convenient.

EXAMPLE 5 Solving a System of Equations Graphically

Solve the system of equations.

$$\begin{cases} y = \ln x & \text{Equation 1} \\ x + y = 1 & \text{Equation 2} \end{cases}$$

Solution

From the graphs of these equations, shown in Figure 8.4, it is clear that there is only one point of intersection. Use the *intersect* feature or the *zoom* and *trace* features of a graphing utility to approximate the solution point as (1, 0). You can confirm this by substituting in *both* equations.

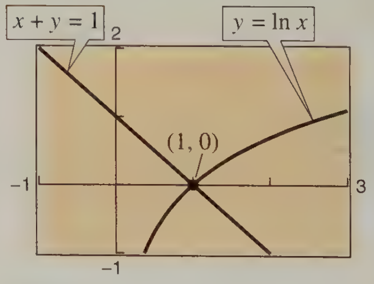


Figure 8.4

Check

$0 = \ln 1$ Equation 1 checks. ✓

$1 + 0 = 1$ Equation 2 checks. ✓

So, the solution is (1, 0).

Applications

The total cost C of producing x units of a product typically has two components: the initial cost and the cost per unit. When enough units have been sold so that the total revenue R equals the total cost, the sales are said to have reached the **break-even point.** You will find that the break-even point corresponds to the point of intersection of the cost and revenue curves.

EXAMPLE 6 Break-Even Analysis

A small business invests $10,000 in equipment to produce a product. Each unit of the product costs $0.65 to produce and is sold for $1.20. How many items must be sold before the business breaks even?

Solution

The total cost of producing x units is

$$\begin{matrix}\text{Total} \\ \text{cost}\end{matrix} = \begin{matrix}\text{Cost per} \\ \text{unit}\end{matrix} \cdot \begin{matrix}\text{Number} \\ \text{of units}\end{matrix} + \begin{matrix}\text{Initial} \\ \text{cost}\end{matrix}$$

$$C = 0.65x + 10{,}000. \qquad \text{Equation 1}$$

The revenue obtained by selling x units is

$$\begin{matrix}\text{Total} \\ \text{revenue}\end{matrix} = \begin{matrix}\text{Price per} \\ \text{unit}\end{matrix} \cdot \begin{matrix}\text{Number} \\ \text{of units}\end{matrix}$$

$$R = 1.2x. \qquad \text{Equation 2}$$

Because the break-even point occurs when $R = C$, you have

$$1.2x = 0.65x + 10{,}000 \qquad \text{Equate R and C.}$$

$$0.55x = 10{,}000 \qquad \text{Subtract } 0.65x \text{ from each side.}$$

$$x = \frac{10{,}000}{0.55} \qquad \text{Divide each side by 0.55.}$$

$$x \approx 18{,}182 \text{ units.} \qquad \text{Use a calculator.}$$

Note in Figure 8.5 that sales less than the break-even point correspond to an overall loss, whereas sales greater than the break-even point correspond to a profit. Verify the break-even point using the *intersect* feature or the *zoom* and *trace* features of a graphing utility.

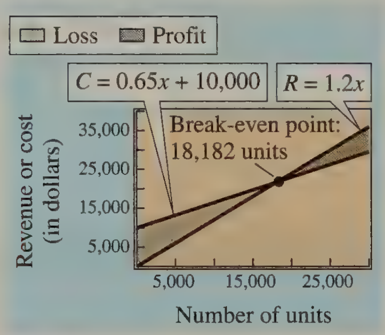


Figure 8.5

Another way to view the solution in Example 6 is to consider the profit function

$$P = R - C.$$

The break-even point occurs when the profit is 0, which is the same as saying that $R = C$.

EXAMPLE 7 State Populations

From 1990 to 1997, the population of Arizona was increasing at a faster rate than the population of Alabama. Two models that approximate the populations P (in thousands) are

$$\begin{cases} P = 3631.2 + 131.7t & \text{Arizona} \\ P = 4055.6 + 39.9t & \text{Alabama} \end{cases}$$

where $t = 0$ represents 1990. According to these two models, when would you expect the population of Arizona to have exceeded the population of Alabama? (Source: U.S. Bureau of the Census)

Algebraic Solution

Because the first equation has already been solved for P in terms of t, you can substitute this value into the second equation and solve for t, as follows.

$$3631.2 + 131.7t = 4055.6 + 39.9t$$

$$131.7t - 39.9t = 4055.6 - 3631.2$$

$$91.8t = 424.4$$

$$t \approx 4.6$$

So, from the given models, you would expect that the population of Arizona exceeded the population of Alabama sometime during 1994.

Graphical Solution

Use a graphing utility to graph $y_1 = 3631.2 + 131.7x$ and $y_2 = 4055.6 + 39.9x$ in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features of the graphing utility to approximate the point of intersection of the graphs. In Figure 8.6, the point of intersection occurs when $x \approx 4.6$.

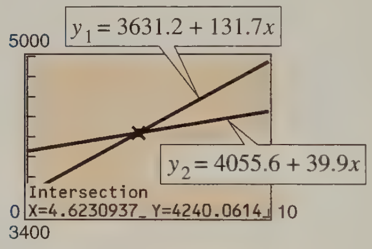


Figure 8.6

So, from the graph, it appears that the population of Arizona exceeded the population of Alabama sometime during 1994.

Writing About Math *Points of Intersection*

In this section, you learned that the graphs of two equations can intersect at zero, one, or more points. Use a graphing utility to graph each of the following systems. Decide whether the system has no solution, one solution, two solutions, or more than two solutions.

a. $\begin{cases} -0.5x^2 + 0.25x + y = -1.5 \\ \phantom{-0.5x^2 + {}} 1.75x + y = -2.625 \end{cases}$ **b.** $\begin{cases} -0.5x^2 + 0.25x + y = -1.5 \\ - 1.25x + y = -4.5 \end{cases}$

c. $\begin{cases} -0.5x^2 + 0.25x + y = -1.5 \\ \phantom{-0.5x^2 + {}} 3.5x + y = 2.25 \end{cases}$

Create three other systems of equations, one with no solution, one with one solution, and one with two solutions. Write a short paragraph that explains how you created these systems. Include a description of how you can check that the systems have the appropriate number of solutions.

8.1 Exercises

In Exercises 1–4, decide whether each ordered pair is a solution of the system of equations.

1. $\begin{cases} 4x - y = 1 \\ 6x + y = -6 \end{cases}$ (a) $(0, -3)$ (b) $(-1, -5)$ (c) $\left(-\frac{3}{2}, 3\right)$ (d) $\left(-\frac{1}{2}, -3\right)$

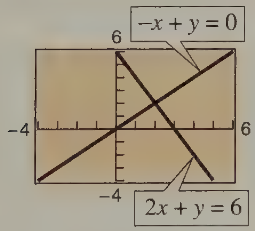
2. $\begin{cases} 4x^2 + y = 3 \\ -x - y = 11 \end{cases}$ (a) $(2, -13)$ (b) $(-2, -9)$ (c) $\left(-\frac{3}{2}, 6\right)$ (d) $\left(-\frac{7}{4}, -\frac{37}{4}\right)$

3. $\begin{cases} y = -2e^x \\ 3x - y = 2 \end{cases}$ (a) $(-2, 0)$ (b) $(0, -2)$ (c) $(0, -3)$ (d) $(-1, -5)$

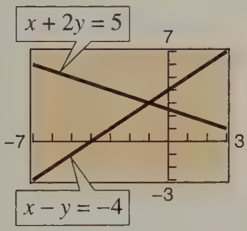
4. $\begin{cases} -\log_{10} x + 3 = y \\ \frac{1}{9}x + y = \frac{28}{9} \end{cases}$ (a) $(100, 1)$ (b) $(10, 2)$ (c) $(1, 3)$ (d) $(1, 1)$

In Exercises 5–14, solve the system by the method of substitution. Check your solution graphically.

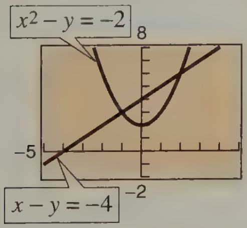
5. $\begin{cases} 2x + y = 6 \\ -x + y = 0 \end{cases}$



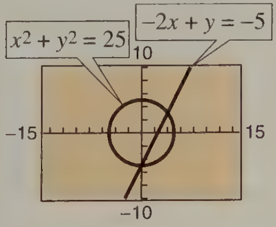
6. $\begin{cases} x - y = -4 \\ x + 2y = 5 \end{cases}$



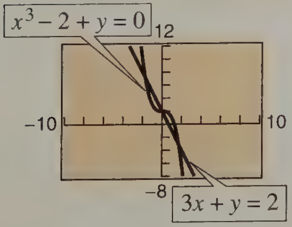
7. $\begin{cases} x - y = -4 \\ x^2 - y = -2 \end{cases}$



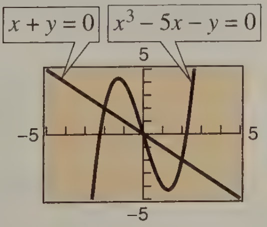
8. $\begin{cases} -2x + y = -5 \\ x^2 + y^2 = 25 \end{cases}$



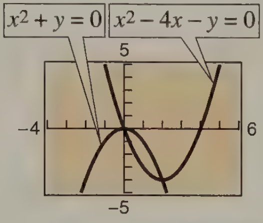
9. $\begin{cases} 3x + y = 2 \\ x^3 - 2 + y = 0 \end{cases}$



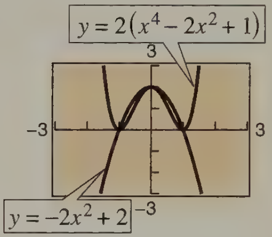
10. $\begin{cases} x + y = 0 \\ x^3 - 5x - y = 0 \end{cases}$



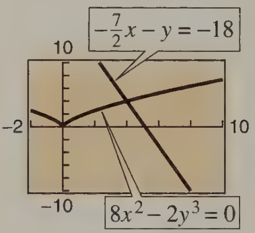
11. $\begin{cases} x^2 + y = 0 \\ x^2 - 4x - y = 0 \end{cases}$



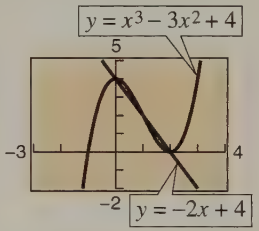
12. $\begin{cases} y = -2x^2 + 2 \\ y = 2(x^4 - 2x^2 + 1) \end{cases}$



13. $\begin{cases} -\frac{7}{2}x - y = -18 \\ 8x^2 - 2y^3 = 0 \end{cases}$



14. $\begin{cases} y = x^3 - 3x^2 + 4 \\ y = -2x + 4 \end{cases}$



In Exercises 15–28, solve the system by substitution. Use a graphing utility to verify your result.

15. $\begin{cases} x - y = 0 \\ 5x - 3y = 10 \end{cases}$

16. $\begin{cases} x + 2y = 1 \\ 5x - 4y = -23 \end{cases}$

17. $\begin{cases} 2x - y + 2 = 0 \\ 4x + y - 5 = 0 \end{cases}$

18. $\begin{cases} 6x - 3y - 4 = 0 \\ x + 2y - 4 = 0 \end{cases}$

19. $\begin{cases} 1.5x + 0.8y = 2.3 \\ 0.3x - 0.2y = 0.1 \end{cases}$

20. $\begin{cases} 0.5x + 3.2y = 9.0 \\ 0.2x - 1.6y = -3.6 \end{cases}$

21. $\begin{cases} \frac{1}{5}x + \frac{1}{2}y = 8 \\ x + y = 20 \end{cases}$

22. $\begin{cases} \frac{1}{2}x + \frac{3}{4}y = 10 \\ \frac{3}{4}x - y = 4 \end{cases}$

23. $\begin{cases} 8x + 4y = 7 \\ 2x + y = 0 \end{cases}$

24. $\begin{cases} 2x - y = 4 \\ -4x + 2y = -12 \end{cases}$

25. $\begin{cases} -\frac{5}{3}x + y = 5 \\ -5x + 3y = 6 \end{cases}$

26. $\begin{cases} -\frac{2}{3}x + y = -2 \\ 2x - 3y = 6 \end{cases}$

27. $\begin{cases} x - y = 0 \\ 2x + y = 0 \end{cases}$

28. $\begin{cases} x - 2y = 0 \\ 3x - y = 0 \end{cases}$

The *Interactive* CD-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered Section and Review Exercises. They also provide Tutorial Exercises, which link to Guided Examples for additional help.

In Exercises 29–36, solve the system of equations graphically. Verify your solutions algebraically.

29. $\begin{cases} -x + 2y = 2 \\ 3x + y = 15 \end{cases}$

30. $\begin{cases} x + y = 0 \\ 3x - 2y = 10 \end{cases}$

31. $\begin{cases} x - 3y = -2 \\ 5x + 3y = 17 \end{cases}$

32. $\begin{cases} -x + 2y = 1 \\ x - y = 2 \end{cases}$

33. $\begin{cases} x + y = 4 \\ x^2 + y^2 - 4x = 0 \end{cases}$

34. $\begin{cases} -x + y = 3 \\ x^2 - 6x - 27 + y^2 = 0 \end{cases}$

35. $\begin{cases} x - y + 3 = 0 \\ x^2 - 4x + 7 = y \end{cases}$

36. $\begin{cases} y^2 - 4x + 11 = 0 \\ -\frac{1}{2}x + y = -\frac{1}{2} \end{cases}$

In Exercises 37–50, use a graphing utility to approximate all points of intersection of the graph of the system of equations. Verify your solutions by checking them in the original system.

37. $\begin{cases} 7x + 8y = 24 \\ x - 8y = 8 \end{cases}$

38. $\begin{cases} x - y = 0 \\ 5x - 2y = 6 \end{cases}$

39. $\begin{cases} 2x - y + 3 = 0 \\ x^2 + y^2 - 4x = 0 \end{cases}$

40. $\begin{cases} 3x - 2y = 0 \\ x^2 + y^2 = 4 \end{cases}$

41. $\begin{cases} x^2 + y^2 = 8 \\ y = x^2 \end{cases}$

42. $\begin{cases} x^2 + y^2 = 25 \\ (x - 8)^2 + y^2 = 41 \end{cases}$

43. $\begin{cases} y = e^x \\ x - y + 1 = 0 \end{cases}$

44. $\begin{cases} y = -4e^{-x} \\ y + 3x + 8 = 0 \end{cases}$

45. $\begin{cases} x + 2y = 8 \\ y = \log_2 x \end{cases}$

46. $\begin{cases} y = -2 + \ln(x - 1) \\ 3y + 2x = 9 \end{cases}$

47. $\begin{cases} y = \sqrt{x} \\ y = x \end{cases}$

48. $\begin{cases} x - y = 3 \\ x - y^2 = 1 \end{cases}$

49. $\begin{cases} x^2 + y^2 = 169 \\ x^2 - 8y = 104 \end{cases}$

50. $\begin{cases} x^2 + y^2 = 4 \\ 2x^2 - y = 2 \end{cases}$

In Exercises 51–62, solve the system graphically or algebraically. Explain your choice of method.

51. $\begin{cases} y = 2x \\ y = x^2 + 1 \end{cases}$

52. $\begin{cases} x + y = 4 \\ x^2 + y = 2 \end{cases}$

53. $\begin{cases} 3x - 7y + 6 = 0 \\ x^2 - y^2 = 4 \end{cases}$

54. $\begin{cases} x^2 + y^2 = 25 \\ 2x + y = 10 \end{cases}$

55. $\begin{cases} x - 2y = 4 \\ x^2 - y = 0 \end{cases}$

56. $\begin{cases} y = (x + 1)^3 \\ y = \sqrt{x - 1} \end{cases}$

57. $\begin{cases} y - e^{-x} = 1 \\ y - \ln x = 3 \end{cases}$

58. $\begin{cases} x^2 + y = 4 \\ e^x - y = 0 \end{cases}$

59. $\begin{cases} y = x^3 - 2x^2 + 1 \\ y = 1 - x^2 \end{cases}$

60. $\begin{cases} y = x^3 - 2x^2 + x - 1 \\ y = -x^2 + 3x - 1 \end{cases}$

61. $\begin{cases} xy - 1 = 0 \\ 2x - 4y + 7 = 0 \end{cases}$

62. $\begin{cases} x - 2y = 1 \\ y = \sqrt{x - 1} \end{cases}$

Break-Even Analysis **In Exercises 63–66, use a graphing utility to graph the cost and revenue functions in the same viewing window. Find the sales x necessary to break even ($R = C$) and the corresponding revenue R obtained by selling x units. (Round to the nearest whole unit.)**

	Cost	*Revenue*
63.	$C = 8650x + 250{,}000$	$R = 9950x$
64.	$C = 2.65x + 350{,}000$	$R = 4.15x$
65.	$C = 5.5\sqrt{x} + 10{,}000$	$R = 3.29x$
66.	$C = 7.8\sqrt{x} + 18{,}500$	$R = 12.84x$

67. ***Break-Even Point*** **A small business invests \$16,000 to produce an item that will sell for \$5.95. Each unit can be produced for \$3.45.**

(a) Write cost and revenue functions for x units produced and sold.

(b) Use a graphing utility to graph the cost and revenue functions in the same viewing window. Use the graph to approximate the number of units that must be sold to break even.

(c) Verify the result of part (b) algebraically.

68. ***Break-Even Point*** A small business has an initial investment of \$5000. The unit cost of the product is \$21.60, and the selling price is \$34.10.

(a) Write cost and revenue functions for x units produced and sold.

(b) Use a graphing utility to graph the cost and revenue functions in the same viewing window. Use the graph to approximate the number of units that must be sold to break even.

(c) Verify the result of part (b) algebraically.

69. ***Choice of Two Jobs*** You are offered two different jobs selling dental supplies.

- One company offers a straight commission of 6% of sales.
- The other company offers a salary of \$250 per week plus 3% of sales.

How much would you have to sell in a week in order to make the straight commission offer better?

70. ***Choice of Two Jobs*** You are offered two different jobs selling college textbooks.

- One company offers an annual salary of \$25,000 plus a year-end bonus of 1% of your total sales.
- The other company offers an annual salary of \$20,000 plus a year-end bonus of 2% of your total sales.

Determine the annual sales that make the second offer better.

71. ***Finance*** A total of \$20,000 is invested in two funds paying 6.5% and 8.5% simple interest. The 6.5% investment has a lower risk. The investor wants a yearly interest check of \$1600 from the investment.

(a) Write a system of equations in which one equation represents the total amount invested and the other equation represents the \$1600 required in interest. Let x and y represent the amounts invested at 6.5% and 8.5%, respectively.

(b) Use a graphing utility to graph the two equations in the same viewing window. As the amount invested at 6.5% increases, how do the amount invested at 8.5% and the amount of interest each change? Explain.

(c) What is the most that can be invested at 6.5% to meet the requirement of \$1600 per year in interest?

72. ***Finance*** A total of \$25,000 is invested in two funds paying 6% and 8.5% simple interest. The 6% investment has a lower risk. The investor wants a yearly interest income of \$2000 from the two investments.

(a) Write a system of equations in which one equation represents the total amount invested and the other equation represents the \$2000 required in interest. Let x and y represent the amounts invested at 6% and 8.5%, respectively.

(b) Use a graphing utility to graph the two equations in the same viewing window. As the amount invested at 6% increases, how do the amount invested at 8.5% and the amount of interest each change? Explain.

(c) What is the most that can be invested at 6% to meet the requirement of \$2000 per year in interest?

73. ***Log Volume*** You are offered two different rules for estimating the number of board feet in a log that is 16 feet long. A board foot is a unit of measure for lumber equal to a board one foot square and one inch thick. One rule is the *Doyle Log Rule* and is modeled by

$$V = (D - 4)^2, \quad 5 \le D \le 40$$

and the other rule is the *Scribner Log Rule* and is modeled by

$$V = 0.79D^2 - 2D - 4, \quad 5 \le D \le 40$$

where D is the diameter of the log and V is its volume in board feet.

(a) Use a graphing utility to graph the log rules in the same viewing window.

(b) For what diameter do the two rules agree?

(c) If you were selling large logs, which rule would you use? Explain your reasoning.

Geometry **In Exercises 74–77, find the dimensions of the rectangle meeting the specified conditions.**

74. The perimeter is 30 meters and the length is 3 meters greater than the width.

75. The perimeter is 280 centimeters and the width is 20 centimeters less than the length.

76. The perimeter is 42 inches and the width is three-fourths the length.

77. The perimeter is 210 feet and the length is $1\frac{1}{2}$ times the width.

78. ***Geometry*** What are the dimensions of a rectangular tract of land if its perimeter is 40 miles and its area is 96 square miles?

79. ***Geometry*** What are the dimensions of an isosceles right triangle with a 2-inch hypotenuse and an area of 1 square inch?

80. ***Data Analysis*** The table shows the amount y (in millions of short tons) of paperboard produced in the United States in the years 1993 through 1996. (Source: American Forest and Paper Association)

Year	1993	1994	1995	1996
y	43.1	45.7	46.6	47.9

(a) Use the regression capabilities of a graphing utility to find a linear model and a quadratic model that represent the data in the interval from 1993 through 1996. (Let $t = 3$ represent 1993.)

(b) Use the graphing utility to graph the data and the two models in the same viewing window.

(c) Approximate the points of intersection of the graphs of the models.

(d) Use the models to estimate newsprint production in 1998. Which model do you think gives the more accurate estimate? Explain.

81. ***Data Analysis*** The table shows the average hourly earnings E of production workers in manufacturing industries in the United States for the years 1992 to 1997. (Source: U.S. Bureau of Labor Statistics)

Year	1992	1993	1994
E	\$11.46	\$11.74	\$12.06

Year	1995	1996	1997
E	\$12.37	\$12.78	\$13.17

Let t represent the time in years, with $t = 2$ corresponding to 1992.

(a) Use the regression capabilities of a graphing utility to fit a linear model and a quadratic model to the data.

(b) Use the graphing utility to graph the data and the two models in the same viewing window.

(c) Approximate the points of intersections of the graphs of the models.

(d) Use the models to estimate the average hourly earnings in 2000. Which model do you think gives the more accurate estimate? Explain.

Synthesis

True or False? **In Exercises 82 and 83, determine whether the statement is true or false. Justify your answer.**

82. In order to solve a system of equations by substitution, you must always solve for y in one of the two equations and then back-substitute.

83. If a system consists of a parabola and a circle, then it can have at most two solutions.

84. ***Think About It*** When solving a system of equations by substitution, how do you recognize that the system has no solution?

85. ***Writing*** Write a brief paragraph describing any advantages of substitution over the graphical method of solving a system of equations.

86. ***Exploration*** Find an equation of a line whose graph intersects the graph of the parabola $y = x^2$ at the following numbers of points. (There are many correct answers.)

(a) Two points

(b) One point

(c) No points

87. ***Conjecture***

(a) Use a graphing utility to graph the system of equations

$$\begin{cases} y = b^x \\ y = x^b \end{cases}$$

for $b = 2$ and $b = 4$.

(b) For a fixed value of $b > 1$, make a conjecture about the number of points of intersection of the graphs in part (a).

Review

In Exercises 88–93, find the general form of the equation of the line through the two points.

88. $(-2, 7), (5, 5)$

89. $(3.5, 4), (10, 6)$

90. $(6, 3), (10, 3)$

91. $(4, -2), (4, 5)$

92. $\left(\frac{3}{5}, 0\right), (4, 6)$

93. $\left(-\frac{7}{3}, 8\right), \left(\frac{5}{2}, \frac{1}{2}\right)$

In Exercises 94–97, find the domain of the function and identify any horizontal or vertical asymptotes.

94. $f(x) = \dfrac{5}{x - 6}$

95. $f(x) = \dfrac{2x - 7}{3x + 2}$

96. $f(x) = \dfrac{x^2 + 2}{x^2 - 16}$

97. $f(x) = 3 - \dfrac{2}{x^2}$

In Exercises 98–101, solve the equation. Round your result to three decimal places.

98. $6^{7-x} = 1249$

99. $8(10^{2x}) = 28$

100. $e^{2x} - 9e^x + 18 = 0$

101. $\log_{10}(x + 3) - \log_{10} x = \log_{10}(x + 8)$

8.2 Systems of Linear Equations in Two Variables

The Method of Elimination

In Section 8.1, you studied two methods for solving a system of equations: substitution and graphing. Now you will study the **method of elimination.** The key step in this method is to obtain, for one of the variables, coefficients that differ only in sign so that *adding* the equations eliminates the variable.

$$\begin{array}{ll}\left\{\begin{array}{r}3x + 5y = 7\\ -3x - 2y = -1\end{array}\right. & \begin{array}{l}\text{Equation 1}\\ \text{Equation 2}\end{array}\\ \hline 3y = 6 & \text{Add equations.}\end{array}$$

Note that by adding the two equations, you eliminate the variable x and obtain a single equation in y. Solving this equation for y produces $y = 2$, which you can then back-substitute into one of the original equations to solve for x.

EXAMPLE 1 The Method of Elimination

Solve the system of linear equations.

$$\begin{cases}3x + 2y = 4 & \text{Equation 1}\\ 5x - 2y = 8 & \text{Equation 2}\end{cases}$$

Solution

You can eliminate y by adding the two equations.

$$\begin{array}{rll}3x + 2y &= 4 & \text{Write Equation 1.}\\ 5x - 2y &= 8 & \text{Write Equation 2.}\\ \hline 8x &= 12 & \text{Add equations.}\end{array}$$

So, $x = \frac{3}{2}$. By back-substituting, you can solve for y.

$$\begin{array}{rll}3x + 2y &= 4 & \text{Write Equation 1.}\\ 3\left(\frac{3}{2}\right) + 2y &= 4 & \text{Substitute } \frac{3}{2} \text{ for } x.\\ y &= -\frac{1}{4} & \text{Solve for } y.\end{array}$$

The solution is $\left(\frac{3}{2}, -\frac{1}{4}\right)$. You can check the solution *algebraically* by substituting into the original system, or graphically as shown in Section 8.1.

Check

$$\begin{array}{rll}3\left(\frac{3}{2}\right) + 2\left(\ \frac{1}{4}\right) &\stackrel{?}{=} 4 & \text{Substitute into Equation 1.}\\ \frac{9}{2} - \frac{1}{2} &= 4 & \text{Equation 1 checks. } \checkmark\\ 5\left(\frac{3}{2}\right) - 2\left(-\frac{1}{4}\right) &\stackrel{?}{=} 8 & \text{Substitute into Equation 2.}\\ \frac{15}{2} + \frac{1}{2} &= 8 & \text{Equation 2 checks. } \checkmark\end{array}$$

Try using substitution to solve the system given in Example 1. Which method is easier? Many people find that the method of elimination is more efficient.

What You Should Learn:

- How to use the method of elimination to solve systems of linear equations in two variables
- How to graphically interpret the number of solutions of systems of linear equations in two variables
- How to use systems of linear equations in two variables to model and solve real-life problems

Why You Should Learn It:

The method of elimination is one method of solving a system of linear equations. For instance, Exercise 72 on page 585 shows how to use a system of linear equations to recover information about types of shoes that were sold in a shoe store.

Frank Siteman/PhotoEdit

The Method of Elimination

To use the **method of elimination** to solve a system of two linear equations in x and y, perform the following steps.

1. Obtain coefficients for x (or y) that differ only in sign by multiplying all terms of one or both equations by suitably chosen constants.
2. Add the equations to eliminate one variable; solve the resulting equation.
3. Back-substitute the value obtained in Step 2 into either of the original equations and solve for the other variable.
4. Check your solution in both of the original equations.

EXAMPLE 2 The Method of Elimination

Solve the system of linear equations.

$$\begin{cases} 5x + 3y = 9 & \text{Equation 1} \\ 2x - 4y = 14 & \text{Equation 2} \end{cases}$$

Algebraic Solution

You can obtain coefficients that differ only in sign by multiplying Equation 1 by 4 and multiplying Equation 2 by 3.

$$\begin{array}{rclrcl} 5x + 3y = 9 & \Rightarrow & 20x + 12y = 36 & & \text{Multiply Equation 1 by 4.} \\ \underline{2x - 4y = 14} & \Rightarrow & \underline{6x - 12y = 42} & & \text{Multiply Equation 2 by 3.} \\ & & 26x \qquad = 78 & & \text{Add equations.} \end{array}$$

From this equation, you can see that $x = 3$. By back-substituting this value of x into Equation 2, you can solve for y.

$$\begin{array}{rcll} 2x - 4y &=& 14 & \text{Write Equation 2.} \\ 2(3) - 4y &=& 14 & \text{Substitute 3 for } x. \\ -4y &=& 8 & \text{Combine like terms.} \\ y &=& -2 & \text{Solve for } y. \end{array}$$

The solution is $(3, -2)$. You can check the solution algebraically by substituting into the original system.

Graphical Solution

Solve each equation for y. Then use a graphing utility to graph $y_1 = 3 - \frac{5}{3}x$ and $y_2 = -\frac{7}{2} + \frac{1}{2}x$ in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features to approximate the point of intersection of the graphs. From the graphs in Figure 8.7, the point of intersection is $(3, -2)$, so the approximate solution is $(3, -2)$. You can determine that this is the exact solution by checking $(3, -2)$ in both equations.

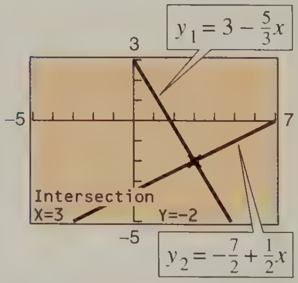


Figure 8.7

In Example 2, the two systems of linear equations

$$\begin{cases} 5x + 3y = 9 \\ 2x - 4y = 14 \end{cases} \quad \text{and} \quad \begin{cases} 20x + 12y = 36 \\ 6x - 12y = 42 \end{cases}$$

are called **equivalent systems** because they have precisely the same solution set. The operations that can be performed on a system of linear equations to produce an equivalent system are (1) interchanging any two equations, (2) multiplying an equation by a nonzero constant, and (3) adding a multiple of one equation to any other equation in the system.

Graphical Interpretation of Two-Variable Systems

It is possible for a *general* system of equations to have exactly one solution, two or more solutions, or no solution. If a system of *linear* equations has two different solutions, it must have an *infinite* number of solutions. To see why this is true, consider the following graphical interpretations of a system of two linear equations in two variables. (Remember that the graph of a linear equation in two variables is a straight line.)

Graphical Interpretation of Solutions

For a system of two linear equations in two variables, the number of solutions is one of the following.

Number of Solutions	*Graphical Interpretation*
1. Exactly one solution	The two lines intersect at one point.
2. Infinitely many solutions	The two lines are identical.
3. No solution	The two lines are parallel.

A system of linear equations is **consistent** if it has at least one solution. It is **inconsistent** if it has no solution.

Exploration

Use a graphing utility to graph each system of equations.

a. $\begin{cases} y = 5x + 1 \\ y - x = -5 \end{cases}$

b. $\begin{cases} 3y = 4x - 1 \\ -8x + 2 = -6y \end{cases}$

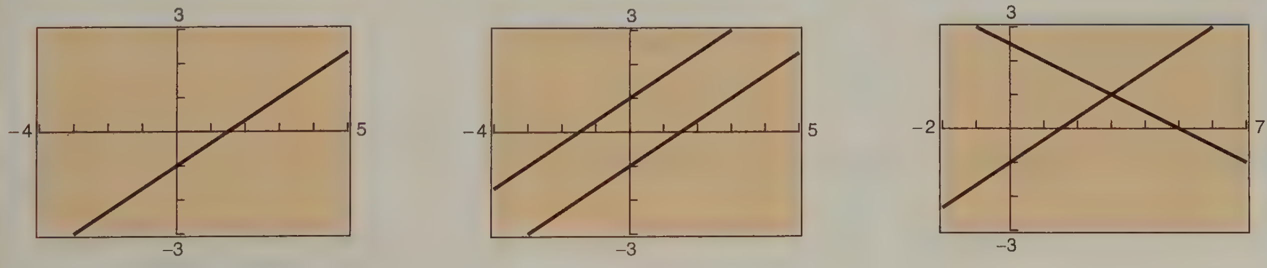
c. $\begin{cases} 2y = -x + 3 \\ -4 = y + \frac{1}{2}x \end{cases}$

Determine the number of solutions each system has. Explain your reasoning.

EXAMPLE 3 Recognizing Graphs of Linear Systems

Match each system of linear equations (a, b, c) with its graph (i, ii, iii) in Figure 8.8. Describe the number of solutions. Then state whether the system is consistent or inconsistent.

a. $\begin{cases} 2x - 3y = 3 \\ -4x + 6y = 6 \end{cases}$ **b.** $\begin{cases} 2x - 3y = 3 \\ x + 2y = 5 \end{cases}$ **c.** $\begin{cases} 2x - 3y = 3 \\ -4x + 6y = -6 \end{cases}$



i. ii. iii.

Figure 8.8

A computer simulation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Solution

a. The graph is a pair of parallel lines (ii). The lines have no point of intersection, so the system has no solution. The system is inconsistent.

b. The graph is a pair of intersecting lines (iii). The lines have one point of intersection, so the system has exactly one solution. The system is consistent.

c. The graph is a pair of lines that coincide (i). The lines have infinitely many points of intersection, so the system has infinitely many solutions. The system is consistent.

In Examples 4 and 5, you can use the method of elimination to determine that a system of linear equations has no solution or infinitely many solutions.

EXAMPLE 4 The Method of Elimination: No-Solution Case

Solve the system of linear equations.

$$\begin{cases} x - 2y = 3 & \text{Equation 1} \\ -2x + 4y = 1 & \text{Equation 2} \end{cases}$$

Algebraic Solution

To obtain coefficients that differ only in sign, multiply Equation 1 by 2.

$$\begin{array}{rcr} x - 2y = 3 & \Rightarrow & 2x - 4y = 6 \\ \underline{-2x + 4y = 1} & \Rightarrow & \underline{-2x + 4y = 1} \\ & & 0 = 7 \end{array}$$

By adding the equations, you obtain $0 = 7$. Because there are no values of x and y for which $0 = 7$, this is a false statement. So, you can conclude that the system is inconsistent and has no solution.

Graphical Solution

Solve each equation for y: $y_1 = -\frac{3}{2} + \frac{1}{2}x \qquad y_2 = \frac{1}{4} + \frac{1}{2}x$

Notice that the lines have the same slope, so they are parallel. You can use a graphing utility to verify this by graphing both equations as shown in Figure 8.9. Then try using the *intersect* feature to find a point of intersection. Because the graphing utility cannot find a point of intersection, you will get an error. Use the *zoom* feature and then try using the *intersect* feature again. Because repeated zooming yields the same error message, you can conclude that the lines are parallel, and, therefore, the system has no solution.

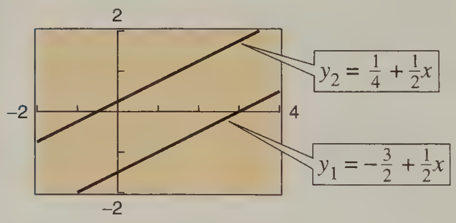


Figure 8.9

In Example 4, note that the occurrence of a false statement, such as $0 = 7$, indicates that the system has no solution. In the next example, note that the occurrence of a statement that is true for all values of the variables—in this case, $0 = 0$—indicates that the system has infinitely many solutions.

EXAMPLE 5 The Method of Elimination: Infinitely Many Solutions Case

Solve the system of linear equations: $\begin{cases} 2x - y = 1 & \text{Equation 1} \\ 4x - 2y = 2 & \text{Equation 2} \end{cases}$

Solution

To obtain coefficients that differ only in sign, multiply Equation 2 by $-\frac{1}{2}$.

$$\begin{array}{rcrl} 2x - y = 1 & \Rightarrow & 2x - y = 1 & \text{Write Equation 1.} \\ \underline{4x - 2y = 2} & \Rightarrow & \underline{-2x + y = -1} & \text{Multiply Equation 2 by } -\tfrac{1}{2}. \\ & & 0 = 0 & \text{Add equations.} \end{array}$$

Because $0 = 0$ for all values of x and y, the two equations turn out to be equivalent (have the same solution set). You can conclude that the system has infinitely many solutions. The solution set consists of all points (x, y) lying on the line $2x - y = 1$, as shown in Figure 8.10.

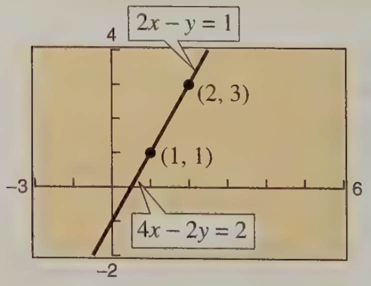


Figure 8.10

Example 6 illustrates a strategy for solving a system of linear equations that has decimal coefficients.

EXAMPLE 6 A Linear System Having Decimal Coefficients

Solve the system of linear equations.

$$\begin{cases} 0.02x - 0.05y = -0.38 & \text{Equation 1} \\ 0.03x + 0.04y = 1.04 & \text{Equation 2} \end{cases}$$

Solution

Because the coefficients are two-place decimals, begin by multiplying each equation by 100 to produce a system with integer coefficients.

$$\begin{cases} 2x - 5y = -38 & \text{Revised Equation 1} \\ 3x + 4y = 104 & \text{Revised Equation 2} \end{cases}$$

Now, to obtain coefficients that differ only in sign, multiply Revised Equation 1 by 3 and multiply Revised Equation 2 by -2.

$$\begin{array}{rcrl} 2x - 5y = -38 & \Longrightarrow & 6x - 15y = -114 & \text{Multiply Revised Equation 1 by 3.} \\ \underline{3x + 4y = 104} & \Longrightarrow & \underline{-6x - 8y = -208} & \text{Multiply Revised Equation 2 by } -2. \\ & & -23y = -322 & \text{Add equations.} \end{array}$$

So, you can conclude that

$$y = \frac{-322}{-23} = 14.$$

Back-substituting this value into Revised Equation 2 produces the following.

$$\begin{aligned} 3x + 4y &= 104 && \text{Write revised Equation 2.} \\ 3x + 4(14) &= 104 && \text{Substitute 14 for } y. \\ 3x &= 48 && \text{Combine like terms.} \\ x &= 16 && \text{Solve for } x. \end{aligned}$$

The solution is (16, 14). Check this in the original system, as follows.

Check

$$\begin{aligned} 0.02(16) - 0.05(14) &\overset{?}{=} -0.38 && \text{Substitute into Equation 1.} \\ 0.32 - 0.70 &= -0.38 && \text{Equation 1 checks. } \checkmark \\ 0.03(16) + 0.04(14) &\overset{?}{=} 1.04 && \text{Substitute into Equation 2.} \\ 0.48 + 0.56 &= 1.04 && \text{Equation 2 checks. } \checkmark \end{aligned}$$

STUDY T!P

The general solution of the linear system

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

is $x = (ce - bf)/(ae - db)$ and $y = (af - cd)/(ae - db)$. If $ae - db = 0$, the system does not have a unique solution. A program called SOLVE for solving such a system is given on our website, *college.hmco.com.* Try using this program to check the solution of the system in Example 6.

The *Interactive* CD-ROM and *Internet* versions of this text show every example with its solution; clicking on the *Try It!* button brings up similar problems. Guided Examples and Integrated Examples show step-by-step solutions to additional examples. Integrated Examples are related to several concepts in the section.

Applications

At this point, you may be asking the question "How can I tell which application problems can be solved using a system of linear equations?" The answer comes from the following considerations.

1. Does the problem involve more than one unknown quantity?
2. Are there two (or more) equations or conditions to be satisfied?

If one or both of these conditions occur, the appropriate mathematical model for the problem may be a system of linear equations. Example 7 shows how to construct such a model.

EXAMPLE 7 An Application of a Linear System

An airplane flying into a headwind travels the 2000-mile flying distance between two cities in 4 hours and 24 minutes. On the return flight, the same distance is traveled in 4 hours. Find the airspeed of the plane and the speed of the wind, assuming that both remain constant.

Solution

The two unknown quantities are the speeds of the wind and the plane. If r_1 is the speed of the plane and r_2 is the speed of the wind, then

$r_1 - r_2 =$ speed of the plane *against* the wind

$r_1 + r_2 =$ speed of the plane *with* the wind

as shown in Figure 8.11. Using the formula

Distance = (rate)(time)

for these two speeds, you obtain the following equations.

$$2000 = (r_1 - r_2)\left(4 + \frac{24}{60}\right)$$

$$2000 = (r_1 + r_2)(4)$$

These two equations simplify as follows.

$$\begin{cases} 5000 = 11r_1 - 11r_2 & \text{Equation 1} \\ 500 = r_1 + r_2 & \text{Equation 2} \end{cases}$$

By elimination, the solution is

$$r_1 = \frac{5250}{11} \approx 477.27 \text{ miles per hour} \quad \text{Speed of plane}$$

$$r_2 = \frac{250}{11} \approx 22.73 \text{ miles per hour.} \quad \text{Speed of wind}$$

Check this solution in the original statement of the problem.

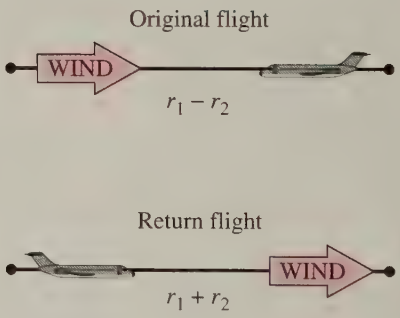
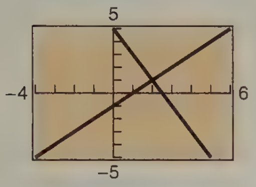


Figure 8.11

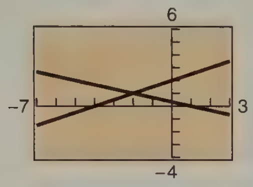
8.2 Exercises

In Exercises 1–10, solve the system of equations by the method of elimination. Label each graphed line with its equation.

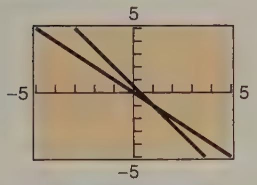
1. $\begin{cases} 2x + y = 5 \\ x - y = 1 \end{cases}$



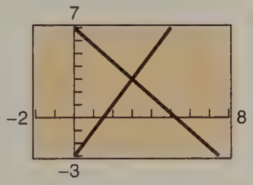
2. $\begin{cases} x + 3y = 1 \\ -x + 2y = 4 \end{cases}$



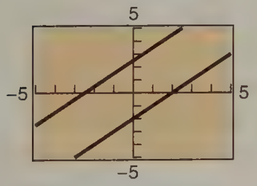
3. $\begin{cases} x + y = 0 \\ 3x + 2y = 1 \end{cases}$



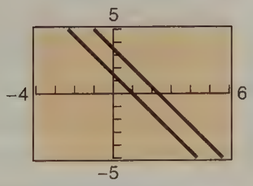
4. $\begin{cases} 2x - y = 3 \\ 4x + 3y = 21 \end{cases}$



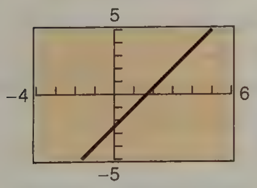
5. $\begin{cases} x - y = 2 \\ -2x + 2y = 5 \end{cases}$



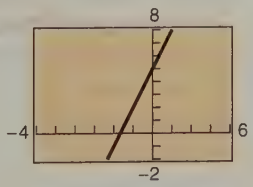
6. $\begin{cases} 3x + 2y = 3 \\ 6x + 4y = 14 \end{cases}$



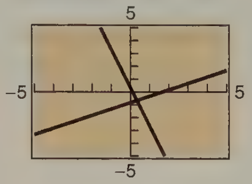
7. $\begin{cases} 3x - 2y = 5 \\ -6x + 4y = -10 \end{cases}$



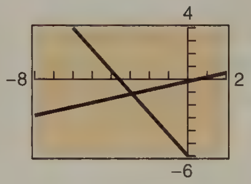
8. $\begin{cases} 9x - 3y = -15 \\ -3x + y = 5 \end{cases}$



9. $\begin{cases} 9x + 3y = 1 \\ 3x - 6y = 5 \end{cases}$



10. $\begin{cases} 5x + 3y = -18 \\ 2x - 6y = 1 \end{cases}$



In Exercises 11–20, solve the system by the method of elimination and check any solution algebraically.

11. $\begin{cases} x + 2y = 4 \\ x - 2y = 1 \end{cases}$

12. $\begin{cases} 3x - 5y = 2 \\ 2x + 5y = 13 \end{cases}$

13. $\begin{cases} 2x + 3y = 18 \\ 5x - y = 11 \end{cases}$

14. $\begin{cases} x + 7y = 12 \\ 3x - 5y = 10 \end{cases}$

15. $\begin{cases} 3x + 2y = 10 \\ 2x + 5y = 3 \end{cases}$

16. $\begin{cases} 2r + 4s = 5 \\ 16r + 50s = 55 \end{cases}$

17. $\begin{cases} 5u + 6v = 24 \\ 3u + 5v = 18 \end{cases}$

18. $\begin{cases} 3x + 11y = 4 \\ -2x - 5y = 9 \end{cases}$

19. $\begin{cases} 1.8x + 1.2y = 4 \\ 9x + 6y = 3 \end{cases}$

20. $\begin{cases} 3.1x - 2.9y = -10.2 \\ 31x - 12y = 34 \end{cases}$

In Exercises 21–32, solve the system by the method of elimination and verify any solution using a graphing utility.

21. $\begin{cases} 4b + 3m = 3 \\ 3b + 11m = 13 \end{cases}$

22. $\begin{cases} 2x + 5y = 8 \\ 5x + 8y = 10 \end{cases}$

23. $\begin{cases} \dfrac{x}{4} + \dfrac{y}{6} = 1 \\ \dfrac{1}{2}x - \dfrac{1}{2}y = \dfrac{1}{6} \end{cases}$

24. $\begin{cases} \dfrac{2}{3}x + \dfrac{1}{6}y = \dfrac{2}{3} \\ \dfrac{10}{3}x - \dfrac{5}{6}y = \dfrac{10}{3} \end{cases}$

25. $\begin{cases} \frac{3}{4}x + y = \frac{1}{8} \\ \frac{9}{4}x + 3y = \frac{3}{8} \end{cases}$

26. $\begin{cases} \frac{2}{5}x - \frac{3}{2}y = 4 \\ \frac{1}{5}x - \frac{3}{4}y = -2 \end{cases}$

27. $\begin{cases} \dfrac{x+3}{4} + \dfrac{y-1}{3} = 1 \\ 2x - y = 12 \end{cases}$

28. $\begin{cases} \dfrac{x-1}{2} + \dfrac{y+2}{3} = 4 \\ x - 2y = 5 \end{cases}$

29. $\begin{cases} 2.5x - 3y = 1.5 \\ 2x - 2.4y = 1.2 \end{cases}$

30. $\begin{cases} 0.2x - 0.5y = -27.8 \\ 0.3x + 0.4y = 68.7 \end{cases}$

31. $\begin{cases} 0.02x - 0.05y = -0.19 \\ 0.03x + 0.04y = 0.52 \end{cases}$

32. $\begin{cases} 0.05x - 0.03y = 0.21 \\ 0.07x + 0.02y = 0.16 \end{cases}$

In Exercises 33–40, use a graphing utility to graph the lines in the system. Use the graphs to determine whether the system is consistent or inconsistent. If the system is consistent, determine the solution. Verify your result algebraically.

33. $\begin{cases} 2x - 5y = 0 \\ x - y = 3 \end{cases}$

34. $\begin{cases} 2x + y = 5 \\ x - 2y = -1 \end{cases}$

35. $\begin{cases} \frac{3}{5}x - y = 3 \\ -3x + 5y = 9 \end{cases}$

36. $\begin{cases} 4x - 6y = 9 \\ \frac{16}{3}x - 8y = 12 \end{cases}$

37. $\begin{cases} x + 7y = 2 \\ 4x - y = 9 \end{cases}$

38. $\begin{cases} 8x - 14y = 5 \\ 2x - 3.5y = 1.25 \end{cases}$

39. $\begin{cases} -x + 7y = 3 \\ -\frac{1}{7}x + y = 5 \end{cases}$

40. $\begin{cases} -7x + 6y = -4 \\ \frac{7}{6}x + y = -1 \end{cases}$

In Exercises 41–48, use a graphing utility to graph the two equations. Use the graphs to approximate the solution of the system.

41. $\begin{cases} 8x + 9y = 42 \\ 6x - y = 16 \end{cases}$

42. $\begin{cases} 4y = -8 \\ 7x - 2y = 25 \end{cases}$

43. $\begin{cases} \frac{3}{2}x - \frac{1}{5}y = 8 \\ -2x + 3y = 3 \end{cases}$

44. $\begin{cases} \frac{3}{4}x - \frac{5}{2}y = -9 \\ -x + 6y = 28 \end{cases}$

45. $\begin{cases} 0.5x + 2.2y = 9 \\ 6x + 0.4y = -22 \end{cases}$

46. $\begin{cases} 2.4x + 3.8y = -17.6 \\ 4x - 0.2y = -3.2 \end{cases}$

47. $\begin{cases} 7x - 2y = 24 \\ 5x + 6y = -20 \end{cases}$

48. $\begin{cases} 10x - 13y = -20 \\ 8x + 11y = -16 \end{cases}$

In Exercises 49–56, use any method to solve the system.

49. $\begin{cases} 3x - 5y = 7 \\ 2x + y = 9 \end{cases}$

50. $\begin{cases} -x + 3y = 17 \\ 4x + 3y = 7 \end{cases}$

51. $\begin{cases} y = 2x - 5 \\ y = 5x - 11 \end{cases}$

52. $\begin{cases} 7x + 3y = 16 \\ y = x + 2 \end{cases}$

53. $\begin{cases} x - 5y = 21 \\ 6x + 5y = 21 \end{cases}$

54. $\begin{cases} y = -3x - 8 \\ y = 15 - 2x \end{cases}$

55. $\begin{cases} -2x + 8y = 19 \\ y = x - 3 \end{cases}$

56. $\begin{cases} 4x - 3y = 6 \\ -5x + 7y = -1 \end{cases}$

Exploration **In Exercises 57–60, find a system of linear equations that has the given solution. (There are many correct answers.)**

57. $(6, 3)$

58. $(8, -2)$

59. $\left(3, \frac{5}{2}\right)$

60. $\left(-\frac{2}{3}, -10\right)$

Supply and Demand **In Exercises 61–64, find the *point of equilibrium* of the demand and supply equations. The point of equilibrium is defined by the price p and the number of units x that satisfy both the demand and supply equations.**

	Demand	*Supply*
61.	$p = 50 - 0.5x$	$p = 0.125x$
62.	$p = 100 - 0.05x$	$p = 25 + 0.1x$
63.	$p = 140 - 0.00002x$	$p = 80 + 0.00001x$
64.	$p = 400 - 0.0002x$	$p = 225 + 0.0005x$

65. ***Airplane Speed*** An airplane flying into a headwind travels the 1800-mile flying distance between two cities in 3 hours and 36 minutes. On the return flight, the distance is traveled in 3 hours. Find the airspeed of the plane and the speed of the wind, assuming that both remain constant.

66. ***Airplane Speed*** Two planes start from the same airport and fly in opposite directions. The second plane starts $\frac{1}{2}$ hour after the first plane, but its speed is 80 kilometers per hour faster. Find the airspeed of each plane if 2 hours after the first plane departs the planes are 3200 kilometers apart.

67. ***Acid Mixture*** Ten liters of a 30% acid solution is obtained by mixing a 20% and a 50% solution.

(a) Write a system of equations. One equation represents the amount of final mixture required and the other represents the amount of acid in the final mixture. Let x and y represent the amounts of 20% and 50% solutions, respectively.

(b) Use a graphing utility to graph the two equations in part (a) in the same viewing window. As the amount of the 20% solution increases, how does the amount of the 50% solution change?

(c) How much of each solution is required to obtain the specified concentration in the final mixture?

68. ***Fuel Mixture*** Five hundred gallons of 89-octane gasoline is obtained by mixing 87-octane gasoline with 92-octane gasoline.

(a) Write a system of equations. One equation represents the amount of final mixture required and the other represents the amount of 87- and

92-octane gasoline in the final mixture. Let x and y represent the number of 87-octane gallons and 92-octane gallons, respectively.

(b) Use a graphing utility to graph the two equations in part (a). As the amount of 87-octane gasoline increases, how does the amount of 92-octane gasoline change?

(c) How much of each type of gasoline is required to obtain the 500 gallons of 89-octane gasoline?

69. ***Finance*** A total of \$12,000 is invested in two corporate bonds that pay 7.5% and 9% simple interest. The investor wants an annual interest income of \$990 from the investments. What is the most that can be invested in the 7.5% bond?

70. ***Finance*** A total of \$32,000 is invested in two municipal bonds that pay 5.75% and 6.25% simple interest. The investor wants an annual interest income of \$1900 from the investments. What is the most that can be invested in the 5.75% bond?

71. ***Ticket Sales*** Five hundred tickets were sold for one performance of a play. The tickets for adults and children sold for \$7.50 and \$4.00, respectively, and the total receipts for the performance were \$3312.50. How many of each kind of ticket were sold?

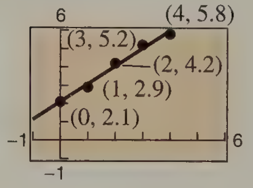
72. ***Shoe Sales*** On Saturday night, the manager of a shoe store evaluates the receipts of the previous week's sales. Two hundred and forty pairs of two different styles of tennis shoes were sold. One style sold for \$66.95 and the other sold for \$84.95. The total receipts were \$17,652. The cash register that was supposed to record the number of each type of shoe sold malfunctioned. Can you recover the information? If so, how many shoes of each type were sold?

73. ***Driving Distances*** In a trip of 300 kilometers, two people drive. One person drives three times as far as the other. Find the distance that each person drives.

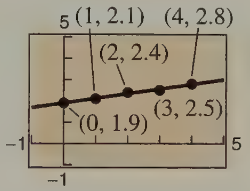
74. ***Truck Scheduling*** A contractor hires two trucking companies to haul 1600 tons of crushed stone for a highway construction project. The contracts state that one company is to haul four times as much as the other. Find the amount hauled by each company.

Fitting a Line to Data **In Exercises 75–78, find the least squares regression line $y = ax + b$ for the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ by solving the system for a and b. Then use the regression capabilities of a graphing utility to confirm your result. (For an explanation of how the coefficients of a and b in the system are obtained, see Appendix B.)**

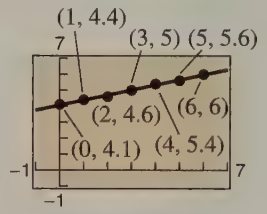
75. $\begin{cases} 5b + 10a = 20.2 \\ 10b + 30a = 50.1 \end{cases}$



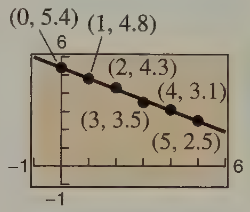
76. $\begin{cases} 5b + 10a = 11.7 \\ 10b + 30a = 25.6 \end{cases}$



77. $\begin{cases} 7b + 21a = 35.1 \\ 21b + 91a = 114.2 \end{cases}$



78. $\begin{cases} 6b + 15a = 23.6 \\ 15b + 55a = 48.8 \end{cases}$



79. ***Data Analysis*** A farmer used four test plots to determine the relationship between wheat yield (in bushels per acre) and the amount of fertilizer (in hundreds of pounds per acre). The results are shown in the table.

Fertilizer, x	1.0	1.5	2.0	2.5
Yield, y	32	41	48	53

(a) Find the least squares regression line $y = ax + b$ for the data by solving the system for a and b.

$$\begin{cases} 4b + 7.0a = 174 \\ 7b + 13.5a = 322 \end{cases}$$

(b) Use the linear regression capabilities of a graphing utility to confirm the result.

(c) Plot the data and the linear regression model.

(d) Use the line to estimate the yield for a fertilizer application of 160 pounds per acre.

80. ***Data Analysis*** A store manager wants to know the demand for a certain product as a function of the price. The daily sales for the different prices of the product are shown in the table.

Price, x	\$1.00	\$1.25	\$1.50
Demand, y	450	375	330

(a) Find the least squares regression line $y = ax + b$ for the data by solving the system for a and b.

$$\begin{cases} 3.00b + 3.7500a = 1155.00 \\ 3.75b + 4.8125a = 1413.75 \end{cases}$$

(b) Use the linear regression capabilities of a graphing utility to confirm the result.

(c) Plot the data and the linear regression model.

(d) Then, use the line to predict the demand when the price is \$1.40.

Synthesis

True or False? **In Exercises 81–83, determine whether the statement is true or false. Justify your answer.**

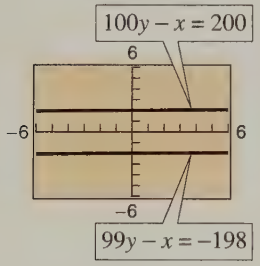
81. If a system of linear equations has two distinct solutions, then it has an infinite number of solutions.

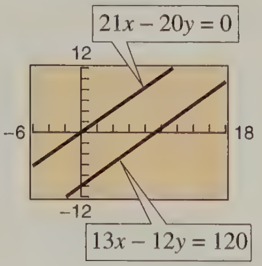
82. Solving a system of equations graphically will always give an exact solution.

83. If a system of linear equations has no solution, then the lines must be parallel.

Think About It **In Exercises 84 and 85, the graphs of the two equations appear to be parallel. Yet, when the system is solved algebraically, it is found that the system does have a solution. Find the solution and explain why it does not appear on the portion of the graph that is shown.**

84. $\begin{cases} 100y - x = 200 \\ 99y - x = -198 \end{cases}$ **85.** $\begin{cases} 21x - 20y = 0 \\ 13x - 12y = 120 \end{cases}$





86. ***Writing*** Briefly explain whether or not it is possible for a consistent system of linear equations to have exactly two solutions.

87. ***Think About It*** Give examples of (a) a system of linear equations that has no solution and (b) a system that has an infinite number of solutions.

Exploration **In Exercises 88 and 89, find the value of *k* such that the system of equations is inconsistent.**

88. $\begin{cases} 4x - 8y = -3 \\ 2x + ky = 16 \end{cases}$ **89.** $\begin{cases} 15x + 3y = 6 \\ -10x + ky = 9 \end{cases}$

Advanced Applications **In Exercises 90–93, solve the system of equations for *u* and *v*. While solving for these variables, consider the transcendental functions as constants. (Systems of this type are found in a course in differential equations.)**

90. $\begin{cases} u \sin x + v \cos x = 0 \\ u \cos x - v \sin x = \sec x \end{cases}$

91. $\begin{cases} u \cos 2x + v \sin 2x = 0 \\ u(-2 \sin 2x) + v(2 \cos 2x) = \csc 2x \end{cases}$

92. $\begin{cases} ue^x + vxe^x = 0 \\ ue^x + v(x + 1)e^x = e^x \ln x \end{cases}$

93. $\begin{cases} ue^{2x} + vxe^{2x} = 0 \\ u(2e^{2x}) + v(2x + 1)e^{2x} = \dfrac{e^{2x}}{x} \end{cases}$

Review

In Exercises 94–97, use a graphing utility to graph the functions. Use the graph to determine the domain and range.

94. $f(x) = x^2 - 2x$ **95.** $g(x) = \sqrt[3]{x - 2}$

96. $h(x) = \sqrt{25 - x^2}$ **97.** $s(t) = \dfrac{4}{9 - t^2}$

In Exercises 98–105, solve the inequality and graph the solution on a real number line.

98. $-11 - 6x \geq 33$ **99.** $2(x - 3) > -5x + 1$

100. $8x - 15 \leq -4(2x - 1)$

101. $-6 \leq 3x - 10 < 6$

102. $|x - 8| < 10$ **103.** $|x + 10| \geq -3$

104. $2x^2 + 3x - 35 < 0$ **105.** $3x^2 + 12x > 0$

In Exercises 106–109, write the expression as the logarithm of a single quantity.

106. $\ln x + \ln 6$ **107.** $\ln x - 5 \ln(x + 3)$

108. $\log_9 12 - \log_9 x$ **109.** $\frac{1}{4} \log_6 3x$

In Exercises 110 and 111, solve the system by the method of substitution.

110. $\begin{cases} 4x - y = -16 \\ -5x + 3y = 6 \end{cases}$ **111.** $\begin{cases} -12x + 9y = 51 \\ -x - 7y = -19 \end{cases}$

8.3 Multivariable Linear Systems

Row-Echelon Form and Back-Substitution

The method of elimination can be applied to a system of linear equations in more than two variables. When elimination is used to solve a system of linear equations, the goal is to rewrite the system in a form to which back-substitution can be applied. Consider the following two systems of linear equations.

System of 3 Linear Equations in 3 Variables (See Example 2):

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$

Equivalent System in Row-Echelon Form (See Example 1):

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$

The second system is said to be in **row-echelon form,** which means that it has a "stair-step" pattern with leading coefficients of 1. After comparing the two systems, it should be clear that it is easier to solve the system in row-echelon form.

EXAMPLE 1 Using Back-Substitution

Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ y + 3z = 5 & \text{Equation 2} \\ z = 2 & \text{Equation 3} \end{cases}$$

Solution

From Equation 3, you know the value of z. To solve for y, substitute $z = 2$ into Equation 2 to obtain

$$y + 3(2) = 5 \qquad \text{Substitute 2 for } z.$$

$$y = -1. \qquad \text{Solve for } y.$$

Finally, substitute $y = -1$ and $z = 2$ into Equation 1 to obtain

$$x - 2(-1) + 3(2) = 9 \qquad \text{Substitute } -1 \text{ for } y \text{ and 2 for } z.$$

$$x = 1. \qquad \text{Solve for } x.$$

The solution is $x = 1$, $y = -1$, and $z = 2$, which can be written as the **ordered triple** $(1, -1, 2)$. Check this in the original system of equations as follows.

Check $1 - 2(-1) + 3(2) = 9$ ✓

$-1 + 3(2) = 5$ ✓

$2 = 2$ ✓

What You Should Learn:

- How to recognize linear systems in row-echelon form and use back-substitution to solve the systems
- How to use Gaussian Elimination to solve systems of linear equations
- How to solve nonsquare systems of linear equations
- How to graphically interpret three-variable systems
- How to use systems of linear equations to write partial fraction decompositions of rational expressions
- How to use systems of linear equations in three or more variables to model and solve real-life problems

Why You Should Learn It:

Systems of linear equations in three or more variables can be used to model and solve real-life problems. For instance, Exercise 109 on page 601 shows how to use a system of linear equations to analyze an automobile's braking system.

Andy Sacks/Tony Stone Images

Gaussian Elimination

Two systems of equations are *equivalent* if they have the same solution set. To solve a system that is not in row-echelon form, first convert it to an *equivalent* system that *is* in row-echelon form by using one or more of the elementary row operations shown below. This process is called **Gaussian elimination,** after the German mathematician Carl Friedrich Gauss (1777–1855).

Elementary Row Operations

1. Interchange two equations.
2. Multiply one of the equations by a nonzero constant.
3. Add a multiple of one equation to another equation.

EXAMPLE 2 Using Gaussian Elimination to Solve a System

Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ -x + 3y = -4 & \text{Equation 2} \\ 2x - 5y + 5z = 17 & \text{Equation 3} \end{cases}$$

Solution

Because the leading coefficient of the first equation is 1, you can begin by saving the x in the upper left position and eliminating the other x's from the first column.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2x - 5y + 5z = 17 \end{cases}$$

Adding the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ -y - z = -1 \end{cases}$$

Adding -2 times the first equation to the third equation produces a new third equation.

Now that all but the first x have been eliminated from the first column, go to work on the second column. (You need to eliminate y from the third equation.)

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2z = 4 \end{cases}$$

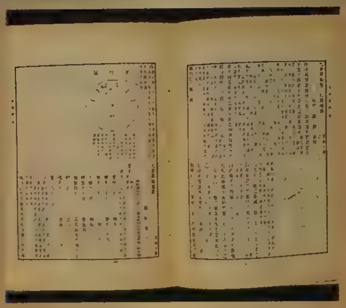
Adding the second equation to the third equation produces a new third equation.

Finally, you need a coefficient of 1 for z in the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$

Multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

This is the same system that was solved in Example 1. As in that example, you can conclude that the solution is $x = 1$, $y = -1$, and $z = 2$, written as $(1, -1, 2)$. Check this in the original system of the equations.



Christopher Lui/China Stock

One of the most influential Chinese mathematics books was the *Chui-chang suan-shu* or *Nine Chapters on the Mathematical Art* (written in approximately 250 B.C.). Chapter Eight contained solutions of systems of linear equations using positive and negative numbers. One such system was

$$\begin{cases} 3x + 2y + 3z = 39 \\ 2x + 3y + 3z = 34. \\ 2x + 2y + 3z = 26 \end{cases}$$

This system was solved using column operations on a matrix. Matrices (plural for matrix) will be discussed in the next chapter.

STUDY TIP

The goal of Gaussian elimination is to use elementary row operations on a system in order to isolate one variable. You can then solve for the value of the variable and use back-substitution to find the values of the remaining variables.

The next example involves an inconsistent system—one that has no solution. The key to recognizing an inconsistent system is that at some stage in the elimination process, you obtain a false statement such as $0 = -2$.

EXAMPLE 3 An Inconsistent System

Solve the system of linear equations.

$$\begin{cases} x - 3y + z = 1 & \text{Equation 1} \\ 2x - y - 2z = 2 & \text{Equation 2} \\ x + 2y - 3z = -1 & \text{Equation 3} \end{cases}$$

Solution

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ x + 2y - 3z = -1 \end{cases}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 5y - 4z = -2 \end{cases}$$

Adding -1 times the first equation to the third equation produces a new third equation.

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 0 = -2 \end{cases}$$

Adding -1 times the second equation to the third equation produces a new third equation.

Because $0 = -2$ is a false statement, you can conclude that this system is inconsistent and therefore has no solution. Moreover, because this system is equivalent to the original system, you can conclude that the original system also has no solution.

As with a system of linear equations in two variables, the number of solutions of a system of linear equations in more than two variables must fall into one of three categories.

The Number of Solutions of a Linear System

For a system of linear equations, exactly one of the following is true.

1. There is exactly one solution.
2. There are infinitely many solutions.
3. There is no solution.

A system of linear equations is called *consistent* if it has at least one solution. A consistent system with exactly one solution is **independent.** A consistent system with infinitely many solutions is **dependent.** A system of linear equations is called *inconsistent* if it has no solution.

EXAMPLE 4 A System with Infinitely Many Solutions

Solve the system of linear equations.

$$\begin{cases} x + y - 3z = -1 & \text{Equation 1} \\ y - z = 0 & \text{Equation 2} \\ -x + 2y = 1 & \text{Equation 3} \end{cases}$$

Solution

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ 3y - 3z = 0 \end{cases}$$

Adding the first equation to the third equation produces a new third equation.

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ 0 = 0 \end{cases}$$

Adding -3 times the second equation to the third equation produces a new third equation.

This means that Equation 3 depends on Equations 1 and 2 in the sense that it gives us no additional information about the variables. So, the original system is equivalent to the system

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \end{cases}.$$

In the last equation, solve for y in terms of z to obtain $y = z$. Back-substituting for y into the previous equation produces $x = 2z - 1$. Finally, letting $z = a$, where a is a real number, the solutions to the given system are all of the form

$$x = 2a - 1, \qquad y = a, \qquad \text{and} \qquad z = a.$$

So, every ordered triple of the form

$$(2a - 1, a, a), \qquad a \text{ is a real number}$$

is a solution of the system.

STUDY T!P

There are an infinite number of solutions to Example 4, but they all are of a specific form. By selecting, for example, a-values of 0, 1, and 3, you can verify that $(-1, 0, 0)$, $(1, 1, 1)$, and $(5, 3, 3)$ are specific solutions. It is incorrect to say simply that the solution to Example 4 is "infinite." You must also specify the form of the solutions.

In Example 4, there are other ways to write the same infinite set of solutions. For instance, the solutions could have been written as

$$\left(b, \tfrac{1}{2}(b + 1), \tfrac{1}{2}(b + 1)\right), \qquad b \text{ is a real number.}$$

To convince yourself that this description produces the same set of solutions, consider the following.

Substitution	*Solution*	
$a = 0$	$(2(0) - 1, 0, 0) = (-1, 0, 0)$	Same solution
$b = -1$	$\left(-1, \tfrac{1}{2}(-1 + 1), \tfrac{1}{2}(-1 + 1)\right) = (-1, 0, 0)$	
$a = 1$	$(2(1) - 1, 1, 1) = (1, 1, 1)$	Same solution
$b = 1$	$\left(1, \tfrac{1}{2}(1 + 1), \tfrac{1}{2}(1 + 1)\right) = (1, 1, 1)$	
$a = 2$	$(2(2) - 1, 2, 2) = (3, 2, 2)$	Same solution
$b = 3$	$\left(3, \tfrac{1}{2}(3 + 1), \tfrac{1}{2}(3 + 1)\right) = (3, 2, 2)$	

Nonsquare Systems

So far, each system of linear equations has been *square*, which means that the number of equations is equal to the number of variables. In a **nonsquare system of equations,** the number of equations differs from the number of variables. A system of linear equations cannot have a unique solution unless there are at least as many equations as there are variables in the system.

EXAMPLE 5 A System with Fewer Equations Than Variables

Solve the system of linear equations.

$$\begin{cases} x - 2y + z = 2 & \text{Equation 1} \\ 2x - y - z = 1 & \text{Equation 2} \end{cases}$$

Solution

Begin by rewriting the system in row-echelon form.

$$\begin{cases} x - 2y + z = 2 \\ 3y - 3z = -3 \end{cases}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 2y + z = 2 \\ y - z = -1 \end{cases}$$

Multiplying the second equation by $\frac{1}{3}$ produces a new second equation.

Solving for y in terms of z, you get $y = z - 1$, and back-substitution into Equation 1 yields

$$x - 2(z - 1) + z = 2$$
$$x - 2z + 2 + z = 2$$
$$x = z.$$

Finally, by letting $z = a$ where a is a real number, you have the solution

$$x = a, \qquad y = a - 1, \qquad \text{and} \qquad z = a.$$

So, every ordered triple of the form

$$(a, a - 1, a), \qquad a \text{ is a real number}$$

is a solution of the system.

In Example 5, try choosing some values of a to obtain different solutions of the system, such as $(1, 0, 1)$, $(2, 1, 2)$, and $(3, 2, 3)$. Then check each of the solutions in the original system as follows.

Check: $(1, 0, 1)$	*Check:* $(2, 1, 2)$	*Check:* $(3, 2, 3)$
$1 - 2(0) + 1 = 2$	$2 - 2(1) + 2 = 2$	$3 - 2(2) + 3 = 2$
$2 = 2$ ✓	$2 = 2$ ✓	$2 = 2$ ✓
$2(1) - 0 - 1 = 1$	$2(2) - 1 - 2 = 1$	$2(3) - 2 - 3 = 1$
$1 = 1$ ✓	$1 = 1$ ✓	$1 = 1$ ✓

Graphical Interpretation of Three-Variable Systems

Solutions of equations with three variables can be pictured with a **three-dimensional coordinate system.** To construct such a system, begin with the xy-coordinate plane in a horizontal position. Then draw the z-axis as a vertical line through the origin.

Every ordered triple (x, y, z) corresponds to a point on the three-dimensional coordinate system. For instance, the points corresponding to

$$(-2, 5, 4), \qquad (2, -5, 3), \qquad \text{and} \qquad (3, 3, -2)$$

are shown in Figure 8.12.

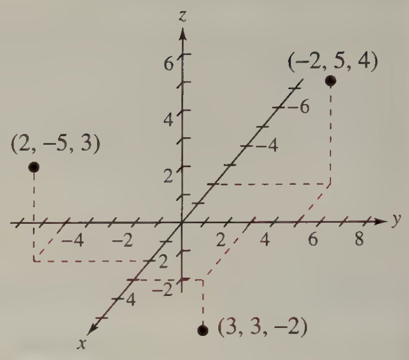


Figure 8.12

The **graph of an equation in three variables** consists of all points (x, y, z) that are solutions of the equation. The graph of a linear equation in three variables is a *plane*. Sketching graphs on a three-dimensional coordinate system is difficult because the sketch itself is only two-dimensional.

One technique for sketching a plane is to find the three points at which the plane intersects the axes. For instance, the plane

$$3x + 2y + 4z = 12$$

intersects the x-axis at the point $(4, 0, 0)$, the y-axis at the point $(0, 6, 0)$, and the z-axis at the point $(0, 0, 3)$.

By plotting these three points, connecting them with line segments, and shading the resulting triangular region, you can sketch a portion of the graph, as shown in Figure 8.13.

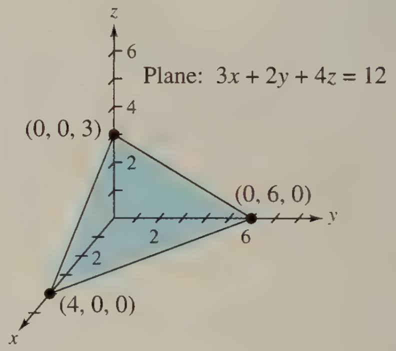
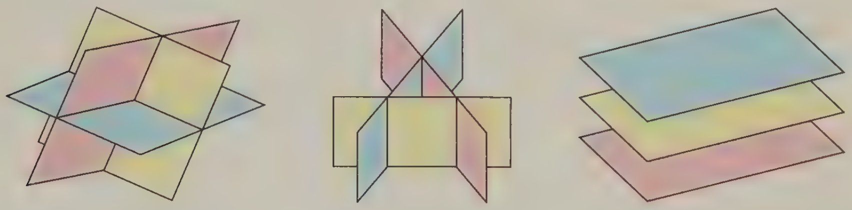
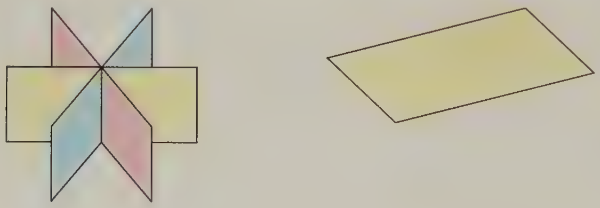


Figure 8.13

The graph of a system of three linear equations in three variables consists of *three* planes. When these planes intersect in a single point, the system has exactly one solution. [See Figure 8.14(a).] When the three planes have no point in common, the system has no solution. [See Figures 8.14(b) and (c).] When the three planes intersect in a line or a plane, the system has infinitely many solutions. [See Figures 8.14(d) and (e).]



(a) Solution: one point **(b) Solution: none** **(c) Solution: none**



(d) Solution: one line **(e) Solution: one plane**

Figure 8.14

Partial Fraction Decomposition and Other Applications

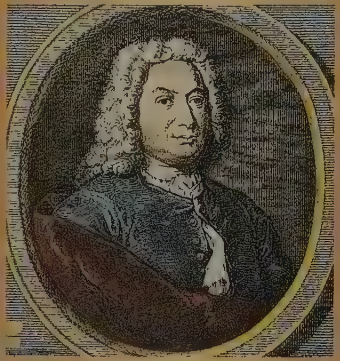
A rational expression can be often written as the sum of two or more simpler rational expressions. For example, the rational expression

$$\frac{x+7}{x^2-x-6}$$

can be written as the sum of two fractions with linear denominators. That is,

$$\frac{x+7}{x^2-x-6} = \overbrace{\underbrace{\frac{2}{x-3}}_{\text{Partial fraction}} + \underbrace{\frac{-1}{x+2}}_{\text{Partial fraction}}}^{\text{Partial fraction decomposition of } \frac{x+7}{x^2-x-6}}.$$

Each fraction on the right side of the equation is a **partial fraction,** and together they make up the **partial fraction decomposition** of the left side.



The Granger Collection

The method of partial fractions was introduced by **John Bernoulli (1667–1748),** a Swiss mathematician who was instrumental in the early development of calculus. Bernoulli was a professor at the University of Basel and taught many outstanding students, the most famous of whom was Leonhard Euler.

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. *Divide if improper:* If $N(x)/D(x)$ is an improper fraction (degree of $N(x) \geq$ degree of $D(x)$), divide the denominator into the numerator to obtain

   $$\frac{N(x)}{D(x)} = (\text{polynomial}) + \frac{N_1(x)}{D(x)}$$

   and apply Steps 2, 3, and 4 (below) to the proper rational expression $N_1(x)/D(x)$.

2. *Factor denominator:* Completely factor the denominator into factors of the form

   $$(px+q)^m \qquad \text{and} \qquad (ax^2+bx+c)^n$$

   where (ax^2+bx+c) is irreducible.

3. *Linear factors:* For *each* factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of m fractions.

   $$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}$$

4. *Quadratic factors:* For *each* factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following sum of n fractions.

   $$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

One of the most important applications of partial fractions is in calculus. If you go on to take a course in calculus, you will learn how partial fractions can be used in a calculus operation called antidifferentiation.

EXAMPLE 6 Partial Fraction Decomposition: Distinct Linear Factors

Write the partial fraction decomposition for

$$\frac{x+7}{x^2-x-6}.$$

Solution

Because $x^2 - x - 6 = (x-3)(x+2)$, you should include one partial fraction with a constant numerator for each linear factor of the denominator and write

$$\frac{x+7}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}.$$

Multiplying both sides of this equation by the least common denominator, $(x-3)(x+2)$, leads to the **basic equation**

$$\begin{aligned} x+7 &= A(x+2) + B(x-3) && \text{Basic equation} \\ &= Ax + 2A + Bx - 3B && \\ &= (A+B)x + 2A - 3B. && \text{Write in polynomial form.} \end{aligned}$$

By equating coefficients of like terms on opposite sides of the equation, you obtain the following system of linear equations.

$$\begin{cases} A + B = 1 \\ 2A - 3B = 7 \end{cases}$$

You can solve the system of linear equations as follows.

$$\begin{array}{rclrll} A + B = 1 & \Rightarrow & 3A + 3B = 3 & & \text{Multiply Equation 1 by 3.} \\ 2A - 3B = 7 & \Rightarrow & \underline{2A - 3B = 7} & & \text{Equation 2} \\ & & 5A \qquad = 10 & & \text{Add equations} \end{array}$$

From this equation, you can see that $A = 2$. By back-substituting this value of A into Equation 1, you can determine that $B = -1$.

Therefore, the partial fraction decomposition is

$$\frac{x+7}{x^2-x-6} = \frac{2}{x-3} - \frac{1}{x+2}.$$

Check this result by combining the two partial fractions on the right side of the equation.

You can graphically check the decomposition found in Example 1. To do this, graph

$$y_1 = \frac{x+7}{x^2-x-6} \quad \text{and} \quad y_2 = \frac{2}{x-3} - \frac{1}{x+2}$$

in the same viewing window. Their graphs should be identical, as shown in Figure 8.15.

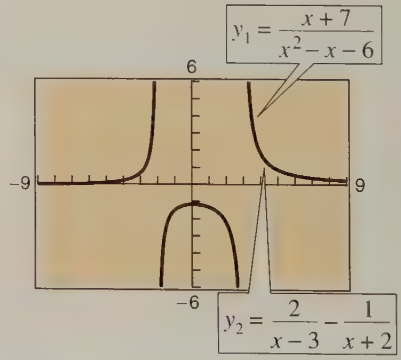


Figure 8.15

The next example shows how to find the partial fraction decomposition for a rational function whose denominator has a repeated linear factor.

EXAMPLE 7 Partial Fraction Decomposition: Repeated Linear Factors

Write the partial fraction decomposition for

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}.$$

Solution

Because the denominator factors as

$$\begin{aligned} x^3 + 2x^2 + x &= x(x^2 + 2x + 1) \\ &= x(x + 1)^2 \end{aligned}$$

you should include one partial fraction with a constant numerator for each power of x and $(x + 1)$ and write

$$\frac{5x^2 + 20x + 6}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}.$$

Multiplying by the LCD, $x(x + 1)^2$, leads to the basic equation

$$\begin{aligned} 5x^2 + 20x + 6 &= A(x + 1)^2 + Bx(x + 1) + Cx && \text{Basic equation} \\ &= Ax^2 + 2Ax + A + Bx^2 + Bx + Cx \\ &= (A + B)x^2 + (2A + B + C)x + A. && \text{Polynomial form} \end{aligned}$$

By equating coefficients of like terms on opposite sides of the equation, you obtain the following system of linear equations.

$$\begin{cases} A + B = 5 \\ 2A + B + C = 20 \\ A = 6 \end{cases}$$

Substituting 6 for A in the first equation produces

$$\begin{aligned} 6 + B &= 5 \\ B &= -1. \end{aligned}$$

Substituting 6 for A and -1 for B in the second equation produces

$$\begin{aligned} 2(6) + (-1) + C &= 20 \\ C &= 9. \end{aligned}$$

Therefore, the partial fraction decomposition is

$$\frac{5x^2 + 20x + 6}{x(x + 1)^2} = \frac{6}{x} - \frac{1}{x + 1} + \frac{9}{(x + 1)^2}.$$

Exploration

Partial fraction decomposition is practical only for rational functions whose denominators factor "nicely." For example, the factorization of the denominator $x^2 - x - 5$ is

$$\left(x - \frac{1 - \sqrt{21}}{2}\right)\left(x - \frac{1 + \sqrt{21}}{2}\right).$$

Write the basic equation and try to complete the decomposition for

$$\frac{x + 7}{x^2 - x - 5}.$$

What problems do you encounter?

EXAMPLE 8 Vertical Motion

The height at time t of an object that is moving in a (vertical) line with constant acceleration a is given by the *position equation* $s = \frac{1}{2}at^2 + v_0t + s_0$. The height s is measured in feet, t is measured in seconds, v_0 is the initial velocity (in feet per second) at $t = 0$, and s_0 is the initial height. Find the values of a, v_0, and s_0, if $s = 52$ at $t = 1$, $s = 52$ at $t = 2$, and $s = 20$ at $t = 3$, as shown in Figure 8.16.

Solution

You can obtain three linear equations in a, v_0, and s_0 as follows.

When $t = 1$: $\frac{1}{2}a(1)^2 + v_0(1) + s_0 = 52 \Rightarrow a + 2v_0 + 2s_0 = 104$

When $t = 2$: $\frac{1}{2}a(2)^2 + v_0(2) + s_0 = 52 \Rightarrow 2a + 2v_0 + s_0 = 52$

When $t = 3$: $\frac{1}{2}a(3)^2 + v_0(3) + s_0 = 20 \Rightarrow 9a + 6v_0 + 2s_0 = 40$

Solving this system yields $a = -32$, $v_0 = 48$, and $s_0 = 20$.

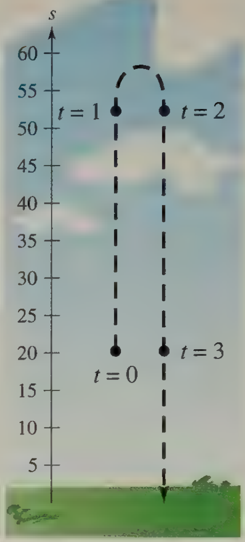


Figure 8.16

EXAMPLE 9 Data Analysis: Curve-Fitting

Find a quadratic equation, $y = ax^2 + bx + c$, whose graph passes through the points $(-1, 3)$, $(1, 1)$, and $(2, 6)$.

Solution

Because the graph of $y = ax^2 + bx + c$ passes through the points $(-1, 3)$, $(1, 1)$, and $(2, 6)$, you can write the following.

When $x = -1, y = 3$: $a(-1)^2 + b(-1) + c = 3$

When $x = 1, y = 1$: $a(1)^2 + b(1) + c = 1$

When $x = 2, y = 6$: $a(2)^2 + b(2) + c = 6$

This produces the following system of linear equations.

$$\begin{cases} a - b + c = 3 & \text{Equation 1} \\ a + b + c = 1 & \text{Equation 2} \\ 4a + 2b + c = 6 & \text{Equation 3} \end{cases}$$

The solution of this system is $a = 2$, $b = -1$, and $c = 0$. So, the equation of the parabola is $y = 2x^2 - x$, as shown in Figure 8.17. From the graph, you can check to see that the parabola passes through the given points $(-1, 3)$, $(1, 1)$, and $(2, 6)$.

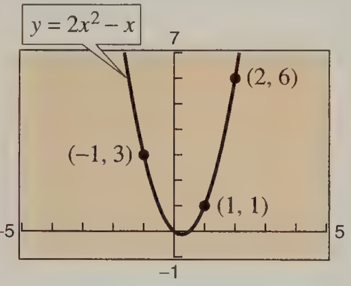


Figure 8.17

STUDY TIP

When you use a system of linear equations to solve an application, it is wise to interpret your solution in the context of the problem to see if it makes sense. For instance, in Example 8 the solution results in a position equation of

$$s = -16t^2 + 48t + 20$$

which implies that the object was thrown upward at a velocity of 48 feet per second from a height of 20 feet. The object undergoes a constant downward acceleration of 32 feet per second squared. (Physics will tell you that this is the value of the acceleration due to gravity.)

8.3 Exercises

In Exercises 1–4, decide whether each ordered triple is a solution of the system of equations.

1. $$\begin{cases} 3x - y + z = 1 \\ 2x - 3z = -14 \\ 5y + 2z = 8 \end{cases}$$

(a) $(2, 5, 0)$ (b) $(-2, 0, 4)$

(c) $(0, -1, 3)$ (d) $(-1, 0, 4)$

2. $$\begin{cases} 3x + 4y - z = 17 \\ 5x - y + 2z = -2 \\ 2x - 3y + 7z = -21 \end{cases}$$

(a) $(3, 2, 0)$ (b) $(1, 3, -2)$

(c) $(4, 1, -3)$ (d) $(1, 5, -1)$

3. $$\begin{cases} 4x + y - z = 0 \\ -8x - 6y + z = -\frac{7}{4} \\ 3x - y = -\frac{9}{4} \end{cases}$$

(a) $(0, 1, 1)$ (b) $\left(-\frac{3}{2}, \frac{5}{4}, -\frac{5}{4}\right)$

(c) $\left(-\frac{1}{2}, \frac{3}{4}, -\frac{5}{4}\right)$ (d) $\left(-\frac{1}{2}, 2, 0\right)$

4. $$\begin{cases} -4x - y - 8z = -6 \\ y + z = 0 \\ 4x - 7y = 6 \end{cases}$$

(a) $(-2, -2, 2)$ (b) $\left(-\frac{33}{2}, -10, 10\right)$

(c) $\left(\frac{1}{8}, -\frac{1}{2}, \frac{1}{2}\right)$ (d) $\left(-\frac{11}{2}, -4, 4\right)$

In Exercises 5–10, use back-substitution to solve the system of linear equations.

5. $$\begin{cases} 2x - y + 5z = 24 \\ y + 2z = 4 \\ z = 6 \end{cases}$$

6. $$\begin{cases} 4x - 3y - 2z = 21 \\ 6y - 5z = -10 \\ z = -4 \end{cases}$$

7. $$\begin{cases} 2x + y - 3z = 10 \\ y = -2 \\ y - z = 4 \end{cases}$$

8. $$\begin{cases} x = 8 \\ 2x + 3y = 10 \\ x - y + 2z = 22 \end{cases}$$

9. $$\begin{cases} 4x - 2y + z = 8 \\ 2z = 4 \\ -y + z = 4 \end{cases}$$

10. $$\begin{cases} 5x - 8z = 22 \\ 3y - 5z = 10 \\ z = -4 \end{cases}$$

In Exercises 11 and 12, perform the row operation and write the equivalent system.

11. Add Equation 1 to Equation 2.

$$\begin{cases} x - 2y + 3z = 5 \\ -x + 3y - 5z = 4 \\ 2x - 3z = 0 \end{cases} \quad \begin{matrix} \text{Equation 1} \\ \text{Equation 2} \\ \text{Equation 3} \end{matrix}$$

What did this operation accomplish?

12. Add -2 times Equation 1 to Equation 3.

$$\begin{cases} x - 2y + 3z = 5 \\ -x + 3y - 5z = 4 \\ 2x - 3z = 0 \end{cases} \quad \begin{matrix} \text{Equation 1} \\ \text{Equation 2} \\ \text{Equation 3} \end{matrix}$$

What did this operation accomplish?

In Exercises 13–40, solve the system of linear equations and check any solution algebraically.

13. $$\begin{cases} x + y + z = 6 \\ 2x - y + z = -1 \\ 3x - z = -7 \end{cases}$$

14. $$\begin{cases} x + y + z = 2 \\ -x + 3y + 2z = 8 \\ 4x + y = 4 \end{cases}$$

15. $$\begin{cases} 2x + 2z = 6 \\ 5x + 3y = 11 \\ 3y - 4z = 1 \end{cases}$$

16. $$\begin{cases} 4x + y - 3z = 11 \\ 2x - 3y + 2z = 9 \\ x + y + z = -3 \end{cases}$$

17. $$\begin{cases} 6y + 4z = -18 \\ 3x + 3y = 9 \\ 2x - 3z = 12 \end{cases}$$

18. $$\begin{cases} 2x + 4y + z = -4 \\ 2x - 4y + 6z = 13 \\ 4x - 2y + z = 6 \end{cases}$$

19. $$\begin{cases} 3x - 2y + 4z = 1 \\ x + y - 2z = 3 \\ 2x - 3y + 6z = 8 \end{cases}$$

20. $$\begin{cases} 5x - 3y + 2z = 3 \\ 2x + 4y - z = 7 \\ x - 11y + 4z = 3 \end{cases}$$

21. $$\begin{cases} 3x + 3y + 5z = 1 \\ 3x + 5y + 9z = 0 \\ 5x + 9y + 17z = 0 \end{cases}$$

22. $$\begin{cases} 2x + y + 3z = 1 \\ 2x + 6y + 8z = 3 \\ 6x + 8y + 18z = 5 \end{cases}$$

23. $$\begin{cases} x + 2y - 7z = -4 \\ 2x + y + z = 13 \\ 3x + 9y - 36z = -33 \end{cases}$$

24. $$\begin{cases} 2x + y - 3z = 4 \\ 4x + 2z = 10 \\ -2x + 3y - 13z = -8 \end{cases}$$

25. $\begin{cases} 3x - 3y + 6z = 6 \\ x + 2y - z = 5 \\ 5x - 8y + 13z = 7 \end{cases}$

26. $\begin{cases} x + 4z = 13 \\ 4x - 2y + z = 7 \\ 2x - 2y - 7z = -19 \end{cases}$

27. $\begin{cases} x - 2y + 3z = 4 \\ 3x - y + 2z = 0 \\ x + 3y - 4z = -2 \end{cases}$

28. $\begin{cases} -x + 3y + z = 4 \\ 4x - 2y - 5z = -7 \\ 2x + 4y - 3z = 12 \end{cases}$

29. $\begin{cases} x + 4z = 1 \\ x + y + 10z = 10 \\ 2x - y + 2z = -5 \end{cases}$

30. $\begin{cases} 3x - 2y - 6z = -4 \\ -3x + 2y + 6z = 1 \\ x - y - 5z = -3 \end{cases}$

31. $\begin{cases} 2x + 3y = 0 \\ 4x + 3y - z = 0 \\ 8x + 3y + 3z = 0 \end{cases}$

32. $\begin{cases} 4x + 3y + 17z = 0 \\ 5x + 4y + 22z = 0 \\ 4x + 2y + 19z = 0 \end{cases}$

33. $\begin{cases} x - 2y + 5z = 2 \\ 4x - z = 0 \end{cases}$

34. $\begin{cases} x - 3y + 2z = 18 \\ 5x - 13y + 12z = 80 \end{cases}$

35. $\begin{cases} 2x - 3y + z = -2 \\ -4x + 9y = 7 \end{cases}$

36. $\begin{cases} 2x + 3y + 3z = 7 \\ 4x + 18y + 15z = 44 \end{cases}$

37. $\begin{cases} 12x + 5y + z = 0 \\ 23x + 4y - z = 0 \end{cases}$

38. $\begin{cases} 10x - 3y + 2z = 0 \\ 19x - 5y - z = 0 \end{cases}$

39. $\begin{cases} x + 3w = 4 \\ 2y - z - w = 0 \\ 3y - 2w = 1 \\ 2x - y + 4z = 5 \end{cases}$

40. $\begin{cases} x + y + z + w = 6 \\ 2x + 3y - w = 0 \\ -3x + 4y + z + 2w = 4 \\ x + 2y - z + w = 0 \end{cases}$

Exploration **In Exercises 41–44, find a system of linear equations that has the given solution. (There are many correct answers.)**

41. $(4, -1, 2)$

42. $(-5, -2, 1)$

43. $\left(3, -\frac{1}{2}, \frac{7}{4}\right)$

44. $\left(-\frac{3}{2}, 4, -7\right)$

Three-Dimensional Graphics **In Exercises 45–48, sketch the plane represented by the linear equation. Then list four points that lie in the plane.**

45. $2x + 3y + 4z = 12$

46. $x + y + z = 6$

47. $2x + y + z = 4$

48. $x + 2y + 2z = 6$

In Exercises 49–54, write the form of the partial fraction decomposition of the rational expression. Do not solve for the constants.

49. $\dfrac{7}{x^2 - 14x}$

50. $\dfrac{x - 2}{x^2 + 4x + 3}$

51. $\dfrac{12}{x^3 - 10x^2}$

52. $\dfrac{x^2 - 3x + 2}{4x^3 + 11x^2}$

53. $\dfrac{4x^2 + 3}{(x - 5)^3}$

54. $\dfrac{6x + 5}{(x + 2)^4}$

In Exercises 55–72, write the partial fraction decomposition for the rational expression. Check your result algebraically by combining the fractions.

55. $\dfrac{1}{x^2 - 1}$

56. $\dfrac{1}{4x^2 - 9}$

57. $\dfrac{1}{x^2 + x}$

58. $\dfrac{3}{x^2 - 3x}$

59. $\dfrac{1}{2x^2 + x}$

60. $\dfrac{5}{x^2 + x - 6}$

61. $\dfrac{3}{x^2 + x - 2}$

62. $\dfrac{x + 1}{x^2 + 4x + 3}$

63. $\dfrac{x^2 + 12x + 12}{x^3 - 4x}$

64. $\dfrac{x + 2}{x(x - 4)}$

65. $\dfrac{4x^2 + 2x - 1}{x^2(x + 1)}$

66. $\dfrac{2x - 3}{(x - 1)^2}$

67. $\dfrac{3x}{(x - 3)^2}$

68. $\dfrac{6x^2 + 1}{x^2(x - 1)^3}$

69. $\dfrac{2x^3 - x^2 + x + 5}{x^2 + 3x + 2}$

70. $\dfrac{x^3 + 2x^2 - x + 1}{x^2 + 3x - 4}$

71. $\dfrac{x^4}{(x - 1)^3}$

72. $\dfrac{4x^4}{(2x - 1)^3}$

In Exercises 73–78, write the partial fraction decomposition for the rational expression. Use a graphing utility to check your result graphically.

73. $\dfrac{5-x}{2x^2+x-1}$

74. $\dfrac{3x^2-7x-2}{x^3-x}$

75. $\dfrac{x-1}{x^3+x^2}$

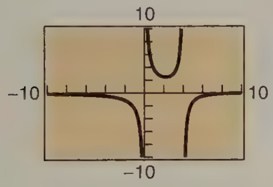
76. $\dfrac{4x^2-1}{2x(x+1)^2}$

77. $\dfrac{2x^3-4x^2-15x+5}{x^2-2x-8}$

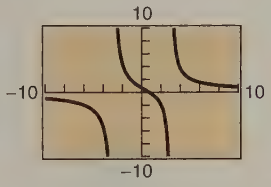
78. $\dfrac{x^3-x+3}{x^2+x-2}$

Graphical Analysis **In Exercises 79 and 80, write the partial fraction decomposition for the rational function. Identify the graph of the rational function and the graphs of each term of its decomposition. State any relationship between the vertical asymptotes of the rational function and the vertical asymptotes of the terms of the decomposition.**

79. $y = \dfrac{x-12}{x(x-4)}$



80. $y = \dfrac{2(4x-3)}{x^2-9}$



Vertical Motion **In Exercises 81–84, find the position equation $s = \frac{1}{2}at^2 + v_0t + s_0$ for an object at the given heights moving vertically at the specified times.**

81. At $t = 1$ second, $s = 128$ feet
At $t = 2$ seconds, $s = 80$ feet
At $t = 3$ seconds, $s = 0$ feet

82. At $t = 1$ second, $s = 48$ feet
At $t = 2$ seconds, $s = 64$ feet
At $t = 3$ seconds, $s = 48$ feet

83. At $t = 1$ second, $s = 452$ feet
At $t = 2$ seconds, $s = 260$ feet
At $t = 3$ seconds, $s = 116$ feet

84. At $t = 1$ second, $s = 132$ feet
At $t = 2$ seconds, $s = 100$ feet
At $t = 3$ seconds, $s = 36$ feet

In Exercises 85–88, find the equation of the parabola

$$y = ax^2 + bx + c$$

that passes through the given points. To verify your result, use a graphing utility to plot the points and graph the parabola.

85. $(0, 0), (2, -2), (4, 0)$

86. $(0, 3), (1, 4), (2, 3)$

87. $(2, 0), (3, -1), (4, 0)$

88. $(1, 3), (2, 2), (3, -3)$

In Exercises 89–92, find the equation of the circle

$$x^2 + y^2 + Dx + Ey + F = 0$$

that passes through the given points. To verify your result, use a graphing utility to plot the points and graph the circle.

89. $(0, 0), (2, 2), (4, 0)$

90. $(0, 0), (0, 6), (3, 3)$

91. $(-3, -1), (2, 4), (-6, 8)$

92. $(-6, -1), (-4, 3), (2, -5)$

93. ***Investments*** An inheritance of \$16,000 was divided among three investments yielding a total of \$990 in interest per year. The interest rates were 5%, 6%, and 7%. Find the amount in each investment if the 5% and 6% investments were \$3000 and \$2000 less than the 7% investment.

94. ***Investments*** A total of \$1520 a year is received in interest from three investments. The interest rates for the three investments are 5%, 7%, and 8%. The 5% investment is half of the 7% investment, and the 7% investment is \$1500 less than the 8% investment. Find the amount in each investment.

95. ***Borrowing*** A small corporation borrowed \$775,000 to expand its product line. Some of the money was borrowed at 8%, some at 9%, and some at 10%. How much was borrowed at each rate if the annual interest was \$67,000 and the amount borrowed at 8% was four times the amount borrowed at 10%?

96. ***Borrowing*** A small corporation borrowed \$800,000 to expand its product line. Some of the money was borrowed at 8%, some at 9%, and some at 10%. How much was borrowed at each rate if the annual interest was \$67,000 and the amount borrowed at 8% was five times the amount borrowed at 10%?

Investment Portfolio **In Exercises 97 and 98, consider an investor with a portfolio totaling $500,000 that is invested in certificates of deposit, municipal bonds, blue-chip stocks, and growth or speculative stocks. How much is put in each type of investment?**

97. The certificates of deposit pay 10% annually, and the municipal bonds pay 8% annually. Over a 5-year period, the investor expects the blue-chip stocks to return 12% annually and the growth stocks to return 13% annually. The investor wants a combined annual return of 10% as well as only one-fourth of the portfolio invested in stocks.

98. The certificates of deposit pay 9% annually, and the municipal bonds pay 5% annually. Over a 5-year period, the investor expects the blue-chip stocks to return 12% annually and the growth stocks to return 14% annually. The investor wants a combined annual return of 10% and also wants to have only one-fourth of the portfolio invested in stocks.

99. ***Crop Spraying*** A mixture of 12 liters of chemical A, 16 liters of chemical B, and 26 liters of chemical C is required to kill a certain destructive crop insect. Commercial spray X contains 1, 2, and 2 parts, respectively, of these chemicals. Commercial spray Y contains only chemical C. Commercial spray Z contains only chemicals A and B in equal amounts. How much of each type of commercial spray is needed to get the desired mixture?

100. ***Chemistry*** A chemist needs 10 liters of a 25% acid solution. The solution is to be mixed from three solutions whose concentrations are 10%, 20%, and 50%. How many liters of each solution should the chemist use to satisfy the following?

(a) Use as little as possible of the 50% solution.

(b) Use as much as possible of the 50% solution.

(c) Use 2 liters of the 50% solution.

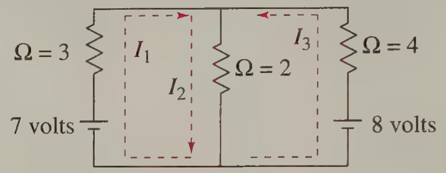
101. ***Truck Scheduling*** A small company that manufactures products A and B has an order for 15 units of product A and 16 units of product B. The company has trucks of three different sizes that can haul the products A and B in the amounts shown in the table. How many trucks of each size are needed to deliver the order? Give *two* possible solutions.

Truck	Large	Medium	Small
Product A	6	4	0
Product B	3	4	3

102. ***Electrical Networks*** When Kirchhoff's Laws are applied to the electrical network in the figure, the currents I_1, I_2, and I_3 are the solution of the system

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 3I_1 + 2I_2 = 7 \\ 2I_2 + 4I_3 = 8 \end{cases}$$

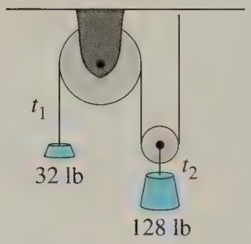
Find the currents.



103. ***Pulley System*** A system of pulleys is loaded with 128-pound and 32-pound weights. The tensions t_1 and t_2 in the ropes and the acceleration a of the 32-pound weight are found by solving the system

$$\begin{cases} t_1 - 2t_2 = 0 \\ t_1 - 2a = 128 \\ t_2 + a = 32 \end{cases}$$

where t_1 and t_2 are measured in pounds and a is in feet per second squared. Solve the system.



104. ***Pulley System*** If the 32-pound weight is replaced with a 64-pound weight in the pulley system described in Exercise 103, the new pulley system is modeled by the following system of equations.

$$\begin{cases} t_1 - 2t_2 = 0 \\ t_1 - 2a = 128 \\ t_2 + 2a = 64 \end{cases}$$

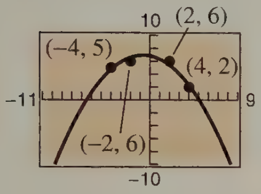
Solve the system, and use your answer for the acceleration to describe what (if anything) is happening in the system.

Fitting a Parabola **In Exercises 105–108, find the least squares regression parabola $y = ax^2 + bx + c$ for the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ by solv-**

ing the following system of linear equations for a, b, and c. Then use the regression capabilities of a graphing utility to confirm the result. (For an explanation of how the coefficients of a, b, and c in the system are obtained, see Appendix B.)

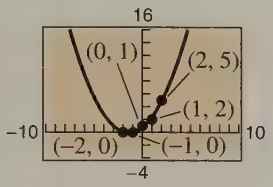
105.

$$\begin{cases} 4c + 40a = 19 \\ 40b = -12 \\ 40c + 544a = 160 \end{cases}$$



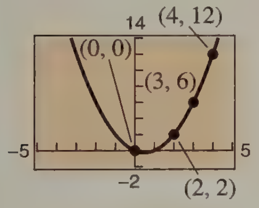
106.

$$\begin{cases} 5c + 10a = 8 \\ 10b = 12 \\ 10c + 34a = 22 \end{cases}$$



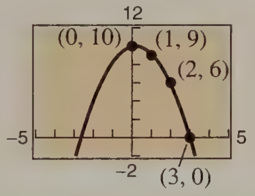
107.

$$\begin{cases} 4c + 9b + 29a = 20 \\ 9c + 29b + 99a = 70 \\ 29c + 99b + 353a = 254 \end{cases}$$



108.

$$\begin{cases} 4c + 6b + 14a = 25 \\ 6c + 14b + 36a = 21 \\ 14c + 36b + 98a = 33 \end{cases}$$



109. ***Analyzing Data*** During the testing of a new braking system on an automobile, the speeds (in miles per hour) and the stopping distances (in feet) were recorded as follows.

Speed, x	20	30	40	50	60
Stopping Distance, y	25	55	105	188	300

(a) Find the least squares regression parabola for the data. Use a graphing utility to graph the parabola and the data in the same viewing window.

(b) Use the model to estimate the stopping distance for a speed of 70 miles per hour.

110. ***Analyzing Data*** A wildlife management team studied the reproduction rates of deer in five 5-acre tracts of a wildlife preserve. For each tract, the number of females and the percent of females that had offspring the following year were recorded. The results are shown in the table.

Number, x	80	100	120	140	160
Percent, y	80	75	68	55	30

(a) Find the least squares regression parabola for the data.

(b) Use a graphing utility to graph the parabola and the data in the same viewing window.

(c) Use the model to predict the percent of females that would have offspring if $x = 170$.

111. ***Exhaust Temperatures*** The magnitude of the range of exhaust temperatures (in degrees Fahrenheit) in an experimental diesel engine is approximated by

$$R = \frac{2000(4 - 3x)}{(11 - 7x)(7 - 4x)}, \quad 0 \le x \le 1$$

where x is the relative load.

(a) Write the partial fraction decomposition for the rational function.

(b) The decomposition in part (a) is the difference of two fractions. The absolute values of the terms give the expected maximum and minimum temperatures of the exhaust gases. Use a graphing utility to graph each term.

112. ***Cellular Phone Revenues*** The amount of service revenue R (in billions of dollars) for cellular phones in the United States for the years 1991 through 1997 can be approximated by the model

$$R = \frac{-2617 + 4423t}{t(411 - 38t)}, \quad 1 \le t \le 7$$

where t represents the year with $t = 1$ corresponding to 1991. Write the partial fraction decomposition for the rational function. (Source: Cellular Telecommunications Industry Association)

Synthesis

True or False? **In Exercises 113–115, determine whether the statement is true or false. Justify your answer.**

113. The system $\begin{cases} x + 4y - 5z = 8 \\ 2y + z = 5 \\ z = 1 \end{cases}$ is in row-echelon form.

114. If a system of three linear equations is inconsistent, then its graph has no common point.

115. For the rational expression

$$\frac{x}{(x+10)(x-10)^2}$$

the partial fraction decomposition is of the form

$$\frac{A}{x+10}+\frac{B}{(x-10)^2}.$$

116. ***Error Analysis*** Suppose you are tutoring a student in algebra. In trying to find a partial fraction decomposition, your student writes the following.

$$\frac{x^2+1}{x(x-1)}=\frac{A}{x}+\frac{B}{x-1}$$

$$x^2+1=A(x-1)+Bx \qquad \text{Basic equation}$$

$$x^2+1=(A+B)x-A$$

Your student then forms the following system of linear equations.

$$\begin{cases} A+B=0 \\ -A = 1 \end{cases}$$

Solve the system and check the partial fraction decomposition it yields. Has your student worked the problem correctly? If not, what went wrong?

In Exercises 117–120, write the partial fraction decomposition for the rational expression. Check your result algebraically. Then assign a value to the constant a and check the result graphically.

117. $\dfrac{1}{a^2-x^2}$

118. $\dfrac{1}{(x+1)(a-x)}$

119. $\dfrac{1}{y(a-y)}$

120. $\dfrac{1}{x(x+a)}$

121. ***Think About It*** Are the two systems of equations equivalent? Give reasons for your answer.

$$\begin{cases} x+3y-z=6 \\ 2x-y+2z=1 \\ 3x+2y-z=2 \end{cases} \qquad \begin{cases} x+3y-z=6 \\ -7y+4z=1 \\ -7y-4z=-16 \end{cases}$$

122. ***Think About It*** When using Gaussian elimination to solve a system of linear equations, how can you recognize that it has no solution? Give an example that illustrates your answer.

Advanced Applications **In Exercises 123–126, find values of x, y, and λ that satisfy the system. These systems arise in certain optimization problems in calculus; λ is called a Lagrange multiplier.**

123. $\begin{cases} y+\lambda=0 \\ x+\lambda=0 \\ x+y-10=0 \end{cases}$

124. $\begin{cases} 2x+\lambda=0 \\ 2y+\lambda=0 \\ x+y-4=0 \end{cases}$

125. $\begin{cases} 2x-2x\lambda=0 \\ -2y+\lambda=0 \\ y-x^2=0 \end{cases}$

126. $\begin{cases} 2+2x+2\lambda=0 \\ 2x+1+\lambda=0 \\ 2x+y-100=0 \end{cases}$

Review

In Exercises 127–134, sketch the graph of the function.

127. $f(x)=-3x+7$

128. $f(x)=6-x$

129. $f(x)=-2x^2$

130. $f(x)=\frac{1}{4}x^2+1$

131. $f(x)=x^2-9x+18$

132. $f(x)=2x^2-9x-5$

133. $f(x)=-x^2(x-3)$

134. $f(x)=\frac{1}{2}x^3-1$

In Exercises 135–138, (a) determine the real zeros of f and (b) sketch the graph of f.

135. $f(x)=x^3+x^2-12x$

136. $f(x)=-8x^4+32x^2$

137. $f(x)=2x^3+5x^2-21x-36$

138. $f(x)=6x^3-29x^2-6x+5$

In Exercises 139–142, use a graphing utility to create a table of values. Then sketch the graph of the equation by hand.

139. $y=4^{-x-4}-5$

140. $y=\left(\frac{5}{2}\right)^{x-1}-4$

141. $y=2.9^{0.8x}-3$

142. $y=-3.5^{x+2}-6$

In Exercises 143–146, solve the system by elimination.

143. $\begin{cases} 3x+3y=7 \\ 3x+5y=3 \end{cases}$

144. $\begin{cases} 6x-5y=3 \\ 10x-12y=5 \end{cases}$

145. $\begin{cases} 2x+y=120 \\ x+2y=120 \end{cases}$

146. $\begin{cases} 12x+42y=-17 \\ 30x-18y=19 \end{cases}$

8.4 Systems of Inequalities

The Graph of an Inequality

The statements $3x - 2y < 6$ and $2x^2 + 3y^2 \geq 6$ are inequalities in two variables. An ordered pair (a, b) is a **solution of an inequality** in x and y if the inequality is true when a and b are substituted for x and y, respectively. The **graph of an inequality** is the collection of all solutions of the inequality. To sketch the graph of an inequality, begin by sketching the graph of the *corresponding equation.* The graph of the equation will normally separate the plane into two or more regions. In each such region, one of the following must be true.

1. *All* points in the region are solutions of the inequality.
2. *No* point in the region is a solution of the inequality.

So, you can determine whether the points in an entire region satisfy the inequality by simply testing *one* point in the region.

Sketching the Graph of an Inequality in Two Variables

1. Replace the inequality sign with an equal sign, and sketch the graph of the corresponding equation. (Use a dashed line for < or > and a solid line for ≤ or ≥.)
2. Test one point in each of the regions formed by the graph in Step 1. If the point satisfies the inequality, shade the entire region to denote that every point in the region satisfies the inequality.

EXAMPLE 1 Sketching the Graph of an Inequality

To sketch the graph of $y \geq x^2 - 1$, begin by graphing the corresponding *equation* $y = x^2 - 1$, which is a parabola, as shown in Figure 8.18. By testing the point $(0, 0)$ *above* the parabola and the point $(0, -2)$ *below* the parabola, you can see that $(0, 0)$ satisfies the inequality because $0 \geq 0^2 - 1$ and that $(0, -2)$ does not satisfy the inequality because $-2 \not\geq 0^2 - 1$. So, the points that satisfy the inequality are those lying above (or on) the parabola.

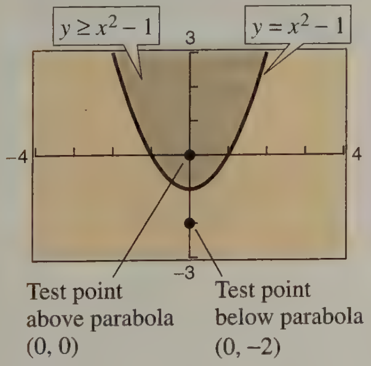


Figure 8.18

What You Should Learn:

- How to sketch graphs of inequalities in two variables
- How to solve systems of inequalities
- How to use systems of inequalities in two variables to model and solve real-life problems

Why You Should Learn It:

Systems of inequalities in two variables can be used to model and solve real-life problems. For instance, Exercise 69 on page 611 shows how to use a system of inequalities to analyze the compositions of dietary supplements.

Tom McCarthy/PhotoEdit

The inequality given in Example 1 is a nonlinear inequality in two variables. Most of the following examples involve **linear inequalities** such as $ax + by < c$. The graph of a linear inequality is a half-plane lying on one side of the line $ax + by = c$.

EXAMPLE 2 Sketching the Graphs of Linear Inequalities

Sketch the graph of each linear inequality.

a. $x > -2$ **b.** $y \le 3$

Solution

a. The graph of the corresponding equation $x = -2$ is a vertical line. The points that satisfy the inequality $x > -2$ are those lying to the right of (but not on) this line, as shown in Figure 8.19.

b. The graph of the corresponding equation $y = 3$ is a horizontal line. The points that satisfy the inequality $y \le 3$ are those lying below (or on) this line, as shown in Figure 8.20.

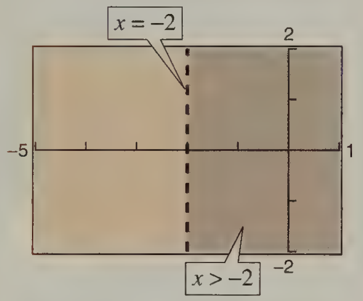


Figure 8.19

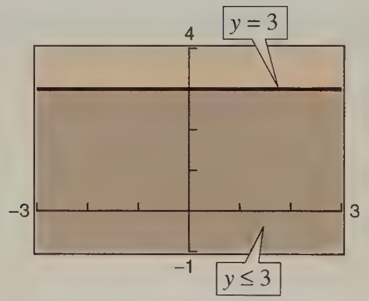


Figure 8.20

STUDY TIP

A graphing utility can be used to graph an inequality. For instance, to graph $y \ge x - 2$, enter $y = x - 2$ and use the *shade* feature of the graphing utility to shade the correct part of the graph. Consult your user's manual for instructions.

EXAMPLE 3 Sketching the Graph of a Linear Inequality

Sketch the graph of $x - y < 2$.

Solution

The graph of the corresponding equation $x - y = 2$ is a line, as shown in Figure 8.21. Because the origin $(0, 0)$ satisfies the inequality, the graph consists of the half-plane lying above the line. (Try checking a point below the line. Regardless of which point below the line you choose, you will see that it does not satisfy the inequality.)

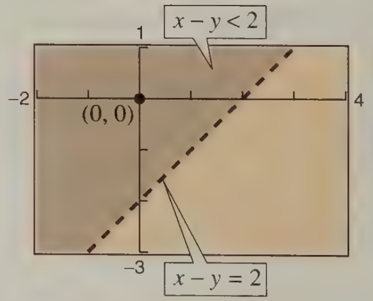


Figure 8.21

To graph a linear inequality, it can help to write the inequality in slope-intercept form. For instance, by writing $x - y < 2$ in Example 3 in the form

$$y > x - 2$$

you can see that the solution points lie *above* the line $y = x - 2$ (or $x - y = 2$), as shown in Figure 8.21.

Systems of Inequalities

Many practical problems in business, science, and engineering involve systems of linear inequalities. A **solution of a system of inequalities** in x and y is a point (x, y) that satisfies each inequality in the system.

To sketch the graph of a system of inequalities in two variables, first sketch the graph of each individual inequality (on the same coordinate system) and then find the region that is *common* to every graph in the system. For systems of *linear* inequalities, it is helpful to find the vertices of the solution region.

EXAMPLE 4 Solving a System of Inequalities

Sketch the graph (and label the vertices) of the solution set of the system.

$$\begin{cases} x - y < 2 & \text{Inequality 1} \\ x > -2 & \text{Inequality 2} \\ y \le 3 & \text{Inequality 3} \end{cases}$$

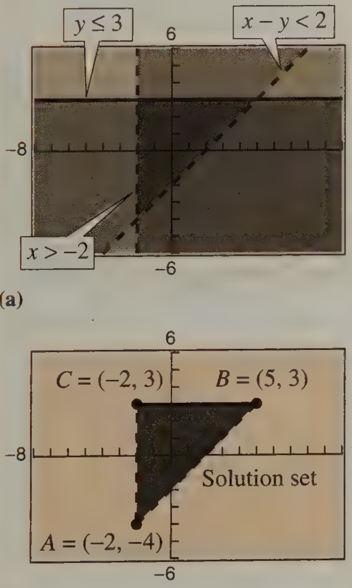
A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Solution

The graphs of these inequalities are shown in Figures 8.19 through 8.21. The triangular region common to all three graphs can be found by superimposing the graphs on the same coordinate system, as shown in Figure 8.22. To find the vertices of the region, solve the three systems of corresponding equations obtained by taking pairs of equations representing the boundaries of the individual regions and solving these pairs of equations.

Vertex A: $(-2, -4)$ *Vertex B:* $(5, 3)$ *Vertex C:* $(-2, 3)$

$$\begin{cases} x - y = 2 \\ x = -2 \end{cases} \qquad \begin{cases} x - y = 2 \\ y = 3 \end{cases} \qquad \begin{cases} x = -2 \\ y = 3 \end{cases}$$



(a)

(b)

Figure 8.22

STUDY T!P

Using different colored pencils to shade the solution of each inequality in a system makes identifying the solution of the system of inequalities easier. The region common to every graph in the system is where all shaded regions overlap. This region represents the solution set of the system.

For the triangular region shown in Figure 8.22, each point of intersection of a pair of boundary lines corresponds to a vertex. With more complicated regions, two border lines can sometimes intersect at a point that is not a vertex of the region, as shown in Figure 8.23. To keep track of which points of intersection are actually vertices of the region, you should sketch the region and refer to your sketch as you find each point of intersection.

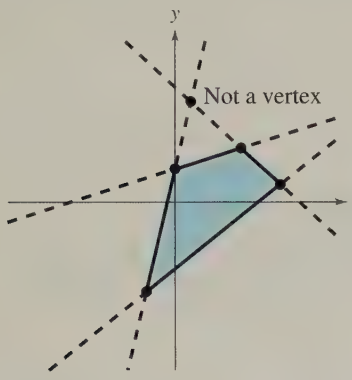


Figure 8.23

EXAMPLE 5 Solving a System of Inequalities

Sketch the region containing all points that satisfy the system.

$$\begin{cases} x^2 - y \le 1 & \text{Inequality 1} \\ -x + y \le 1 & \text{Inequality 2} \end{cases}$$

Solution

As shown in Figure 8.24, the points that satisfy the inequality $x^2 - y \le 1$ are the points lying above (or on) the parabola

$$y = x^2 - 1. \qquad \text{Parabola}$$

The points that satisfy the inequality $-x + y \le 1$ are the points lying below (or on) the line

$$y = x + 1. \qquad \text{Line}$$

To find the points of intersection of the parabola and the line, solve the system of corresponding equations.

$$\begin{cases} x^2 - y = 1 \\ -x + y = 1 \end{cases}$$

Using the method of substitution, you can find the solutions to be $(-1, 0)$ and $(2, 3)$. So, the region containing all points that satisfy the system are indicated by the purple shaded region in Figure 8.24.

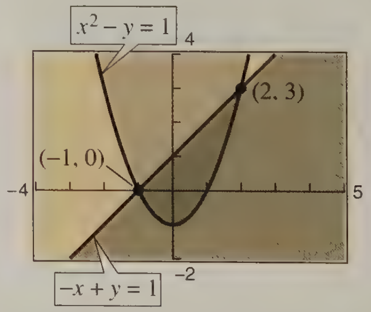


Figure 8.24

When solving a system of inequalities, you should be aware that the system might have no solution, or it might be represented by an unbounded region in the plane. These two possibilities are shown in Examples 6 and 7.

EXAMPLE 6 A System with No Solution

Sketch the solution set of the system.

$$\begin{cases} x + y > 3 & \text{Inequality 1} \\ x + y < -1 & \text{Inequality 2} \end{cases}$$

Solution

From the way the system is written, it is clear that the system has no solutions, because the quantity $(x + y)$ cannot be both less than -1 and greater than 3. Graphically, the inequality $x + y > 3$ is represented by the half-plane lying above the line $x + y = 3$, and the inequality $x + y < -1$ is represented by the half-plane lying below the line $x + y = -1$, as shown in Figure 8.25. These two half-planes have no points in common. So the system of inequalities has no solution.

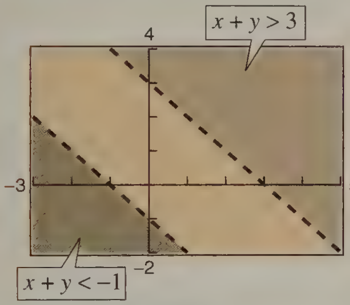


Figure 8.25 *No Solution*

EXAMPLE 7 An Unbounded Solution Set

Sketch the solution set of the system.

$$\begin{cases} x + y < 3 & \text{Inequality 1} \\ x + 2y > 3 & \text{Inequality 2} \end{cases}$$

Solution

The graph of the inequality $x + y < 3$ is the half-plane that lies below the line $x + y = 3$, as shown in Figure 8.26. The graph of the inequality $x + 2y > 3$ is the half-plane that lies above the line $x + 2y = 3$. The intersection of these two half-planes is an *infinite wedge* that has a vertex at $(3, 0)$. This unbounded region representing the solution set has an infinite number of points.

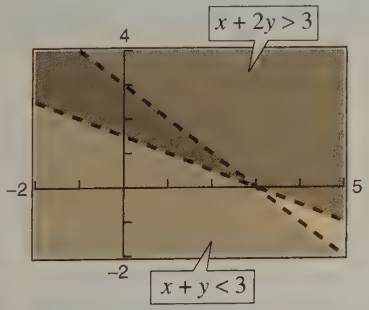


Figure 8.26 *Unbounded Region*

Applications

The next example discusses two concepts that economists call *consumer surplus* and *producer surplus*. As shown in Figure 8.27, the *point of equilibrium* is defined by the price p and the number of units x that satisfy both the demand and supply equations. Consumer surplus is defined as the area of the region that lies *below* the demand curve, *above* the horizontal line passing through the equilibrium point, and to the right of the p-axis. Similarly, the producer surplus is defined as the area of the region that lies *above* the supply curve, *below* the horizontal line passing through the equilibrium point, and to the right of the p-axis. The consumer surplus is a measure of the amount that consumers would have been willing to pay *above what they actually paid*, whereas the producer surplus is a measure of the amount that producers would have been willing to receive *below what they actually received.*

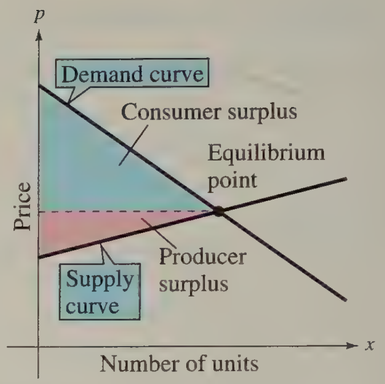


Figure 8.27

EXAMPLE 8 Consumer Surplus and Producer Surplus

The demand and supply functions for a certain type of calculator are

$$\begin{cases} p = 150 - 0.00001x & \text{Demand equation} \\ p = 60 + 0.00002x & \text{Supply equation} \end{cases}$$

where p is the price (in dollars) and x represents the number of units. Find the consumer surplus and producer surplus for these two equations.

Solution

Begin by finding the point of equilibrium by solving the equation

$$60 + 0.00002x = 150 - 0.00001x.$$

In Example 8 in Section 8.2, you saw that the solution is $x = 3{,}000{,}000$, which corresponds to an equilibrium price of $p = \$120$. So, the consumer surplus and producer surplus are the areas of the following triangular regions.

Consumer Surplus

$$\begin{cases} p \le 150 - 0.00001x \\ p \ge 120 \\ x \ge 0 \end{cases}$$

Producer Surplus

$$\begin{cases} p \ge 60 + 0.00002x \\ p \le 120 \\ x \ge 0 \end{cases}$$

In Figure 8.28, you can see that these regions are triangles. So, the consumer and producer surpluses are

$$\text{Consumer surplus} = \tfrac{1}{2}(\text{base})(\text{height})$$
$$= \tfrac{1}{2}(3{,}000{,}000)(30)$$
$$= \$45{,}000{,}000$$

$$\text{Producer surplus} = \tfrac{1}{2}(\text{base})(\text{height})$$
$$= \tfrac{1}{2}(3{,}000{,}000)(60)$$
$$= \$90{,}000{,}000$$

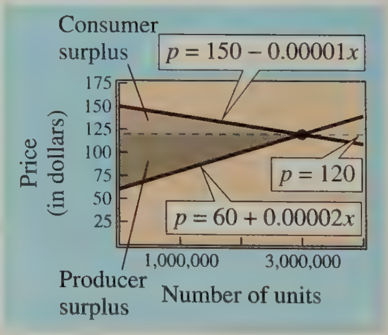


Figure 8.28

EXAMPLE 9 Nutrition

The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Set up a system of linear inequalities that must be satisfied in order to meet the minimum daily requirements for calories and vitamins.

Solution

Begin by letting x and y represent the following.

x = number of cups of dietary drink X

y = number of cups of dietary drink Y

To meet the minimum daily requirements, the following inequalities must be satisfied.

$$\begin{cases} 60x + 60y \geq 300 & \text{Calories} \\ 12x + 6y \geq 36 & \text{Vitamin A} \\ 10x + 30y \geq 90 & \text{Vitamin C} \\ x \geq 0 & \\ y \geq 0 & \end{cases}$$

The last two inequalities are included because x and y cannot be negative. The graph of this system of inequalities is shown in Figure 8.29. (More is said about this application in Example 6 in Section 8.5.)

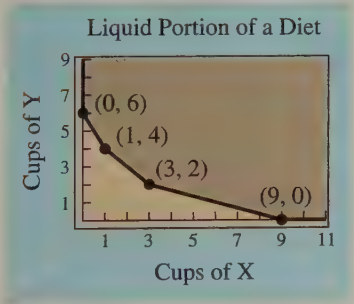


Figure 8.29

STUDY TIP

When using a system of inequalities to represent a real-life application in which the variables cannot be negative, remember to include inequalities for this condition. For instance, in Example 9, x and y cannot be negative, so the inequalities $x \geq 0$ and $y \geq 0$ must be included in the system.

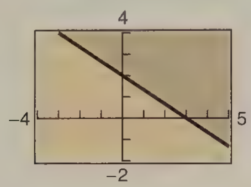
Writing About Math *Creating a System of Inequalities*

Plot the points (0, 0), (4, 0), (3, 2), and (0, 2) in a coordinate plane. Draw the quadrilateral that has these four points as its vertices. Write a system of linear inequalities that has the quadrilateral as its solution. Explain how you found the system of inequalities.

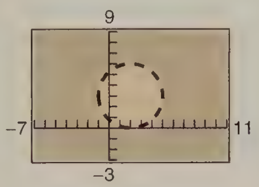
8.4 Exercises

In Exercises 1–8, match the inequality with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

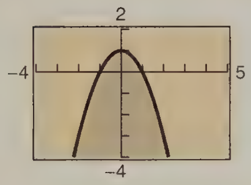
(a)



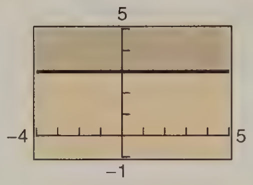
(b)



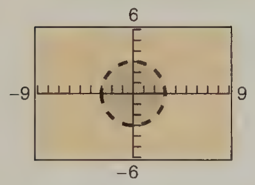
(c)



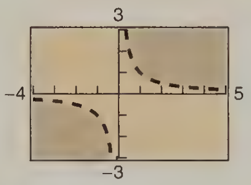
(d)



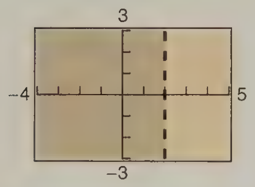
(e)



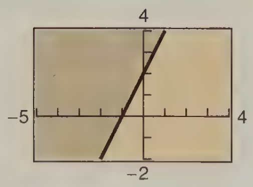
(f)



(g)



(h)



1. $x < 2$

2. $y \geq 3$

3. $2x + 3y \geq 6$

4. $2x - y \leq -2$

5. $x^2 + y^2 < 9$

6. $(x - 2)^2 + (y - 3)^2 > 9$

7. $xy > 1$

8. $y \leq 1 - x^2$

In Exercises 9–20, sketch the graph of the inequality.

9. $x \geq 2$

10. $x \leq -5$

11. $y \geq -1$

12. $y \leq 3$

13. $y < 2 - x$

14. $y > 2x - 4$

15. $2y - x \geq 4$

16. $5x + 3y \geq -15$

17. $y^2 - x < 0$

18. $4x + y^2 > 1$

19. $(x + 1)^2 + y^2 < 9$

20. $(x - 1)^2 + (y - 4)^2 > 9$

In Exercises 21–32, use a graphing utility to graph the inequality. Be sure to shade the region representing the solution.

21. $y \geq \frac{2}{3}x - 1$

22. $y \leq 6 - \frac{3}{2}x$

23. $y < -3.8x + 1.1$

24. $y \geq -20.74 + 2.66x$

25. $x^2 + 5y - 10 \leq 0$

26. $2x^2 - y - 3 > 0$

27. $y \leq \dfrac{1}{1 + x^2}$

28. $y > \dfrac{-10}{x^2 + x + 4}$

29. $y < \ln x$

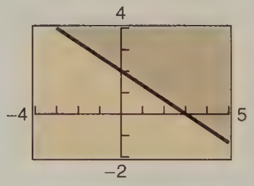
30. $y \geq 4 - \ln(x + 5)$

31. $y > 3^{-x-4}$

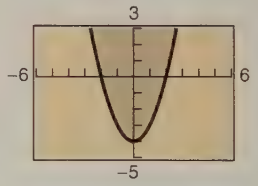
32. $y \leq 2^{2x-1} - 3$

In Exercises 33–36, write an inequality for the shaded region shown in the graph.

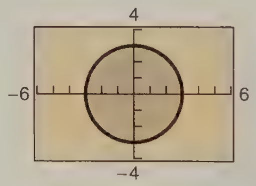
33.



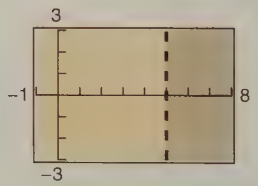
34.



35.



36.



In Exercises 37 and 38, decide whether each ordered pair is a solution of the system of linear inequalities.

37. $\begin{cases} -2x + 5y \geq 3 \\ y < 4 \\ -4x + 2y < 7 \end{cases}$

(a) (0, 2) (b) (−6, 4)
(c) (−8, −2) (d) (−3, 2)

38. $\begin{cases} 3x + y > 1 \\ -y - \frac{1}{2}x^2 \leq -4 \\ -15x + 4y > 0 \end{cases}$

(a) (0, 10) (b) (0, −1)
(c) (2, 9) (d) (−1, 6)

In Exercises 39–50, sketch the graph of the solution of the system of inequalities. Verify your sketch by using a graphing utility to graph the system, shading the region that represents the solution of the system.

39. $\begin{cases} x + y \le 1 \\ -x + y \le 1 \\ y \ge 0 \end{cases}$

40. $\begin{cases} 3x + 2y < 6 \\ x > 0 \\ y > 0 \end{cases}$

41. $\begin{cases} -3x + 2y < 6 \\ x - 4y > -2 \\ 2x + y < 3 \end{cases}$

42. $\begin{cases} x - 7y > -36 \\ 5x + 2y > 5 \\ 6x - 5y > 6 \end{cases}$

43. $\begin{cases} 2x + y > 2 \\ 6x + 3y < 2 \end{cases}$

44. $\begin{cases} x - 2y < -6 \\ 5x - 3y > -9 \end{cases}$

45. $\begin{cases} -x^2 + y \ge 5 \\ \frac{1}{4}x + y < 3 \end{cases}$

46. $\begin{cases} y^2 - 3x \ge 9 \\ x + y \ge -3 \end{cases}$

47. $\begin{cases} x > y^2 \\ x < y + 2 \end{cases}$

48. $\begin{cases} x - y^2 > 0 \\ x - y < 2 \end{cases}$

49. $\begin{cases} x^2 + y^2 \le 9 \\ x^2 + y^2 \ge 1 \end{cases}$

50. $\begin{cases} x < 2y - y^2 \\ 0 < x + y \end{cases}$

In Exercises 51–56, use a graphing utility to graph the inequalities. Be sure to shade the region representing the solution of the system.

51. $\begin{cases} y \le \sqrt{3x} + 1 \\ y \ge x^2 + 1 \end{cases}$

52. $\begin{cases} y < -x^2 + 2x + 3 \\ y > x^2 - 4x + 3 \end{cases}$

53. $\begin{cases} y < x^3 - 2x + 1 \\ y > -2x \\ x \le 1 \end{cases}$

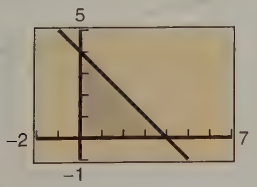
54. $\begin{cases} y \ge x^4 - 2x^2 + 1 \\ y \le 1 - x^2 \end{cases}$

55. $\begin{cases} x^2 y \ge 1 \\ 0 < x \le 4 \\ y \le 4 \end{cases}$

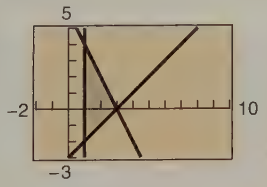
56. $\begin{cases} y \le e^{-x^2/2} \\ y \ge 0 \\ -2 \le x \le 2 \end{cases}$

In Exercises 57–64, find a set of inequalities to describe the region.

57.



58.



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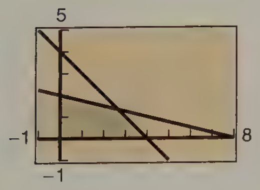
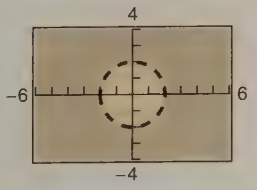


60.



61. Rectangular region with vertices at (2, 1), (5, 1), (5, 7), and (2, 7)

62. Parallelogrammatic region with vertices at (0, 0), (4, 0), (1, 4), and (5, 4)

63. Triangular region with vertices at (0, 0), (5, 0), and (2, 3)

64. Triangular region with vertices at (−1, 0), (1, 0), and (0, 1)

In Exercises 65–69, (a) find a system of inequalities that models the problem and (b) use a graphing utility to graph the systems, shading the region that represents the solution of the system.

65. ***Investment*** A person plans to invest some or all of \$20,000 in two different interest-bearing accounts. Each account is to contain at least \$5000, and one account should have at least twice the amount that is in the other account.

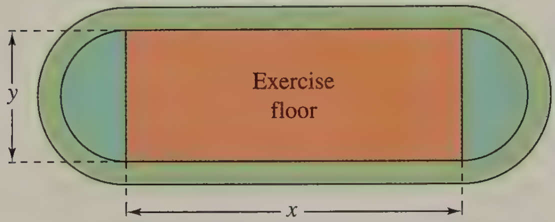
66. ***Concert Ticket Sales*** One type of concert ticket costs \$15 and another costs \$25. The promoter of the concert must sell at least 15,000 tickets, including at least 8000 of the \$15 tickets and at least 4000 of the \$25 tickets, and the gross receipts must total at least \$275,000 in order for the concert to be held.

67. ***Furniture Production*** A furniture company can sell all the tables and chairs it produces. Each table requires 1 hour of assembly and $1\frac{1}{3}$ hours of finishing. Each chair requires $1\frac{1}{2}$ hours of assembly and $1\frac{1}{2}$ hours of finishing. The company's assembly center is available 12 hours per day, and its finishing center is available 15 hours per day.

68. ***Computer Inventory*** A store sells two models of a certain brand of computer. Because of the demand, it is necessary to stock at least twice as many units of model A as units of model B. The costs to the store for the two models are \$800 and \$1200, respectively. The management does not want more than \$20,000 in computer inventory at any one time, and it wants at least four model A computers and two model B computers in inventory at all times.

69. ***Diet Supplement*** A dietitian is asked to design a special diet using two different foods. The minimum daily requirements in the new diet are 280 units of calcium, 160 units of iron, and 180 units of vitamin B. Each ounce of food X contains 20 units of calcium, 15 units of iron, and 10 units of vitamin B. Each ounce of food Y contains 10 units of calcium, 10 units of iron, and 20 units of vitamin B.

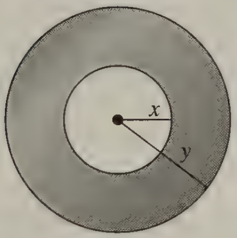
70. ***Physical Fitness Facility*** You plan an exercise facility that has an indoor running track with an exercise floor inside the track. The track must be at least 125 meters long, and the exercise floor must have an area of at least 500 square meters.



(a) Find a system of inequalities describing the requirements of the exercise facility.

(b) Sketch the graph of the system in part (a).

71. ***Graphical Reasoning*** Two concentric circles have radii of x and y meters, where $y > x$. The area between the boundaries of the circles must be at least 10 square meters.



(a) Find an inequality describing the constraints on the circles.

(b) Use a graphing utility to graph the inequality in part (a). Graph the line $y = x$ in the same viewing window.

(c) Identify the graph of the line in relation to the boundary of the inequality. Explain its meaning in the context of the problem.

Consumer and Producer Surpluses **In Exercises 72–75, use a graphing utility to graph the system representing the consumer surplus and producer surplus for the supply and demand equations. Be sure to shade the region representing the solution of the system. Find the consumer surplus and the producer surplus.**

	Demand	*Supply*
72.	$p = 50 - 0.5x$	$p = 0.125x$
73.	$p = 60 - x$	$p = 10 + \frac{7}{3}x$
74.	$p = 300 - x$	$p = 100 + x$
75.	$p = 140 - 0.00002x$	$p = 80 + 0.00001x$

Synthesis

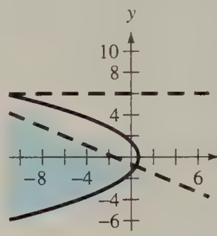
True or False? **In Exercises 76 and 77, determine whether the statement is true or false. Justify your answer.**

76. The area of the figure defined by the system below is 99 square units.

$$\begin{cases} x \ge -3 \\ x \le 6 \\ y \le 5 \\ y \ge -6 \end{cases}$$

77. The graph below shows the solution of the system

$$\begin{cases} y \le 6 \\ -4x - 9y > 6. \\ 3x + y^2 \ge 2 \end{cases}$$



78. ***Think About It*** After graphing the boundary of an inequality in x and y, how do you decide on which side of the boundary the solution set of the inequality lies?

79. ***Writing*** Describe the difference between the solution set of a system of equations and the solution set of a system of inequalities.

Review

In Exercises 80–83, find the equation of the line passing through the two points.

80. $(-2, 6), (4, -4)$ **81.** $(-8, 0), (3, -1)$

82. $\left(\frac{3}{4}, -2\right), \left(-\frac{7}{2}, 5\right)$ **83.** $(3.4, -5.2), (-2.6, 0.8)$

84. ***Mortgage Loans*** The table shows the number of outstanding mortgage loans M (in millions) in the United States for the years 1992 to 1997. (Source: Mortgage Bankers Association of America)

Year	1992	1993	1994	1995	1996	1997
M	42.6	45.3	47.6	49.2	50.1	51.2

Use the regression capabilities of a graphing utility to find a linear model and a quadratic model that represent the data. Let $t = 0$ represent 1990. Use a graphing utility to plot the actual data and the models in the same viewing window. How closely do the models represent the data?

8.5 Linear Programming

What You Should Learn:

- How to solve linear programming problems
- How to use linear programming to model and solve real-life problems

Why You Should Learn It:

Linear programming is a powerful tool used in business and industry to effectively manage resources in order to maximize profits or minimize costs. For instance, Exercise 41 on page 621 shows how to use linear programming to analyze the profitability of two models of compact disc players.

Jeff Greenberg/Unicorn Stock Photos

Linear Programming: A Graphical Approach

Many applications in business and economics involve a process called **optimization,** in which you are asked to find the minimum or maximum value of a quantity. In this section you will study an optimization strategy called **linear programming.**

A two-dimensional linear programming problem consists of a linear **objective function** and a system of linear inequalities called **constraints.** The objective function gives the quantity that is to be maximized (or minimized), and the constraints determine the set of **feasible solutions.** For example, suppose you are asked to maximize the value of

$z = ax + by$ Objective function

subject to a set of constraints that determines the region in Figure 8.30. Because every point in the region satisfies each constraint, it is not clear how you should go about finding the point that yields a maximum value of z. Fortunately, it can be shown that if there is an optimal solution, it must occur at one of the vertices. So, *you can find the maximum value by testing z at each of the vertices.*

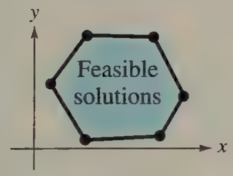


Figure 8.30

Optimal Solution of a Linear Programming Problem

If a linear programming problem has a solution, it must occur at a vertex of the set of feasible solutions. If there is more than one solution, at least one of them must occur at such a vertex. In either case, the value of the objective function is unique.

Here are some guidelines for solving a linear programming problem in two variables in which an objective function is to be maximized *or* minimized.

Solving a Linear Programming Problem

1. Sketch the region corresponding to the system of constraints. (The points inside or on the boundary of the region are *feasible solutions.*)
2. Find the vertices of the region.
3. Test the objective function at each of the vertices and select the values of the variables that optimize the objective function. For a bounded region, both a minimum and a maximum value will exist. (For an unbounded region, *if* an optimal solution exists, it will occur at a vertex.)

EXAMPLE 1 Solving a Linear Programming Problem

Find the maximum value of

$$z = 3x + 2y \qquad \text{Objective function}$$

subject to the following constraints.

$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ x - y &\leq 1 \end{aligned}\right\} \qquad \text{Constraints}$$

Solution

The constraints form the region shown in Figure 8.31. At the four vertices of this region, the objective function has the following values.

$$\text{At } (0, 0): \quad z = 3(0) + 2(0) = 0$$

$$\text{At } (1, 0): \quad z = 3(1) + 2(0) = 3$$

$$\text{At } (2, 1): \quad z = 3(2) + 2(1) = 8 \qquad \text{Maximum value of } z$$

$$\text{At } (0, 2): \quad z = 3(0) + 2(2) = 4$$

So, the maximum value of z is 8, and this occurs when $x = 2$ and $y = 1$.

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

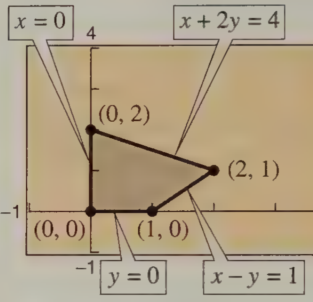


Figure 8.31

In Example 1, try testing some of the *interior* points in the region. You will see that the corresponding values of z are less than 8. Here are some examples.

$$\text{At } (1, 1): \quad z = 3(1) + 2(1) = 5$$

$$\text{At } \left(1, \tfrac{1}{2}\right): \quad z = 3(1) + 2\left(\tfrac{1}{2}\right) = 4$$

$$\text{At } \left(\tfrac{1}{2}, \tfrac{3}{2}\right): \quad z = 3\left(\tfrac{1}{2}\right) + 2\left(\tfrac{3}{2}\right) = \tfrac{9}{2}$$

To see why the maximum value of the objective function in Example 1 must occur at a vertex, consider writing the objective function in the form

$$y = -\frac{3}{2}x + \frac{z}{2} \qquad \text{Family of lines}$$

where $z/2$ is the y-intercept of the objective function. This equation represents a family of lines, each of slope $-\frac{3}{2}$. Of these infinitely many lines, you want the one that has the largest z-value while still intersecting the region determined by the constraints. In other words, of all the lines with a slope of $-\frac{3}{2}$, you want the one that has the largest y-intercept *and* intersects the given region, as shown in Figure 8.32. It should be clear that such a line will pass through one (or more) of the vertices of the region.

STUDY TIP

Remember that a vertex of a region can be found using a system of linear equations. The system will consist of the equations of the lines passing through the vertex.

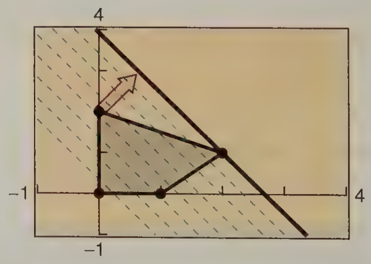


Figure 8.32

The next example shows that the same basic procedure can be used to solve a problem in which the objective function is to be *minimized*.

EXAMPLE 2 Minimizing an Objective Function

Find the minimum value of

$$z = 5x + 7y \qquad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the following constraints.

$$\left.\begin{aligned} 2x + 3y &\geq 6 \\ 3x - y &\leq 15 \\ -x + y &\leq 4 \\ 2x + 5y &\leq 27 \end{aligned}\right\} \quad \text{Constraints}$$

Solution

The region bounded by the constraints is shown in Figure 8.33. By testing the objective function at each vertex, you obtain the following.

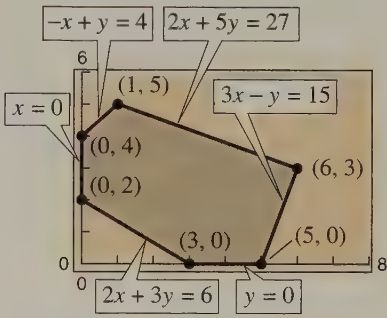


Figure 8.33

At (0, 2): $z = 5(0) + 7(2) = 14$ Minimum value of z

At (0, 4): $z = 5(0) + 7(4) = 28$

At (1, 5): $z = 5(1) + 7(5) = 40$

At (6, 3): $z = 5(6) + 7(3) = 51$

At (5, 0): $z = 5(5) + 7(0) = 25$

At (3, 0): $z = 5(3) + 7(0) = 15$

So, the minimum value of z is 14, and this occurs when $x = 0$ and $y = 2$.

EXAMPLE 3 Maximizing an Objective Function

Find the maximum value of

$$z = 5x + 7y \qquad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the following constraints.

$$\left.\begin{aligned} 2x + 3y &\geq 6 \\ 3x - y &\leq 15 \\ -x + y &\leq 4 \\ 2x + 5y &\leq 27 \end{aligned}\right\} \quad \text{Constraints}$$

Solution

This linear programming problem is identical to that given in Example 2 above, except that the objective function is *maximized* instead of minimized. Using the values of z at the vertices shown above, you can conclude that the maximum value of z is 51, and that this value occurs when $x = 6$ and $y = 3$.

It is possible for the maximum (or minimum) value in a linear programming problem to occur at *two* different vertices. For instance, at the vertices of the region shown in Figure 8.34, the objective function

$z = 2x + 2y$ Objective function

has the following values.

At (0, 0): $z = 2(0) + 2(0) = 0$

At (0, 4): $z = 2(0) + 2(4) = 8$

At (2, 4): $z = 2(2) + 2(4) = 12$ Maximum value of z

At (5, 1): $z = 2(5) + 2(1) = 12$ Maximum value of z

At (5, 0): $z = 2(5) + 2(0) = 10$

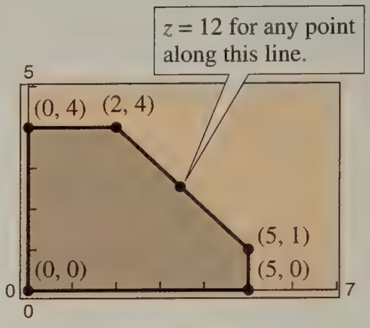


Figure 8.34

In this case, you can conclude that the objective function has a maximum value (of 12) not only at the vertices (2, 4) and (5, 1), but also at *any point on the line segment connecting these two vertices*, as shown in Figure 8.34. Note that by rewriting the objective function as

$$y = -x + \frac{1}{2}z$$

you can see that it has the same slope as the line through the vertices (2, 4) and (5, 1).

Some linear programming problems have no optimal solution. This can occur if the region determined by the constraints is *unbounded*.

EXAMPLE 4 An Unbounded Region

Find the maximum value of

$z = 4x + 2y$ Objective function

where $x \geq 0$ and $y \geq 0$, subject to the following constraints.

$$\left.\begin{aligned} x + 2y &\geq 4 \\ 3x + y &\geq 7 \\ -x + 2y &\leq 7 \end{aligned}\right\} \quad \text{Constraints}$$

Solution

The region determined by the constraints is shown in Figure 8.35. For this unbounded region, there is no maximum value of z. To see this, note that the point $(x, 0)$ lies in the region for all values of $x \geq 4$. By choosing large values of x, you can obtain values of $z = 4(x) + 2(0) = 4x$ that are as large as you want. So, there is no maximum value of z. For the vertices of the region, the objective function has the following values. So, there *is* a minimum value of $z = 10$, which occurs at the vertex (2, 1).

At (1, 4): $z = 4(1) + 2(4) = 12$

At (2, 1): $z = 4(2) + 2(1) = 10$ Minimum value of z

At (4, 0): $z = 4(4) + 2(0) = 16$

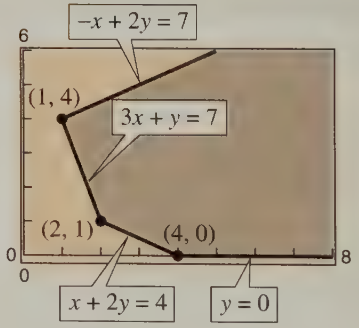


Figure 8.35

Applications

Example 5 shows how linear programming can be used to find the maximum profit in a business application.

EXAMPLE 5 Maximum Profit

A manufacturer wants to maximize the profit for two products. Product I yields a profit of \$1.50 per unit, and product II yields a profit of \$2.00 per unit. Market tests and available resources have indicated the following constraints.

1. The combined production level should not exceed 1200 units per month.
2. The demand for product II is no more than half the demand for product I.
3. The production level of product I is less than or equal to 600 units plus three times the production level of product II.

Solution

If you let x be the number of units of product I and y be the number of units of product II, the objective function (for the combined profit) is

$P = 1.5x + 2y.$ Objective function

The three constraints translate into the following linear inequalities.

1. $x + y \le 1200$ → $x + y \le 1200$

2. $y \le \frac{1}{2}x$ → $-x + 2y \le 0$

3. $x \le 3y + 600$ → $x - 3y \le 600$

Because neither x nor y can be negative, you also have the two additional constraints of $x \ge 0$ and $y \ge 0$. Figure 8.36 shows the region determined by the constraints. To find the maximum profit, test the value of P at the vertices of the region.

At (0, 0): $P = 1.5(0) + 2(0)$

$= 0$

At (800, 400): $P = 1.5(800) + 2(400)$

$= 2000$ Maximum profit

At (1050, 150): $P = 1.5(1050) + 2(150)$

$= 1875$

At (600, 0): $P = 1.5(600) + 2(0)$

$= 900$

So, the maximum profit is \$2000, and it occurs when the monthly production consists of 800 units of product I and 400 units of product II.

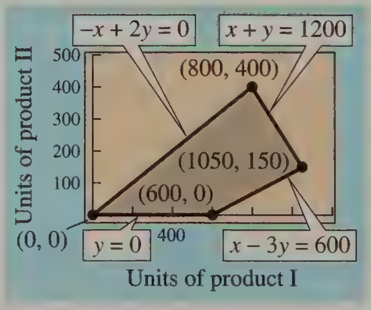


Figure 8.36

In Example 5, suppose the manufacturer improved the production of product I so that it yielded a profit of \$2.50 per unit. This would change the vertex at which the maximum profit would occur. In this case, a maximum profit of \$2925 would result from producing 1050 units of product I and 150 units of product II.

EXAMPLE 6 Minimum Cost

The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X costs \$0.12 and provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y costs \$0.15 and provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. How many cups of each drink should be consumed each day to minimize the cost and still meet the daily requirements?

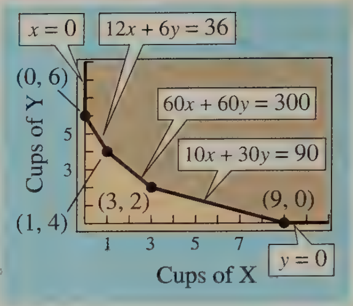


Figure 8.37

Solution

As in Example 9 on page 609, let x be the number of cups of dietary drink X and let y be the number of cups of dietary drink Y.

$$\left.\begin{array}{ll} \textit{For Calories:} & 60x + 60y \geq 300 \\ \textit{For Vitamin A:} & 12x + 6y \geq 36 \\ \textit{For Vitamin C:} & 10x + 30y \geq 90 \\ & x \geq 0 \\ & y \geq 0 \end{array}\right\} \quad \text{Constraints}$$

The cost C is given by $C = 0.12x + 0.15y$. Objective function

The graph of the region determined by the constraints is shown in Figure 8.37. To determine the minimum cost, test C at each vertex of the region.

At $(0, 6)$: $C = 0.12(0) + 0.15(6) = 0.90$

At $(1, 4)$: $C = 0.12(1) + 0.15(4) = 0.72$

At $(3, 2)$: $C = 0.12(3) + 0.15(2) = 0.66$ Minimum value of C

At $(9, 0)$: $C = 0.12(9) + 0.15(0) = 1.08$

So, the minimum cost is \$0.66 per day, and this occurs when three cups of drink X and two cups of drink Y are consumed each day.

Writing About Math *Creating a Linear Programming Problem*

Sketch the region determined by the constraints.

$$\left.\begin{array}{r} x \geq 0 \\ y \geq 0 \\ x + 2y \leq 10 \\ x + y \leq 7 \end{array}\right\} \quad \text{Constraints}$$

Find, if possible, an objective function of the form $z = ax + by$ that has a maximum at the indicated vertex of the region.

a. Maximum at $(0, 5)$

b. Maximum at $(4, 3)$

c. Maximum at $(7, 0)$

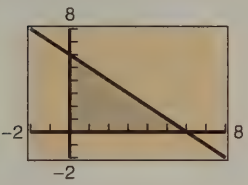
d. Maximum at $(0, 0)$

Write a short paragraph explaining how you found each objective function.

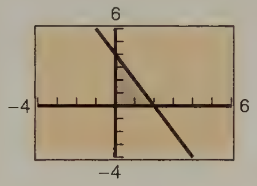
8.5 Exercises

In Exercises 1–12, find the minimum and maximum values of the objective function and where they occur, subject to the indicated constraints. (For each exercise, the graph of the region determined by the constraints is provided.)

1. Objective function:
$z = 3x + 5y$
Constraints:
$x \geq 0$
$y \geq 0$
$x + y \leq 6$



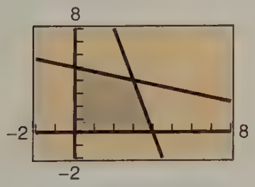
2. Objective function:
$z = 2x + 8y$
Constraints:
$x \geq 0$
$y \geq 0$
$2x + y \leq 4$



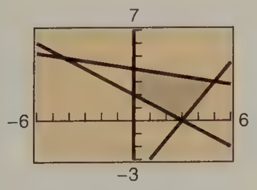
3. Objective function:
$z = 10x + 7y$
Constraints:
(See Exercise 1.)

4. Objective function:
$z = 7x + 3y$
Constraints:
(See Exercise 2.)

5. Objective function:
$z = 5x + 2y$
Constraints:
$x \geq 0$
$y \geq 0$
$x + 3y \leq 15$
$4x + y \leq 16$



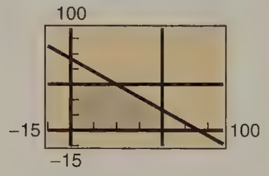
6. Objective function:
$z = 4x + 3y$
Constraints:
$x \geq 0$
$2x + 3y \geq 6$
$3x - 2y \leq 9$
$x + 5y \leq 20$



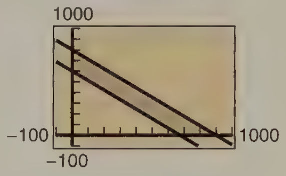
7. Objective function:
$z = 5x + 0.5y$
Constraints:
(See Exercise 5.)

8. Objective function:
$z = x + 6y$
Constraints:
(See Exercise 6.)

9. Objective function:
$z = 10x + 7y$
Constraints:
$0 \leq x \leq 60$
$0 \leq y \leq 45$
$5x + 6y \leq 420$



10. Objective function:
$z = 50x + 35y$
Constraints:
$x \geq 0$
$y \geq 0$
$8x + 9y \leq 7200$
$8x + 9y \geq 5400$



11. Objective function:
$z = 25x + 30y$
Constraints:
(See Exercise 9.)

12. Objective function:
$z = 16x + 18y$
Constraints:
(See Exercise 10.)

In Exercises 13–20, sketch the constraint region. Then find the minimum and maximum values of the objective function and where they occur, subject to the constraints.

13. Objective function:
$z = 6x + 10y$
Constraints:
$x \geq 0$
$y \geq 0$
$2x + 5y \leq 10$

14. Objective function:
$z = 4x + 6y$
Constraints:
$-x + y \leq 11$
$x + y \leq 27$
$2x + 5y \leq 90$
$x \geq 0$
$y \geq 0$

15. Objective function:
$z = 9x + 24y$
Constraints:
(See Exercise 13.)

16. Objective function:
$z = 7x + 2y$
Constraints:
(See Exercise 14.)

17. Objective function:
$z = 4x + 3y$
Constraints:
$x \geq 0$
$2x + 3y \geq 6$
$3x - 2y \leq 9$
$x + 5y \leq 20$

18. Objective function:
$z = 4x + 5y$
Constraints:
$x \geq 0$
$y \geq 0$
$2x + 2y \leq 10$
$x + 2y \leq 6$

19. Objective function:

$z = 3x + 7y$

Constraints:
(See Exercise 17.)

20. Objective function:

$z = 2x - y$

Constraints:
(See Exercise 18.)

In Exercises 21–26, use a graphing utility to sketch the region determined by the constraints. Then find the minimum and maximum values of the objective function and where they occur, subject to the constraints.

21. Objective function:

$z = 4x + y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 40 \\ 2x + 3y &\geq 72 \end{aligned}$$

22. Objective function:

$z = x$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + 3y &\leq 60 \\ 2x + y &\leq 28 \\ 4x + y &\leq 48 \end{aligned}$$

23. Objective function:

$z = x + 4y$

Constraints:
(See Exercise 21.)

24. Objective function:

$z = y$

Constraints:
(See Exercise 22.)

25. Objective function:

$z = 2x + 3y$

Constraints:
(See Exercise 21.)

26. Objective function:

$z = 3x + 2y$

Constraints:
(See Exercise 22.)

Exploration **In Exercises 27–30, perform the following: (a) use a graphing utility to graph the region bounded by the following constraints.**

$$\begin{aligned} 3x + y &\leq 15 \\ 4x + 3y &\leq 30 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

(b) Graph the objective function for the given maximum value of z in the same viewing window as the graph of the constraints. (c) Use the graph to determine the feasible point or points that yield the maximum. Explain how you arrived at your answer.

	Objective Function	*Maximum*
27.	$z = 2x + y$	$z = 12$
28.	$z = 5x + y$	$z = 25$
29.	$z = x + y$	$z = 10$
30.	$z = 3x + y$	$z = 15$

Exploration **In Exercises 31–34, perform the following: (a) use a graphing utility to graph the region bounded by the following constraints.**

$$\begin{aligned} x + 4y &\leq 20 \\ x + y &\leq 8 \\ 3x + 2y &\leq 21 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

(b) Graph the objective function for the given maximum value of z in the same viewing window as the graph of the constraints. (c) Use the graph to determine the feasible point or points that yield the maximum. Explain how you arrived at your answer.

	Objective Function	*Maximum*
31.	$z = x + 5y$	$z = 25$
32.	$z = 2x + 4y$	$z = 24$
33.	$z = 4x + 5y$	$z = 36$
34.	$z = 4x + y$	$z = 28$

In Exercises 35–40, the linear programming problem has an unusual characteristic. Sketch a graph of the solution region for the problem and describe the unusual characteristic. The objective function is to be maximized in each case.

35. Objective function:

$z = 2.5x + y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 3x + 5y &\leq 15 \\ 5x + 2y &\leq 10 \end{aligned}$$

36. Objective function:

$z = x + y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ -x + y &\leq 1 \\ -x + 2y &\leq 4 \end{aligned}$$

37. Objective function:

$z = -x + 2y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x &\leq 10 \\ x + y &\leq 7 \end{aligned}$$

38. Objective function:

$z = x + y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ -x + y &\leq 0 \\ -3x + y &\geq 3 \end{aligned}$$

39. Objective function:

$z = 3x + 4y$

Constraints:

$x \geq 0$

$y \geq 0$

$x + y \leq 1$

$2x + y \leq 4$

40. Objective function:

$z = x + 2y$

Constraints:

$x \geq 0$

$y \geq 0$

$x + 2y \leq 4$

$2x + y \leq 4$

41. ***Maximum Profit*** A merchant plans to sell two models of compact disc players at costs of \$250 and \$400. The \$250 model yields a profit of \$45, and the \$400 model yields a profit of \$50. The merchant estimates that the total monthly demand will not exceed 250 units. The merchant does not want to invest more than \$70,000 in inventory for these products. Find the number of units of each model that should be stocked in order to maximize profit. What is the maximum profit?

42. ***Maximum Profit*** A fruit grower has 150 acres of land available to raise two crops, A and B. It takes 1 day to trim an acre of crop A and 2 days to trim an acre of crop B, and there are 240 days per year available for trimming. It takes 0.3 day to pick an acre of crop A and 0.1 day to pick an acre of crop B, and there are 30 days available for picking. The profits are \$140 per acre for crop A and \$235 per acre for crop B. Find the number of acres of each fruit that should be planted to maximize profit. What is the maximum profit?

43. ***Minimum Cost*** Two gasolines, type A (\$1.13 per gallon) and type B (\$1.28 per gallon), have octane ratings of 80 and 92, respectively. Determine the blend of minimum cost with an octane rating of at least 90. What is the minimum cost? (*Hint:* Let x be the fraction of each gallon that is type A and let y be the fraction that is type B.)

44. ***Maximum Revenue*** An accounting firm has 900 hours of staff time and 100 hours of reviewing time available each week. The firm charges \$2000 for an audit and \$300 for a tax return. Each audit requires 100 hours of staff time and 10 hours of review time. Each tax return requires 12.5 hours of staff time and 2.5 hours of review time. What numbers of audits and tax returns will yield the maximum revenue? What is the maximum revenue?

45. ***Maximum Revenue*** The accounting firm in Exercise 44 lowers its charge for an audit to \$1000. What numbers of audits and tax returns will yield the maximum revenue? What is the maximum revenue?

46. ***Maximum Profit*** A manufacturer produces two models of bicycles. The amounts of time (in hours) required for assembling, painting, and packaging the two models are as follows.

	Model A	Model B
Assembling	2	2.5
Painting	4	1
Packaging	1	0.75

The total amounts of time available for assembling, painting, and packaging are 4000, 4800, and 1500 hours, respectively. The profits per unit are \$45 for model A and \$50 for model B. How many of each model should be produced to maximize profit? What is the maximum profit?

47. ***Maximum Profit*** A manufacturer produces two models of snowboards. The amounts of time (in hours) required for assembling, painting, and packaging the two models are as follows.

	Model A	Model B
Assembling	2.5	3
Painting	2	1
Packaging	0.75	1.25

The total amounts of time available for assembling, painting, and packaging are 4000, 2500, and 1500 hours, respectively. The profits per unit are \$50 for model A and \$52 for model B. How many of each model should be produced to maximize profit? What is the maximum profit?

48. ***Minimum Cost*** A farming cooperative mixes two brands of cattle feed. Brand X costs \$25 per bag and contains 2 units of nutritional element A, 2 units of element B, and 2 units of element C. Brand Y costs \$20 per bag and contains 1 unit of nutritional element A, 9 units of element B, and 3 units of element C. The minimum requirements for nutrients A, B, and C are 12 units, 36 units, and 24 units, respectively. Find the number of bags of each brand that should be mixed to produce a mixture having a minimum cost per bag. What is the minimum cost per mixed bag?

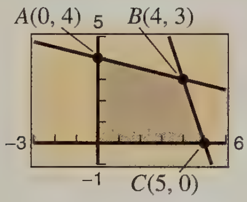
Synthesis

True or False? **In Exercises 49 and 50, determine whether the statement is true or false. Justify your answer.**

49. If an objective function has a maximum value at the adjacent vertices $(4, 7)$ and $(8, 3)$, you can conclude that it also has a maximum value at the points $(4.5, 6.5)$ and $(7.8, 3.2)$.

50. When solving a linear programming problem, if the objective function has a maximum value at two adjacent vertices, you can assume that there is an infinite number of points that will produce the maximum value.

Think About It **In Exercises 51–54, find an objective function that has a maximum or minimum value at the indicated vertex of the constraint region shown below. (There are many correct answers.)**



51. The maximum occurs at vertex A.

52. The maximum occurs at vertex B.

53. The maximum occurs at vertex C.

54. The minimum occurs at vertex C.

In Exercises 55 and 56, determine values of t such that the objective function has a maximum value at each indicated vertex.

55. Objective function:

$z = 3x + ty$

Constraints:

$x \geq 0$

$y \geq 0$

$x + 3y \leq 15$

$4x + y \leq 16$

(a) $(0, 5)$

(b) $(3, 4)$

56. Objective function:

$z = 3x + ty$

Constraints:

$x \geq 0$

$y \geq 0$

$x + 2y \leq 4$

$x - y \leq 1$

(a) $(2, 1)$

(b) $(0, 2)$

Review

In Exercises 57–60, simplify the compound fraction.

57. $\dfrac{\left(\dfrac{9}{x}\right)}{\left(\dfrac{6}{x} + 2\right)}$

58. $\dfrac{\left(1 + \dfrac{2}{x}\right)}{\left(x - \dfrac{4}{x}\right)}$

59. $\dfrac{\left(\dfrac{4}{x^2 - 9} + \dfrac{2}{x - 2}\right)}{\left(\dfrac{1}{x + 3} + \dfrac{1}{x - 3}\right)}$

60. $\dfrac{\left(\dfrac{1}{x + 1} + \dfrac{1}{2}\right)}{\left(\dfrac{3}{2x^2 + 4x + 2}\right)}$

In Exercises 61–66, solve the equation algebraically. Round your result to three decimal places.

61. $e^{2x} + 2e^x - 15 = 0$

62. $e^{2x} - 10e^x + 24 = 0$

63. $8(62 - e^{x/4}) = 192$

64. $\dfrac{150}{e^{-x} - 4} = 75$

65. $7 \ln 3x = 12$

66. $\ln(x + 9)^2 = 2$

In Exercises 67–70, use any method to solve the system.

67. $\begin{cases} -3x + 5y = -9 \\ x - 4y = 10 \end{cases}$

68. $\begin{cases} 5x - y = -13 \\ -x + 2y = 17 \end{cases}$

69. $\begin{cases} 4x + 5y = 6 \\ 5x - 11y = -27 \end{cases}$

70. $\begin{cases} -\frac{3}{2}x + 2y = 9 \\ 5x - \frac{1}{2}y = 44 \end{cases}$

In Exercises 71–74, sketch the graph of the solution of the system of inequalities. Use a graphing utility to verify your solution.

71. $\begin{cases} 2x + y \leq 7 \\ y \leq 4 \\ x \geq -1 \\ y \geq 0 \end{cases}$

72. $\begin{cases} x + y \leq 6 \\ y \geq -2 \\ x \geq -5 \end{cases}$

73. $\begin{cases} x^2 + y \leq 1 \\ y - 3x \geq -3 \\ y \geq -3 \end{cases}$

74. $\begin{cases} x^2 + y^2 \leq 9 \\ x + y \leq 3 \\ x \geq 0 \end{cases}$

8 Chapter Summary

What did you learn?

Section 8.1

	Review Exercises
☐ How to use the method of substitution to solve systems of equations in two variables	1–6
☐ How to solve systems of equations graphically	7–12
☐ How to use systems of equations to model and solve real-life problems	13–16

Section 8.2

☐ How to use the method of elimination to solve systems of linear equations in two variables	17–24
☐ How to graphically interpret the number of solutions of systems of linear equations in two variables	25–30
☐ How to use systems of linear equations in two variables to model and solve real-life problems	31–34

Section 8.3

☐ How to recognize linear systems in row-echelon form and use back-substitution to solve the systems	35, 36
☐ How to use Gaussian elimination to solve systems of linear equations	37–42
☐ How to solve nonsquare systems of linear equations	43, 44
☐ How to graphically interpret three-variable systems	45, 46
☐ How to use systems of linear equations to write partial fraction decompositions of rational expressions	47–52
☐ How to use systems of linear equations in three or more variables to model and solve real-life problems	53, 54

Section 8.4

☐ How to sketch graphs of inequalities in two variables	55–60
☐ How to solve systems of inequalities	61–70
☐ How to use systems of inequalities in two variables to model and solve real-life problems	71–76

Section 8.5

☐ How to solve linear programming problems	77–80
☐ How to use linear programming to model and solve real-life problems	81, 82

8 Review Exercises

8.1 **In Exercises 1–6, solve the system by the method of substitution.**

1. $\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$

2. $\begin{cases} 2x = 3(y - 1) \\ y = x \end{cases}$

3. $\begin{cases} x^2 - y^2 = 9 \\ x - y = 1 \end{cases}$

4. $\begin{cases} x^2 + y^2 = 169 \\ 3x + 2y = 39 \end{cases}$

5. $\begin{cases} y = 2x^2 \\ y = x^4 - 2x^2 \end{cases}$

6. $\begin{cases} x = y + 3 \\ x = y^2 + 1 \end{cases}$

In Exercises 7–12, use a graphing utility to solve the system of equations. If you cannot identify the exact solution, find the solution accurate to two decimal places.

7. $\begin{cases} 5x + 6y = 7 \\ -x - 4y = 0 \end{cases}$

8. $\begin{cases} 8x - 3y = -3 \\ 2x + 5y = 28 \end{cases}$

9. $\begin{cases} y^2 - 2y + x = 0 \\ x + y = 0 \end{cases}$

10. $\begin{cases} y = 2x^2 - 4x + 1 \\ y = x^2 - 4x + 3 \end{cases}$

11. $\begin{cases} y = 2(6 - x) \\ y = 2^{x-2} \end{cases}$

12. $\begin{cases} y = \ln(x - 1) - 3 \\ y = 4 - \frac{1}{2}x \end{cases}$

13. ***Break-Even Point*** You set up a business and make an initial investment of \$10,000. The unit cost of the product is \$2.85 and the selling price is \$4.95. How many units must you sell to break even?

14. ***Choice of Two Jobs*** You are offered two sales jobs. One company offers an annual salary of \$22,500 plus a year-end bonus of 1.5% of your total sales. The other company offers a salary of \$20,000 plus a year-end bonus of 2% of total sales. What amount of sales will make the second offer better? Explain.

15. ***Geometry*** The perimeter of a rectangle is 480 meters and its length is 1.5 times its width. Find the dimensions of the rectangle.

16. ***Geometry*** The perimeter of a rectangle is 68 feet and its width is $\frac{8}{9}$ times its length. Find the dimensions of the rectangle.

8.2 **In Exercises 17–24, solve by elimination.**

17. $\begin{cases} 2x - y = 2 \\ 6x + 8y = 39 \end{cases}$

18. $\begin{cases} 40x + 30y = 24 \\ 20x - 50y = -14 \end{cases}$

19. $\begin{cases} \frac{1}{5}x + \frac{3}{10}y = \frac{7}{50} \\ \frac{2}{5}x + \frac{1}{2}y = \frac{1}{5} \end{cases}$

20. $\begin{cases} \frac{5}{12}x - \frac{3}{4}y = \frac{25}{4} \\ -x + \frac{7}{8}y = -\frac{38}{5} \end{cases}$

21. $\begin{cases} 3x - 2y = 0 \\ 3x + 2(y + 5) = 10 \end{cases}$

22. $\begin{cases} 7x + 12y = 63 \\ 2x + 3y = 15 \end{cases}$

23. $\begin{cases} 1.25x - 2y = 3.5 \\ 5x - 8y = 14 \end{cases}$

24. $\begin{cases} 1.5x + 2.5y = 8.5 \\ 6x + 10y = 24 \end{cases}$

In Exercises 25–30, use a graphing utility to graph the system. Use the graph to determine whether the system is consistent or inconsistent. If the system is consistent, determine the solution. Verify your results algebraically.

25. $\begin{cases} 3x + 2y = 0 \\ x - y = 4 \end{cases}$

26. $\begin{cases} x + y = 6 \\ -2y = -12 + 2x \end{cases}$

27. $\begin{cases} \frac{1}{4}x - \frac{1}{5}y = 2 \\ -5x + 4y = 8 \end{cases}$

28. $\begin{cases} \frac{7}{2}x - 7y = -1 \\ -x + 2y = 4 \end{cases}$

29. $\begin{cases} 8x - 2y = 11 \\ -4x + y = -5.5 \end{cases}$

30. $\begin{cases} -x + 3.2y = 10.4 \\ 3x - 9.6y = 6.4 \end{cases}$

31. ***Flying Speeds*** Two planes leave Pittsburgh and Philadelphia at the same time, each going to the other city. One plane flies 25 miles per hour faster than the other. Find the air speed of each plane if the cities are 275 miles apart and the planes pass each other after 40 minutes of flying time.

32. ***Finance*** A total of \$46,000 is invested in two corporate bonds that pay 6.75% and 7.25% simple interest. The investor wants an annual interest income of \$3245 from the investments. What is the most that can be invested in the 6.75% bond?

Supply and Demand **In Exercises 33 and 34, find the point of equilibrium.**

	Demand Function	*Supply Function*
33.	$p = 37 - 0.0002x$	$p = 22 + 0.00001x$
34.	$p = 120 - 0.0001x$	$p = 45 + 0.0002x$

8.3 **In Exercises 35 and 36, use back-substitution to solve the system of linear equations.**

35. $\begin{cases} x - 4y + 3z = 3 \\ -y + z = -1 \\ z = -5 \end{cases}$

36. $\begin{cases} x - 7y + 8z = 85 \\ y - 9z = -35 \\ z = 3 \end{cases}$

In Exercises 37–40, solve the system of linear equations and check any solution algebraically.

37. $\begin{cases} x + 3y - z = 13 \\ 2x - 5z = 23 \\ 4x - y - 2z = 14 \end{cases}$

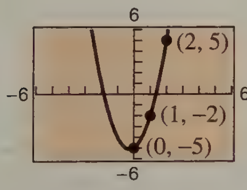
38. $\begin{cases} x + 2y + 6z = 4 \\ -3x + 2y - z = -4 \\ 4x + 2z = 16 \end{cases}$

39. $\begin{cases} x - 2y + z = -6 \\ 2x - 3y = -7 \\ -x + 3y - 3z = 11 \end{cases}$

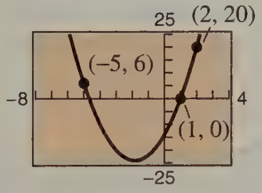
40. $\begin{cases} 2x + 6z = -9 \\ 3x - 2y + 11z = -16 \\ 3x - y + 7z = -11 \end{cases}$

In Exercises 41 and 42, find the equation of the parabola $y = ax^2 + bx + c$ that passes through the given points. Use a graphing utility to verify your result.

41.



42.



In Exercises 43 and 44, solve the nonsquare system of equations.

43. $\begin{cases} 5x - 12y + 7z = 16 \\ 3x - 7y + 4z = 9 \end{cases}$

44. $\begin{cases} 2x + 5y - 19z = 34 \\ 3x + 8y - 31z = 54 \end{cases}$

In Exercises 45 and 46, sketch the plane represented by the linear equation. Then list four points that lie in the plane.

45. $2x - 4y + z = 8$

46. $3x + 3y - z = 9$

In Exercises 47–52, write the partial fraction decomposition for the rational expression.

47. $\dfrac{4 - x}{x^2 + 6x + 8}$

48. $\dfrac{-x}{x^2 + 3x + 2}$

49. $\dfrac{x^2}{x^2 + 2x - 15}$

50. $\dfrac{9}{x^2 - 9}$

51. $\dfrac{x^2 + 2x}{x^3 - x^2 + x - 1}$

52. $\dfrac{3x^3 + 4x}{(x^2 + 1)^2}$

53. ***Agriculture*** A mixture of 6 gallons of chemical A, 8 gallons of chemical B, and 13 gallons of chemical C is required to kill a certain destructive crop insect. Commercial spray X contains 1, 2, and 2 parts, respectively, of these chemicals. Commercial spray Y contains only chemical C. Commercial spray Z contains chemicals A, B, and C in equal amounts. How much of each type of commercial spray is needed to get the desired mixture?

54. ***Investments*** An inheritance of \$20,000 was divided among three investments yielding \$1780 in interest per year. The interest rates for the three investments were 7%, 9%, and 11%. Find the amount placed in each investment if the second and third were \$3000 and \$1000 less than the first, respectively.

8.4 **In Exercises 55–60, sketch the graph of the inequality.**

55. $x \le 6$

56. $y \ge -10$

57. $y \le 5 - \frac{1}{2}x$

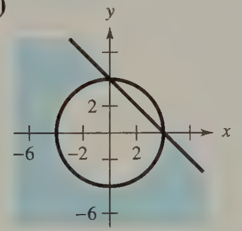
58. $3y - x \ge 7$

59. $y - 4x^2 > -1$

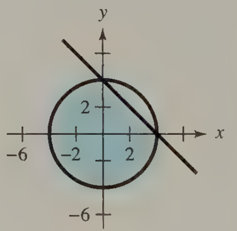
60. $y \ge 2e^x - 6$

In Exercises 61–64, match the system of inequalities with the graph of its solution. [The graphs are labeled (a), (b), (c), and (d).]

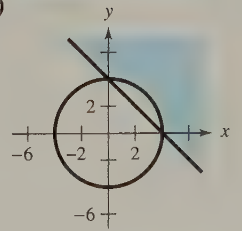
(a)



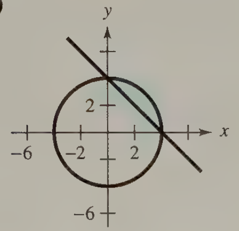
(b)



(c)



(d)



61. $\begin{cases} x^2 + y^2 \le 16 \\ x + y \ge 4 \end{cases}$

62. $\begin{cases} x^2 + y^2 \le 16 \\ x + y \le 4 \end{cases}$

63. $\begin{cases} x^2 + y^2 \ge 16 \\ x + y \ge 4 \end{cases}$

64. $\begin{cases} x^2 + y^2 \ge 16 \\ x + y \le 4 \end{cases}$

In Exercises 65–70, sketch a graph of the solution set of the system of inequalities. Use a graphing utility to verify your result.

65. $\begin{cases} x + 2y \le 160 \\ 3x + y \le 180 \\ x \ge 0 \\ y \ge 0 \end{cases}$

66. $\begin{cases} 2x + 3y \le 24 \\ 2x + y \le 16 \\ x \ge 0 \\ y \ge 0 \end{cases}$

67. $\begin{cases} y < x + 1 \\ y > x^2 - 1 \end{cases}$

68. $\begin{cases} y \le 6 - 2x - x^2 \\ y \ge x + 6 \end{cases}$

69. $\begin{cases} 2x - 3y \ge 0 \\ 2x - y \le 8 \\ y \ge 0 \end{cases}$

70. $\begin{cases} x^2 + y^2 \le 9 \\ (x - 3)^2 + y^2 \le 9 \end{cases}$

In Exercises 71 and 72, derive a set of inequalities to describe the region.

71. *Parallelogram:* Vertices at $(1, 5)$, $(3, 1)$, $(6, 10)$, $(8, 6)$

72. *Triangle:* Vertices at $(1, 2)$, $(6, 7)$, $(8, 1)$

In Exercises 73 and 74, find a system of inequalities that models the description. Use a graphing utility to graph and shade the solution of the system.

73. ***Fruit Distribution*** A Pennsylvania fruit grower has at most 1500 bushels of apples that are to be divided between markets in Harrisburg and philadelphia. These two markets need at least 400 bushels and 600 bushels, respectively.

74. ***Inventory Costs*** A warehouse operator has 24,000 square feet of floor space in which to store two products. Each unit of product I requires 20 square feet of floor space and costs \$12 per day to store. Each unit of product II requires 30 square feet of floor space and costs \$8 per day to store. The total storage cost per day cannot exceed \$12,400.

In Exercises 75 and 76, use a graphing utility to shade the regions representing the consumer and producer surpluses for the equations. Find the surpluses.

	Demand	*Supply*
75.	$p = 160 - 0.0001x$	$p = 70 + 0.0002x$
76.	$p = 130 - 0.0002x$	$p = 30 + 0.0003x$

8.5 **In Exercises 77–80, use a graphing utility to graph the region determined by the constraints. Then find the minimum or maximum values of the objective function and where they occur, subject to the constraints.**

77. Maximize:

$z = 3x + 4y$

Constraints:

$\begin{aligned} x &\ge 0 \\ y &\ge 0 \\ 2x + 5y &\le 50 \\ 4x + y &\le 28 \end{aligned}$

78. Minimize:

$z = 10x + 7y$

Constraints:

$\begin{aligned} x &\ge 0 \\ y &\ge 0 \\ 2x + y &\ge 100 \\ x + y &\ge 75 \end{aligned}$

79. Minimize:

$z = 1.75x + 2.25y$

Constraints:

$\begin{aligned} x &\ge 0 \\ y &\ge 0 \\ 2x + y &\ge 25 \\ 3x + 2y &\ge 45 \end{aligned}$

80. Maximize:

$z = 50x + 70y$

Constraints:

$\begin{aligned} x &\ge 0 \\ y &\ge 0 \\ x + 2y &\le 1500 \\ 5x + 2y &\le 3500 \end{aligned}$

81. ***Maximum Revenue*** A student is working part time as a cosmetologist to pay college expenses. The student may work no more than 24 hours per week. Haircuts cost \$17 and require an average of 20 minutes, and permanents cost \$60 and require an average of 1 hour and 10 minutes. What combination of haircuts and/or permanent will yield a maximum revenue? What is the maximum revenue?

82. ***Minimum Cost*** Two gasolines, type A and type B, have octane ratings of 80 and 92, respectively. Type A costs \$1.25 per gallon and type B costs \$1.55 per gallon. Determine the blend of minimum cost with an octane rating of at least 88. (*Hint:* Let x be the fraction of each gallon that is type A and let y be the fraction that is type B.) What is the minimum cost?

Synthesis

True or False? **In Exercises 83 and 84, determine whether the statement is true or false. Justify your answer.**

83. The system $\begin{cases} y \le 5 \\ y \ge -2 \\ y \ge \frac{7}{2}x - 9 \\ y \ge -\frac{7}{2}x + 26 \end{cases}$ represents a region shaped like an isosceles trapezoid.

84. It is possible for an objective function of a linear programming problem to have exactly ten maximum value points.

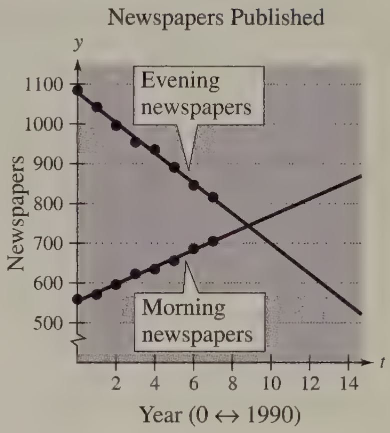
Chapter Project *Fitting Models to Data*

In this project, you will find and use models relating to newspaper circulation in the United States by using *least squares regression analysis.* This procedure can be performed with a computer or graphing utility.

The numbers of morning and evening newspapers published in the United States from 1990 through 1997 are shown in the table.

a. Use the regression capabilities of a graphing utility to find linear models for both sets of data. Then use the models to estimate the numbers of morning and evening newspapers published in 2003. In the table, $t = 0$ represents 1990. (Source: Editor and Publisher Company)

Year, t	0	1	2	3	4	5	6	7
Morning	559	571	596	623	635	656	686	705
Evening	1084	1042	996	954	935	891	846	816



b. Use the models you found in part (a) to predict the year when the number of morning papers published is equal to the number of evening papers published. Use the graph of the models shown to verify your result.

Questions for Further Exploration

1. The numbers (in millions) of morning newspapers sold each day in the United States from 1990 to 1997 are shown in the table below. In the table $t = 0$ represents 1990. Find a linear model that represents the data. Use your model to estimate the number of morning papers sold each day in 2003.

Year, t	0	1	2	3
Morning	41.3	41.5	42.4	43.1

Year, t	4	5	6	7
Morning	43.4	44.3	44.8	45.4

2. From 1990 through 1997, the number of morning newspapers published and sold increased. Did the average circulation per morning paper (number sold ÷ number published) increase or decrease? Explain.

3. If you had the opportunity to invest in a company that published only one type of newspaper (morning or evening), which would you choose? Explain your reasoning.

8 Chapter Test

Take this test as you would take a test in class. After you are done, check your work against the answers in the back of the book.

The *Interactive* CD-ROM and *Internet* versions of this text provide answers to the Chapter Tests and Cumulative Tests. They also offer Chapter Pre-Tests (that test key skills and concepts covered in previous chapters) and Chapter Post-Tests, both of which have randomly generated exercises with diagnostic capabilities.

In Exercises 1–3, solve the system by the method of substitution.

1. $\begin{cases} x - y = 6 \\ 3x + 5y = 2 \end{cases}$ **2.** $\begin{cases} y = x - 1 \\ y = (x - 1)^3 \end{cases}$ **3.** $\begin{cases} 4x - y^2 = 7 \\ x - y = 3 \end{cases}$

In Exercises 4–6, solve the system graphically.

4. $\begin{cases} 4x - 3y = -15 \\ 4x + 3y = -9 \end{cases}$ **5.** $\begin{cases} y = 16 - x^2 \\ y = x + 4 \end{cases}$ **6.** $\begin{cases} y - \ln x = 8 \\ 3x + y + 10 = 21 \end{cases}$

In Exercises 7–9, solve the linear system by elimination.

7. $\begin{cases} 2x + 5y = -11 \\ 5x - y = 19 \end{cases}$ **8.** $\begin{cases} x - 2y + 3z = -5 \\ 2x - z = -4 \\ 3y + z = 17 \end{cases}$ **9.** $\begin{cases} 5x + 5y - z = 0 \\ 10x + 5y + 2z = 0 \\ 5x + 15y - 9z = 0 \end{cases}$

10. Find a system of linear equations that has the solution $\left(\frac{4}{3}, -8\right)$.

11. Find a system of linear equations that has the solution $\left(-\frac{1}{2}, 5, -\frac{9}{4}\right)$.

12. Find the equation of the parabola $y = ax^2 + bx + c$ passing through the points $(0, 6)$, $(-2, 2)$, and $\left(3, \frac{9}{2}\right)$.

13. Write the partial fraction decomposition for the rational expression $\dfrac{5x - 2}{(x - 1)^2}$.

In Exercises 14–16, sketch the graph of the inequality.

14. $x \geq -7$ **15.** $-4x + 6y < -11$ **16.** $(x - 4)^2 + y^2 < 16$

In Exercises 17–19, graph the inequalities and shade the region representing the solution.

17. $\begin{cases} 2x + y \leq 6 \\ 2x - y \leq 0 \\ x \geq -5 \end{cases}$ **18.** $\begin{cases} y < -x^4 + x^2 + 4 \\ y > 4x \end{cases}$ **19.** $\begin{cases} x^2 + y^2 \leq 16 \\ x \geq 1 \\ y \geq -3 \end{cases}$

20. Derive a set of inequalities to describe the region in the figure at the right.

21. Find the maximum value of the objective function $z = 20x + 12y$ subject to the constraints $x \geq 0$, $y \geq 0$, $x + 4y \leq 32$, and $3x + 2y \leq 36$.

22. A merchant plans to sell two models of compact disc players at costs of \$275 and \$400. The \$275 model yields a profit of \$55 and the \$400 model yields a profit of \$75. The merchant estimates that the total monthly demand will not exceed 300 units. The merchant does not want to invest more than \$100,000 in inventory for these products. Find the number of units of each model that should be stocked in order to maximize profit.

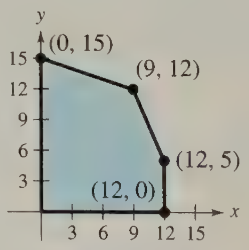


FIGURE FOR 20

Answers to Odd-Numbered Exercises and Tests

Chapter P

Section P.1 *(page 9)*

1. (a) $5, 1$ (b) $-9, 5, 0, 1, -4, -1$
(c) $-9, -\frac{7}{2}, 5, \frac{2}{3}, 0, 1, -4, -1$ (d) $\sqrt{2}$

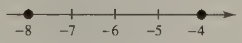
3. (a) $1, 20$ (b) $-13, 1, -10, 20$
(c) $2.01, 0.666 \ldots, -13, 1, -10, 20$
(d) $0.010110111 \ldots$

5. (a) $\frac{6}{3}, 3$ (b) $\frac{6}{3}, -2, 3, -3$
(c) $-\frac{1}{3}, \frac{6}{3}, -7.5, -2, 3, -3$ (d) $-\pi, \frac{1}{2}\sqrt{2}$

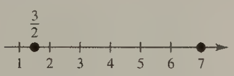
7. 0.625 **9.** $0.\overline{123}$ **11.** $\frac{23}{5}$ **13.** $\frac{13}{2}$

15. $-\frac{183}{100}$ **17.** $-1 < 2.5$

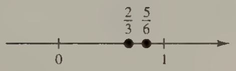
19. $-4 > -8$



21. $\frac{3}{2} < 7$

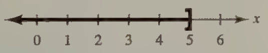


23. $\frac{5}{6} > \frac{2}{3}$



25. (a) $x \leq 5$ is the set of all real numbers less than or equal to 5.

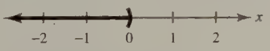
(b)



(c) Unbounded

27. (a) $x < 0$ is the set of all negative real numbers.

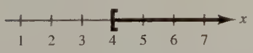
(b)



(c) Unbounded

29. (a) $x \geq 4$ is the set of all real numbers greater than or equal to 4.

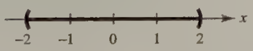
(b)



(c) Unbounded

31. (a) $-2 < x < 2$ is the set of all real numbers greater than -2 and less than 2.

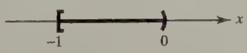
(b)



(c) Bounded

33. (a) $-1 \leq x < 0$ is the set of all negative real numbers greater than or equal to -1.

(b)



(c) Bounded

35. $\frac{127}{90}, \frac{584}{413}, \frac{7071}{5000}, \sqrt{2}, \frac{47}{33}$ **37.** $x < 0; (-\infty, 0)$

39. $y \geq 0; [0, \infty)$ **41.** $12 \leq c \leq 32; [12, 32]$

43. $W > 45; (45, \infty)$

45. The set of all real numbers greater than -6

47. 10 **49.** $\pi - 3 \approx 0.1416$ **51.** -1 **53.** -9

55. 1 for $x > -2$; undefined for $x = -2$; -1 for $x < -2$

57. $|-3| > -|-3|$ **59.** $-5 = -|5|$

61. $-|-2| = -|2|$ **63.** 4 **65.** 51

67. $\frac{5}{2}$ **69.** $\frac{128}{75}$

71. (a) $-A$ is negative. (b) $B - A$ is negative.

73. $|x - 5| \leq 3$ **75.** $|y - 0| \geq 6$ **77.** 11 miles

79. The temperature dropped 23°.

81. $|\$113{,}356 - \$112{,}700| = \$656 > \$500$
$0.05(\$112{,}700) = \$5635$

Because the actual expenses differ from the budget by more than \$500, there is failure to meet the "budget variance test."

83. $|\$37{,}335 - \$37{,}640| = \$305 < \$500$
$0.05(\$37{,}640) = \$1882$

Because the difference between the actual expenses and the budget is less than \$500 and less than 5% of the budgeted amount, there is compliance with the "budget variance test."

85. $y = \$92.5$ billion, $|y - x| = \$0.3$ billion

There was a surplus of \$0.3 billion.

87. $y = \$517.1$ billion, $|y - x| = \$73.8$ billion

There was a deficit of \$73.8 billion.

89. $y = \$1351.8$ billion, $|y - x| = \$163.9$ billion

There was a deficit of \$163.9 billion.

91. Terms: $7x, 4$; Coefficient: 7

93. Terms: $3x^2, -8x, -11$; Coefficients: $3, -8$

95. Terms: $4x^3, \frac{x}{2}, -5$; Coefficients: $4, \frac{1}{2}$

97. (a) -10 (b) -6 **99.** (a) 14 (b) 2

101. (a) Division by 0 is undefined. (b) 0

103. Commutative Property of Addition

105. Multiplicative Inverse Property

107. Distributive Property

109. Multiplicative Identity Property

111. Associative and Commutative Properties of Multiplication

113. 0 **115.** Division by 0 is undefined. **117.** $\frac{1}{2}$

119. $\frac{3}{8}$ **121.** $\frac{11x}{12}$ **123.** 48 **125.** -2.57

127. 1.56

129. (a)

n	1	0.5	0.01	0.0001	0.000001
$5/n$	5	10	500	50,000	5,000,000

(b) $5/n$ approaches ∞ as n approaches 0.

131. (a) No. If u is negative while v is positive, or vice versa, the expressions will not be equal.

(b) $|u + v| \leq |u| + |v|$

133. Answers will vary. Natural numbers are the integers from 1 to infinity. A rational number can be expressed as the ratio of two integers; an irrational number cannot.

135. False. $\frac{3 + 5}{4} = 2 = \frac{3}{4} + \frac{5}{4}$, but $\frac{4}{3} + \frac{4}{5} = \frac{32}{15} \neq \frac{4}{3 + 5}$.

Section P. 2 *(page 21)*

1. (a) 48 (b) 81 **3.** (a) 729 (b) -9
5. (a) 243 (b) $-\frac{3}{4}$ **7.** (a) $\frac{5}{6}$ (b) 4 **9.** -1600
11. 2.125 **13.** $\frac{7}{16}$ **15.** -108 **17.** $\frac{3}{2}$
19. (a) $-125z^3$ (b) $5x^6$ **21.** (a) $\frac{7}{x}$ (b) $\frac{4}{3}(x + y)^2$
23. (a) 1 (b) $\frac{1}{4x^4}$ **25.** (a) $\frac{x^2}{y^2}$ (b) 1
27. $64^{1/3} = 4$ **29.** $\sqrt{196} = 14$ **31.** $(-216)^{1/3} = -6$
33. $8^{3/4} = 27$ **35.** 3 **37.** 3 **39.** -125 **41.** $\frac{1}{8}$
43. -4 **45.** -7.225 **47.** 21.316 **49.** 14.499
51. (a) $2\sqrt{2}$ (b) $2\sqrt[3]{3}$ **53.** (a) $3y^2\sqrt{6x}$ (b) $\frac{4a^2}{|b|}\sqrt{2}$
55. 625 **57.** $\frac{2}{x}$ **59.** $\frac{1}{x^3}, x > 0$
61. (a) $\frac{\sqrt{3}}{3}$ (b) $4\sqrt[3]{4}$
63. (a) $\frac{x(5 + \sqrt{3})}{11}$ (b) $3(\sqrt{6} - \sqrt{5})$ **65.** $\frac{2}{\sqrt{2}}$
67. $\frac{2}{3(\sqrt{5} - \sqrt{3})}$ **69.** (a) $\sqrt{3}$ (b) $\sqrt[3]{(x + 1)^2}$
71. (a) $2\sqrt[4]{2}$ (b) $\sqrt[8]{2x}$ **73.** (a) $34\sqrt{2}$ (b) $22\sqrt{2}$
75. (a) $13\sqrt{x + 1}$ (b) $18\sqrt{5x}$
77. $\sqrt{5} + \sqrt{3} > \sqrt{5 + 3}$ **79.** $5 > \sqrt{3^2 + 2^2}$
81. 5.73×10^7 **83.** 8.99×10^{-5} **85.** 604,800,000
87. 0.0000000000000000001602
89. (a) 954.448 (b) 3.077×10^{10}
91. (a) 67,082.039 (b) 39.791
93. (a) 50,000 (b) 200,000
95. When any positive integer is squared, the units digit is 0, 1, 4, 5, 6, or 9. Therefore, $\sqrt{5233}$ is not an integer.
97. $r = 0.280$ **99.** $t = 13.29$ seconds
101. $8\frac{1}{3}$ minutes **103.** True. $x^{k+1}/x = x^k x/x = x^k$.
105. $1 = \frac{a^n}{a^n} = a^{n-n} = a^0$

Section P.3 *(page 32)*

1. d **3.** b **5.** f
7. Answers will vary, but first term is $-2x^3$.
9. Answers will vary, but first term has form $-ax^4$, $a > 0$.
11. Degree: 2; Leading coefficient: 5
13. Degree: 7; Leading coefficient: 1
15. Degree: 5; Leading coefficient: -2
17. Polynomial: $-2x^3 + 7x + 10$
19. Not a polynomial because of the operation of division.
21. Polynomial: $-x^4 + x^3 + x^2 - x$
23. $-2x - 10$ **25.** $3x^3 - 2x + 2$
27. $8.1x^3 + 29.7x^2 + 11$ **29.** $3x^3 - 6x^2 + 3x$
31. $-15z^2 + 5z$ **33.** $-4x^4 + 4x$ **35.** $7.5x^3 + 15x$
37. $-\frac{1}{4}x^2 - 6x$ **39.** $x^2 + 7x + 12$
41. $6x^2 - 7x - 5$ **43.** $4x^2 - 20xy + 25y^2$
45. $x^2 - 100$ **47.** $x^2 - 4y^2$ **49.** $m^2 - 6m + 9 - n^2$
51. $x^2 + 2xy + y^2 - 6y - 6x + 9$ **53.** $4r^4 - 25$
55. $x^3 + 3x^2 + 3x + 1$ **57.** $8x^3 - 12x^2y + 6xy^2 - y^3$
59. $\frac{1}{4}x^2 - 5x + 25$ **61.** $\frac{1}{16}x^2 - 9$
63. $5.76x^2 + 14.4x + 9$ **65.** $2.25x^2 - 16$
67. $2x^2 + 2x$ **69.** $u^4 - 16$ **71.** $3(x + 2)$
73. $2x(x^2 - 3)$ **75.** $(x - 5)(3x + 8)$
77. $(x - 6)(x + 6)$ **79.** $(4y - 3)(4y + 3)$
81. $\left(2x - \frac{1}{3}\right)\left(2x + \frac{1}{3}\right)$
83. $[(x - 1) - 2][(x - 1) + 2] = (x - 3)(x + 1)$
85. $(x - 2)^2$ **87.** $\left(x + \frac{1}{2}\right)^2$ **89.** $(2t + 1)^2$
91. $\left(3t + \frac{1}{4}\right)^2$ **93.** $(x - 1)(x + 2)$
95. $(s - 2)(s - 3)$ **97.** $-(y - 4)(y + 5)$
99. $(3x - 2)(x - 1)$ **101.** $(2x + 1)(x - 1)$
103. $(5x + 1)(x + 5)$ **105.** $-(5u - 2)(u + 3)$
107. $(x - 2)(x^2 + 2x + 4)$ **109.** $(y + 4)(y^2 - 4y + 16)$
111. $\left(x - \frac{2}{3}\right)\left(x^2 + \frac{2}{3}x + \frac{4}{9}\right)$ **113.** $(2x - 1)(4x^2 + 2x + 1)$

115. $\left(\frac{1}{2}x + 1\right)\left(\frac{1}{4}x^2 - \frac{1}{2}x + 1\right)$ **117.** $(x - 1)(x^2 + 2)$

119. $(2x - 1)(x^2 - 3)$ **121.** $x(x - 3)(x + 3)$

123. $x^2(x - 4)$ **125.** $(x - 1)^2$ **127.** $(2x - 1)^2$

129. $-2x(x - 2)(x + 1)$ **131.** $(9x + 1)(x + 1)$

133. $\frac{1}{96}(3x + 2)(4x - 3)$ **135.** $(3x + 1)(x^2 + 5)$

137. $x(x^2 + 1)(x - 4)$ **139.** $-z(z + 10)$

141. $(x + 1)^2(x - 1)^2$ **143.** $2(t - 2)(t^2 + 2t + 4)$

145. $(2x - 1)(6x - 1)$ **147.** $-(x + 1)(x - 3)(x + 9)$

149. $7(x^2 + 1)(3x^2 - 1)$ **151.** $-2x(x - 5)^3(x + 5)$

153. $-(x^2 + 1)^4\left(\frac{x^2}{2} + 1\right)$

155. (a) $500r^2 + 1000r + 500$

(b)

r	$2\frac{1}{2}\%$	3%	4%
$500(1 + r)^2$	525.31	530.45	540.80

r	$4\frac{1}{2}\%$	5%
$500(1 + r)^2$	546.01	551.25

(c) Amount increases with increasing r.

157. $V = x(15 - 2x)\left(\frac{45 - 3x}{2}\right)$

$= \frac{3}{2}x(x - 15)(2x - 15)$

x (cm)	3	5	7
V (cu cm)	486	375	84

159. (a) $T = 0.14x^2 - 3.33x + 58.40$

(b)

x (mi/hr)	30	40	55
T (ft)	84.50	149.20	298.75

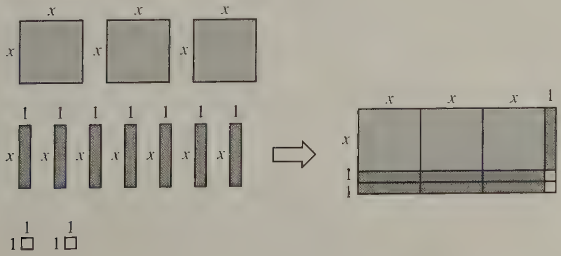
(c) Stopping distance increases at an accelerating rate as speed increases.

161. $(x + 1)(x + 4) = x(x + 4) + 1(x + 4)$

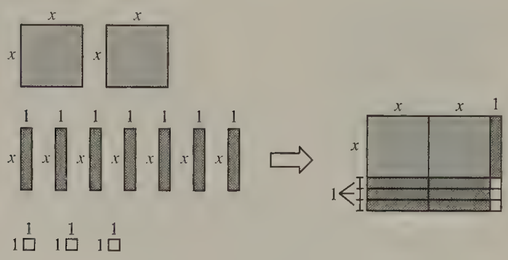
Distributive Property

163. b **165.** a

167.



169.



171. $4\pi(r + 1)$ **173.** $4(6 - x)(6 + x)$

175. $-14, 14, -2, 2$ **177.** $-51, 51, -15, 15, -27, 27$

179. 2, -3 (Answers will vary.)

181. 3, -8 (Answers will vary.)

183. $9x^2 - 9x - 54 = 9(x^2 - x - 6)$

$= 9(x + 2)(x - 3)$

185. $kx(Q - x)$

187. False. $(x^2 - 1)(x^2 + 1)$ becomes a fourth-degree polynomial.

189. False. Counterexample:

$x^2 + 2^2 \neq (x + 2)^2$ if $x = 3$.

191. n

193. (a) Yes. The sum of two polynomials will have the same degree as the polynomial of greater degree unless the polynomials have equal degree and their leading coefficients are opposites.

(b) No. Same reasoning as in (a).

(c) No. Same reasoning as in (a).

195. A polynomial is in factored form if it is written as a product, not as a sum.

Section P.4 *(page 44)*

1. All real numbers **3.** All nonnegative real numbers

5. All real numbers x such that $x \neq 2$

7. All real numbers x such that $x \geq -1$

9. $3x,\ x \neq 0$ **11.** $\frac{3x}{2},\ x \neq 0$ **13.** $\frac{3y}{y + 1},\ x \neq 0$

15. $-\frac{4y}{5},\ y \neq \frac{1}{2}$ **17.** $-\frac{1}{2},\ x \neq 5$

19. $y - 4,\ y \neq -4$ **21.** $\frac{x(x + 3)}{x - 2},\ x \neq -2$

23. $\frac{y - 4}{y + 6},\ y \neq 3$ **25.** $-(x^2 + 1),\ x \neq 2$ **27.** $z - 2$

29.

x	0	1	2	3	4	5	6
$\dfrac{x^2-2x-3}{x-3}$	1	2	3	Undef.	5	6	7
$x+1$	1	2	3	4	5	6	7

The expressions are equivalent except at $x = 3$.

31. Only common factors of the numerator and denominator can be canceled. In this case, factors of terms were incorrectly canceled.

33. $\dfrac{\pi}{4}$ **35.** $\dfrac{1}{5(x-2)}, \ x \neq 1$ **37.** $\dfrac{r+1}{r}, \ r \neq 1$

39. $\dfrac{t-3}{(t+3)(t-2)}, \ t \neq -2$ **41.** $\dfrac{3}{2}, \ x \neq -y$

43. $\dfrac{x+5}{x-1}$ **45.** $\dfrac{6x+13}{x+3}$ **47.** $-\dfrac{2}{x-2}$

49. $-\dfrac{x^2+3}{(x+1)(x-2)(x-3)}$ **51.** $\dfrac{2-x}{x^2+1}, \ x \neq 0$

53. $x^{-2}(x^7-2)$ **55.** $x^{-1/2}(3x^2-2)$ **57.** $-\dfrac{1}{(x^2+1)^5}$

59. $\dfrac{2x^3-2x^2-5}{(x-1)^{1/2}}$ **61.** $\dfrac{1}{2}, \ x \neq 2$

63. $x(x+1), \ x \neq -1, 0$ **65.** $-\dfrac{2x+h}{x^2(x+h)^2}, \ h \neq 0$

67. $\dfrac{2x-1}{2x}, \ x > 0$ **69.** $\dfrac{3x-1}{3}, \ x \neq 0$

71. $\dfrac{1}{\sqrt{x+2}+\sqrt{x}}$

73. (a) $\dfrac{1}{16}$ minute (b) $\dfrac{x}{16}$ minute(s) (c) $\dfrac{60}{16} = \dfrac{15}{4}$ minutes

75. $\dfrac{11x}{30}$ **77.** $\dfrac{x}{2(2x+1)}$

79. (a) $r = 9.09\%$ (b) $\dfrac{288(MN-P)}{N(MN+12P)}$; 9.09%

81. (a)

t	0	2	4	6	8	10
T	75	55.9	48.3	45	43.3	42.3

t	12	14	16	18	20	22
T	41.7	41.3	41.1	40.9	40.7	40.6

(b) 40

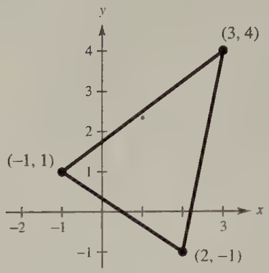
83. False. The domain of the left-hand side is $x^n \neq 1$.

85. False. Example: LCD of $1/x$ and $1/x^2$ is x^2, not x^3.

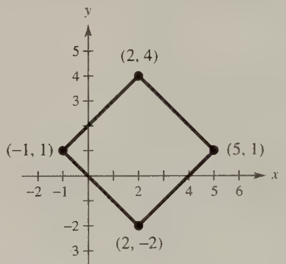
87. No. $\dfrac{ax-b}{b-ax}$ is undefined for values of a, b, and x such that $b = ax$.

Section P.5 *(page 54)*

1.



3.



5. A: $(2, 6)$; B: $(-6, -2)$; C: $(4, -4)$; D: $(-3, 2)$

7. A: $(0, 5)$; B: $(-3, -6)$; C: $(1, -4.5)$; D: $(-4, 2)$

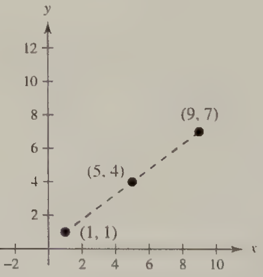
9. $(-3, 4)$ **11.** $(-5, -5)$ **13.** Quadrant IV

15. Quadrant II **17.** Quadrant III or IV

19. Quadrant III **21.** Quadrants I and III

23. $(0, 1), (4, 2), (1, 4)$ **25.** 8 **27.** 5 **29.** 13

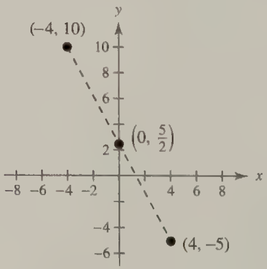
31. (a)



(b) 10

(c) $(5, 4)$

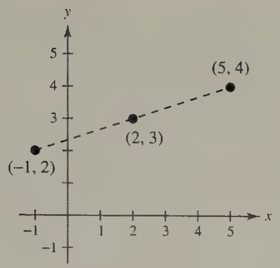
33. (a)



(b) 17

(c) $\left(0, \frac{5}{2}\right)$

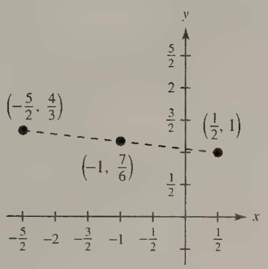
35. (a)



(b) $2\sqrt{10}$

(c) $(2, 3)$

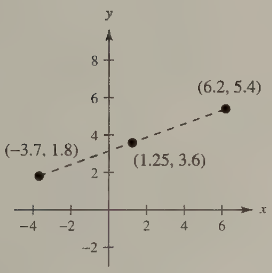
37. (a)



(b) $\dfrac{\sqrt{82}}{3}$

(c) $\left(-1, \dfrac{7}{6}\right)$

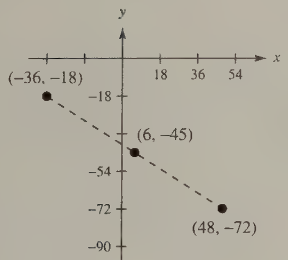
39. (a)



(b) $\sqrt{110.97}$

(c) $(1.25, 3.6)$

41. (a)



(b) $6\sqrt{277}$

(c) $(6, -45)$

43. (a) 4, 3, 5 (b) $4^2 + 3^2 = 5^2$

45. (a) 10, 3, $\sqrt{109}$ (b) $10^2 + 3^2 = (\sqrt{109})^2$

47. \$630,000 **49.** $(\sqrt{5})^2 + (\sqrt{45})^2 = (\sqrt{50})^2$

51. Opposite sides have equal lengths of $2\sqrt{5}$ and $\sqrt{85}$.

53. $(2x_m - x_1, 2y_m - y_1)$;

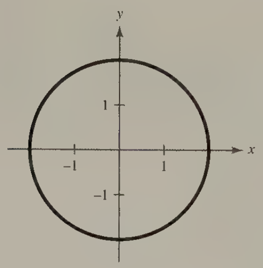
(a) $(7, 0)$ (b) $(9, -3)$

55. $x^2 + y^2 = 9$ **57.** $(x - 2)^2 + (y + 1)^2 = 16$

59. $(x + 1)^2 + (y - 2)^2 = 5$

61. $(x - 3)^2 + (y - 4)^2 = 25$

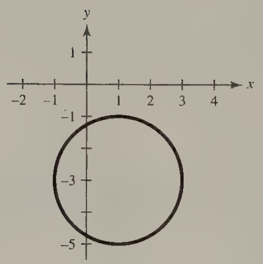
63.



Center: $(0, 0)$

Radius $= 2$

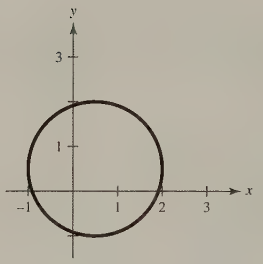
65.



Center: $(1, -3)$

Radius $= 2$

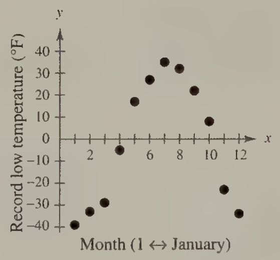
67.



Center: $\left(\frac{1}{2}, \frac{1}{2}\right)$

Radius $= \frac{3}{2}$

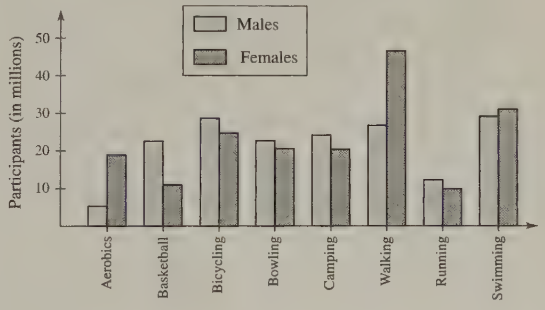
69.



71. \$1.66; 1996 **73.** 167% **75.** 65

77. Answers will vary; for example, corn ≈ 14%, soybeans ≈ 25%, wheat ≈ 29%.

79.

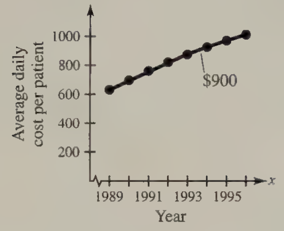


81. (a) ≈52%

(b) No. The trend limits the amount of funds available for capital improvements in industry. (Answers will vary.)

83. (a) During 1993

(b)



85. ≈180.28 kilometers

87. (a) Answers will vary; from 1991 on, the data are nearly in a straight line. Estimate: 7 new members in 2001.

(b) Answers will vary; Elections for inclusion in the Hall of Fame began in 1986.

89. $1245.05 million

91. True. The lengths of the sides from $(-8, 4)$ to $(2, 11)$ and from $(2, 11)$ to $(-5, 1)$ are both $\sqrt{149}$.

93. 0; 0

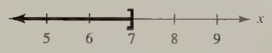
Review Exercises *(page 60)*

1. (a) 11 (b) 11, −14

(c) $11, -14, -\frac{8}{9}, \frac{5}{2}, 0.4$ (d) $\sqrt{6}$

3. $0.8\overline{3} < 0.875$

5. The set consists of all real numbers less than or equal to 7.



7. 122 **9.** $|x - 7| \geq 4$ **11.** $|y + 30| < 5$

13. (a) −13 (b) 27 **15.** (a) −18 (b) −12

17. Associative Property of Addition

19. Commutative Property of Multiplication **21.** $\frac{14}{9}$

23. $\frac{1}{24}$ **25.** $\frac{47x}{60}$ **27.** (a) $-8z^3$ (b) $3a^3b^2$

29. (a) $\frac{3u^5}{v^4}$ (b) m^{-2} **31.** 3.3674×10^{10}

33. 483,300,000 **35.** 78 **37.** $2x^2$ **39.** $\frac{3}{4}$

41. $\frac{x}{3}\sqrt[3]{2}$ **43.** $2\sqrt{2}$ **45.** $3\sqrt{3x}$ **47.** $\sqrt{2x}(2x + 1)$

49. $192\sqrt{2}$ in.² **51.** $\frac{\sqrt{x} + 1}{x - 1}$ **53.** $\frac{-3}{\sqrt{2} + \sqrt{11}}$

55. $\sqrt[4]{16}$ **57.** $\frac{1}{16}$ **59.** $(x - 1)^{1/12}$

61. $-2x^4 + x^3 + x^2 - x - 10$ **63.** $-2y^2 + 11y - 8$

65. $-3x^4 - 4x^3 - 9x^2 - x - 13$

67. $2x^5 + 3x^4 - x^3 - 9x^2 - 15x$

69. $x^2 + 2x - 1 - \frac{2}{x}$ **71.** $49x^2 - 16$

73. $8x^3 - 12x^2 + 6x - 1$ **75.** $x^2 - 2xy + y^2 - 36$

77. $\frac{3}{2}x(x - 3)$ **79.** $2(5x - 1)$ **81.** $(x - 3)(x + 4)$

83. $-3x(2x^3 + x^2 - 4)$

85. $R = x(1600 - 0.50x)$; $p = 1600 - 0.50x$

87. $\left(3x - \frac{1}{5}\right)\left(3x + \frac{1}{5}\right)$ **89.** $(4x - 3)(16x^2 + 12x + 9)$

91. $(x - 7)(x - 2)$ **93.** $(3x + 2)(x + 4)$

95. $(x - 6)(x - 1)(x + 1)$ **97.** $(2x + 5)(x^2 - 7)$

99. All positive real numbers

101. All real numbers x such that $x \geq -12$, $x \neq 0$

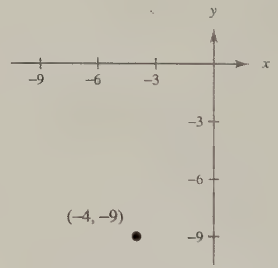
103. $\frac{6y}{y + 2}$, $x \neq 0$ **105.** $\frac{x - 3}{8}$, $x \neq 6$

107. $\frac{x - 1}{x - 3}$, $x \neq -1, \frac{1}{2}$ **109.** $\frac{2(x + 3)}{x(x - 1)}$, $x \neq -3, \frac{3}{2}$

111. $\frac{2x^3 - 4x^2 - 15x + 5}{(x - 4)(x + 2)}$ **113.** $\frac{3x}{(x - 1)(x^2 + x + 1)}$

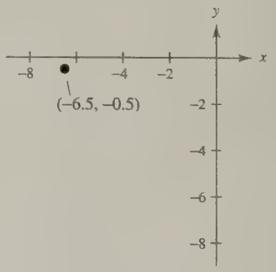
115. $\frac{y - x}{x + y}$, $x \neq 0, y \neq 0, x \neq y$

117.



Quadrant III

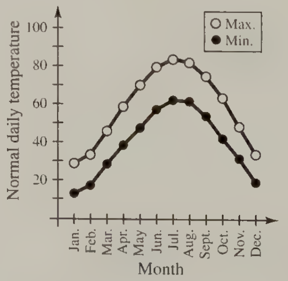
119.



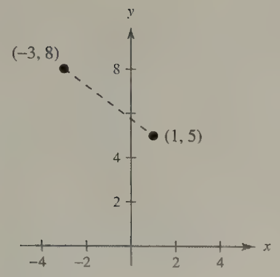
Quadrant III

121. Quadrant I or III

123.

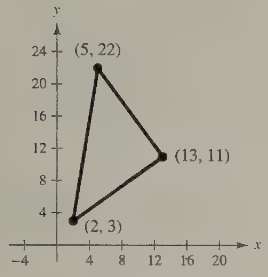


125.



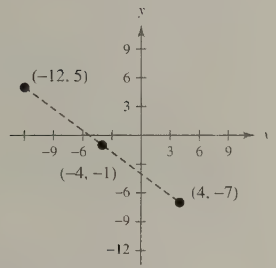
Distance: 5

127.



$2(\sqrt{185})^2 = (\sqrt{370})^2$

129. Midpoint: $(-4, -1)$



131. (a) \$352.35 million

(b) The estimate is \$2.95 million over the actual revenue, which is less than a 1% error.

133. $(x - 3)^2 + (y - 2)^2 = 65$

135. False. $(a + b)^2 = a^2 + 2ab + b^2$

137. The numerator and denominator are not equivalent expressions. Factor -1 from numerator: $\dfrac{-1(1 - x)}{(1 - x)} = -1.$

139. You must raise 2 to the fourth power as well.
$(2x)^4 = 2^4x^4 = 16x^4$

141. Operations under the radical should be performed first.
$\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

143. The sum of the square roots of two terms is not equivalent to the square root of the sum of the terms.

Chapter Test *(page 65)*

1. $-\frac{10}{3} > -|-4|$ **2.** 56 **3.** (a) -18 (b) $\frac{4}{27}$

4. (a) $-\frac{27}{125}$ (b) $\frac{8}{729}$ **5.** (a) 25 (b) 6

6. (a) 1.8×10^5 (b) 2.7×10^{13}

7. (a) $12z^8$ (b) $(u - 2)^{-7}$ (c) $\dfrac{3x^2}{y^2}$

8. (a) $15z\sqrt{2z}$ (b) $-10\sqrt{y}$ (c) $\dfrac{2}{v}\sqrt[3]{\dfrac{2}{v^2}}$

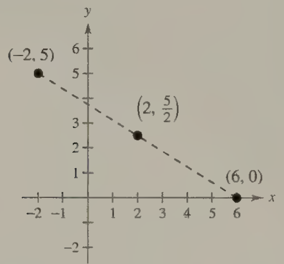
9. $2x^2 - 3x - 5$ **10.** $x^2 - 5$ **11.** $8,\ x \neq 3$

12. $\dfrac{x - 1}{2x},\ x \neq \pm 1$ **13.** $x^2(2x + 1)(x - 2)$

14. $(x - 2)(x + 2)^2$ **15.** $(2x - 3)(4x^2 + 6x + 9)$

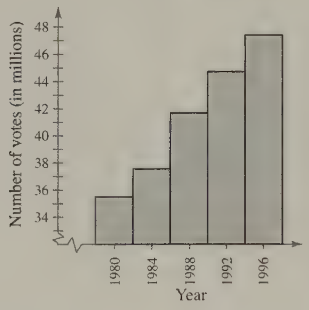
16. (a) $4\sqrt[3]{4}$ (b) $-3(1 + \sqrt{3})$ **17.** $\frac{5}{6}\sqrt{3}x^2$

18. Midpoint: $\left(2, \frac{5}{2}\right)$;
Distance: $\sqrt{89}$



19. $l(45 - l)$; Width $= 45 - l$

20.



Chapter 1

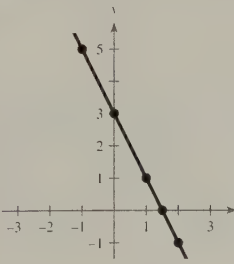
Section 1.1 *(page 75)*

1. (a) Yes (b) Yes **3.** (a) No (b) Yes

5. (a) No (b) Yes **7.** (a) Yes (b) Yes

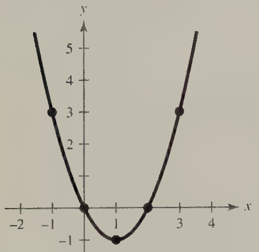
9.

x	-1	0	1	$\frac{3}{2}$	2
y	5	3	1	0	-1



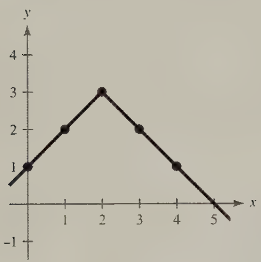
11.

x	-1	0	1	2	3
y	3	0	-1	0	3



13.

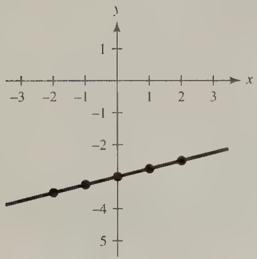
x	0	1	2	3	4
y	1	2	3	2	1



15. (a)

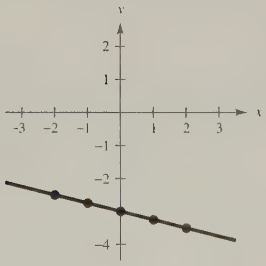
x	-2	-1	0	1	2
y	$-\frac{7}{2}$	$-\frac{13}{4}$	-3	$-\frac{11}{4}$	$-\frac{5}{2}$

(b)



(c)

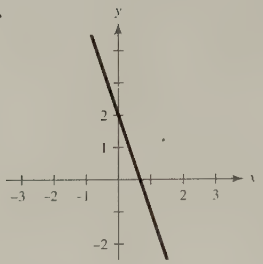
x	-2	-1	0	1	2
y	$-\frac{5}{2}$	$-\frac{11}{4}$	-3	$-\frac{13}{4}$	$-\frac{7}{2}$



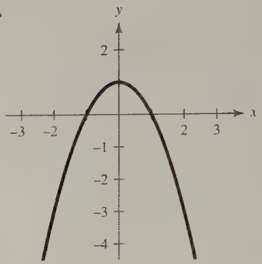
The lines have opposite slopes but the same y-intercept.

17. (d) **19.** (f) **21.** (a)

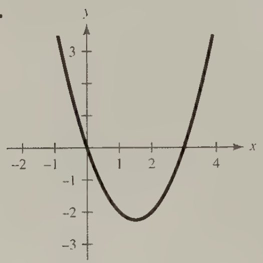
23.



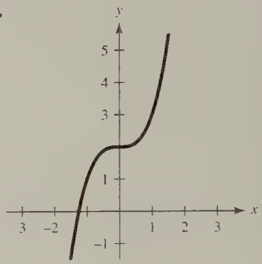
25.



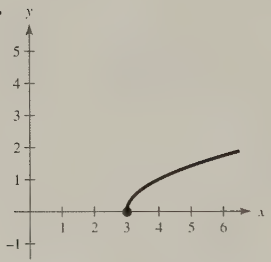
27.



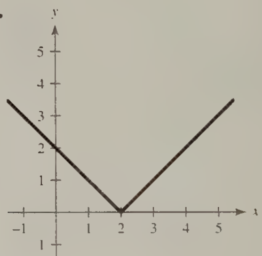
29.



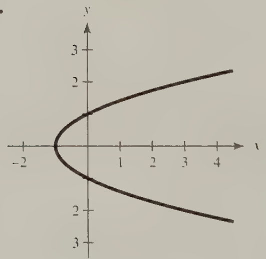
31.



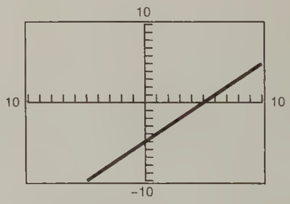
33.



35.

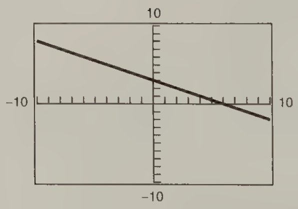


37.



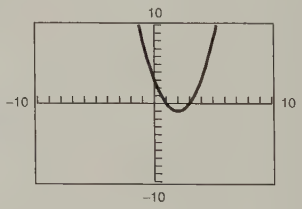
Intercepts: $(5, 0)$, $(0, -5)$

39.



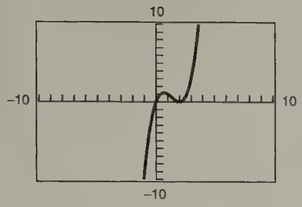
Intercepts: $(6, 0)$, $(0, 3)$

41.



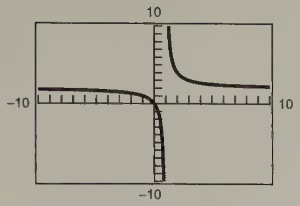
Intercepts: $(3, 0)$, $(1, 0)$, $(0, 3)$

43.



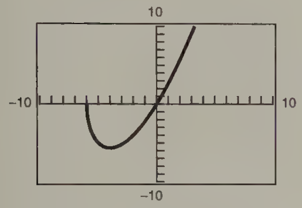
Intercepts: (0, 0), (2, 0)

45.



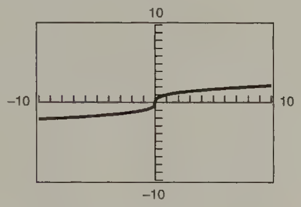
Intercept: (0, 0)

47.



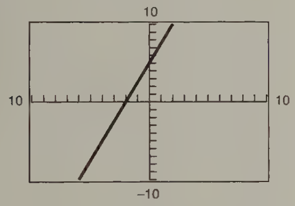
Intercepts: (−6, 0), (0, 0)

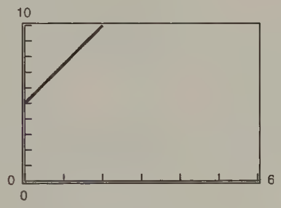
49.



Intercept: (0, 0)

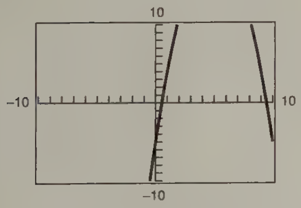
51.

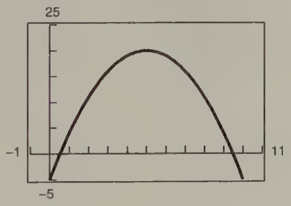




The standard setting gives a more complete graph.

53.





The specified setting gives a more complete graph.

55.

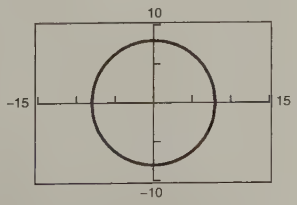
Xmin = -5
Xmax = 5
Xscl = 1
Ymin = -30
Ymax = 10
Yscl = 5

57.

Xmin = -30
Xmax = 30
Xscl = 5
Ymin = -10
Ymax = 50
Yscl = 5

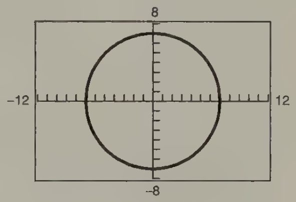
59. $y_1 = \sqrt{64 - x^2}$

$y_2 = -\sqrt{64 - x^2}$



61. $y_1 = \sqrt{49 - x^2}$

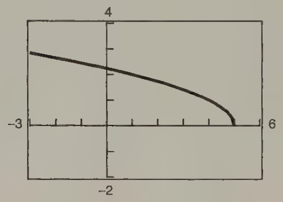
$y_2 = -\sqrt{49 - x^2}$



63. The graphs are identical. Distributive Property

65. The graphs are identical. Associative Property of Multiplication

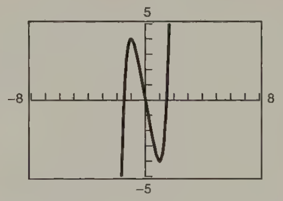
67.



(a) (2, 1.73)

(b) (−4, 3)

69.



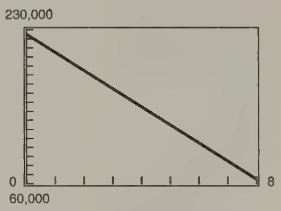
(a) (−0.5, 2.47)

(b) (1, −4), (−1.65, −4)

71. (a)

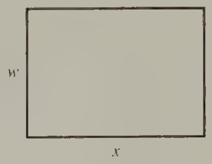
Xmin = 0
Xmax = 8
Xscl = 1
Ymin = 60000
Ymax = 230000
Yscl = 10000

(b)



(c) $109,000 (d) $178,000

73. (a)



(b) $2w + 2x = 12$

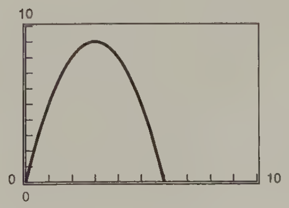
$w + x = 6$

$w = 6 - x$

$A = xw$

$A = x(6 - x)$

(c)



(d) $A \approx 5.4$

(e) $x = 3$, $w = 3$

75. (a) The life expectancy of a child born in 1950

(b) 1973 (c) ≈66.3 years (d) ≈78.7 years

77. (a)

x	10	20	30	40	50
y	107.33	26.56	11.60	6.36	3.94

x	60	70	80	90	100
y	2.62	1.83	1.31	0.96	0.71

(b) $x = 45.64$ (c) $y = 1.103$

(d) They are inversely related.

79. False. $x = 0$ has an infinite number of x-intercepts.

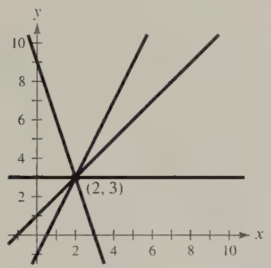
81. Answers will vary. **83.** $-51\sqrt{y}$ **85.** 1000

87. $12x^3 - 4x^2 + 44x$ **89.** $-3x^4 + 8x^2 - 5$

Section 1.2 *(page 87)*

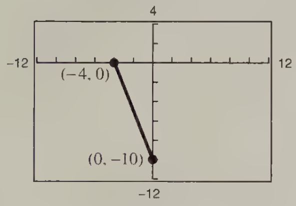
1. (a) L_2 (b) L_3 (c) L_1

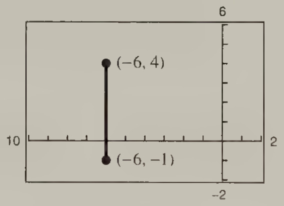
3.



5. $\frac{3}{2}$ **7.** 0 **9.** -4

11. $m = -\frac{5}{2}$ **13.** m is undefined.

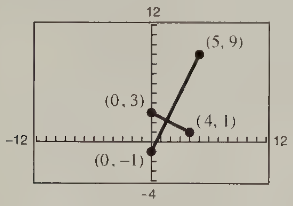


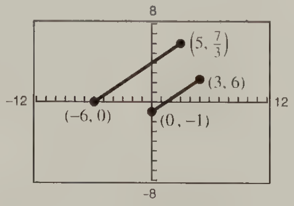


15. (0, 1), (3, 1), (−1, 1) **17.** (−6, 2), (−4, 6), (−3, 8)

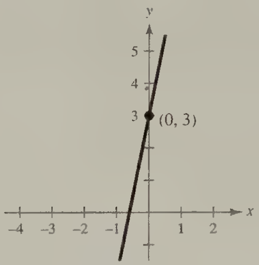
19. (3, −4), (5, −3), (9, −1)

21. Perpendicular **23.** Parallel

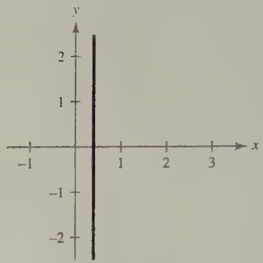




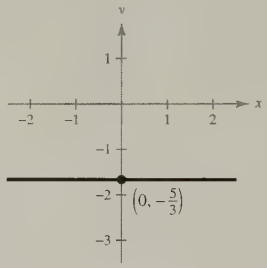
25. (a) $m = 5$;
Intercept: (0, 3)
(b)



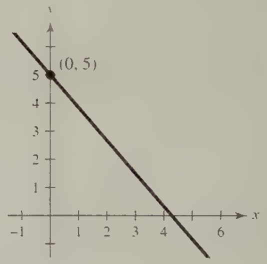
27. (a) m is undefined.
There is no y-intercept.
(b)



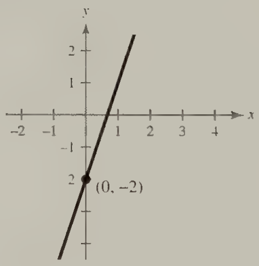
29. (a) $m = 0$;
Intercept: $\left(0, -\frac{5}{3}\right)$
(b)



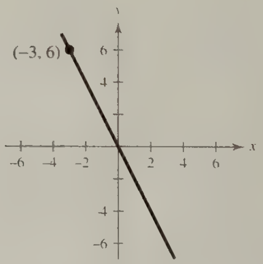
31. (a) $m = -\frac{7}{6}$;
Intercept: (0, 5)
(b)



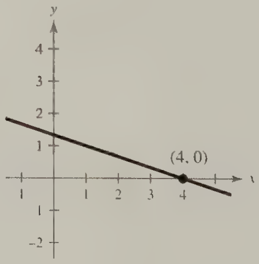
33. $3x - y - 2 = 0$



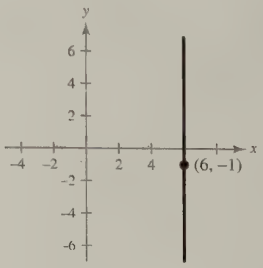
35. $2x + y = 0$



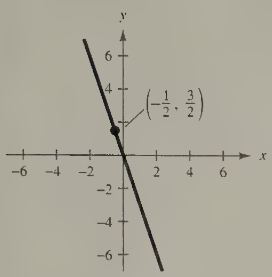
37. $x + 3y - 4 = 0$



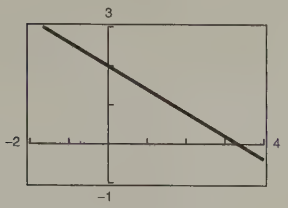
39. $x - 6 = 0$



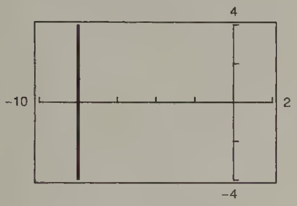
41. $3x + y = 0$



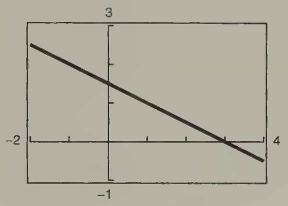
43. $3x + 5y - 10 = 0$



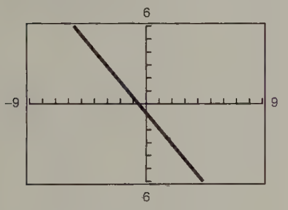
45. $x + 8 = 0$



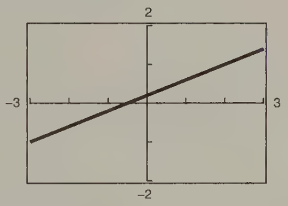
47. $x + 2y - 3 = 0$



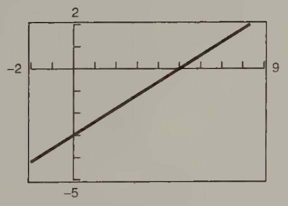
49. $30x + 25y + 18 = 0$



51. $2x - 5y + 1 = 0$



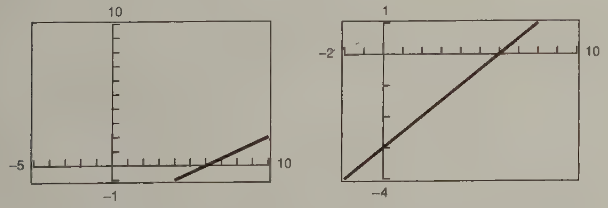
53.



x- and y-intercepts

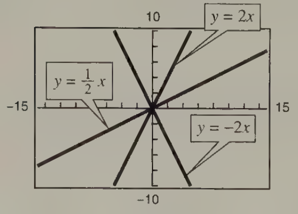
55. $3x + 2y - 6 = 0$ **57.** $12x + 3y + 2 = 0$

59.



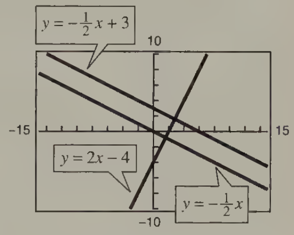
The second setting gives a more complete graph.

61.



$y = \frac{1}{2}x$ and $y = -2x$ are perpendicular.

63.



$y = -\frac{1}{2}x$ and $y = -\frac{1}{2}x + 3$ are parallel. Both are perpendicular to $y = 2x - 4$.

65. (a) $2x - y - 3 = 0$ (b) $x + 2y - 4 = 0$

67. (a) $-6x - 8y + 3 = 0$ (b) $96x - 72y + 127 = 0$

69. (a) $10x - 10y + 43 = 0$ (b) $10x + 10y - 93 = 0$

71. $3x - 2y - 1 = 0$

73. (a) Sales increase of \$135

(b) No sales increase

(c) Sales decrease of \$40

75. (a) Greatest increases per share: 1990 and 1996

Greatest decrease per share: 1997

(b) $37x - 1000y + 943 = 0$

(c) Slope is the average increase per share per year.

(d) \$1.46. Answers will vary.

77. 16,667 feet **79.** $125t - V + 2415 = 0$

81. $2000t + V - 22{,}400 = 0$

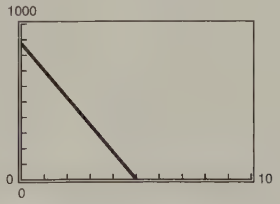
83. (b); slope $= -10$; the amount owed decreases by \$10 per week.

85. (a); slope $= 0.25$; expenses increase by \$0.25 per mile.

87. $F = \frac{9}{5}C + 32$ **89.** \$39,500

91. (a) $V = -175t + 875$

(b)



t	0	1	2	3	4	5
V	875	700	525	350	175	0

93. (a) $C = 16.75t + 36{,}500$ (b) $R = 27t$

(c) $P = 10.25t - 36{,}500$ (d) $t \approx 3561$ hours

95. (a) Answers will vary. Example: $y = 92.84t + 487.82$

(b) Answers will vary. Example: \$1,601,000

(c) Average increase per year

97. False. The line through $(10, -3)$ and $(2, -9)$ is $y = \frac{3}{4}x + \frac{9}{4}$, and $\left(-12, -\frac{37}{2}\right)$ is not on this line.

99. -4

101. No. The slopes of two perpendicular lines have opposite signs (assuming that neither line is vertical or horizontal).

103. Yes. $-10x^2 + 3x + 1$

105. Yes. $-2x^4 - x^3 + 2x^2 + 2$ **107.** No.

109. $(x - 4)(x - 7)$ **111.** $(3x - 1)(x - 5)$

113. $(x + 8)^2 + (y + 5)^2 = \frac{9}{16}$

115. $(x - 2)^2 + (y - 1)^2 = 52$

Section 1.3 *(page 100)*

1. Yes **3.** No

5. Yes. Each input value is matched with one output value.

7. No. The same input value is matched with two different output values.

9. (a) Function

(b) Not a function because the element 1 in A corresponds to two elements, -2 and 1, in B.

(c) Function (d) Not a function because the element 2 in A corresponds to no element in B.

11. Each is a function. To each year there corresponds one and only one circulation.

13. Not a function **15.** Function **17.** Function

19. Not a function **21.** Function **23.** Not a function

25. (a) $\frac{1}{5}$ (b) 1 (c) $\frac{1}{4t + 1}$ (d) $\frac{1}{x + c + 1}$

27. (a) -1 (b) -9 (c) $2x - 5$

29. (a) 0 (b) -0.75 (c) $x^2 + 2x$

31. (a) 1 (b) 2.5 (c) $3 - 2|x|$

33. (a) $-\frac{1}{9}$ (b) Undefined (c) $\frac{1}{y^2 + 6y}$

35. (a) 1 (b) -1 (c) 1 **37.** (a) -1 (b) 2 (c) 6

39.

x	-2	-1	0	1	2
$f(x)$	1	-2	-3	-2	1

41.

t	-5	-4	-3	-2	-1
$h(t)$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1

43.

x	-2	-1	0	1	2
$f(x)$	5	$\frac{9}{2}$	4	1	0

45. 5 **47.** $\frac{4}{3}$ **49.** ± 3 **51.** ± 4 **53.** $2, -1$

55. 0, 3 **57.** All real numbers x

59. All real numbers t except $t = 0$

61. All real numbers y such that $y \geq 10$

63. All real numbers x such that $-1 \leq x \leq 1$

65. All real numbers x except $x = 0, -2$

67. All real numbers s such that $s \neq 4$ and $s \geq 1$

69. All real numbers x except $x = 0$

71. $\{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$

73. $\{(-2, 0), (-1, 1), (0, \sqrt{2}), (1, \sqrt{3}), (2, 2)\}$

75. $g(x) = -2x^2; c = -2$ **77.** $r(x) = \frac{32}{x}; c = 32$

79. $2, c \neq 0$ **81.** $3 + h, h \neq 0$

83. $3x^2 + 3xc + c^2, c \neq 0$ **85.** $-\frac{1}{t}, t \neq 1$

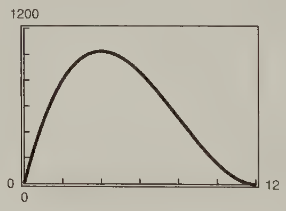
87. $A = \frac{C^2}{4\pi}$ **89.** $A = \frac{s^2}{2}$

91. (a)

Height, x	Width	Volume, V
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$
3	$24 - 2(3)$	$3[24 - 2(3)]^2 = 972$
4	$24 - 2(4)$	$4[24 - 2(4)]^2 = 1024$
5	$24 - 2(5)$	$5[24 - 2(5)]^2 = 980$
6	$24 - 2(6)$	$6[24 - 2(6)]^2 = 864$

Maximum when $x = 4$

(b) $V = x(24 - 2x)^2, 0 < x < 12$



(c) $x = 9, V = 324; x = 10, V = 160$

93. $A = \frac{x^2}{2(x - 2)}, x > 2$

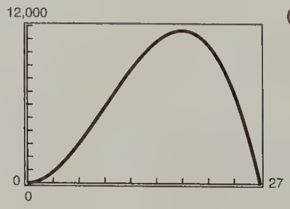
95. (a) $V = x^2y$

$= x^2(108 - 4x)$

$= 108x^2 - 4x^3$

(b) $0 < x < 27$

(c)



(d) $x = 18$ in., $y = 36$ in.

97. (a) $C = 12.30x + 98{,}000$ (b) $R = 17.98x$
(c) $P = 5.68x - 98{,}000$

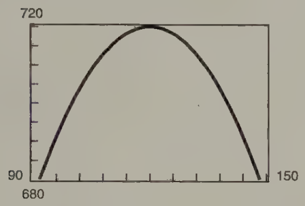
99. (a) $R = \dfrac{240n - n^2}{20}$

(b)

n	90	100	110	120	130	140	150
$R(n)$	\$675	\$700	\$715	\$720	\$715	\$700	\$675

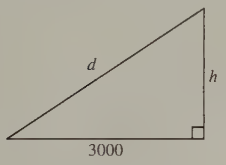
The revenue is maximum when $n = 120$.

(c)



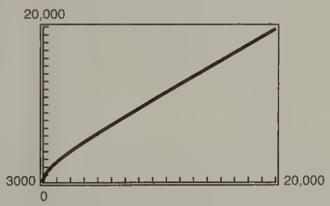
The revenue is maximum when $n = 120$.

101. (a)



(b) $h = \sqrt{d^2 - 3000^2}$; $d \geq 3000$

(c)



Xmin = 3000
Xmax = 20,000
Xscl = 1000
Ymin = 0
Ymax = 20,000
Yscl = 1000

(d) ≈9539 feet

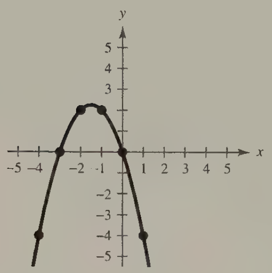
103. False. Range is $[-1, \infty)$.

105. No. 3 corresponds to both u and v.

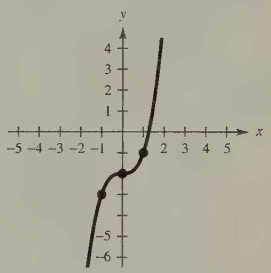
107. Function notation is a convenient way of referencing the value of a function for a specific domain value.

109. $\dfrac{x^2 - x - 3}{(x - 4)(x + 5)(x - 1)}$ **111.** $\dfrac{x + 7}{x - 7}, \; x \neq 9$

113.



115.



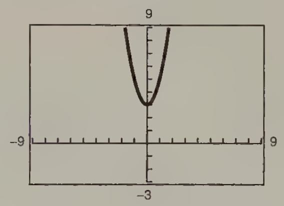
Section 1.4 *(page 114)*

1. Domain: $(-\infty, \infty)$; Range: $(-\infty, 1]$

3. Domain: $(-\infty, -1], [1, \infty)$; Range: $[0, \infty)$

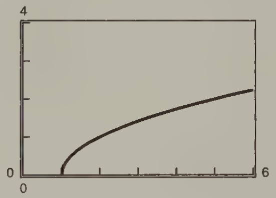
5. Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

7.



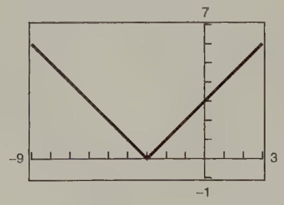
Domain: $(-\infty, \infty)$; Range: $[3, \infty)$

9.



Domain: $[1, \infty)$; Range: $[0, \infty)$

11.



Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

13. Function. Graph the given function over the window shown in the figure.

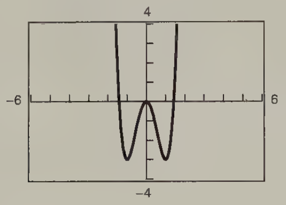
15. Not a function. Solve for y and graph the resulting two functions.

17. Function. Solve for y and graph the resulting function.

19. (a) Increasing on $(-\infty, \infty)$ (b) Odd function

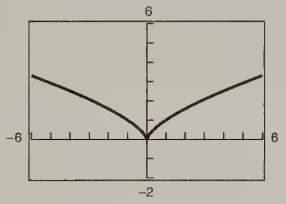
21. (a) Increasing on $(-\infty, 0), (2, \infty)$
Decreasing on $(0, 2)$
(b) Neither even nor odd

23. (a)



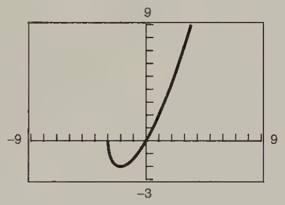
(b) Increasing on $(-1, 0), (1, \infty)$
Decreasing on $(-\infty, -1), (0, 1)$
(c) Even function

25. (a)



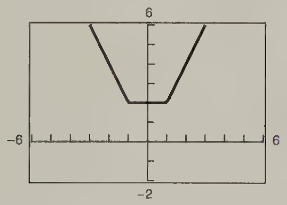
(b) Decreasing on $(-\infty, 0)$; Increasing on $(0, \infty)$
(c) Even function

27. (a)



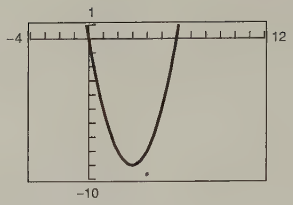
(b) Increasing on $(-2, \infty)$; Decreasing on $(-3, -2)$
(c) Neither even nor odd

29. (a)



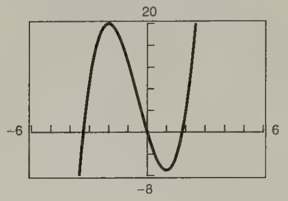
(b) Decreasing on $(-\infty, -1)$; Constant on $(-1, 1)$;
Increasing on $(1, \infty)$
(c) Even function

31.



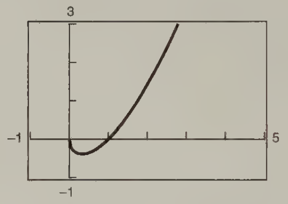
Relative minimum: $(3, -9)$

33.



Relative minimum: $(1, -7)$
Relative maximum: $(-2, 20)$

35.



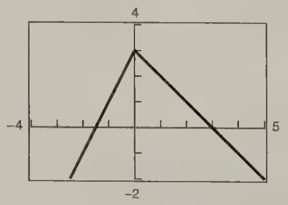
Minimum: $(0.33, -0.38)$

37. (a) Answers will vary.
(b) Relative minimum at $(2, -9)$
(c) Answers will vary.

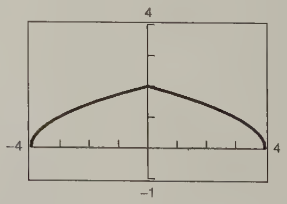
39. (a) Answers will vary.
(b) Relative minimum at $(1.63, -8.71)$
Relative maximum at $(-1.63, 8.71)$
(c) Answers will vary.

41. (a) Answers will vary. (b) Relative minimum at $(4, 0)$
(c) Answers will vary.

43.



45.

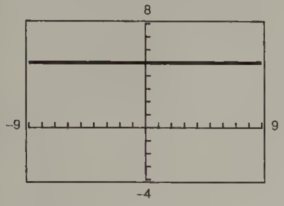


47. Neither even nor odd **49.** Odd function
51. Odd function **53.** Even function
55. (a) $\left(\frac{3}{2}, 4\right)$ (b) $\left(\frac{3}{2}, -4\right)$

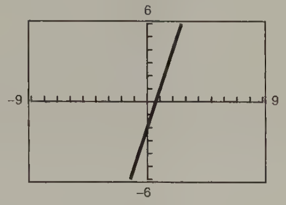
57. (a) $(-4, 9)$ (b) $(-4, -9)$

59. (a) $(-x, -y)$ (b) $(-x, y)$

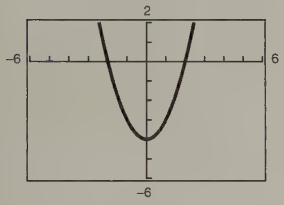
61. Even function



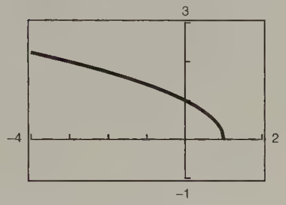
63. Neither even nor odd



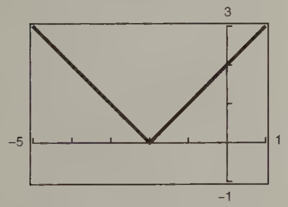
65. Even function



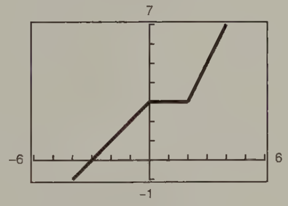
67. Neither even nor odd



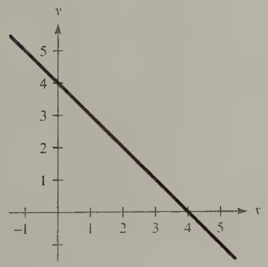
69. Neither even nor odd



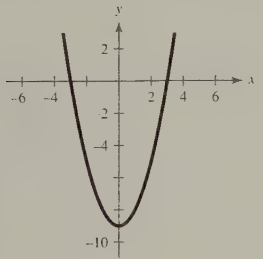
71. Neither even nor odd



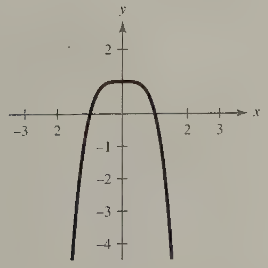
73. $(-\infty, 4]$



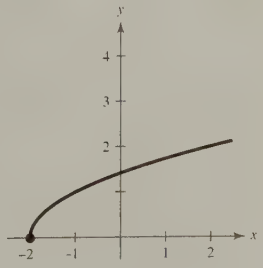
75. $(-\infty, -3], [3, \infty)$



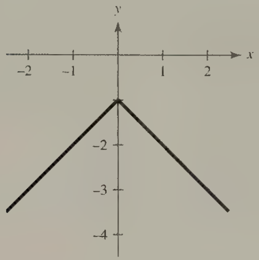
77. $[-1, 1]$



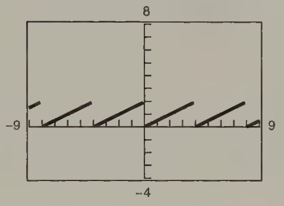
79. $[-2, \infty)$



81. $f(x) < 0$ for all x



83.



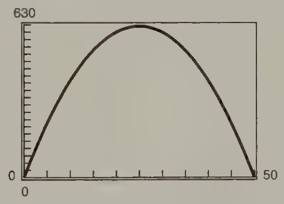
Domain: $(-\infty, \infty)$

Range: $[0, 2)$

Sawtooth pattern

85. (a) Answers will vary.

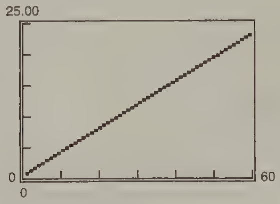
(b)



(c) 625 square meters; 25 × 25 meters

87. (a) C_2 is the appropriate model. The cost of the first minute is \$1.05 and the cost increases \$0.38 when the next minute begins, etc.

(b)



\$7.89

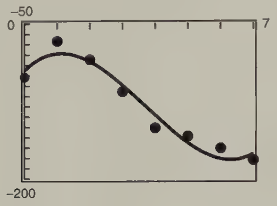
89. $h = -x^2 + 4x - 3,\ 1 \le x \le 3$

91. $h = 2x - x^2,\ 0 \le x \le 2$ **93.** $L = \frac{1}{2}y^2, 0 \le y \le 4$

95. (a) $y = 1.473x^3 - 16.411x^2 + 31.24x - 95.2$

(b) Domain: $[0, 7]$

(c)



(d) Most accurate: 1992; Least accurate: 1991

(e) Yes. The cubic curve is starting an upswing for future years.

97. False. Counterexample: $f(x) = \sqrt{1 + x^2}$

99. If $y = a_{2n+1}x^{2n+1} + a_{2n-1}x^{2n-1} + \cdot \cdot \cdot + a_3x^3 + a_1x$, each exponent is odd. Then

$$f(-x) = -a_{2n+1}x^{2n+1} - a_{2n-1}x^{2n-1} - \cdot \cdot \cdot - a_3x^3 - a_1x,$$

which is equal to $-f(x)$. Therefore, by definition, the original function is odd.

101. (a) Even. g is a reflection in the x-axis.

(b) Even. g is a reflection in the y-axis.

(c) Even. g is a vertical shift downward.

(d) Neither even nor odd. g is shifted to the right and reflected in the x-axis.

103. No. x is not a function of y because horizontal lines can be drawn to intersect the graph twice, so each y-value corresponds to two distinct x-values if $-5 < y < 5$.

105. (a) $d = 10$ (b) Midpoint: $(-1, 3)$

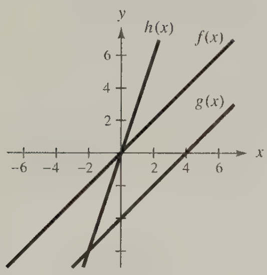
107. (a) $d = \dfrac{\sqrt{733}}{4}$ (b) Midpoint: $\left(-\dfrac{21}{8}, \dfrac{5}{12}\right)$

109. (a) -17 (b) 1 (c) $-x^2 + 3x + 1$

111. (a) 6 (b) -55 (c) $\frac{1}{9}$ **113.** $-h - 6,\ h \neq 0$

Section 1.5 *(page 124)*

1.



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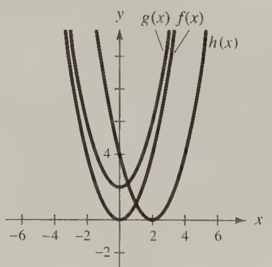

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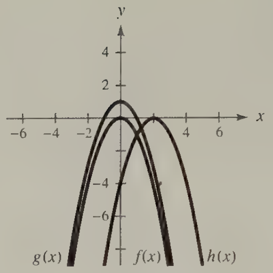

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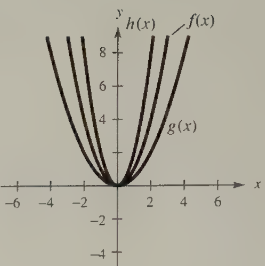
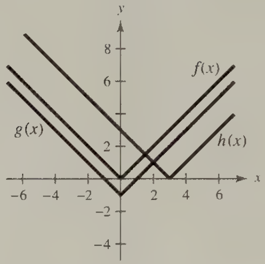
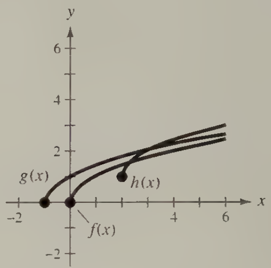


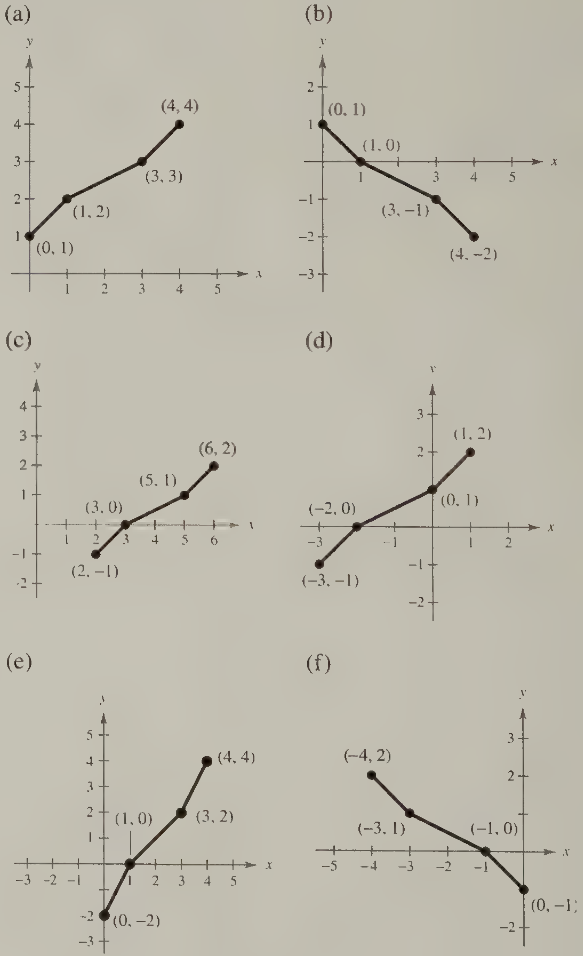
9.



11.



13.



15. Vertical shrink of $y = x$

$y = \frac{1}{2}x$

17. Constant function

$y = 7$

19. Reflection in the x-axis and a vertical shift of $y = \sqrt{x}$

$y = 1 - \sqrt{x}$

21. Horizontal shift of $y = |x|$

$y = |x + 2|$

23. Vertical shift of $y = x^2$

$y = x^2 - 1$

25. Reflection in the x-axis of $y = x^3$ followed by a vertical shift

$y = 1 - x^3$

27. Vertical shift 2 units upward

29. Horizontal shift 2 units to the right

31. Vertical stretch

33. Horizontal shift 2 units to the left

35. Reflection in the x-axis

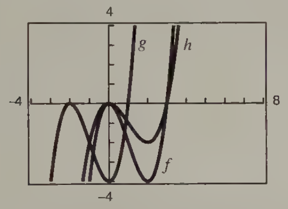
37. Vertical shrink

39. Reflection in the x-axis and vertical shift 4 units upward

41. Horizontal shift 2 units to the left and vertical shrink

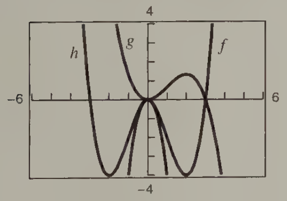
43. Vertical shrink and vertical shift 2 units upward

45.



g is a horizontal shift and h is a vertical shrink.

47.



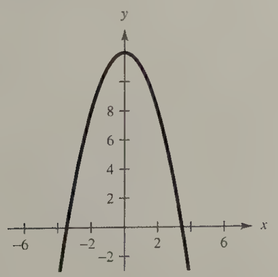
g is a vertical shrink and a reflection in the x-axis and h is a reflection in the y-axis.

49. $g(x) = -(x^3 - 3x^2) + 1$

51. (a) $f(x) = x^2$

(b) Reflection in the x-axis and vertical shift 12 units upward

(c)

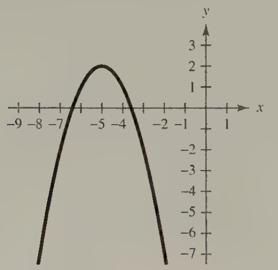


(d) $g(x) = 12 - f(x)$

53. (a) $f(x) = x^2$

(b) Horizontal shift 5 units to the left, reflection in the x-axis, and vertical shift 2 units upward

(c)

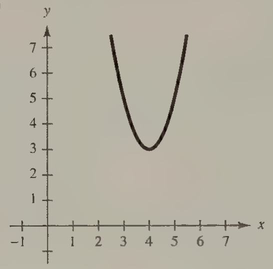


(d) $g(x) = 2 - f(x + 5)$

55. (a) $f(x) = x^2$

(b) Horizontal shift 4 units to the right, vertical stretch, and vertical shift 3 units upward

(c)

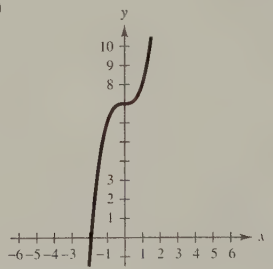


(d) $g(x) = 3 + 2f(x - 4)$

57. (a) $f(x) = x^3$

(b) Vertical shift 7 units upward

(c)

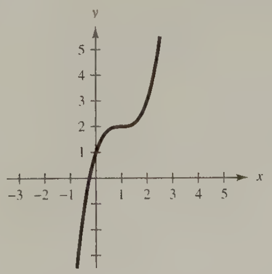


(d) $g(x) = f(x) + 7$

59. (a) $f(x) = x^3$

(b) Horizontal shift 1 unit to the right and vertical shift 2 units upward

(c)

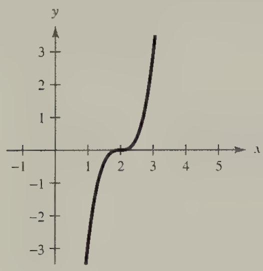


(d) $g(x) = f(x - 1) + 2$

61. (a) $f(x) = x^3$

(b) Horizontal shift 2 units to the right and vertical stretch

(c)

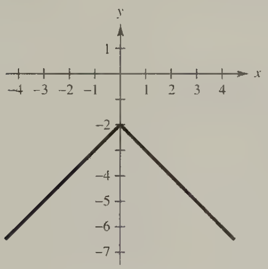


(d) $g(x) = 3f(x - 2)$

63. (a) $f(x) = |x|$

(b) Reflection in the x-axis and vertical shift 2 units downward

(c)

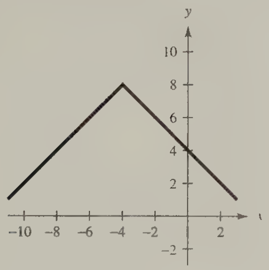


(d) $g(x) = -f(x) - 2$

65. (a) $f(x) = |x|$

(b) Horizontal shift 4 units to the left, reflection in the x-axis, and vertical shift 8 units upward

(c)

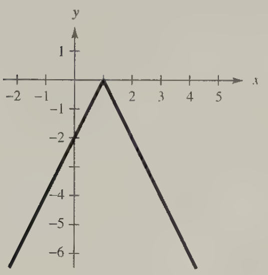


(d) $g(x) = -f(x + 4) + 8$

67. (a) $f(x) = |x|$

(b) Horizontal shift 1 unit to the right, reflection in the x-axis, and vertical stretch

(c)

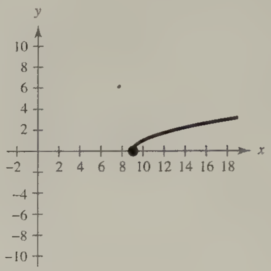


(d) $g(x) = -2f(x - 1)$

69. (a) $f(x) = \sqrt{x}$

(b) Horizontal shift 9 units to the right

(c)

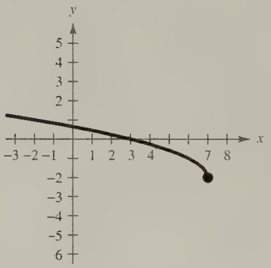


(d) $g(x) = f(x - 9)$

71. (a) $f(x) = \sqrt{x}$

(b) Reflection in the y-axis, horizontal shift 7 units to the right, and vertical shift 2 units downward

(c)

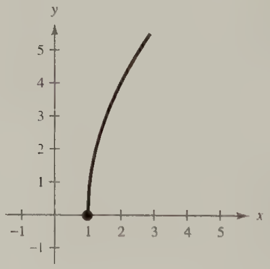


(d) $g(x) = f(7 - x) - 2$

73. (a) $f(x) = \sqrt{x}$

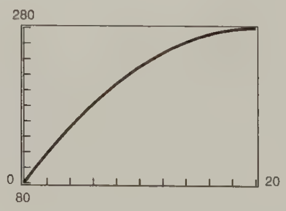
(b) Horizontal shift 1 unit to the right and vertical stretch 4 units

(c)



(d) $g(x) = 4f(x - 1)$

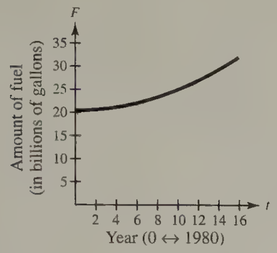
75. (a)



(b) $P(x) = 55 + 20x - 0.5x^2$; vertical shift

(c) $P(x) = 80 + \frac{1}{5}x - \frac{x^2}{20{,}000}$; horizontal stretch

77. (a) Vertical shrink and vertical shift



(b) $G(t) = 0.04t^2 + 0.8t + 24.46$; $G(t) = F(t + 10)$

79. (a) To each time t there corresponds one and only one temperature T.

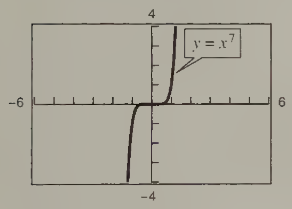
(b) 60°, 72°

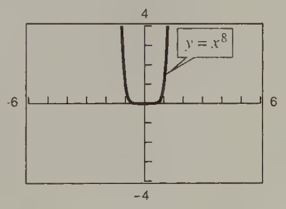
(c) All the temperature changes would be 1 hour later.

(d) The temperature would be decreased by 1 degree.

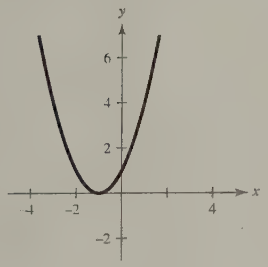
81. False. The point $(-1, 28)$ does not lie on the graph of $g(x) = -(x - 6)^2 + 3$.

83. $y = x^7$ resembles the cubic graph and $y = x^8$ resembles the quadratic graph. Both are steeper on $(-\infty, -1)$ and $(1, \infty)$ and both are closer to 0 on $-1 < x < 1$.





85.



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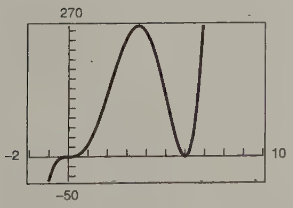
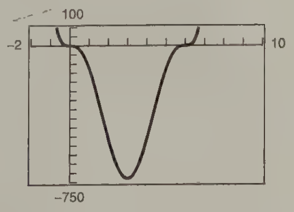


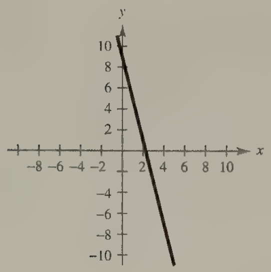
89.



91. All real numbers $x \geq 5$ except $x = 7$

93. All real numbers

95.



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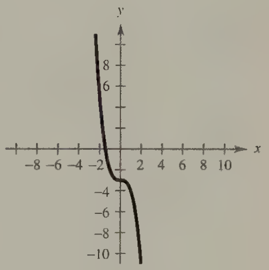
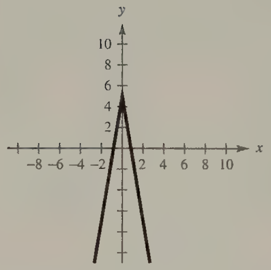


99.



Section 1.6 *(page 134)*

1. (a) $2x$ (b) 2 (c) $x^2 - 1$ (d) $\dfrac{x+1}{x-1}$, $x \neq 1$

(e) All $x \neq 1$

3. (a) $x^2 - x + 1$ (b) $x^2 + x - 1$ (c) $x^2 - x^3$

(d) $\dfrac{x^2}{1-x}$, $x \neq 1$ (e) All $x \neq 1$

5. (a) $x^2 + 5 + \sqrt{1-x}$ (b) $x^2 + 5 - \sqrt{1-x}$

(c) $(x^2 + 5)\sqrt{1-x}$ (d) $\dfrac{x^2+5}{\sqrt{1-x}}$, $x < 1$

(e) $x < 1$

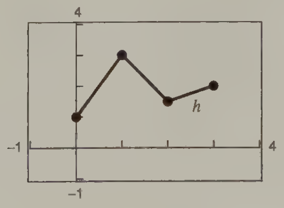
7. (a) $\dfrac{x+1}{x^2}$ (b) $\dfrac{x-1}{x^2}$ (c) $\dfrac{1}{x^3}$ (d) x, $x \neq 0$

(e) $x \neq 0$

9. 9 **11.** 5 **13.** 0 **15.** 26 **17.** $4t^2 - 2t + 5$

19. $-125t^3 - 100t^2 - 5t - 4$ **21.** $\dfrac{t^2+1}{-t-4}$

23.



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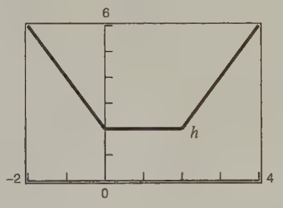

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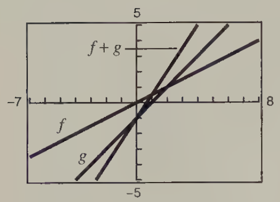

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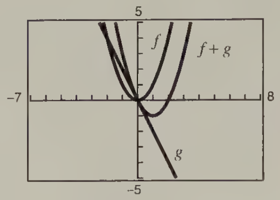

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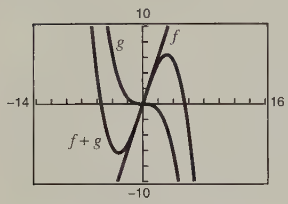
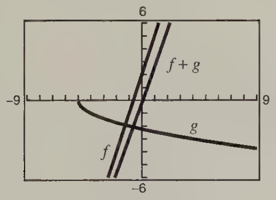


$f(x),\ 0 \le x \le 2;$

$g(x),\ x > 6$

33.



$f(x),\ 0 \le x \le 2;$

$f(x),\ x > 6$

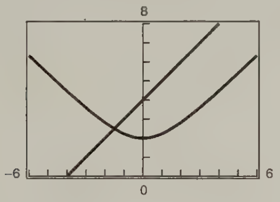
35. (a) $(x - 1)^2$ (b) $x^2 - 1$

37. (a) $20 - 3x$ (b) $-3x$

39. (a) $(f \circ g)(x) = \sqrt{x^2 + 4}$

$(g \circ f)(x) = x + 4,\ x \ge -4$

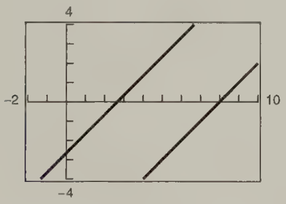
(b)



Not equal

41. (a) $(f \circ g)(x) = x - \frac{8}{3};\ (g \circ f)(x) = x - 8$

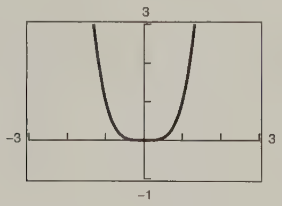
(b)



Not equal

43. (a) $(f \circ g)(x) = x^4;\ (g \circ f)(x) = x^4$

(b)



Equal

45. (a) $(f \circ g)(x) = 24 - 5x;\ (g \circ f)(x) = -5x$

(b) $24 - 5x \ne -5x$

(c)

x	0	1	2	3
$g(x)$	4	3	2	1
$(f \circ g)(x)$	24	19	14	9

x	0	1	2	3
$f(x)$	4	9	14	19
$(g \circ f)(x)$	0	-5	-10	-15

47. (a) $(f \circ g)(x) = \sqrt{x^2 + 1};\ (g \circ f)(x) = x + 1,\ x \ge -6$

(b) $x + 1 \ne \sqrt{x^2 + 1}$

(c)

x	0	1	2	3
$g(x)$	-5	-4	-1	4
$(f \circ g)(x)$	1	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{10}$

x	0	1	2	3
$f(x)$	$\sqrt{6}$	$\sqrt{7}$	$\sqrt{8}$	3
$(g \circ f)(x)$	1	2	3	4

49. (a) $(f \circ g)(x) = |2x + 2|;\ (g \circ f)(x) = 2|x + 3| - 1$

(b) $(f \circ g)(x) = \begin{cases} 2x + 2, & x \ge -1 \\ -2x - 2, & x < -1 \end{cases}$

$(g \circ f)(x) = \begin{cases} 2x + 5, & x \ge -3 \\ -2x - 7, & x < -3 \end{cases}$

$(f \circ g)(x) \ne (g \circ f)(x)$

(c)

x	0	-1	-3	-5
$g(x)$	-1	-3	-7	-11
$(f \circ g)(x)$	2	0	4	8

x	0	-1	-3	-5
$f(x)$	3	2	0	2
$(g \circ f)(x)$	5	3	-1	3

51. (a) 3 (b) 0 **53.** (a) 0 (b) 4

55. (a) 0 (b) 4 **57.** $f(x) = x^2,\ g(x) = 2x + 1$

59. $f(x) = \sqrt[3]{x},\ g(x) = x^2 - 4$

61. $f(x) = \dfrac{1}{x},\ g(x) = x + 2$

63. $f(x) = x^2 + 2x,\ g(x) = x + 4$

65. (a) $x \ge 0$ (b) All real numbers (c) All real numbers

67. (a) All real numbers except $x = 0$ (b) All real numbers
(c) All real numbers except $x = -3$

69. (a) All real numbers except $x = 0$ (b) All real numbers
(c) All real numbers except $x = 1$

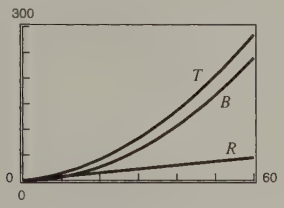
71. $3, h \neq 0$ **73.** $-2x - h,\ h \neq 0$

75. $\dfrac{-4}{x(x+h)},\ h \neq 0$

77. $\dfrac{2}{\sqrt{2(x+h)+1} + \sqrt{2x+1}},\ h \neq 0$

79. (a) $T = \frac{3}{4}x + \frac{1}{15}x^2$

(b)



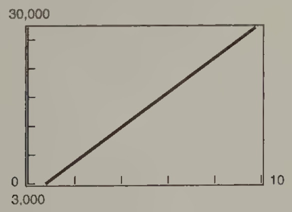
(c) B. For example, $B(60) = 240$ while $R(60)$ is only 45.

81. $y_1 = -0.59x^2 + 7.66x + 144.9$
$y_2 = 16.58x + 245.06$
$y_3 = 1.84x + 21.92$

83. $(A \circ r)(t) = 0.36\pi t^2$
$A \circ r$ represents the area of the circle at time t.

85. (a) $(C \circ x)(t) = 3000t + 750$
$C \circ x$ represents the cost after t production hours.

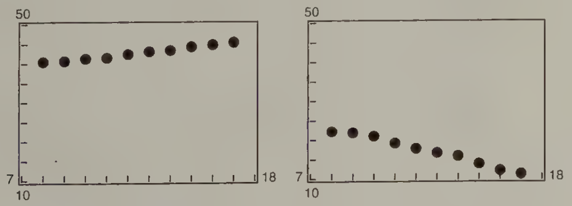
(b)



4.75 hours

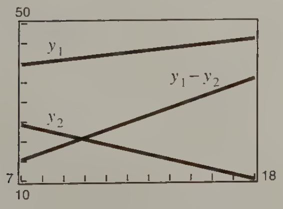
87. $g(f(x))$ represents 3 percent of an amount over $500,000.

89.



Both data sets appear to be linear.

$y_1 = 0.57x + 35.6$; $y_2 = -1.29x + 33.3$



The difference between the morning and evening newspaper circulations is increasing.

91. True **93.** Odd (Proofs will vary.)

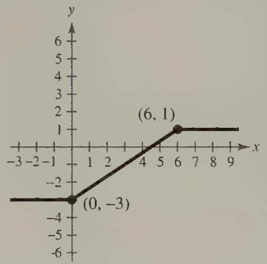
95. $\frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$
$= \frac{1}{2}[f(x) + f(-x) + f(x) - f(-x)]$
$= \frac{1}{2}[2f(x)]$
$= f(x)$

97. $(0, -5), (1, -5), (2, -7)$ (Answers will vary.)

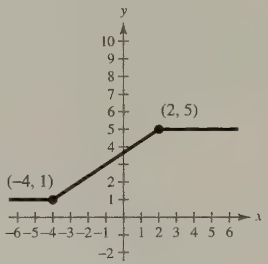
99. $(0, 2\sqrt{6}), (1, \sqrt{23}), (2, 2\sqrt{5})$ (Answers will vary.)

101. $10x - y + 38 = 0$ **103.** $30x + 11y - 34 = 0$

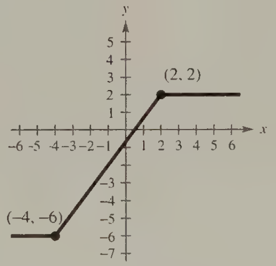
105.



107.



109.



Section 1.7 *(page 145)*

1. (c) **3.** (a) **5.** $f^{-1}(x) = \frac{1}{8}x$

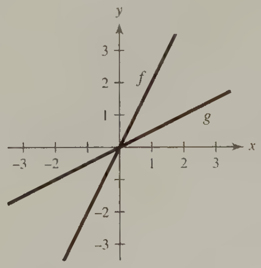
7. $f^{-1}(x) = x - 10$ **9.** $f^{-1}(x) = \frac{1}{2}(x - 1)$

11. $f^{-1}(x) = x^3$

13. (a) $f(g(x)) = f\left(\dfrac{x}{2}\right) = 2\left(\dfrac{x}{2}\right) = x$

$g(f(x)) = g(2x) = \dfrac{(2x)}{2} = x$

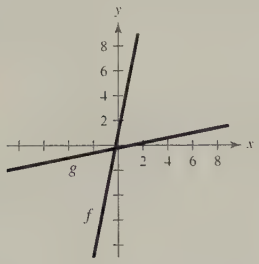
(b)



15. (a) $f(g(x)) = f\left(\frac{x-1}{5}\right) = 5\left(\frac{x-1}{5}\right) + 1 = x$

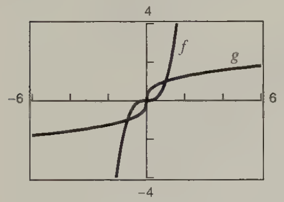
$g(f(x)) = g(5x + 1) = \frac{(5x+1) - 1}{5} = x$

(b)



17. $f(g(x)) = f\left(\sqrt[3]{x}\right) = \left(\sqrt[3]{x}\right)^3 = x$

$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$



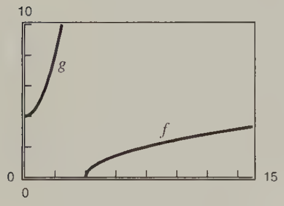
Reflections in the line $y = x$

19. $f(g(x)) = f(x^2 + 4), \quad x \ge 0$

$= \sqrt{(x^2 + 4) - 4} = x$

$g(f(x)) = g\left(\sqrt{x-4}\right)$

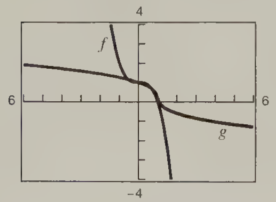
$= \left(\sqrt{x-4}\right)^2 + 4 = x$



Reflections in the line $y = x$

21. $f(g(x)) = f\left(\sqrt[3]{1-x}\right) = 1 - \left(\sqrt[3]{1-x}\right)^3 = x$

$g(f(x)) = g(1 - x^3) = \sqrt[3]{1 - (1 - x^3)} = x$



Reflections in the line $y = x$

23. (a) $f(g(x)) = f\left(-\frac{2x+6}{7}\right)$

$= -\frac{7}{2}\left(-\frac{2x+6}{7}\right) - 3 = x$

$g(f(x)) = g\left(-\frac{7}{2}x - 3\right)$

$= -\frac{2\left(-\frac{7}{2}x - 3\right) + 6}{7} = x$

(b)

x	0	2	-2	6
$f(x)$	-3	-10	4	-24

x	-3	-10	4	-24
$g(x)$	0	2	-2	6

25. (a) $f(g(x)) = f\left(\sqrt[3]{x-5}\right) = \left(\sqrt[3]{x-5}\right)^3 + 5 = x$

$g(f(x)) = g(x^3 + 5) = \sqrt[3]{(x^3 + 5) - 5} = x$

(b)

x	0	1	-1	-2	4
$f(x)$	5	6	4	-3	69

x	5	6	4	-3	69
$g(x)$	0	1	-1	-2	4

27. (a) $f(g(x)) = f(8 + x^2)$

$= -\sqrt{(8 + x^2) - 8}$

$= -\sqrt{x^2} = -(-x) = x, \ x \le 0$

$g(f(x)) = g\left(-\sqrt{x-8}\right)$

$= 8 + \left(-\sqrt{x-8}\right)^2$

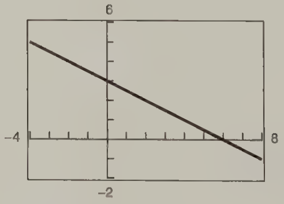
$= 8 + (x - 8) = x, \ x \ge 8$

(b)

x	8	9	12	15
$f(x)$	0	-1	-2	$-\sqrt{7}$

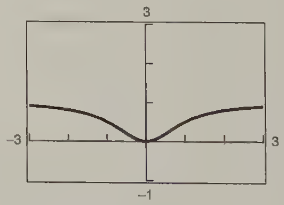
x	0	-1	-2	$-\sqrt{7}$
$g(x)$	8	9	12	15

29.



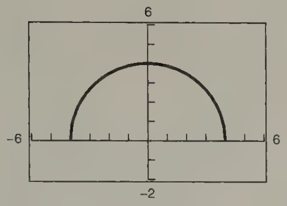
One-to-one

31.



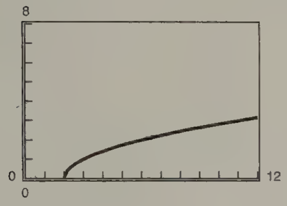
Not one-to-one

33.



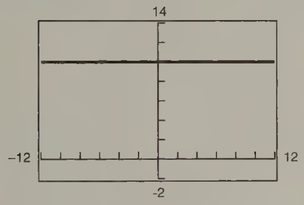
Not one-to-one

35.



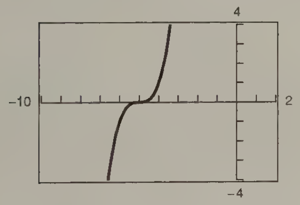
One-to-one

37.



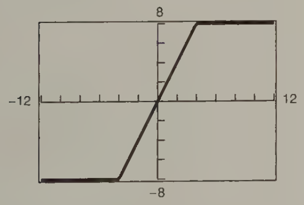
Not one-to-one

39.



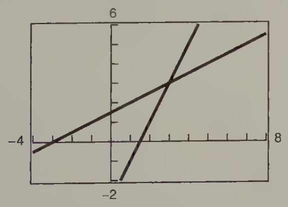
One-to-one

41.



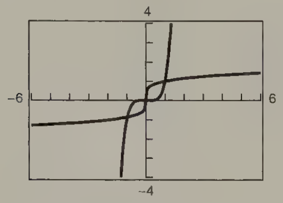
Not one-to-one

43. $f^{-1}(x) = \dfrac{x+3}{2}$



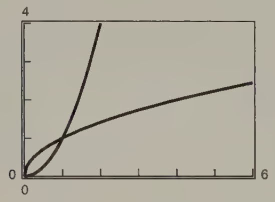
Reflections in the line $y = x$

45. $f^{-1}(x) = \sqrt[5]{x}$



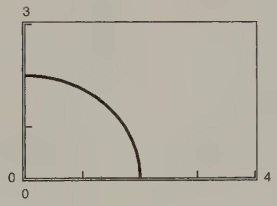
Reflections in the line $y = x$

47. $f^{-1}(x) = x^2,\ x \geq 0$



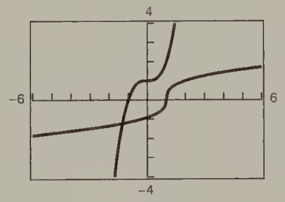
Reflections in the line $y = x$

49. $f^{-1}(x) = \sqrt{4 - x^2},\ 0 \leq x \leq 2$



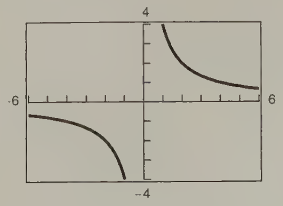
Reflections in the line $y = x$

51. $f^{-1}(x) = x^3 + 1$



Reflections in the line $y = x$

53. $f^{-1}(x) = \dfrac{4}{x}$



Reflections in the line $y = x$

55. Not one-to-one

57. $f^{-1}(x) = \dfrac{5x - 4}{3}$ **59.** $f^{-1}(x) = \sqrt{x} - 3,\ x \ge 0$

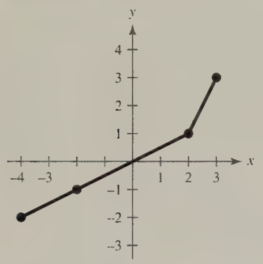
61. Not one-to-one **63.** $f^{-1}(x) = \dfrac{x^2 - 3}{2},\ x \ge 0$

65. Not one-to-one **67.** $f^{-1}(x) = \dfrac{x - b}{a}$

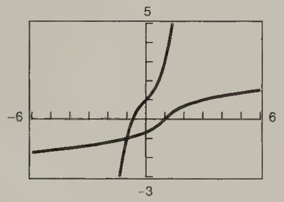
69. $y = \sqrt{x} + 2,\ x \ge 0$ **71.** $y = x - 2,\ x \ge 0$

73.

x	-4	-2	2	3
$f^{-1}(x)$	-2	-1	1	3

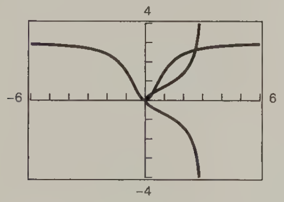


75. (a) and (b)



(c) Inverse function because it satisfies the Vertical Line Test

77. (a) and (b)



(c) Not an inverse function because it does not satisfy the Vertical Line Test

79. 32 **81.** 600 **83.** $2\sqrt[3]{x + 3}$

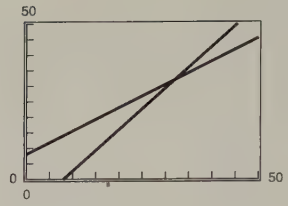
85. $\dfrac{x + 1}{2}$ **87.** $\dfrac{x + 1}{2}$

89. (a) $y = \dfrac{x - 8}{0.75}$

y = number of units produced

x = hourly wage

(b)



(c) \$15.50 (d) 19 units

91. (a) Yes

(b) $f^{-1}(t)$ represents the year new car sales totaled \$$t$ billion.

(c) 5, or 1995

(d) No. The inverse is not a function because f is not one-to-one.

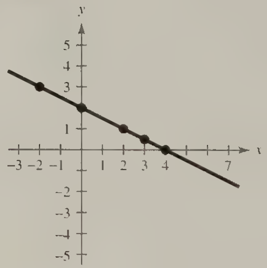
93. True **95.** Answers will vary. **97.** $9x,\ x \ne 0$

99. $-(x + 6),\ x \ne 6$ **101.** 2 **103.** 0

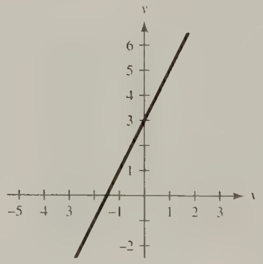
Review Exercises *(page 149)*

1.

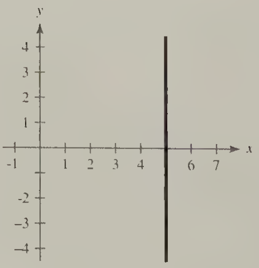
x	-2	0	2	3	4
y	3	2	1	$\frac{1}{2}$	0



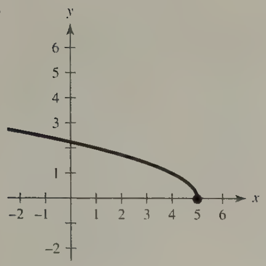
3.



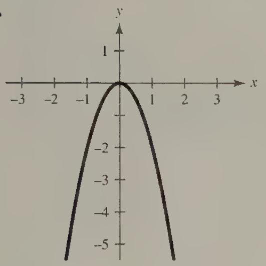
5.



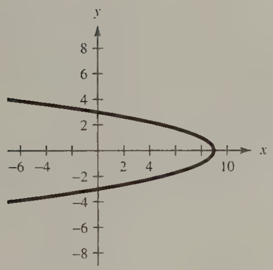
7.



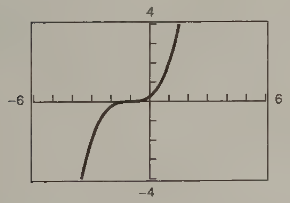
9.



11.

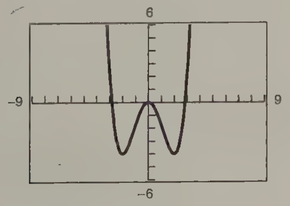


13.



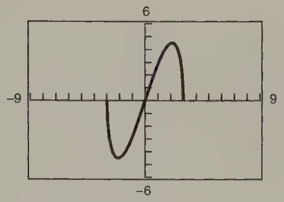
Intercepts: $(-1, 0)$, $\left(0, \frac{1}{4}\right)$

15.



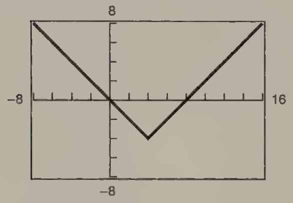
Intercepts: $(0, 0)$, $\left(\pm 2\sqrt{2}, 0\right)$

17.



Intercepts: $(0, 0)$, $(\pm 3, 0)$

19.

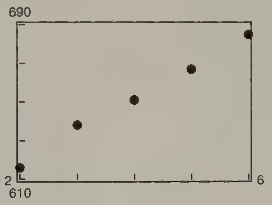


Intercepts: $(0, 0)$, $(8, 0)$

21.

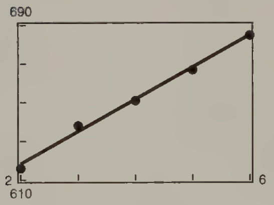
Xmin = -20
Xmax = 50
Xscl = 10
Ymin = -2
Ymax = 1
Yscl = 0.5

23. (a)



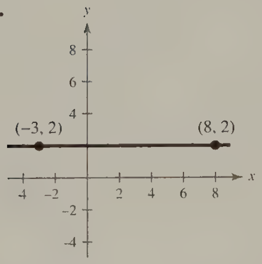
(b) $y = 16.7x + 585$

(c)



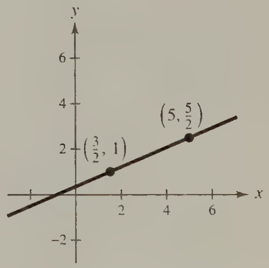
(d) 2000: $752;
2002: $785

25.



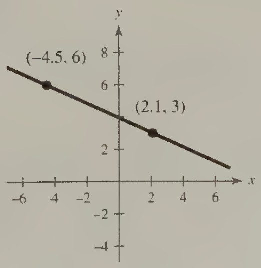
$m = 0$

27.



$m = \frac{3}{7}$

29.



$m = -\frac{5}{11}$

31. $t = \frac{7}{3}$ **33.** $t = 3$

35. (a) $x - 4y - 6 = 0$ (b) $(6, 0), (10, 1), (-2, -2)$

37. (a) $3x - 2y - 10 = 0$ (b) $(4, 1), (2, -2), (-2, 8)$

39. (a) $2x + y + 17 = 0$

(b) $(-5, -7), (-4, -9), (-3, -11)$

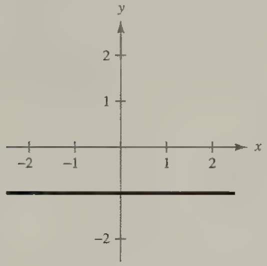
41. (a) $5x + 5y + 24 = 0$ (b) $\left(-5, \frac{1}{5}\right), \left(-4, -\frac{4}{5}\right), \left(-6, \frac{6}{5}\right)$

43. (a) $y = 6$ (b) $(0, 6), (1, 6), (-1, 6)$

45. (a) $x = 10$ (b) $(10, 1), (10, 3), (10, -2)$

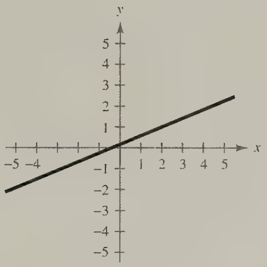
47. (a) $y = -1$

(b)



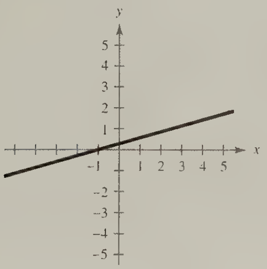
49. (a) $y = \frac{5}{12}x + \frac{1}{6}$

(b)



51. (a) $y = \frac{2}{7}x + \frac{2}{7}$

(b)

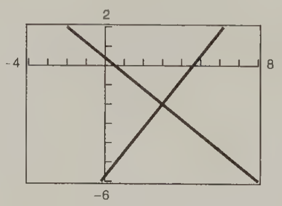


53. $V = 850t + 12{,}500$ **55.** $x + y - 1 = 0$

57. \$210,000

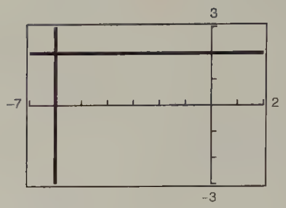
59. (a) $5x - 4y - 23 = 0$

(b) $4x + 5y - 2 = 0$



61. (a) $x = -6$

(b) $y = 2$



63. (a) Not a function because element 20 in A corresponds to two elements, 4 and 6, in B.

(b) Function

(c) Function

(d) Not a function because 30 in A corresponds to no element in B.

65. Not a function **67.** Function

69. (a) 5 (b) 17 (c) $t^4 + 1$ (d) $-x^2 - 1$

71. All real numbers **73.** $[-5, 5]$

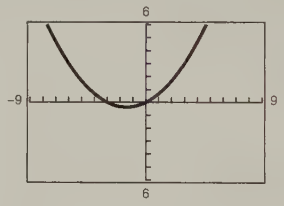
75. All real numbers except $s = 3$

77. (a) $C = 5.35x + 16{,}000$ (b) $P = 2.85x - 16{,}000$

79. Domain: $(-\infty, \infty)$; Range: $(-\infty, 3]$

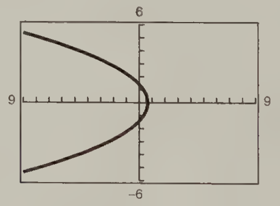
81. Domain: $[-6, 6]$; Range: $[0, 6]$

83. (a)



(b) Function

85. (a)



(b) Not a function

87. Increasing on $(-\infty, -1), (1, \infty)$;

Decreasing on $(-1, 1)$

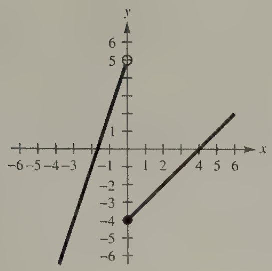
89. Increasing on $(6, \infty)$

91. Relative maximum: $(0, 16)$

Relative minimums: $(-2, 0), (2, 0)$

93. Relative maximum: $(3, 27)$

95.

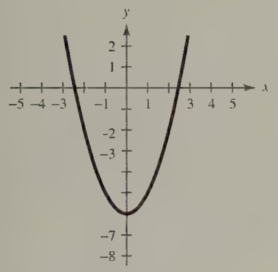


97. Even

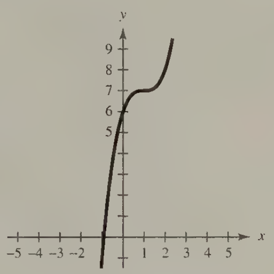
99. Constant function $f(x) = C$. $g(x) = -2$

101. Cubic function $f(x) = x^3$. Reflection in x-axis and vertical shift 2 units downward. $g(x) = -x^3 - 2$

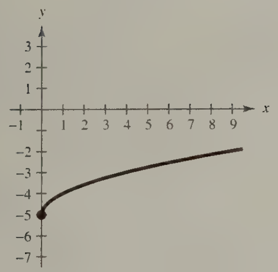
103.



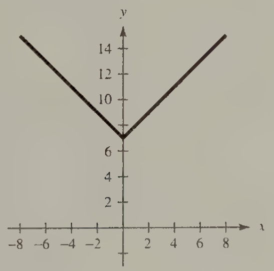
105.



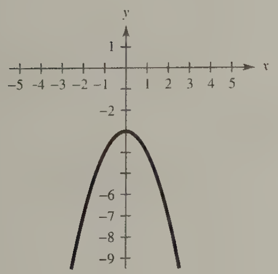
107.



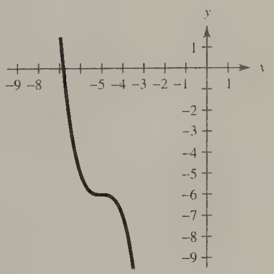
109.



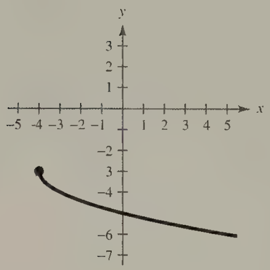
111.



113.



115.



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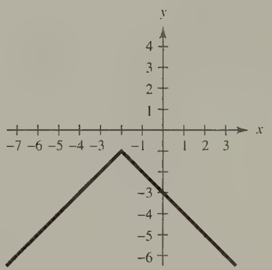

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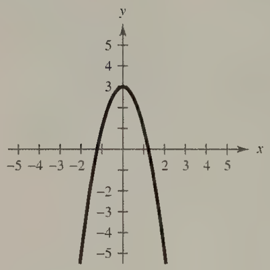
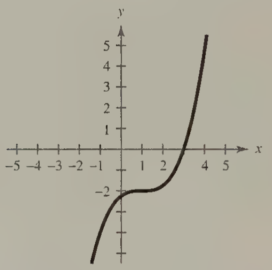
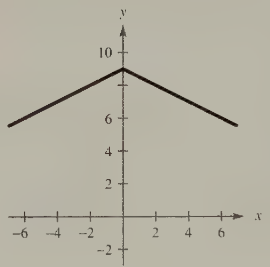


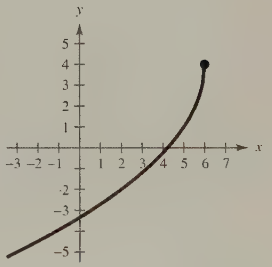
121.



123.



125.



127. −7 **129.** −42 **131.** 5

133. 23 **135.** −97

137. $y_1 = 0.380t^2 + 3.75t + 16.9$

$y_2 = 0.146t^2 + 0.30t + 23.2$

139. $f^{-1}(x) = \frac{x}{6}$; $f(f^{-1}(x)) = f\left(\frac{x}{6}\right) = 6\left(\frac{x}{6}\right) = x$

$f^{-1}(f(x)) = f^{-1}(6x) = \frac{6x}{6} = x$

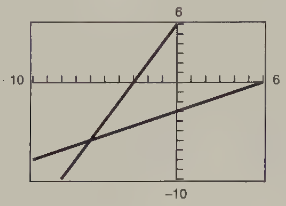
141. $f^{-1}(x) = x + 7$

$f(f^{-1}(x)) = f(x + 7) = (x + 7) - 7 = x$

$f^{-1}(f(x)) = f^{-1}(x - 7) = (x - 7) + 7 = x$

143. (a) $f^{-1}(x) = 2x + 6$

(b)



(c) $f^{-1}(f(x)) = f^{-1}\left(\frac{1}{2}x - 3\right)$

$= 2\left(\frac{1}{2}x - 3\right) + 6$

$= x$

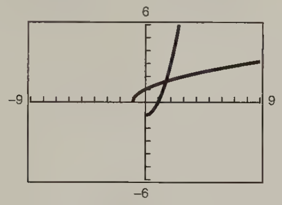
$f(f^{-1}(x)) = f(2x + 6)$

$= \frac{1}{2}(2x + 6) - 3$

$= x$

145. (a) $f^{-1}(x) = x^2 - 1,\ x \geq 0$

(b)



(c) $f^{-1}(f(x)) = f^{-1}\left(\sqrt{x + 1}\right)$

$= (x + 1) - 1$

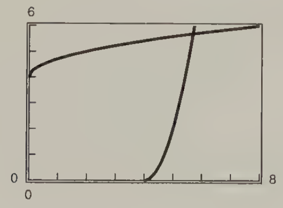
$= x$

$f(f^{-1}(x)) = f(x^2 - 1),\ x \geq 0$

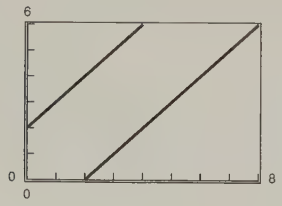
$= \sqrt{x^2 - 1 + 1}$

$= x$

147. $x \geq 4,\ f^{-1}(x) = \sqrt{\frac{x}{2}} + 4$



149. $x \geq 2,\ f^{-1}(x) = x + 2,\ x \geq 0$



151. $f^{-1}(x) = 12x$ **153.** $f^{-1}(x) = \sqrt[3]{\frac{x + 3}{4}}$

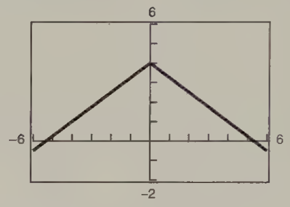
155. $f^{-1}(x) = x^2 - 10,\ x \geq 0$

157. False. It is reflected in the x-axis before it is shifted down.

159. False. $f(x) = 4 - x = f^{-1}(x)$

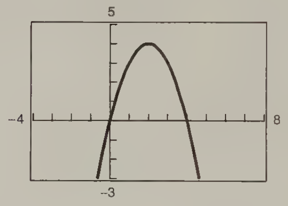
Chapter Test *(page 154)*

1.



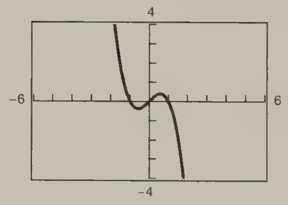
$(0, 4), \left(-\frac{16}{3}, 0\right), \left(\frac{16}{3}, 0\right)$

2.



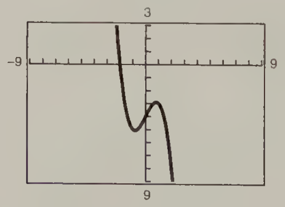
$(0, 0), (4, 0)$

3.



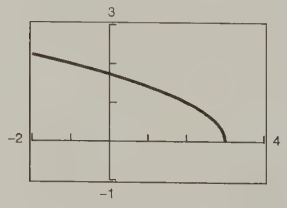
$(0, 0), (1, 0), (-1, 0)$

4.



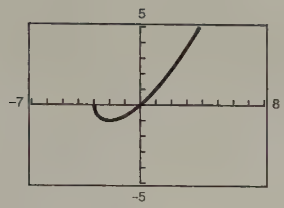
$(-2, 0), (0, -4)$

5.



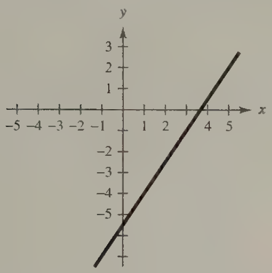
$\left(0, \sqrt{3}\right), (3, 0)$

6.



$(-3, 0), (0, 0)$

7. $(1, -4), (5, 2), (7, 5)$



8. $\left(0, -\frac{9}{2}\right), (3, 0)$ **9.** $2x - 5y + 20 = 0$

10. No. To some x there correspond more than one value of y.

11. 7 **12.** $10 - \sqrt{6 - t}$ **13.** $\dfrac{\sqrt{3 - x} - 1}{2 - x}$

14. $(-\infty, 3]$ **15.** $C = 5.60x + 24{,}000$

$P = 3.60x - 24{,}000$

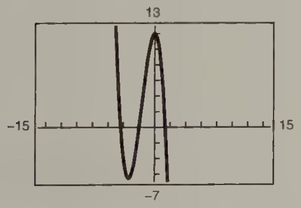
16. Increasing: $(-2, 0), (2, \infty)$

Decreasing: $(-\infty, -2), (0, 2)$

17. Increasing: $(-2, 2)$

Constant: $(-\infty, -2), (2, \infty)$

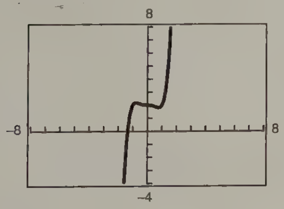
18.



Relative minimum: $(-3.33, -6.52)$

Relative maximum: $(0, 12)$

19.



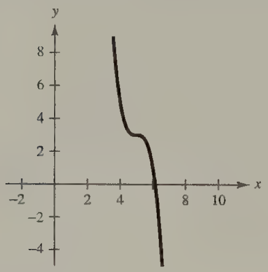
Relative maximum: $(-0.77, 2.19)$

Relative minimum: $(0.77, 1.81)$

20. (a) $f(x) = x^3$

(b) Horizontal shift 5 units to the right, reflection in x-axis, vertical stretch, and then a vertical shift 3 units upward

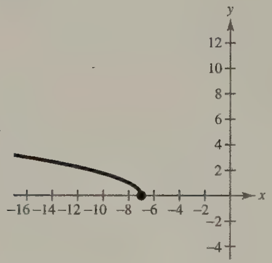
(c)



21. (a) $f(x) = \sqrt{x}$

(b) Reflection in y-axis and a horizontal shift 7 units to the left

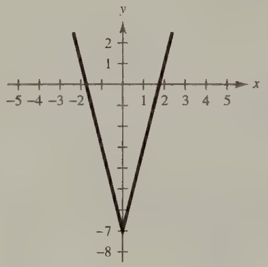
(c)



22. (a) $f(x) = |x|$

(b) Reflection in y-axis (no effect), vertical stretch, and vertical shift 7 units downward

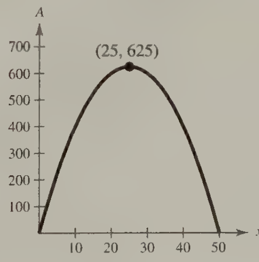
(c)



23. (a) $x^2 - \sqrt{2 - x}, \ (-\infty, 2]$ (b) $\dfrac{x^2}{\sqrt{2 - x}}, \ (-\infty, 2)$

(c) $2 - x, \ (-\infty, 2]$ (d) $2 - x^2, \ [0, \infty)$

24. $A = -x^2 + 50x, \ 0 < x < 50$



Maximum area = 625 square inches

Chapter 2

Section 2.1 *(page 164)*

1. (a) Yes (b) No (c) No (d) No

3. (a) Yes (b) No (c) No (d) No

5. (a) No (b) No (c) No (d) Yes

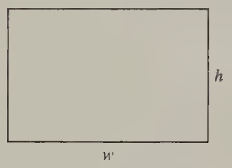
7. Identity **9.** Identity **11.** Conditional **13.** $-\frac{96}{23}$

15. -4 **17.** $-\frac{6}{5}$ **19.** 10 **21.** 4 **23.** 5

25. $\frac{11}{6}$ **27.** $\frac{5}{3}$ **29.** No solution **31.** $h = \frac{2A}{b}$

33. $b = \frac{2A - ah}{h}$ **35.** 61.2 inches **37.** 23,437.5 miles

39. (a)



(b) $h = \frac{2}{3}w$; $P = \frac{10}{3}w$

(c) $w = \frac{9}{10}$ meter; $h = \frac{3}{5}$ meter

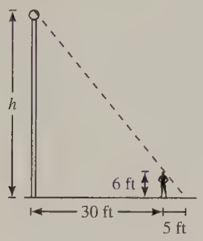
41. 187 points or 93.5%

43. 3 hours at 58 miles per hour; 2 hours and 45 minutes at 52 miles per hour

45. ≈46.3 miles per hour **47.** ≈8.33 minutes

49. $21\frac{1}{3}$ meters

51. (a)

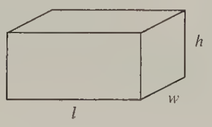


(b) 42 feet

53. $16,666.67 **55.** First three quarters: 4.25%
Last quarter: 2.75%

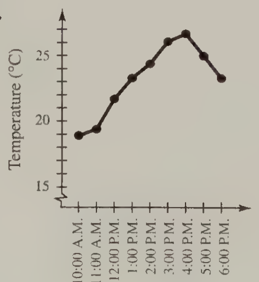
57. 8823 units **59.** $x = 3\frac{2}{3}$ feet

61. (a)

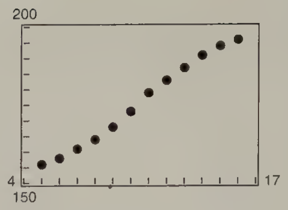


(b) $w = 8$ inches;
$h = 12$ inches;
$l = 24$ inches

63.



65. (a)

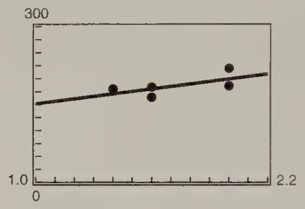


(b) Answers will vary. (c) $y = 4x + 134$

(d) 2000: 214; 2002: 222; 2005: 234

67. (a) $y = 47.77x + 103.8$

(b)



(c) A $1000 advertising increase means a $47.7 thousand = $47,770 sales increase.

(d) $175,460

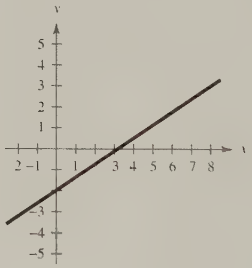
69. False. $(9.5)^3 < \frac{4}{3}\pi(5.9)^3$.

71. Answers will vary. Example: $9x + 27 = 0$.

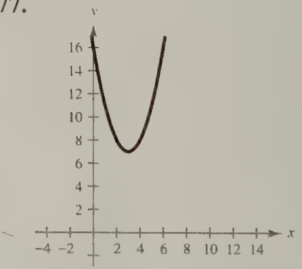
73. Equations with the same solution set

$4x + 16 = 0, 2x + 8 = 0$

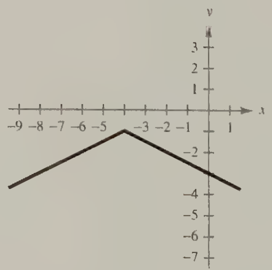
75.



77.



79.

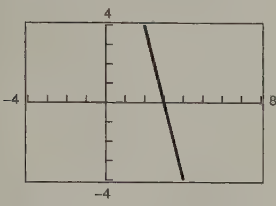


81. -28 **83.** -357

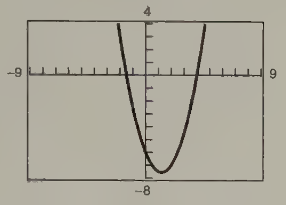
Section 2.2 *(page 176)*

1. $(5, 0), (0, -5)$ **3.** $(-2, 0), (1, 0), (0, -2)$
5. $(-2, 0), (0, 0)$ **7.** No intercepts
9. $(-2, 0), (6, 0), (0, -2)$ **11.** $(1, 0), \left(0, \frac{1}{2}\right)$

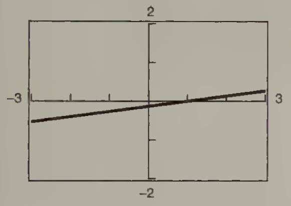
13.



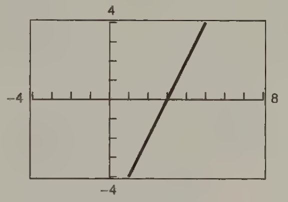
15.



17.

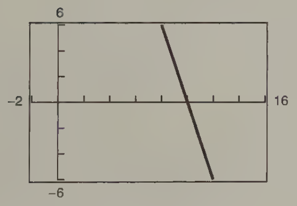


19.



$(3, 0)$

21.



$(10, 0)$

23. $\frac{15}{4}$ **25.** $\frac{89}{13}$

27. 6 **29.** 3, 12 **31.** $1, -1.2$ **33.** $-\frac{5}{8}$
35. 2.172, 7.828 **37.** -1.379 **39.** $0.5, -3, 3$
41. $-0.717, 2.107$ **43.** -1.333 **45.** $-1, 7$

47. (a)

x	-1	0	1	2	3	4
$3.2x - 5.8$	-9	-5.8	-2.6	0.6	3.8	7

(b) $1 < x < 2$

(c)

x	1.5	1.6	1.7	1.8	1.9	2
$3.2x - 5.8$	-1	-0.68	-0.36	-0.04	0.28	0.6

(d) $1.8 < x < 1.9$

To improve accuracy, evaluate the expression in this interval and determine where the sign changes.

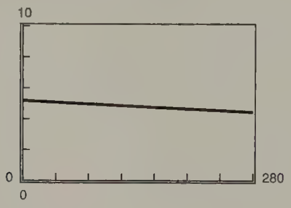
(e) $x = 1.8125$

49. $(1, 1)$ **51.** $(2, 2)$ **53.** $(-1, 3), (2, 6)$ **55.** $(4, 1)$
57. $(1.449, 1.899), (-3.449, -7.899)$
59. $(-2, 8), (1.333, 8)$ **61.** $(0, 0), (-2, 8), (2, 8)$

63. (a) 6.46
(b) $\frac{1.73}{0.27} \approx 6.41$. The second method decreases the accuracy.

65. (a) $T(x) = \dfrac{x}{63} + \dfrac{280 - x}{54}$

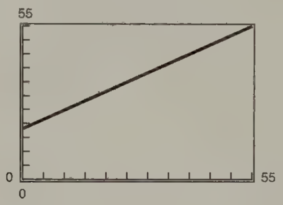
(b) $0 \le x \le 280$



(c) 164.5 miles

67. (a) $C = 0.33(55 - x) + x$

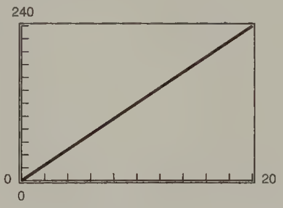
(b) $0 \le x \le 55$



(c) 22.2 gallons

69. (a) $A(x) = 12x$

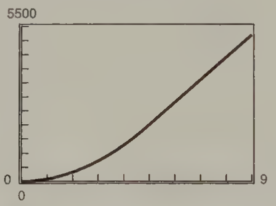
(b)



(c) 16.7

71. (a) 5200 cubic feet (b) $y = \frac{1}{8}x$
(c) Answers will vary.

(d)



(e)

d	3	5	7	9
V	720	2000	3600	5200

(f) 8.5 feet (g) 38,896 gallons

73. 1994. To answer the question algebraically, solve the equation $0.46t + 30.81 = 33$. To answer the question graphically, graph the model $y = 0.46x + 30.81$ and the horizontal line $y = 33$ and determine where they intersect.

75. (a) The time and quantity when utilization of cucumbers equaled that of nectarines and peaches

(b) (3.84, 5.46)

77. True **79.** $\frac{4\sqrt{3}}{5}$ **81.** $\frac{3(8-\sqrt{11})}{53}$

83. $3x^2 + 13x - 30$ **85.** $4x^2 - 81$

87. $6x^4 + 5x^2y - 4y^2$

89. Increasing on $(-\infty, 0)$; Decreasing on $(0, \infty)$

Section 2.3 *(page 186)*

1. $a = -9, b = 4$ **3.** $a = 6, b = 5$ **5.** $4 + 5i$

7. 12 **9.** $-1 - 5i$ **11.** -75 **13.** $0.3i$

15. $11 - i$ **17.** $7 - 3\sqrt{2}i$ **19.** $-14 + 20i$

21. $\frac{1}{6} + \frac{7}{6}i$ **23.** $-4.2 + 7.5i$ **25.** $-2\sqrt{3}$

27. -10 **29.** $5 + i$ **31.** $12 + 30i$ **33.** 24

35. $-9 + 40i$ **37.** $\sqrt{-6}\sqrt{-6} = \sqrt{6}i\sqrt{6}i = 6i^2 = -6$

39. 25 **41.** 41 **43.** 484 **45.** 11 **47.** $-6i$

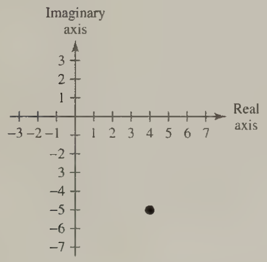
49. $\frac{16}{41} + \frac{20}{41}i$ **51.** $\frac{3}{5} + \frac{4}{5}i$ **53.** $-7 - 6i$

55. $-\frac{9}{1681} + \frac{40}{1681}i$ **57.** $-\frac{1}{2} - \frac{5}{2}i$ **59.** $-1 + 6i$

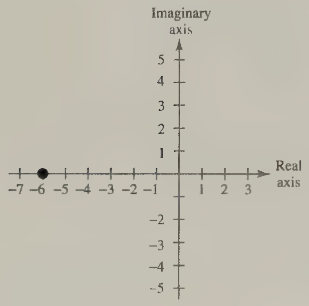
61. $-5i$ **63.** $-375\sqrt{3}i$ **65.** i **67.** $4 + 3i$

69. $6i$

71.



73.



75. Yes. 0, 0, 0, 0, 0, 0

77. Yes.

$0.5i, -0.25 + 0.5i, -0.1875 + 0.25i, -0.0273 + 0.4063i,$

$-0.1643 + 0.4778i, -0.2013 + 0.3430i$

79. No. 1, 2, 5, 26, 677, 458,330

81. (a) 8 (b) 8 (c) 8

83. (a) $3.12 - 0.97i$ (b) $12.82 + 5.28i$ **85.** True

87. (a) 1 (b) i (c) -1 (d) $-i$

89. Answers will vary. **91.** $x^3 + x^2 + 2x - 6$

93. $4x^2 - 20x + 25$ **95.** $(-4, 0), (2, 0), (0, -8)$

97. $(-1, 0), (1, 0), (0, -1)$

Section 2.4 *(page 200)*

1. $2x^2 + 5x - 3 = 0$ **3.** $3x^2 - 60x - 10 = 0$

5. $0, -\frac{1}{2}$ **7.** $4, -2$ **9.** $3, -\frac{1}{2}$ **11.** $2, -6$

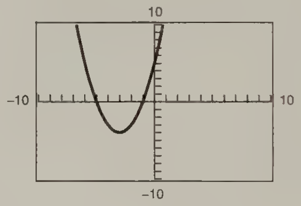
13. $-a - b, -a + b$ **15.** $\pm\sqrt{7}$; ± 2.65

17. $12 \pm 3\sqrt{2}$; 16.24, 7.76 **19.** $\frac{1}{2} \pm \frac{3}{2}\sqrt{2}$; -1.62, 2.62

21. 2 **23.** $-8, 4$ **25.** $-3 \pm \sqrt{7}$ **27.** $1 \pm \frac{\sqrt{6}}{3}$

29. $2 \pm 2\sqrt{3}$

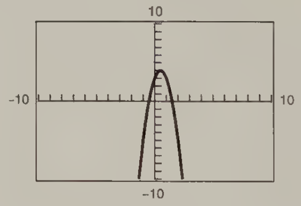
31. (a)



(b) and (c)

$(-1, 0), (-5, 0)$

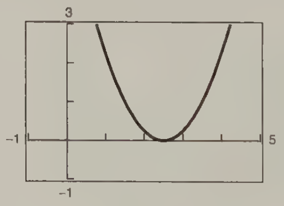
33. (a)



(b) and (c)

$\left(-\frac{1}{2}, 0\right), \left(\frac{3}{2}, 0\right)$

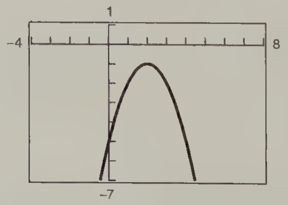
35. (a)



(b) and (c)

$\left(\frac{5}{2}, 0\right)$

37. (a)



(b) and (c)

No x-intercepts; $2 \pm i$

39. No real solutions **41.** Two real solutions

43. No real solutions **45.** $1 \pm \sqrt{3}$ **47.** $-4 \pm 2\sqrt{5}$

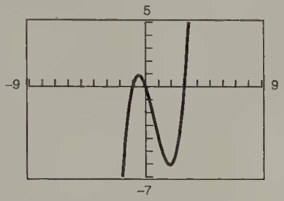
49. $\frac{2}{7}$ **51.** $-\frac{3}{2}, -\frac{5}{2}$ **53.** $1 \pm \sqrt{2}$ **55.** $6, -12$

57. $\frac{1}{2} \pm \sqrt{3}$ **59.** $-\frac{1}{2}$ **61.** $0, \pm\frac{3\sqrt{2}}{2}$ **63.** $\pm 3, \pm 3i$

65. $-3, 0$ **67.** $3, 1, -1$ **69.** $\pm\sqrt{3}, \pm 1$

71. $\pm\frac{1}{2}, \pm 4$ **73.** $-\frac{1}{5}, -\frac{1}{3}$ **75.** $\frac{1}{4}$

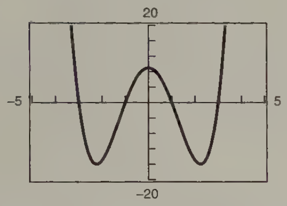
77. (a)



(b) and (c)

$x = 0, 3, -1$

79. (a)

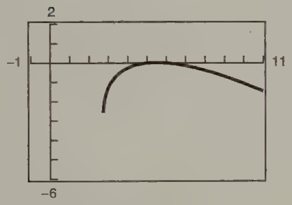


(b) and (c)

$x = \pm 3, \pm 1$

81. 26 **83.** 0 **85.** 9 **87.** $-59, 69$ **89.** 1

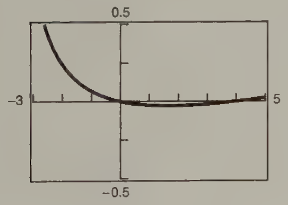
91. (a)



(b) and (c)

$x = 5, 6$

93. (a)



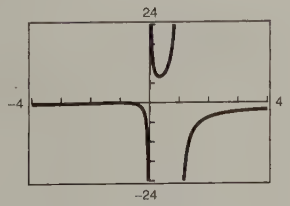
(b) and (c)

$x = 0, 4$

95. $4, -5$ **97.** $\frac{-3 \pm \sqrt{21}}{6}$ **99.** $2, -\frac{3}{2}$

101. $3, -2$ **103.** $\sqrt{3}, -3$

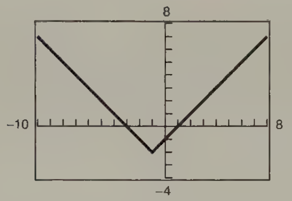
105. (a)



(b) and (c)

$x = -1$

107. (a)



(b) and (c)

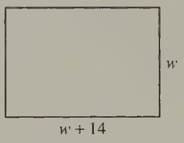
$x = 1, -3$

109. $x^2 + x - 30 = 0$ **111.** $21x^2 + 31x - 42 = 0$

113. $x^3 - 4x^2 - 2x + 8 = 0$ **115.** $x^4 - 3x^2 - 4 = 0$

117. $x = 6$ or -4 **119.** $h = \frac{1}{\pi r}\sqrt{S^2 - \pi^2 r^4}$

121. (a)



(b) $1632 = w^2 + 14w$

(c) Width: 34 feet;
length: 48 feet

123. 14 centimeters × 14 centimeters

125. (a) $s = -16t^2 + 1821$

(b)

t	0	2	4	6	8	10
s	1821	1757	1565	1245	797	221

(c) 10 seconds; ≈10.67 seconds

127. (a) $s = -16t^2 + 1368$ (b) 1112 feet

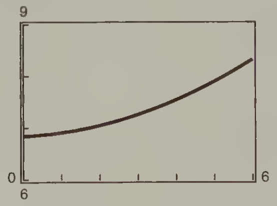
(c) ≈9.25 seconds

129. (a) 1995

(b)

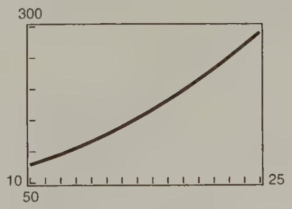
t	0	1	2	3	4	5	6
y	6.84	6.91	7.05	7.26	7.54	7.90	8.33

(c)



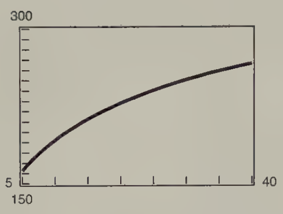
(d) 1991

131. (a)



(b) ≈16.8°C (c) ≈2.5 times

133. (a)



(b) 211.6°F (c) 24.725 pounds per square inch

135. 50,000 units **137.** $\frac{5\sqrt{2}}{2}$ centimeters

139. Eastbound plane: ≈550 miles per hour;

Northbound plane: ≈600 miles per hour

141. False. Both solutions are complex.

143. False. For example, $|x| = x^2 + x - 3$ has two extraneous solutions.

145. (a) $0, -\frac{b}{a}$ (b) 0, 1 **147.** $x(x - 7)(x + 2)$

149. $x^{1/3}(9x + 25)$ **151.** Not a function

153. Function **155.** Not a function

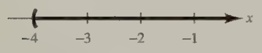
Section 2.5 *(page 214)*

1. (d) **3.** (c)

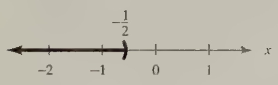
5. (a) Yes (b) No (c) Yes (d) No

7. (a) No (b) Yes (c) Yes (d) No

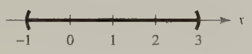
9. $x > -4$



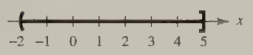
11. $x < -\frac{1}{2}$



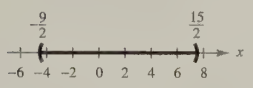
13. $-1 < x < 3$



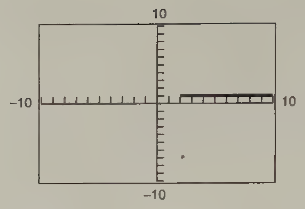
15. $-2 < x \le 5$



17. $-\frac{9}{2} < x < \frac{15}{2}$

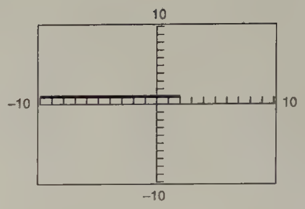


19.



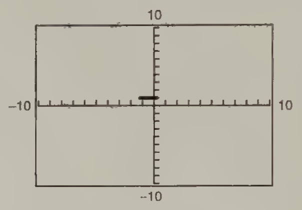
$x > 2$

21.



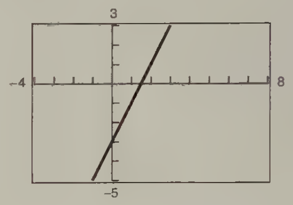
$x \le 2$

23.



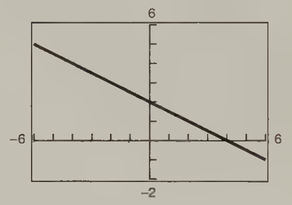
$-\frac{4}{3} < x < \frac{1}{3}$

25.



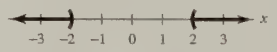
(a) $x \ge 2$ (b) $x \le \frac{3}{2}$

27.

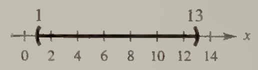


(a) $-2 \le x \le 4$ (b) $x \le 4$

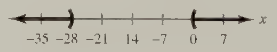
29. $x < -2,\ x > 2$



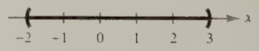
31. $1 < x < 13$



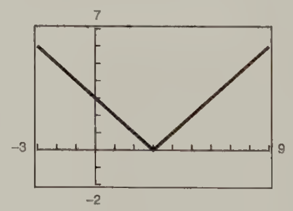
33. $x < -28,\ x > 0$



35. $-2 < x < 3$



37.

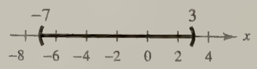


(a) $1 \le x \le 5$

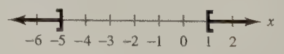
(b) $x \le -1,\ x \ge 7$

39. $|x| \le 3$ **41.** $|x - 7| \ge 3$ **43.** $|x - 12| \le 10$

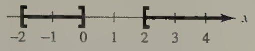
45. $(-7, 3)$



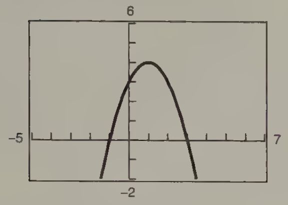
47. $(-\infty, -5], [1, \infty)$



49. $[-2, 0], [2, \infty)$



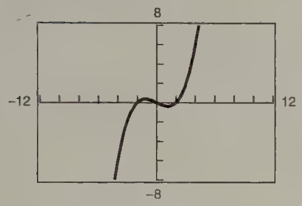
51.



(a) $x \le -1,\ x \ge 3$

(b) $0 \le x \le 2$

53.

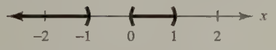


(a) $-2 \le x \le 0,$

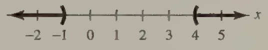
$2 \le x < \infty$

(b) $x \le 4$

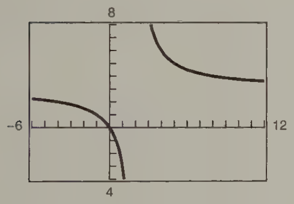
55. $(-\infty, -1), (0, 1)$



57. $(-\infty, -1), (4, \infty)$



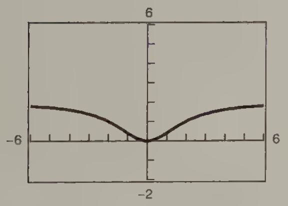
59.



(a) $0 \le x < 2$

(b) $2 < x \le 4$

61.

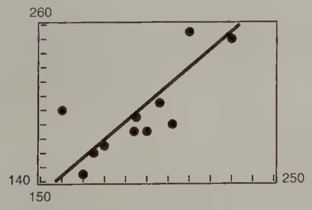


(a) $2 \le |x|$

(b) $-\infty < x < \infty$

63. $[5, \infty)$ **65.** $(-\infty, \infty)$ **67.** $[-\frac{5}{2}, \infty)$

69. (a) and (b)



(c) 186.23 pounds

(d) Answers will vary.

71. $[65.8, 71.2]$

73. (a) $333\frac{1}{3}$ vibrations per second (b) 3.6 millimeters

(c) $1.2 < t < 2.4$ (d) $0 < v < 500$

75. False. $10 \ge -x$

77. (a) a, b

(b)

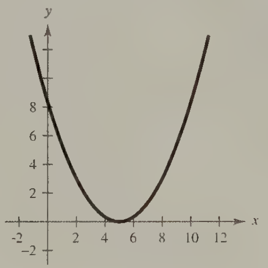
Interval	Expression	Sign
$(-\infty, a)$	$(x - a)$	$-$
	$(x - b)$	$-$
	$(x - a)(x - b)$	$+$
(a, b)	$(x - a)$	$+$
	$(x - b)$	$-$
	$(x - a)(x - b)$	$-$
(b, ∞)	$(x - a)$	$+$
	$(x - b)$	$+$
	$(x - a)(x - b)$	$+$

(c) x-values that are zeros of the function

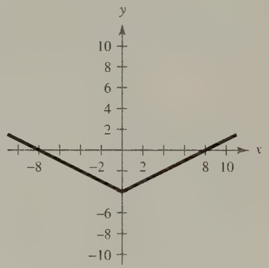
79. $d = \sqrt{106} \approx 10.296$; Midpoint $= (5.5, 0.5)$

81. $d = \sqrt{180} \approx 13.416$; Midpoint $= (-3, 3)$

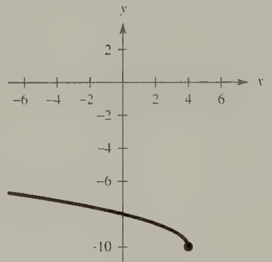
83.



85.



87.



89. $y^{-1} = \dfrac{x - 8}{5}$ **91.** $y^{-1} = x^3 + 7$

Review Exercises *(page 218)*

1. (a) No (b) No (c) No (d) Yes

3. $x = \frac{1}{2}$ **5.** $x = \frac{7}{3}$

7. September: \$325,000; October: \$364,000

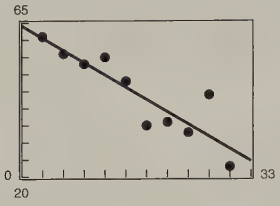
9. $2\frac{6}{7}$ liters **11.** 4 **13.** 56 miles per hour

15. Basketball: $\approx$455.95 cubic inches;

Baseball: $\approx$13.4 cubic inches

17. (a) $y = -1.2x + 64$

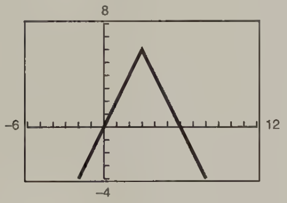
(b)



(c) Eliminate the point (27, 44); $y = -1.4x + 66$

19. (20, 0), (0, −4) **21.** (0, 25), (5, 0), (−5, 0)

23.



(0, 0), (6, 0)

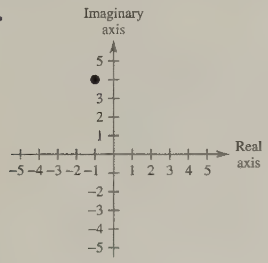
25. $x = 9.4$ **27.** $x = -2.722$ **29.** No solution

31. (5, 2) **33.** (−1, 8) **35.** $3 - 2\sqrt{3}i$ **37.** $1 - 4i$

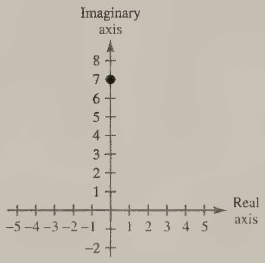
39. $-\sqrt{2}i$ **41.** $17 + 28i$ **43.** $9 + 20i$ **45.** $-16i$

47. $\frac{17}{26} + \frac{7}{26}i$ **49.** $-\frac{7}{625} - \frac{24}{625}i$

51.



53.



55. $-\frac{5}{2}, 3$ **57.** $\pm\frac{5}{4}$ **59.** $-3 \pm 2\sqrt{3}$

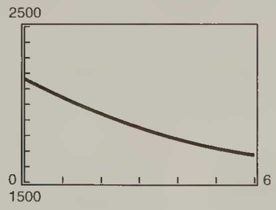
61. $\dfrac{-1 \pm \sqrt{61}}{2}$ **63.** $0, -\dfrac{13}{2}$ **65.** $0, \dfrac{1}{6}$ **67.** $0, \dfrac{3}{2}$

69. 66 **71.** 2 **73.** $\dfrac{38 + 5\sqrt{3}}{24}$ **75.** 79

77. $\pm 2, \pm\frac{2}{3}$ **79.** $\frac{7}{3}$ **81.** −2, 0 **83.** −5, 2

85. 2, 3

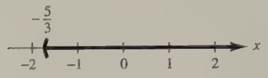
87. (a)



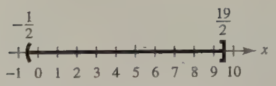
(b) 1991

(c) Eventually the curve begins to rise rapidly. At this point the model is probably not a good predictor.

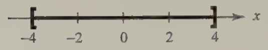
89. $\left(-\frac{5}{3}, \infty\right)$



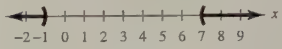
91. $\left(-\frac{1}{2}, \frac{19}{2}\right]$



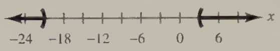
93. [−4, 4]



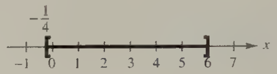
95. $(-\infty, -1), (7, \infty)$



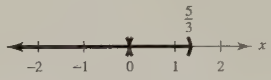
97. $(-\infty, -21), (3, \infty)$



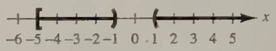
99. $\left[-\frac{1}{4}, 6\right]$



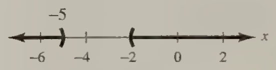
101. $(-\infty, 0), \left(0, \frac{5}{3}\right)$



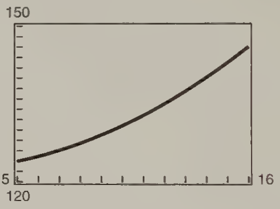
103. $[-5, -1), (1, \infty)$



105. $(-\infty, -5), (-2, \infty)$



107. (a)



(b) 2001

109. False. $(3 + 2i) + (5 - 2i) = 8$

111. An identity is true for all values of the variable and a conditional equation is true for only some values of the variable.

113. (a) Negative (b) Positive

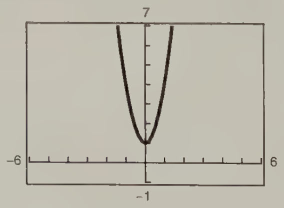
Chapter Test *(page 222)*

1. $x = 3$ **2.** $x = \frac{2}{15}$ **3.** $-9 - 18i$

4. $6 + \left(2\sqrt{5} + \sqrt{14}\right)i$ **5.** $13 + 4i$ **6.** $-17 + 14i$

7. $\frac{43}{37} + \frac{38}{37}i$ **8.** $\frac{36}{41} + \frac{86}{41}i$

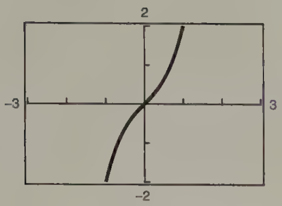
9.



No x-intercepts

No real zeros

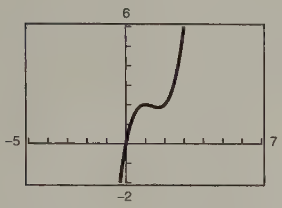
10.



One x-intercept: 0

One real zero: 0

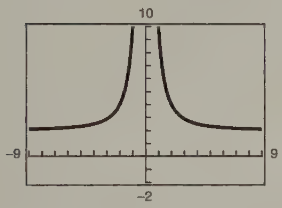
11.



One x-intercept: 0

One real zero: 0

12.



No x-intercepts

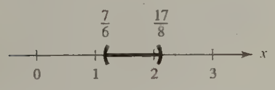
No real zeros

13. $x = 1, 9$ **14.** $x = -6 \pm \sqrt{38}$ **15.** $x = \pm\frac{9}{2}$

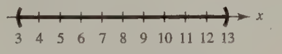
16. $x = -3, \frac{1}{5}$ **17.** $\pm 2, \frac{4}{3}$ **18.** 2 **19.** $\pm\sqrt{58}$

20. $-\frac{5}{2}, \frac{11}{4}$

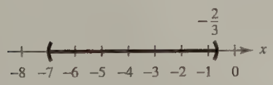
21. $\left(\frac{7}{6}, \frac{17}{8}\right)$



22. $(3, 13)$



23. $\left(-7, -\frac{2}{3}\right)$



24. $C = 16.4t + 402$; 2002 **25.** \$0.10

Cumulative Test for Chapters P–2 *(page 223)*

1. $\frac{7x^3}{16y^5}$ **2.** $9\sqrt{15}$ **3.** $2x^2y\sqrt{7y}$ **4.** $7x - 10$

5. $x^3 - x^2 - 5x + 6$ **6.** $\frac{x-1}{(x+1)(x+3)}$

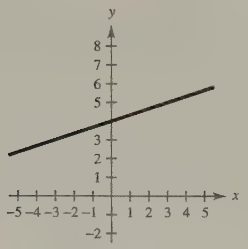
7. $(3 + x)(7 - x)$ **8.** $x(1 + x)(1 - 6x)$

9. $2(3 - 2x)(9 + 6x + 4x^2)$

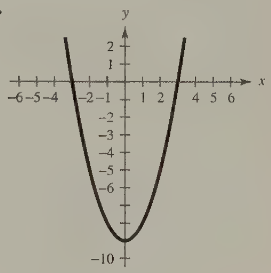
10. Midpoint: $(1.5, -2)$; $d = 2\sqrt{61} \approx 15.62$

11. $\left(x + \frac{1}{2}\right)^2 + (y + 8)^2 = \frac{25}{16}$

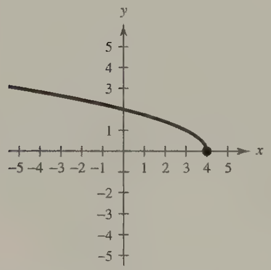
12.



13.



14.



15. (a) $28x + 11y + 52 = 0$

(b) $\left(0, -\frac{52}{11}\right), \left(-\frac{52}{28}, 0\right), \left(-2, \frac{4}{11}\right)$

16. (a) $2x + y = 0$ (b) $(0, 0), (1, -2), (2, -4)$

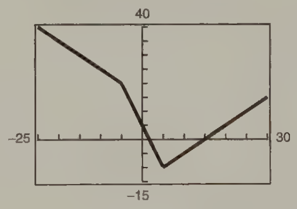
17. (a) $x = -\frac{3}{7}$ (b) $\left(-\frac{3}{7}, 0\right), \left(-\frac{3}{7}, 1\right), \left(-\frac{3}{7}, -3\right)$

18. No. It doesn't pass the Vertical Line Test.

19. (a) $\frac{3}{2}$ (b) Undefined (c) $\frac{s+2}{s}$

20. (a) -13 (b) -14 (c) -8

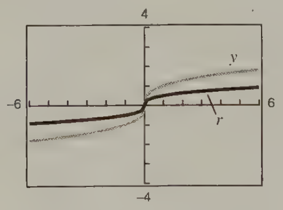
21.



Decreasing on $(-\infty, 5)$

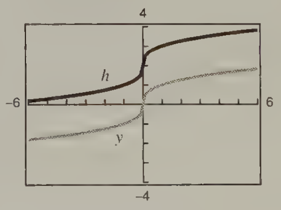
Increasing on $(5, \infty)$

22. (a)



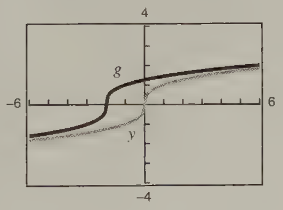
Vertical shrink

(b)



Vertical shift

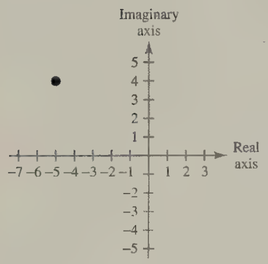
(c)



Horizontal shift

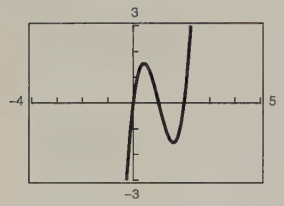
23. -53 **24.** $\frac{197}{16}$ **25.** -79 **26.** 42

27.



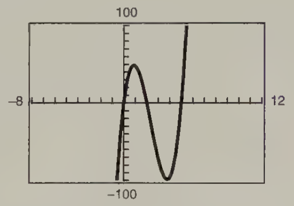
28. $13x^2 - 9x + 6 = 0$

29.



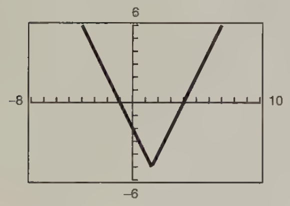
x-intercepts: 0, 1, 2

30.



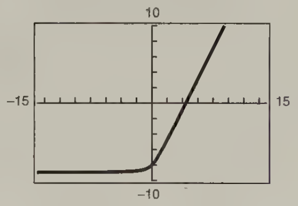
x-intercepts: 0, 2, 5

31.



x-intercepts: $-1, 4$

32.



x-intercept: $4.\overline{44}$

33. $X = \pm\sqrt{R^2 - Z^2}$ **34.** $p = \dfrac{k}{3\pi r^2 L}$ **35.** $C = 4$

36. $C = -\frac{4}{5}$ **37.** $(-\infty, 17.143)$

38. $(-\infty, -3], \left[\frac{5}{2}, \infty\right)$ **39.** $(4, \infty)$

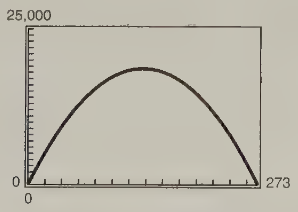
40. $S = 119.76t + 914.6$

2000: \$2112.2 million; 2002: \$2351.7 million;
2004: \$2591.2 million

41. $r = 4.297$ inches

42. (a) $A = x(273 - x)$

(b)



$0 < x < 273$

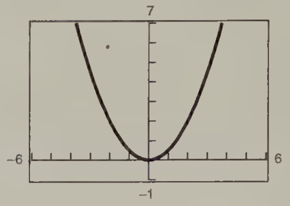
(c) $\approx$76.2 feet × 196.8 feet

Chapter 3

Section 3.1 *(page 233)*

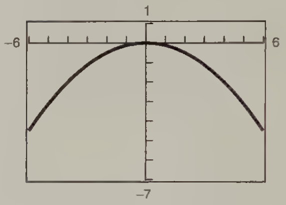
1. (g) **3.** (b) **5.** (f) **7.** (e)

9. (a)



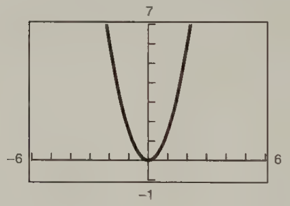
Vertical shrink

(b)



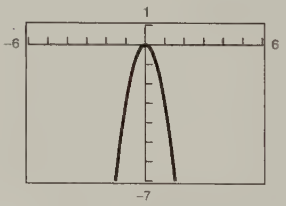
Vertical shrink and reflection in the x-axis

(c)



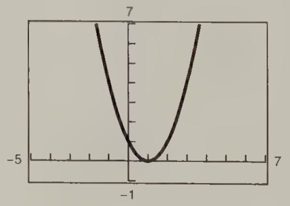
Vertical stretch

(d)



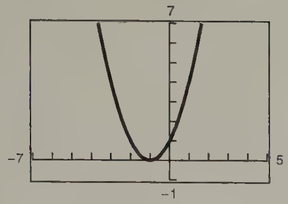
Vertical stretch and reflection in the x-axis

11. (a)



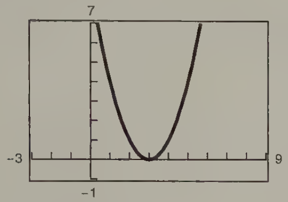
Horizontal shift

(b)



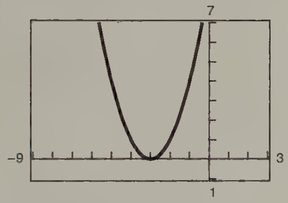
Horizontal shift

(c)



Horizontal shift

(d)

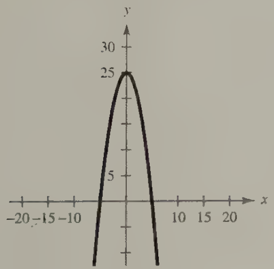


Horizontal shift

13. Vertex: $(0, 25)$

Intercepts:

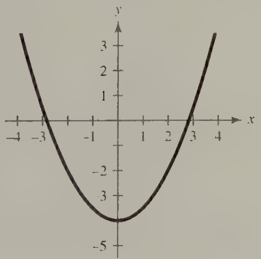
$(0, 25), (\pm 5, 0)$



15. Vertex: $(0, -4)$

Intercepts:

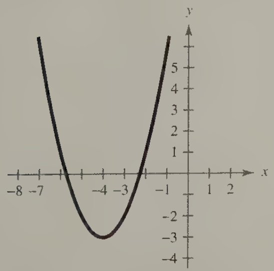
$(\pm 2\sqrt{2}, 0), (0, -4)$



17. Vertex: $(-4, -3)$

Intercepts:

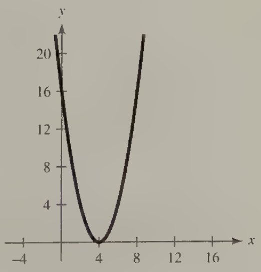
$(\pm\sqrt{3} - 4, 0), (0, 13)$



19. Vertex: $(4, 0)$

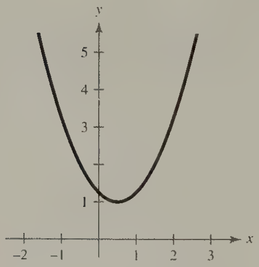
Intercepts:

$(4, 0), (0, 16)$



21. Vertex: $\left(\frac{1}{2}, 1\right)$

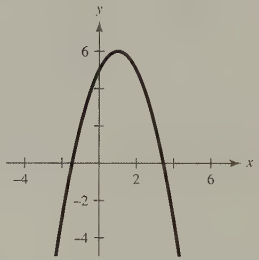
Intercept: $\left(0, \frac{5}{4}\right)$



23. Vertex: $(1, 6)$

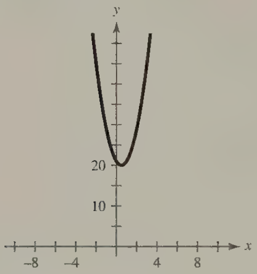
Intercepts:

$(1 \pm \sqrt{6}, 0), (0, 5)$



25. Vertex: $\left(\frac{1}{2}, 20\right)$

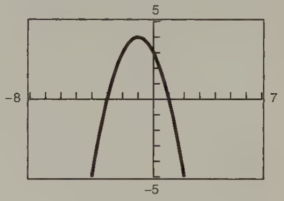
Intercept: $(0, 21)$



27. Vertex: $(-1, 4)$

Intercepts:

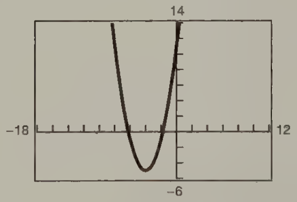
$(1, 0), (-3, 0), (0, 3)$



29. Vertex: $(-4, -5)$

Intercepts:

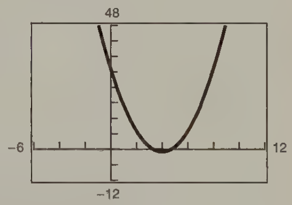
$(-4 \pm \sqrt{5}, 0), (0, 11)$



31. Vertex: $(4, -1)$

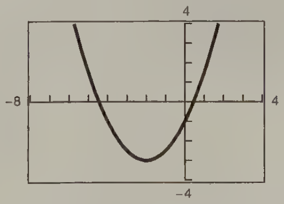
Intercepts:

$\left(4 \pm \frac{1}{2}\sqrt{2}, 0\right), (0, 31)$



33. Vertex: $(-2, -3)$

Intercepts: $(-2 \pm \sqrt{6}, 0), (0, -1)$



35. $y = (x - 1)^2$ **37.** $y = -(x + 1)^2 + 4$

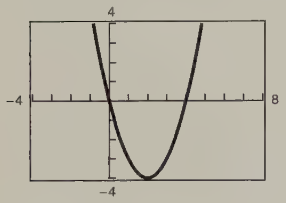
39. $f(x) = (x + 2)^2 + 5$ **41.** $f(x) = -\frac{1}{2}(x - 3)^2 + 4$

43. $f(x) = 2(x + 2)^2 - 2$ **45.** $f(x) = \frac{19}{81}\left(x - \frac{5}{2}\right)^2 - \frac{3}{4}$

47. $(\pm 4, 0)$; They are the same.

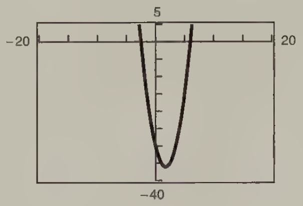
49. $(5, 0), (-1, 0)$; They are the same.

51.



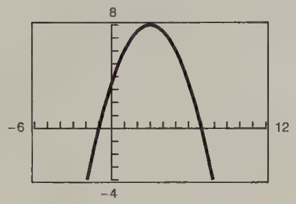
$(0, 0), (4, 0)$;
They are the same.

53.



$\left(-\frac{5}{2}, 0\right), (6, 0)$;
They are the same.

55.



$(7, 0), (-1, 0)$; They are the same.

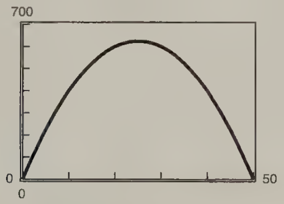
57. $f(x) = x^2 - 2x - 3$
$g(x) = -x^2 + 2x + 3$

59. $f(x) = 2x^2 + 7x + 3$
$g(x) = -2x^2 - 7x - 3$

61. 55, 55 **63.** 12, 6

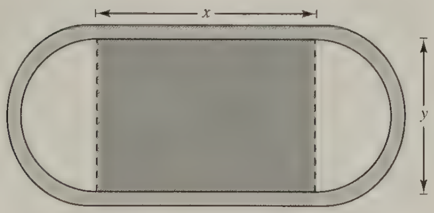
65. (a) $A = x(50 - x),\ 0 < x < 50$

(b)



(c) 25 feet × 25 feet

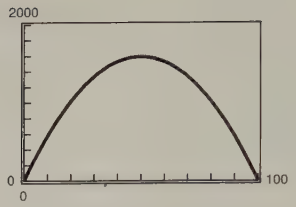
67. (a)



(b) $r = \frac{1}{2}y$; $d = y\pi$ (c) $y = \frac{200 - 2x}{\pi}$

(d) $A = x\left(\frac{200 - 2x}{\pi}\right)$

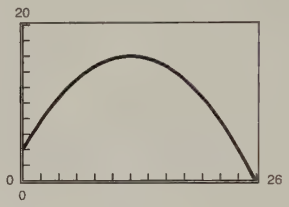
(e)



$x = 50$ meters, $y = \frac{100}{\pi}$ meters

69. 20 **71.** 350,000 units

73. (a)

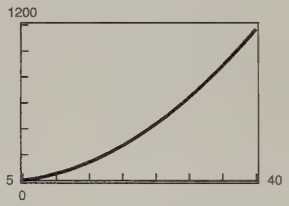


(b) 4 feet

(c) 16 feet

(d) 25.86 feet

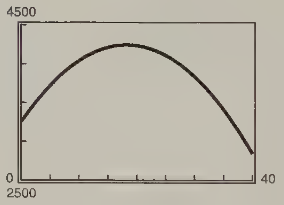
75. (a)



(b) 166.69 board feet

(c) 26.59 inches

77. (a)



(b) 4242 cigarettes per person. Answers will vary.

(c) 9703 cigarettes per smoker per year; 27 cigarettes per smoker per day

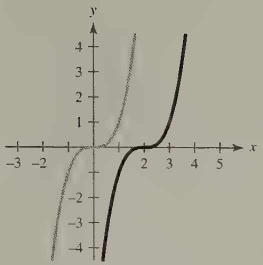
79. True **81.** (a). The profits are positive and rising.

83. (4, 2) **85.** (2, 11) **87.** $20i$ **89.** $52 - 47i$

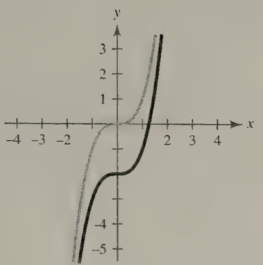
Section 3.2 *(page 246)*

1. (f) **3.** (c) **5.** (e) **7.** (g)

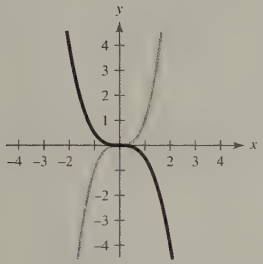
9. (a)



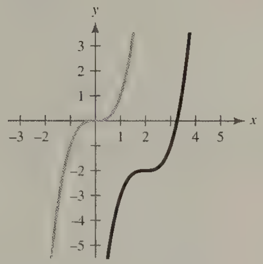
(b)



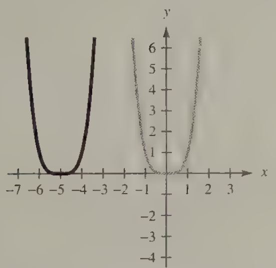
(c)



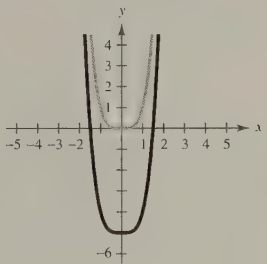
(d)



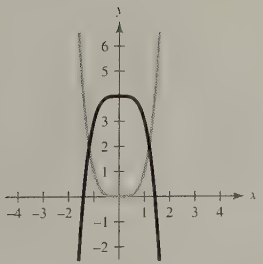
11. (a)



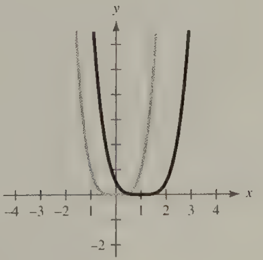
(b)



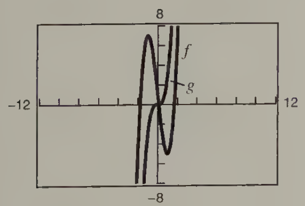
(c)



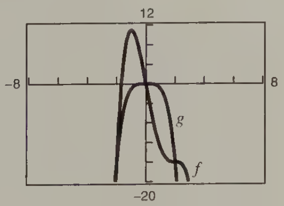
(d)



13.



15.



17. Rises to the left.
Rises to the right.

19. Falls to the left.
Falls to the right.

21. Rises to the left.
Falls to the right.

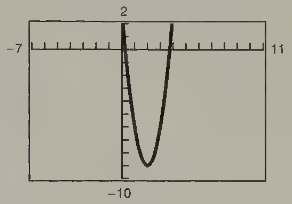
23. Rises to the left.
Falls to the right.

25. Falls to the left.
Falls to the right.

27. ±5 **29.** 3

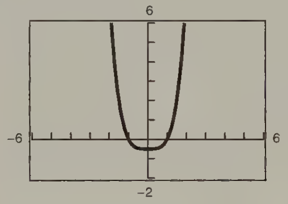
31. 1, −2 **33.** 2, 0 **35.** $\dfrac{-5 \pm \sqrt{37}}{2}$

37. (a)



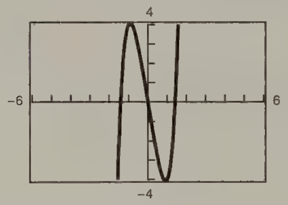
(b) Answers will vary. (c) $(2 - \sqrt{3}, 0), (2 + \sqrt{3}, 0)$

39. (a)



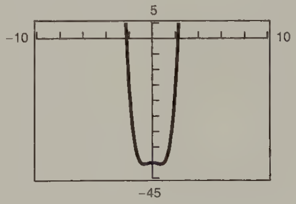
(b) Answers will vary. (c) $(-1, 0), (1, 0)$

41. (a)



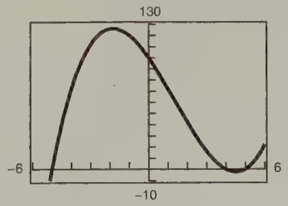
(b) Answers will vary. (c) $(0, 0), (-\sqrt{2}, 0), (\sqrt{2}, 0)$

43. (a)



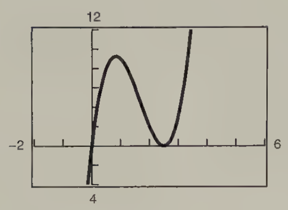
(b) Answers will vary. (c) $(-\sqrt{5}, 0), (\sqrt{5}, 0)$

45. (a)



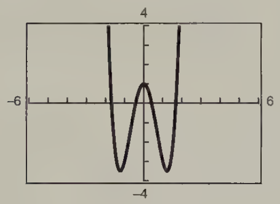
(b) Answers will vary. (c) $(4, 0), (-5, 0), (5, 0)$

47. (a)



(b) Answers will vary. (c) $(0, 0), \left(\frac{5}{2}, 0\right)$

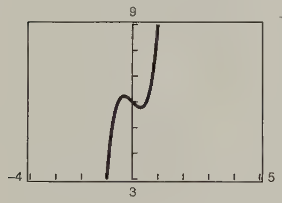
49.



Relative maximum: $(0, 1)$

Relative minimums: $(\pm 1.225, -3.500)$

51.



Relative maximum: $(-0.324, 6.218)$

Relative minimum: $(0.324, 5.782)$

53. $f(x) = x^2 - 12x$ **55.** $f(x) = x^2 + 4x - 12$

57. $f(x) = x^3 + 7x^2 + 12x$

59. $f(x) = x^4 - 4x^3 - 9x^2 + 36x$

61. $f(x) = x^2 - 2x - 2$

63. $f(x) = x^3 - 10x^2 + 27x - 22$

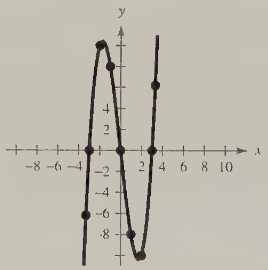
65. (a) Falls to the left, rises to the right

(b) $(0, 0), (3, 0), (-3, 0)$

(c)

x	-3.3	-2	-1	1	2	3.3
y	-6.2	10	8	-8	-10	6.2

(d)



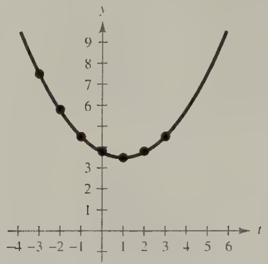
67. (a) Rises to the left and the right

(b) No zeros

(c)

t	-3	-2	-1	0	1	2	3
y	7.5	5.8	4.5	3.8	3.5	3.8	4.5

(d)



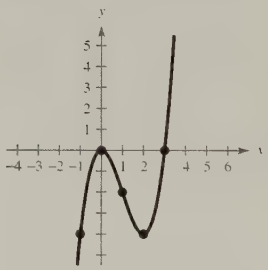
69. (a) Falls to the left, rises to the right

(b) $(0, 0), (3, 0)$

(c)

x	-2.5	-1	1	2	4
y	-34.4	-4	-2	-4	16

(d)



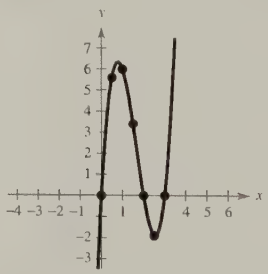
71. (a) Falls to the left, rises to the right

(b) $(0, 0), (2, 0), (3, 0)$

(c)

x	-0.5	0.5	1	1.5	2.5	3.5
y	-13	5.6	6	3.4	-1.9	7.9

(d)



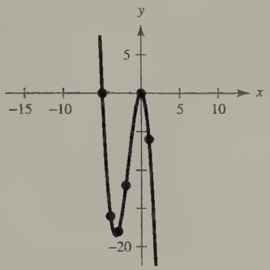
73. (a) Rises to the left, falls to the right

(b) (0, 0), (−5, 0)

(c)

x	−6	−4	−3	−2	1	2
y	36	−16	−18	−12	−6	−28

(d)



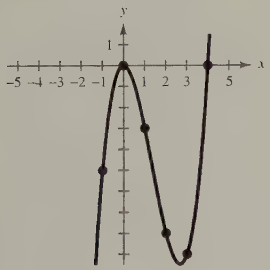
75. (a) Falls to the left, rises to the right

(b) (0, 0), (4, 0)

(c)

x	−2	−1	1	2	3	5
y	−24	−5	−3	−8	−9	25

(d)



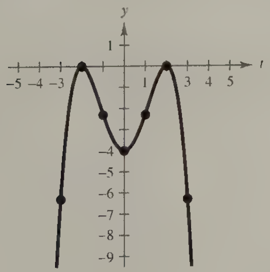
77. (a) Falls to the left and right

(b) (−2, 0), (2, 0)

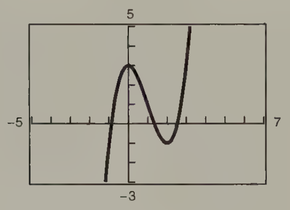
(c)

t	−4	−3	−1	0	1	3
y	−36	−6.3	−2.3	−4	−2.3	−6.3

(d)



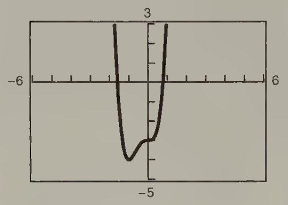
79. (a)



(−1, 0), (1, 2), (2, 3)

(b) −0.879, 1.347, 2.532

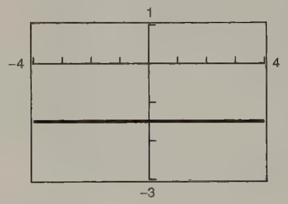
81. (a)



(−2, −1), (0, 1)

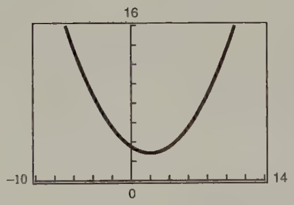
(b) −1.585, 0.779

83.



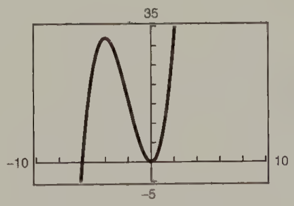
Xmin = -4
Xmax = 4
Xscl = 1
Ymin = -3
Ymax = 1
Yscl = 1

85.



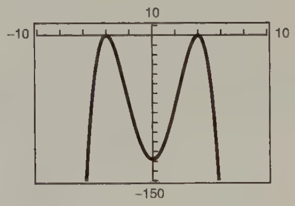
Xmin = -10
Xmax = 14
Xscl = 2
Ymin = 0
Ymax = 16
Yscl = 2

87.



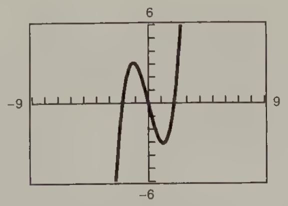
Two x-intercepts

89.



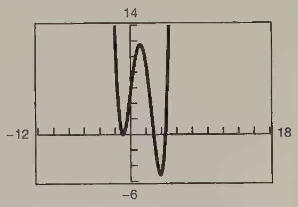
Symmetric to the y-axis
Two x-intercepts

91.



Symmetric to the origin
Three x-intercepts

93.



Three x-intercepts

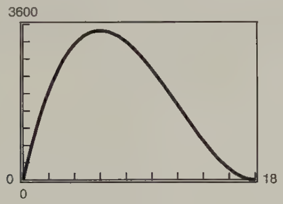
95. (a) and (b)

Height	*Width*	*Volume*
1	$36 - 2(1)$	$1[36 - 2(1)]^2 = 1156$
2	$36 - 2(2)$	$2[36 - 2(2)]^2 = 2048$
3	$36 - 2(3)$	$3[36 - 2(3)]^2 = 2700$
4	$36 - 2(4)$	$4[36 - 2(4)]^2 = 3136$
5	$36 - 2(5)$	$5[36 - 2(5)]^2 = 3380$
6	$36 - 2(6)$	$6[36 - 2(6)]^2 = 3456$
7	$36 - 2(7)$	$7[36 - 2(7)]^2 = 3388$

$5 < x < 7$

(c) Answers will vary. Domain: $0 < x < 18$

(d)

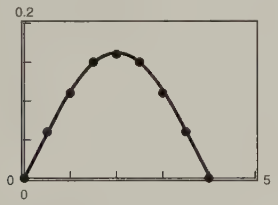


$x = 6$

97. (200, 160)

99. (a) $0.002983x^4 - 0.02386x^3 + 0.0194x^2 + 0.113x$

(b)

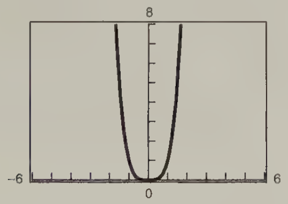


(c) 0. Yes

The model fits well.

101. False. It can have at most three turning points.

103.



(a) Vertical shift of two units; even

(b) Horizontal shift of two units; neither even nor odd

(c) Reflection in the y-axis; even

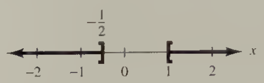
(d) Reflection in the x-axis; even

(e) Vertical shrink; even (f) Vertical shrink; even

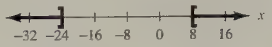
(g) $g(x) = x^3$; odd (h) $g(x) = x^{16}$; even

105. 33 **107.** $-\frac{4}{3} \approx -1.3$ **109.** 72

111. $x \leq -\frac{1}{2}$, $x \geq 1$



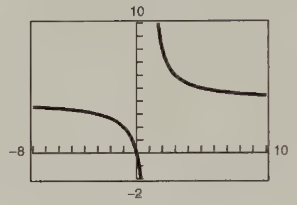
113. $x \leq -24$, $x \geq 8$



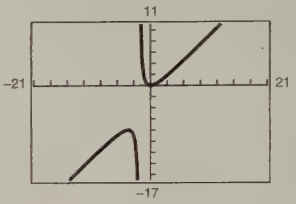
115. $f(x) = \frac{17}{25}x^2 - 8$ **117.** $f(x) = \frac{1}{5}(x + 5)^2 - 2$

Section 3.3 *(page 260)*

1.



3.



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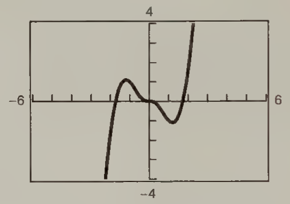


7. $2x + 4$ **9.** $x^2 - 3x + 1$ **11.** $7 - \dfrac{11}{x + 2}$

13. $3x + 5 - \dfrac{2x - 3}{2x^2 + 1}$ **15.** $x^2 + 2x + 4 + \dfrac{2x - 11}{x^2 - 2x + 3}$

17. $2x - \dfrac{17x - 5}{x^2 - 2x + 1}$ **19.** $3x^2 + 2x + 20 + \dfrac{58}{x - 4}$

21. $6x^2 + 25x + 74 + \dfrac{248}{x - 3}$ **23.** $9x^2 - 16$

25. $x^2 - 8x + 64$ **27.** $4x^2 + 14x - 30$

29. $f(x) = (x - 4)(x^2 + 3x - 2) + 3$, $f(4) = 3$

31. $f(x) = (x - \sqrt{2})[x^2 + (3 + \sqrt{2})x + 3\sqrt{2}] - 8$,
$f(\sqrt{2}) = -8$

33. $f(x) = (x - 1 + \sqrt{3})[4x^2 - (2 + 4\sqrt{3})x - (2 + 2\sqrt{3})]$,
$f(1 - \sqrt{3}) = 0$

35. (a) 1 (b) 4 (c) 4 (d) 1954

37. (a) 97 (b) $-\frac{5}{3}$ (c) 17 (d) −199

39. $(x - 2)(x + 3)(x - 1)$
Zeros: $2, -3, 1$

41. $(2x - 1)(x - 5)(x - 2)$
Zeros: $\frac{1}{2}, 5, 2$

43. $(x - \sqrt{2})(x + \sqrt{2})(x + 2)$
Zeros: $\pm\sqrt{2}, -2$

45. (a) Answers will vary. (b) $(2x - 1)$
(c) $(x + 2)(x - 1)(2x - 1)$ (d) $-2, 1, \frac{1}{2}$

47. (a) Answers will vary. (b) $(x-1), (x-2)$
(c) $(x-5)(x+4)(x-1)(x-2)$ (d) $-4, 1, 2, 5$

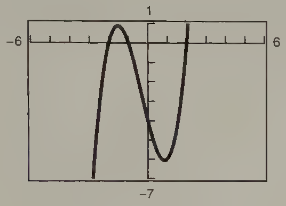
49. (a) Answers will vary. (b) $(x+7)$
(c) $(2x+1)(3x-2)(x+7)$ (d) $-7, -\frac{1}{2}, \frac{2}{3}$

51. $\pm 1, \pm 3$

53. $\pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{9}{2}, \pm\frac{15}{2}, \pm\frac{45}{2}$

55. (a) $\pm 1, \pm 2, \pm 4$

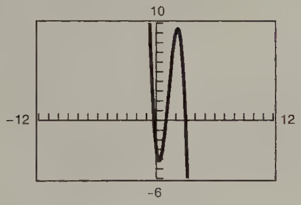
(b)



(c) $-2, -1, 2$

57. (a) $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}$

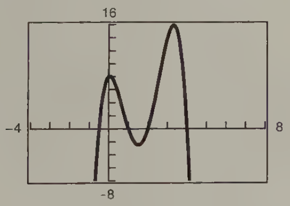
(b)



(c) $-\frac{1}{4}, 1, 3$

59. (a) $\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{2}$

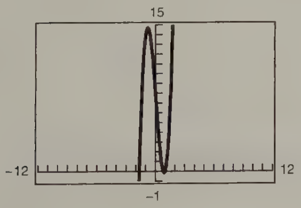
(b)



(c) $-\frac{1}{2}, 1, 2, 4$

61. (a) $\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}, \pm\frac{1}{6}$

(b)



(c) $1, \dfrac{-5 \pm \sqrt{217}}{12}$

63. $-1, 2$ **65.** $0, -1, -3, 4$ **67.** $-2, -\frac{1}{2}, 4$

69. (a) $-2.236, 2, 2.236$ (b) 2
(c) $f(x) = (x-2)(x-\sqrt{5})(x+\sqrt{5})$

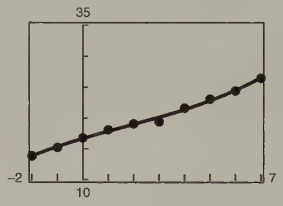
71. (a) $-2, 0.268, 3.732$ (b) -2
(c) $h(t) = (t+2)(t-2+\sqrt{3})(t-2-\sqrt{3})$

73. (a) $0, 3, 4, -1.414, 1.414$ (b) $0, 3, 4$
(c) $h(x) = x(x-3)(x-4)(x+\sqrt{2})(x-\sqrt{2})$

75. and 77. Answers will vary. **79.** $\pm 2, \pm\frac{3}{2}$ **81.** $\pm 1, \frac{1}{4}$

83. (d) **85.** (b)

87. (a) and (b)



$R = 0.01326t^3 - 0.0677t^2 + 1.231t + 16.68$

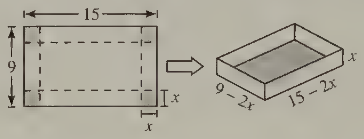
The model is a good fit.

(c)

t	-2	-1	0	1	2	3	4	5	6	7
R	13.84	15.37	16.68	17.86	18.98	20.12	21.37	22.8	24.49	26.53

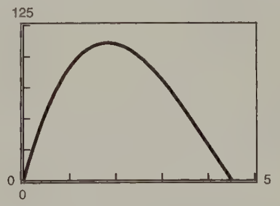
(d) $R(12) = 44.62$. No; the model will turn sharply upward.

89. (a)



(b) $V = x(9-2x)(15-2x)$; Domain: $0 < x < \frac{9}{2}$

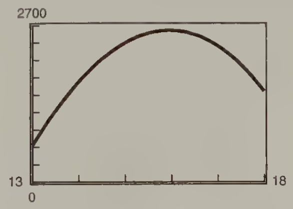
(c)



$1.82 \times 5.36 \times 11.36$

(d) $\frac{1}{2}, \frac{7}{2}, 8$; 8 is not in the domain of V.

91. (a)



(b) 16.89
(c) 16.89

93. False. $-\frac{4}{7}$ is the zero derived from $(7x+4)$.

95. True **97.** $x^{2n} - x^n + 3$

99. Multiply the divisor and the quotient to obtain the dividend. Answers will vary.

101. $3 - 10i$ **103.** $\frac{1}{17}(10 + 23i)$

105. $f(x) = x^3 - 6x^2 - 19x + 24$

107. $f(x) = x^2 - 4x + 1$

Section 3.4 *(page 269)*

1. $0, 6, 6$ **3.** $2, -4, -4, -4$ **5.** $-6, i, -i$

7. $2, -3 + 5i, -3 - 5i$

9. Zeros: $4, -i, i$. One real zero; same

11. Zeros: $\sqrt{2}i, \sqrt{2}i, -\sqrt{2}i, -\sqrt{2}i$. No real zeros; same

13. $2 \pm \sqrt{3}$
$(x - 2 - \sqrt{3})(x - 2 + \sqrt{3})$

15. $6 \pm \sqrt{10}$
$(x - 6 - \sqrt{10})(x - 6 + \sqrt{10})$

17. $\pm 5i$
$(x + 5i)(x - 5i)$

19. $\pm 3, \pm 3i$
$(x + 3)(x - 3)(x + 3i)(x - 3i)$

21. $1 \pm i$
$(z - 1 + i)(z - 1 - i)$

23. $-5, 4 \pm 3i$
$(t + 5)(t - 4 + 3i)(t - 4 - 3i)$

25. $-\frac{3}{4}, 1 \pm \frac{1}{2}i$
$(4x + 3)(x - 1 - \frac{1}{2}i)(x - 1 + \frac{1}{2}i)$

27. $\pm i, \pm 3i$
$(x + i)(x - i)(x + 3i)(x - 3i)$

29. $2, 2, \pm 2i$
$(x - 2)^2(x + 2i)(x - 2i)$

31. $-2, -\frac{1}{2}, \pm i$
$(x + 2)(2x + 1)(x - i)(x + i)$

33. (a) $7 \pm \sqrt{3}$ (b) $(x - 7 - \sqrt{3})(x - 7 + \sqrt{3})$
(c) $(7 \pm \sqrt{3}, 0)$

35. (a) $-7 \pm \sqrt{5}$ (b) $(x + 7 - \sqrt{5})(x + 7 + \sqrt{5})$
(c) $(-7 \pm \sqrt{5}, 0)$

37. (a) $-6, 3 \pm 4i$ (b) $(x + 6)(x - 3 - 4i)(x - 3 + 4i)$
(c) $(-6, 0)$

39. (a) $\pm 4i, \pm 3i$ (b) $(x + 4i)(x - 4i)(x + 3i)(x - 3i)$
(c) None

41. $x^3 - x^2 + 25x - 25$ **43.** $x^3 - 10x^2 + 33x - 34$

45. $x^4 + 37x^2 + 36$ **47.** $x^4 + 8x^3 + 9x^2 - 10x + 100$

49. (a) $(x^2 + 1)(x^2 - 7)$ (b) $(x^2 + 1)(x + \sqrt{7})(x - \sqrt{7})$
(c) $(x + i)(x - i)(x + \sqrt{7})(x - \sqrt{7})$

51. (a) $(x^2 - 6)(x^2 - 2x + 3)$
(b) $(x + \sqrt{6})(x - \sqrt{6})(x^2 - 2x + 3)$
(c) $(x + \sqrt{6})(x - \sqrt{6})(x - 1 - \sqrt{2}i)(x - 1 + \sqrt{2}i)$

53. $-\frac{3}{2}, \pm 5i$ **55.** $-3, 5 \pm 2i$ **57.** $-\frac{2}{3}, 1 \pm \sqrt{3}i$

59. $\frac{3}{4}, \frac{1}{2}(1 \pm \sqrt{5}i)$

61. (a) 1.000, 2.000 (b) 1, 2 (c) $1, 2, -3 \pm \sqrt{2}i$

63. (a) 0.750 (b) $\frac{3}{4}$ (c) $\frac{3}{4}, \frac{1}{2} \pm \frac{\sqrt{5}}{2}i$

65. No. Setting $h = 64$ and solving the resulting equation yields imaginary roots.

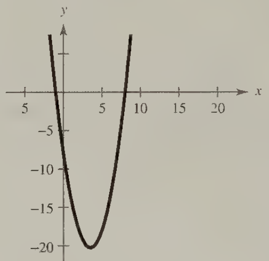
67. False. A polynomial can only have an even number of complex zeros, so one of the zeros of a third-degree polynomial must be real.

69. (a) $0 < k < 4$ (b) $k = 4$ (c) $k < 0$ (d) $k > 4$

71. $f(x) = x^2 + b$ **73.** $\frac{x - 5}{x + 3}, x \neq 3$

75. $x - 5, x \neq 0, 4$

77.

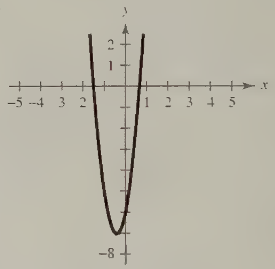


Vertex: $(3.5, -20.25)$

Intercepts:

$(-1, 0), (8, 0), (0, -8)$

79.



Vertex: $\left(-\frac{5}{12}, -7\frac{1}{24}\right)$

Intercepts:

$\left(-\frac{3}{2}, 0\right), \left(\frac{2}{3}, 0\right), (0, -6)$

Section 3.5 *(page 277)*

1. (a)

x	$f(x)$
0.5	-2
0.9	-10
0.99	-100
0.999	-1000

x	$f(x)$
1.5	2
1.1	10
1.01	100
1.001	1000

x	$f(x)$
5	0.25
10	$0.\overline{1}$
100	$0.\overline{01}$
1000	$0.\overline{001}$

x	$f(x)$
-5	$-0.\overline{16}$
-10	$-0.\overline{09}$
-100	$-0.\overline{0099}$
-1000	$-0.\overline{000999}$

(b) Vertical asymptote: $x = 1$
Horizontal asymptote: $y = 0$

(c) Domain: all $x \neq 1$

3. (a)

x	$f(x)$
0.5	3
0.9	27
0.99	297
0.999	2997

x	$f(x)$
1.5	9
1.1	33
1.01	303
1.001	3003

x	$f(x)$
5	3.75
10	$3.\overline{33}$
100	$3.\overline{03}$
1000	$3.\overline{003}$

x	$f(x)$
−5	−2.5
−10	−2.727
−100	−2.97
−1000	−2.997

(b) Vertical asymptote: $x = 1$

Horizontal asymptotes: $y = \pm 3$

(c) Domain: all $x \neq 1$

5. (a)

x	$f(x)$
0.5	−1
0.9	−12.79
0.99	−147.8
0.999	−1498

x	$f(x)$
1.5	5.4
1.1	17.29
1.01	152.3
1.001	1502.3

x	$f(x)$
5	3.125
10	$3.\overline{03}$
100	$3.\overline{0003}$
1000	3

x	$f(x)$
−5	3.125
−10	$3.\overline{03}$
−100	$3.\overline{0003}$
−1000	3.000003

(b) Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: $y = 3$

(c) Domain: all $x \neq \pm 1$

7. (a) **9.** (c) **11.** (b)

13. (a) Domain: all $x \neq 0$

(b) Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$

15. (a) Domain: all $x \neq 3$

(b) Vertical asymptote: $x = 3$

Horizontal asymptote: $y = -1$

17. (a) Domain: all $x \neq \pm 1$

(b) Vertical asymptotes: $x = \pm 1$

19. (a) Domain of f: all $x \neq -2$

Domain of g: all real numbers

(b) Vertical asymptote: None

(c)

x	−4	−3	−2.5	−2	−1.5	−1	0
$f(x)$	−6	−5	−4.5	Undef.	−3.5	−3	−2
$g(x)$	−6	−5	−4.5	−4	−3.5	−3	−2

(d) Values differ only where f is undefined.

21. (a) Domain of f: all $x \neq 0, 3$; Domain of g: all $x \neq 0$

(b) Vertical asymptote: $x = 0$

(c)

x	−1	−0.5	0	0.5	2	3	4
$f(x)$	−1	−2	Undef.	2	$\frac{1}{2}$	Undef.	$\frac{1}{4}$
$g(x)$	−1	−2	Undef.	2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

(d) Values differ only where f is undefined and g is defined.

23. 4; Less than; Greater than

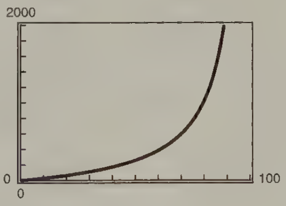
25. 2; Greater than; Less than

27. ± 3 **29.** 7

31. (a) \$28.33 million (b) \$170 million

(c) \$765 million

(d)



Answers will vary.

(e) No. The function is undefined at the 100% level.

33. (a)

M	200	400	600	800	1000
t	0.472	0.596	0.710	0.817	0.916

M	1200	1400	1600	1800	2000
t	1.009	1.096	1.178	1.255	1.328

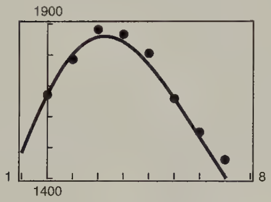
The greater the mass, the more time required per oscillation.

(b) $M \approx 1306$ grams

35. (a) 333 deer, 500 deer, 800 deer

(b) 1500. Because the degrees of the numerator and the denominator are equal, the limiting size is the ratio of the leading coefficients, 60/0.4 = 1500.

37. (a)



(b) $\approx 889{,}000$

(c) Not likely. The graph of the model dips quite low in the future.

39. True. Can be written as

$$\frac{N(x)}{D(x)} = \frac{x^3 - 2x^2 - 5x + 6}{1}.$$

41. $f(x) = \dfrac{1}{1 + x^2}$ **43.** $f(x) = \dfrac{-3(x+1)^2}{x\left(x - \frac{5}{2}\right)}$

45. $x = 5$ **47.** $t = 0, \pm 5\sqrt{2}$

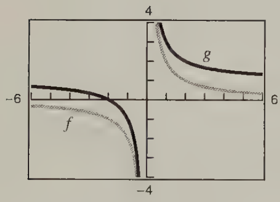
49. $x = \pm\sqrt{15}, \pm\sqrt{15}\,i$ **51.** $x - 7 - \dfrac{6}{x - 3}$

53. $4x - 21 + \dfrac{116}{x + 6}$ **55.** $x^3 + 2x^2 + 36x + 72$

57. $x^3 + 5x^2 + 7x - 13$

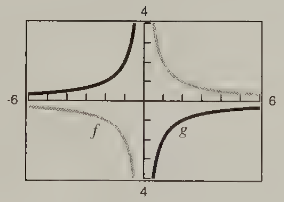
Section 3.6 *(page 286)*

1.



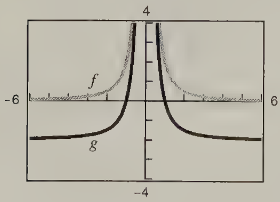
Vertical shift

3.



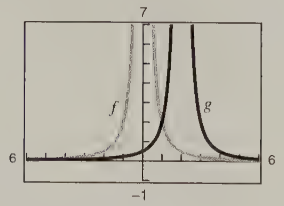
Reflection in the x-axis

5.



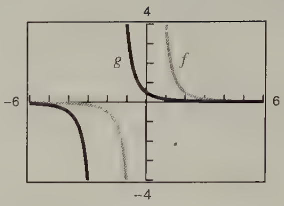
Vertical shift

7.



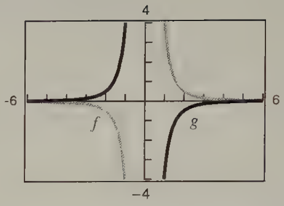
Horizontal shift

9.



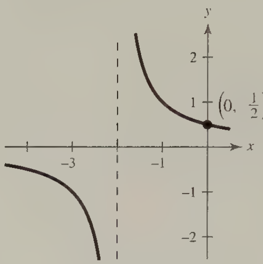
Horizontal shift

11.



Reflection in the x-axis

13.



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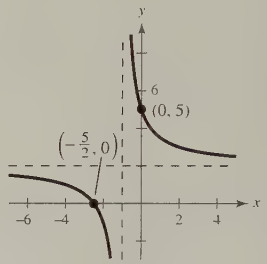

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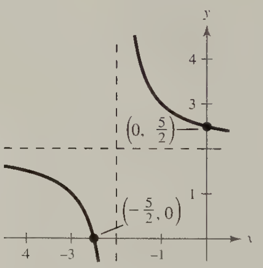

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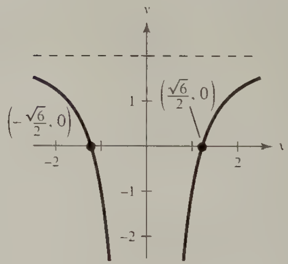

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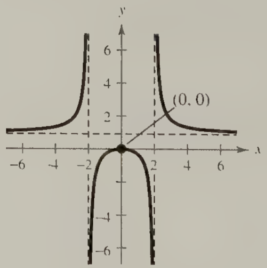

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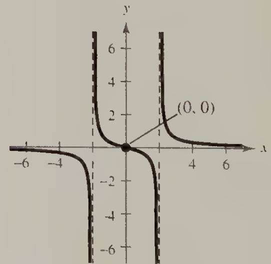

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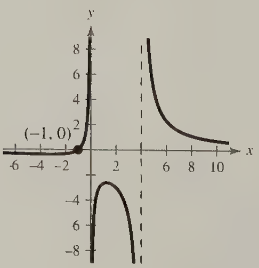

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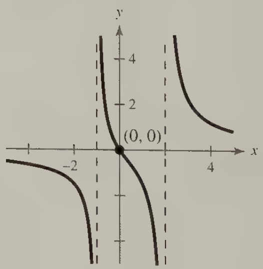

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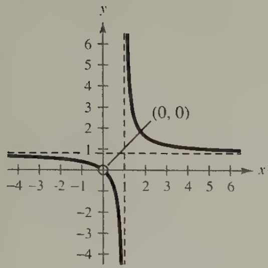
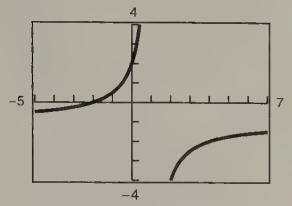


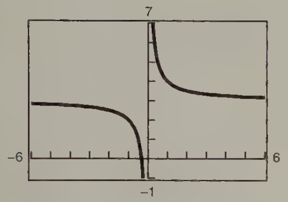
31.



Domain: $(-\infty, 1), (1, \infty)$

$x = 1, y = -1$

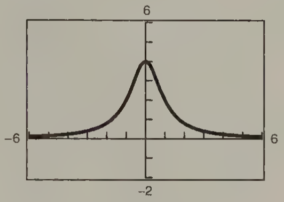
33.



Domain: $(-\infty, 0), (0, \infty)$

$x = 0, y = 3$

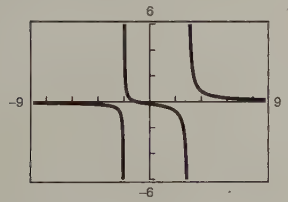
35.



Domain: $(-\infty, \infty)$

$y = 0$

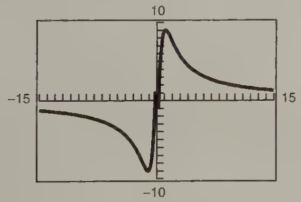
37.



Domain: $(-\infty, -2), (-2, 3), (3, \infty)$

$x = -2, x = 3, y = 0$

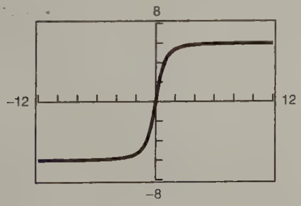
39.



Domain: $(-\infty, 0), (0, \infty)$

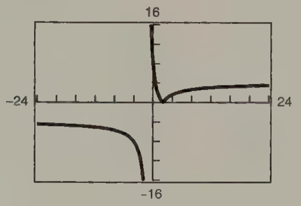
$x = 0, y = 0$

41.



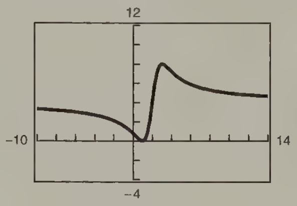
There are two horizontal asymptotes, $y = 6, y = -6$.

43.



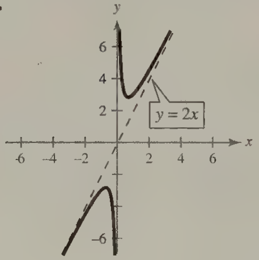
There are two horizontal asymptotes, $y = 4$ and $y = -4$, and one vertical asymptote, $x = -1$.

45.

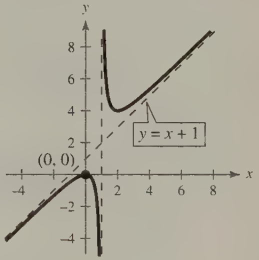


The graph crosses the horizontal asymptote, $y = 4$.

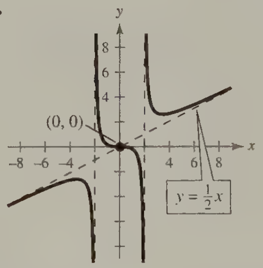
47.



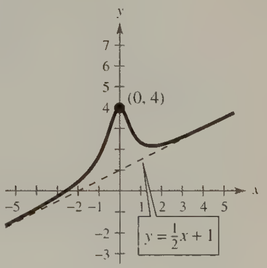
49.



51.

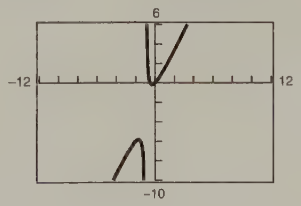


53.



55. $(-1, 0)$ **57.** $(1, 0), (-1, 0)$

59.

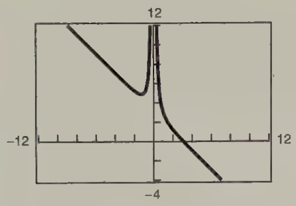


Domain: $(-\infty, -1), (-1, \infty)$

Vertical asymptote: $x = -1$

$y = 2x - 1$

61.

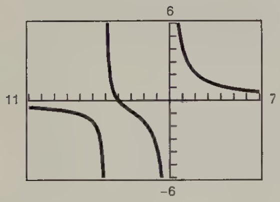


Domain: $(-\infty, 0), (0, \infty)$

Vertical asymptote: $x = 0$

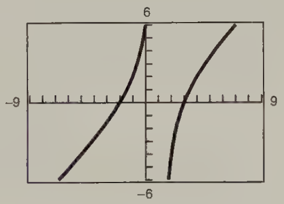
$y = -x + 3$

63.



$(-4, 0)$

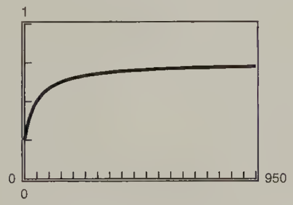
65.



$(3, 0), (-2, 0)$

67. (a) Answers will vary. (b) $[0, 990]$

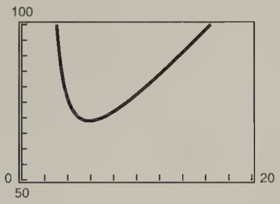
(c)



Increases more slowly; the concentration reaches 74.5% when the tank is full.

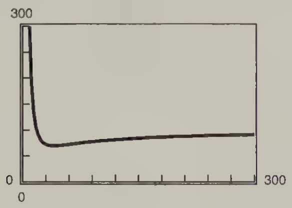
69. (a) Answers will vary. (b) $(2, \infty)$

(c)



5.9×11.8 inches

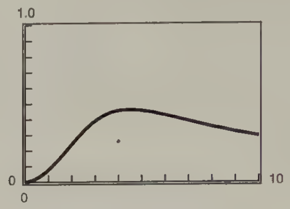
71.



$x \approx 40$

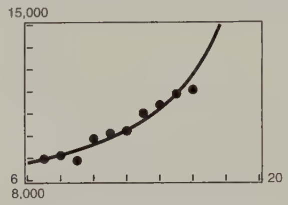
73. (a) $C = 0$. The chemical will eventually dissipate.

(b) $t \approx 4.5$ hours

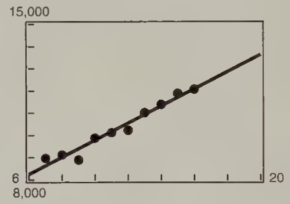


(c) Before ≈ 2.6 hours and after ≈ 8.3 hours

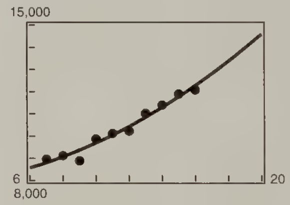
75. (a)



(b) $y = 384.5x + 5938$



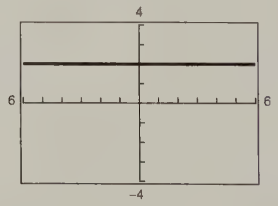
(c) $y = 14.87x^2 + 42.5x + 7781$



(d) Answers will vary. The rational model seems to start climbing too fast. The quadratic model may fit the data a little better than the linear model, is easy to use, and would be most useful.

77. False. A graph with a vertical asymptote is not continuous.

79.



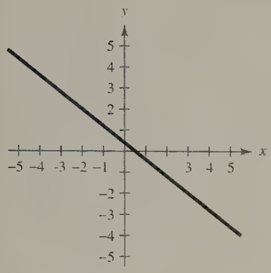
The denominator is a factor of the numerator.

81. No. (Discussions will vary.)

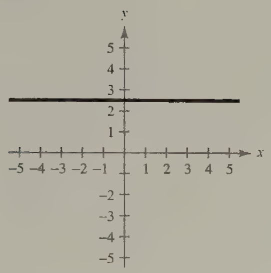
83. $f(x) = \dfrac{x^2 + 2x - 15}{x + 4}$ **85.** $f(x) = \dfrac{-2x - 12}{x - 3}$

87. $\dfrac{1}{16x^4}$ **89.** $\dfrac{1}{x^2}$ **91.** $\dfrac{1}{x^3}$

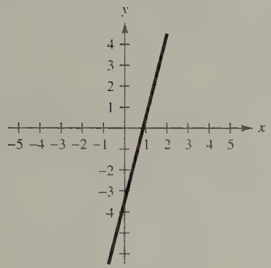
93.



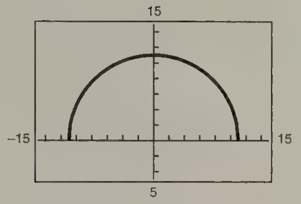
95.



97.



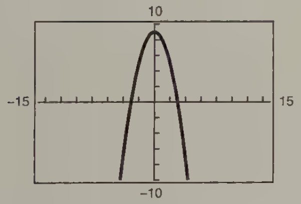
99.



Domain: $-11 \le x \le 11$

Range: $0 \le y \le 11$

101.

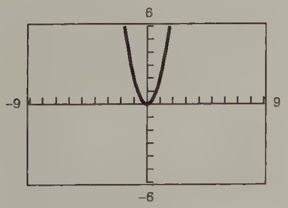


Domain: $(-\infty, \infty)$

Range: $y \le 9$

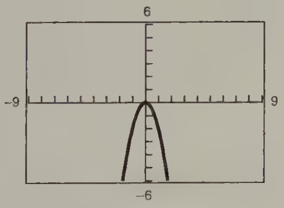
Review Exercises *(page 291)*

1. (a)



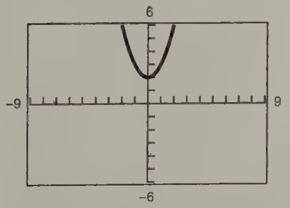
Vertical stretch

(b)



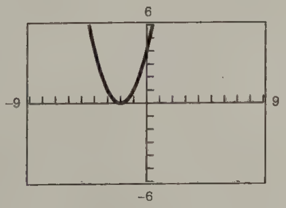
Vertical stretch and reflection in the x-axis

(c)



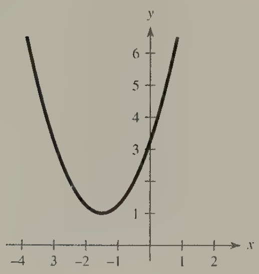
Vertical shift

(d)



Horizontal shift

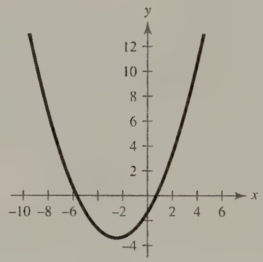
3.



Vertex: $\left(-\frac{3}{2}, 1\right)$

Intercept: $\left(0, \frac{13}{4}\right)$

5.



Vertex: $\left(-\dfrac{5}{2}, -\dfrac{41}{12}\right)$

Intercepts:

$\left(0, -\dfrac{4}{3}\right), \left(\dfrac{-5 \pm \sqrt{41}}{2}, 0\right)$

7. $f(x) = (x - 1)^2 - 4$ **9.** Minimum: $(1, -1)$

11. Maximum: $(3, 9)$ **13.** Maximum: $(1, 3)$

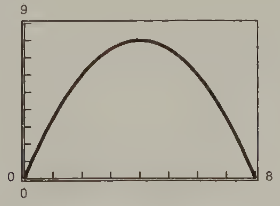
15. Minimum: $\left(-\frac{5}{2}, -\frac{41}{4}\right)$

17. (a)

x	y	Area
1	$4 - \frac{1}{2}(1)$	$(1)\left[4 - \frac{1}{2}(1)\right] = \frac{7}{2}$
2	$4 - \frac{1}{2}(2)$	$(2)\left[4 - \frac{1}{2}(2)\right] = 6$
3	$4 - \frac{1}{2}(3)$	$(3)\left[4 - \frac{1}{2}(3)\right] = \frac{15}{2}$
4	$4 - \frac{1}{2}(4)$	$(4)\left[4 - \frac{1}{2}(4)\right] = 8$
5	$4 - \frac{1}{2}(5)$	$(5)\left[4 - \frac{1}{2}(5)\right] = \frac{15}{2}$
6	$4 - \frac{1}{2}(6)$	$(6)\left[4 - \frac{1}{2}(6)\right] = 6$

(b) $x = 4, y = 2$ (c) $A = x\left(\dfrac{8 - x}{2}\right), \ 0 < x < 8$

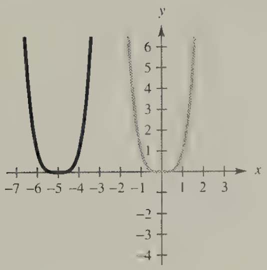
(d)



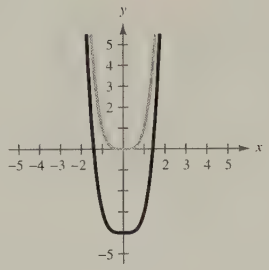
$x = 4, y = 2$

(e) $A = -\frac{1}{2}(x - 4)^2 + 8$

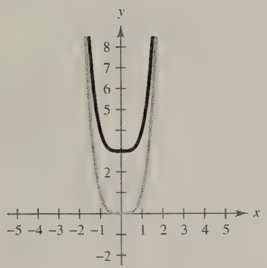
19. (a)



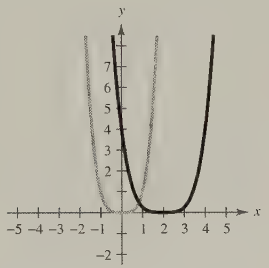
(b)



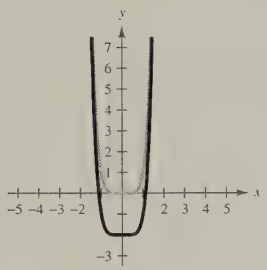
(c)



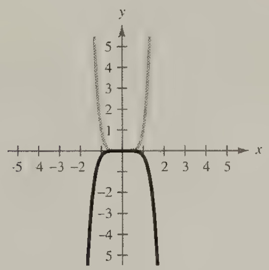
(d)



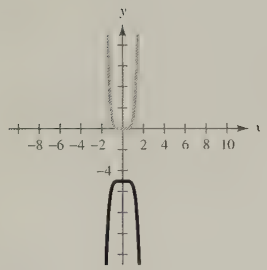
21. (a)



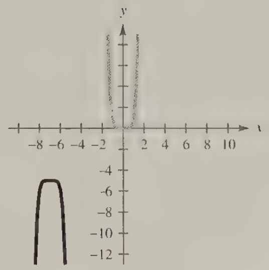
(b)



(c)



(d)



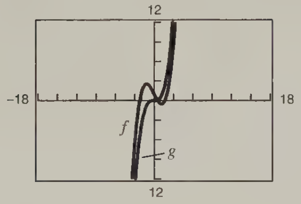
23. Falls to the left.

Falls to the right.

25. Rises to the left.

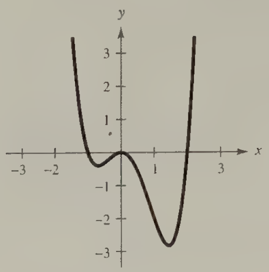
Rises to the right.

27.



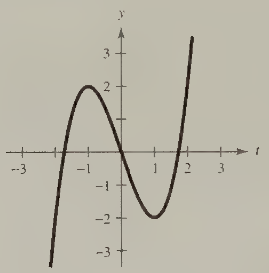
29. (a) $x = -1, 0, 2$

(b)



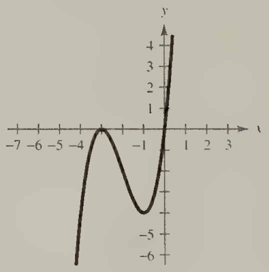
31. (a) $t = 0, \pm\sqrt{3}$

(b)



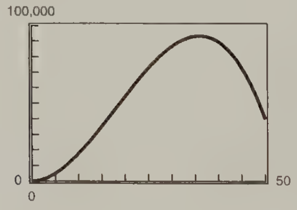
33. (a) $x = -3, 0$

(b)



35. (a) $V = x^2(216 - 4x)$

(b)



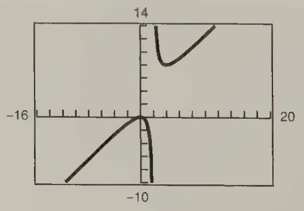
$x = 36$ centimeters, $y = 72$ centimeters

37. (a) $(-3, -2), (-1, 0), (0, 1)$

(b) $(-2.247, 0), (-0.555, 0), (0.802, 0)$

39. (a) $(-3, -2), (2, 3)$ (b) $(-2.570, 0), (2.570, 0)$

41.



43. $8x + 5 + \dfrac{2}{3x - 2}$

45. $x^2 - 2,\ x \neq \pm 1$ **47.** $x^2 - x + 1,\ x \neq 0, -2$

49. $\frac{1}{4}x^3 - \frac{9}{2}x^2 + 9x - 18 + \frac{36}{x + 2}$

51. $6x^3 - 27x,\ x \neq \frac{2}{3}$

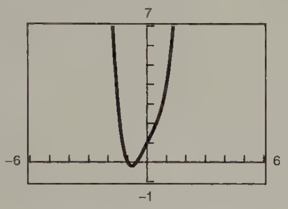
53. (a) No (b) Yes (c) Yes (d) No

55. $6x^4 + 13x^3 + 7x^2 - x - 1$

57. $3x^4 - 14x^3 + 17x^2 - 42x + 24$ **59.** $\frac{3}{4}, 1, 1$

61. $\frac{5}{6}, \pm 2i$ **63.** $-1, \frac{3}{2}, 3, \frac{2}{3}$ **65.** Answers will vary.

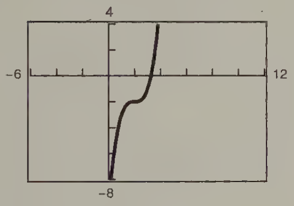
67. (a)



(b) Two

(c) $-1, -0.54$

69. (a)



(b) One

(c) 3.26

71. $2, 1 \pm i$

$(x - 2)(x - 1 - i)(x - 1 + i)$

73. $-4, -1 \pm \sqrt{2}i$

$(x + 4)(x + 1 + \sqrt{2}i)(x + 1 - \sqrt{2}i)$

75. $\pm 3i, \pm 5i$

$(x - 3i)(x + 3i)(x - 5i)(x + 5i)$

77. $f(x) = x^4 + 4x^3 + 29x^2 + 100x + 100$

79. $f(x) = x^4 + 9x^3 + 48x^2 + 78x - 136$

81. $f(x) = 3x^4 - 13x^3 + 5x^2 + 43x + 22$

83. (a) $(x^2 + 4)(x^2 - 2)$

(b) $(x^2 + 4)(x - \sqrt{2})(x + \sqrt{2})$

(c) $(x + 2i)(x - 2i)(x - \sqrt{2})(x + \sqrt{2})$

85. (a) $(x^2 + 9)(x^2 - 2x - 1)$

(b) $(x^2 + 9)(x - 1 + \sqrt{2})(x - 1 - \sqrt{2})$

(c) $(x + 3i)(x - 3i)(x - 1 + \sqrt{2})(x - 1 - \sqrt{2})$

87. Domain: all $x \neq 1$

Vertical asymptote: $x = 1$

Horizontal asymptote: $y = -1$

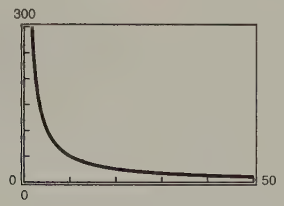
89. Domain: all $x \neq 6, -3$

Vertical asymptotes: $x = 6, x = -3$

Horizontal asymptote: $y = 0$

91. $y = -1$ **93.** $y = 2$ **95.** None **97.** $y = \pm 1$

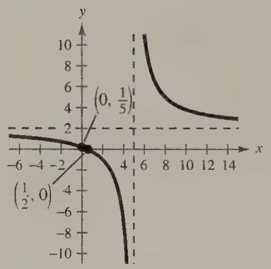
99. (a)



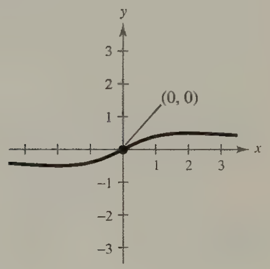
(b) 10.5, 5.5, 1, 0.55

(c) 0.5

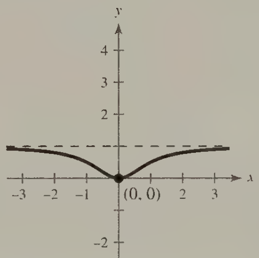
101.



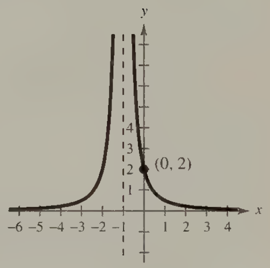
103.



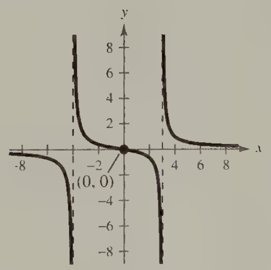
105.



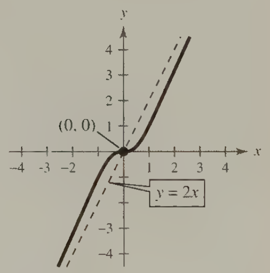
107.



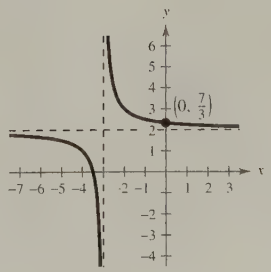
109.



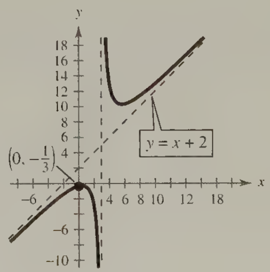
111.



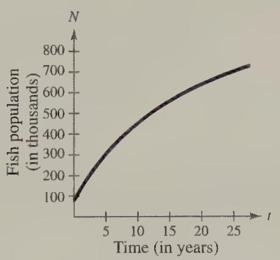
113.



115.



117. (a)



(b) 304,000; 453,333; 702,222

(c) 1,200,000, because N has a horizontal asymptote at $y = 1{,}200{,}000$.

119. False. For the graph of a rational function to have a slant asymptote, the degree of its numerator must be exactly one more than the degree of its denominator.

Chapter Test *(page 296)*

1. (a) Reflection in the x-axis followed by a vertical shift

(b) Horizontal shift

2. Vertex: $(-2, -1)$

Intercepts: $(0, 3), (-3, 0), (-1, 0)$

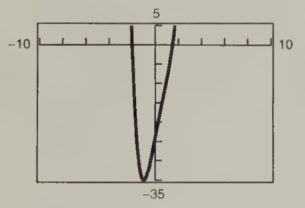
3. $y = (x - 3)^2 - 6$

4. (a) 50 feet

(b) 5. Changing the constant term results in a vertical shift of the graph and therefore changes the maximum height.

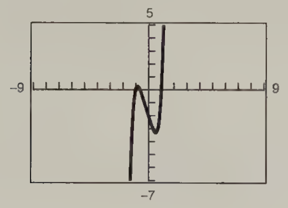
5. $3x + \dfrac{x-1}{x^2+1}$ **6.** $2x^3 + 4x^2 + 3x + 6 + \dfrac{9}{x-2}$

7. $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24, \pm\frac{1}{2}, \pm\frac{3}{2}$



$-2, \frac{3}{2}$

8. $\pm 1, \pm 2, \pm\frac{1}{3}, \pm\frac{2}{3}$



$\pm 1, -\frac{2}{3}$

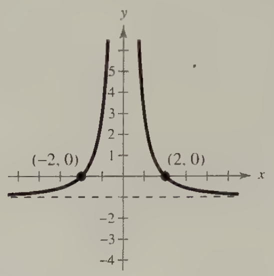
9. $-0.819, 1.380$ **10.** $-1.414, -0.667, 1.414$

11. $f(x) = x^4 - 9x^3 + 28x^2 - 30x$

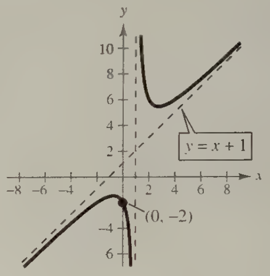
12. $f(x) = x^4 - 6x^3 + 16x^2 - 24x + 16$

13. $f(x) = x^4 + 3x^3 - 8x^2 + 10x$

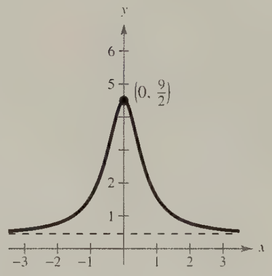
14.



15.



16.



17. $f(x) = \dfrac{4x^2}{x^2 - 9}$ **18.** 24.8 years

Chapter 4

Section 4.1 *(page 307)*

1. 4112.033 **3.** 77,494.076 **5.** 19.568 **7.** 1.649

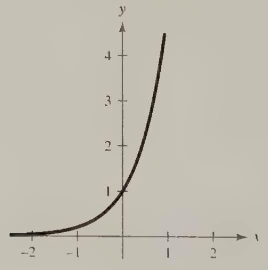
9. 9897.129 **11.** $f(x) = h(x)$

13. $f(x) = g(x) = h(x)$ **15.** (c) **17.** (e) **19.** (g)

21. (a) **23.** Right shift of 5 units

25. Left shift of 4 units and reflection in x-axis

27.

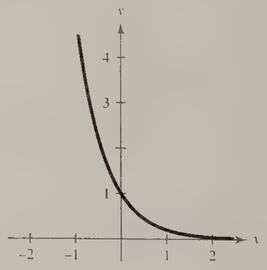


(a) $y = 0$

(b) $(0, 1)$

(c) Increasing

29.

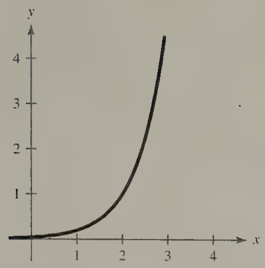


(a) $y = 0$

(b) $(0, 1)$

(c) Decreasing

31.

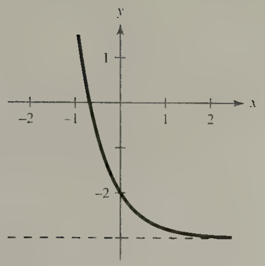


(a) $y = 0$

(b) $\left(0, \frac{1}{25}\right)$

(c) Increasing

33.



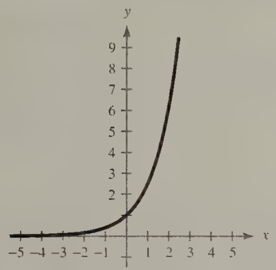
(a) $y = -3$

(b) $(0, -2), (-0.683, 0)$

(c) Decreasing

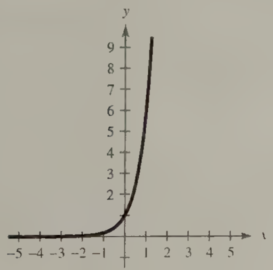
35.

x	-1	0	1	2	3
$f(x)$	0.4	1	2.5	6.3	15.6



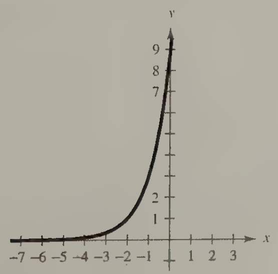
37.

x	-1	0	1	2
$f(x)$	0.2	1	6	36



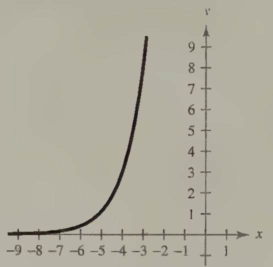
39.

x	-3	-2	0	1
$f(x)$	$\frac{1}{3}$	1	9	27



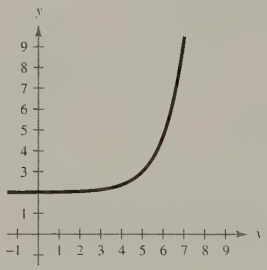
41.

x	-7	-6	-5	-4	-3
$f(x)$	0.1	0.4	1.1	3	8

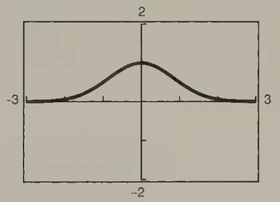


43.

x	2	3	4	5	6	7
$f(x)$	2.0	2.1	2.4	3	4.7	9.3

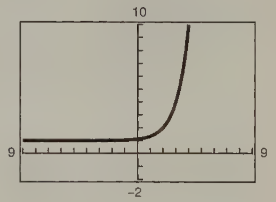


45.



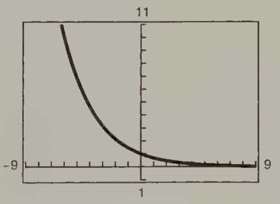
$y = 0$

47.



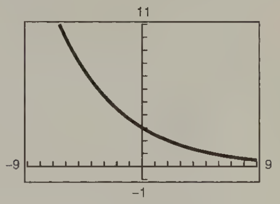
$y = 1$

49.



$y = 0$

51.



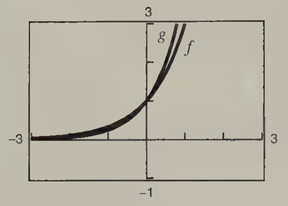
$s(t) = 0$

53. (a)

x	-1	-0.5	0	0.5	1
$f(x)$	0.3333	0.5774	1	1.7321	3
$g(x)$	0.25	0.5	1	2	4

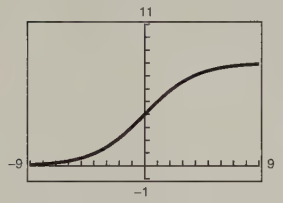
$x < 0$

(b)



(i) $4^x < 3^x$ when $x < 0$ (ii) $4^x > 3^x$ when $x > 0$

55. (a)



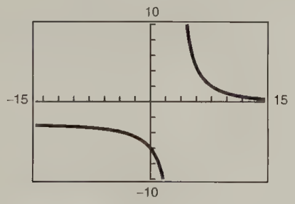
(b)

x	-30	-20	-10	0
$f(x)$	0.0000024	0.00036	0.054	4

x	10	20	30
$f(x)$	7.95	7.9996	7.999998

$y = 0, y = 8$

57. (a)



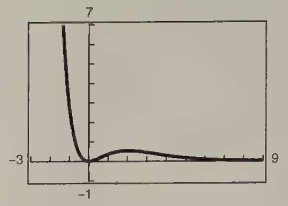
(b)

x	-20	-10	0	3	3.4	3.46
$f(x)$	-3.03	-3.22	-6	-34	-230	-2617

x	3.47	4	5	7	10	15	25
$f(x)$	3516	27	8	2.9	1.1	0.3	0.04

$y = -3, y = 0, x \approx 3.46$

59. (a)



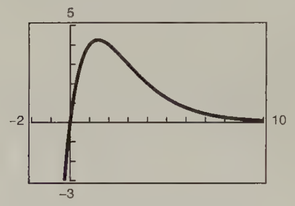
(b) Decreasing on $(-\infty, 0), (2, \infty)$

Increasing on $(0, 2)$

(c) Relative minimum at $(0, 0)$

Relative maximum at $(2, 0.541)$

61. (a)



(b) Increasing on $(-\infty, 1.443)$

Decreasing on $(1.443, \infty)$

(c) Relative maximum at $(1.443, 4.246)$

63.

n	1	2	4	12	365	Continuous
A	5397.3	5477.8	5520.1	5549.1	5563.4	5563.85

65.

n	1	2	4	12	365	Continuous
A	11,652	12,003	12,189	12,317	12,380	12,382.58

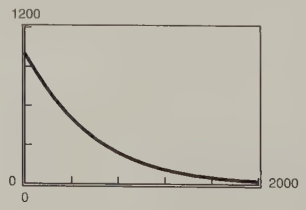
67.

t	1	10	20	30	40	50
A	12,999	26,706	59,436	132,278	294,390	655,178

69.

t	1	10	20	30	40	50
A	12,804	22,946	43,877	83,902	160,435	306,782

71. (a)



(b) \$421.12

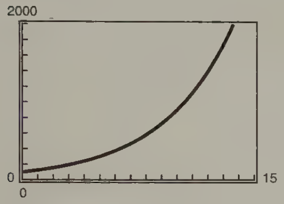
(c) \$350.13

(d)

x	100	200	300	400
P	849.53	717.64	603.25	504.94

x	500	600	700
P	421.12	350.13	290.35

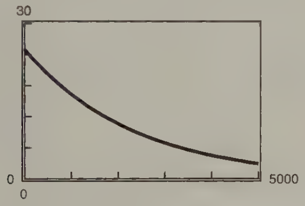
73. (a)



(b) $P(0) = 100$; $P(5) \approx 300$; $P(10) \approx 900$

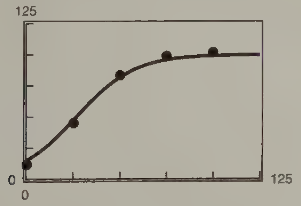
75. (a) 25 units (b) 16.30 units

(c)



(d) Never. The graph has a horizontal asymptote at $Q = 0$.

77. (a) and (b)



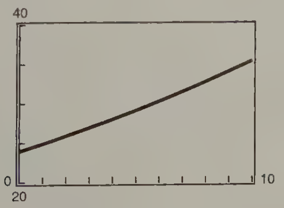
The model fits the data well.

(c)

x	0	25	50	75	100
y	15	47	82	96	99

(d) 64.7% (e) 37.4

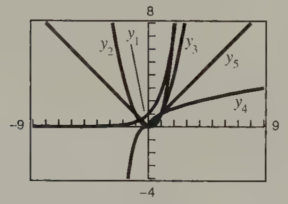
79. (a)



(b) \$35.45

81. True

83.



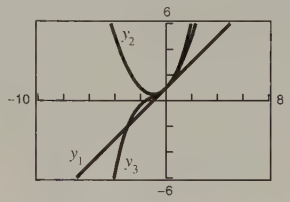
(a) $y = e^x$

(b) The exponential function increases at a faster rate.

(c) It usually implies rapid growth.

85. $1 < \sqrt{2} < 2$, so $2^1 < 2^{\sqrt{2}} < 2^2$

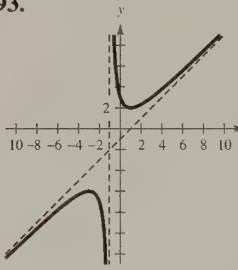
87.



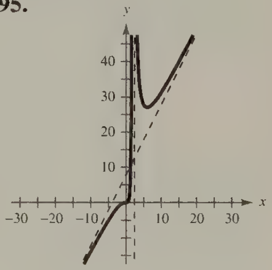
Answers will vary.

89. $f^{-1}(x) = -\frac{3}{2}\left(x - \frac{5}{2}\right)$ **91.** No inverse function

93.



95.



Section 4.2 *(page 318)*

1. $4^3 = 64$ **3.** $7^{-2} = \frac{1}{49}$ **5.** $32^{2/5} = 4$ **7.** $e^0 = 1$

9. $\log_5 125 = 3$ **11.** $\log_{81} 3 = \frac{1}{4}$ **13.** $\log_6 \frac{1}{36} = -2$

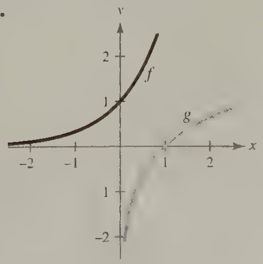
15. $\ln 20.0855\ldots = 3$ **17.** $\ln 13.463\ldots = 2.6$

19. 4 **21.** $-\frac{1}{2}$ **23.** -2 **25.** 9 **27.** 8 **29.** 2

31. 2.538 **33.** -0.097 **35.** 1.746 **37.** 1.869

39. 7.022 **41.** 22.276

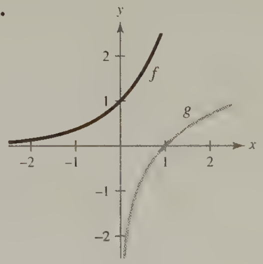
43.



Reflections in the line $y = x$

$g = f^{-1}$

45.



Reflections in the line $y = x$

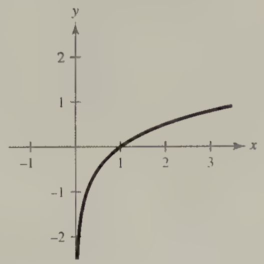
$g = f^{-1}$

47. (c) **49.** (d) **51.** (b)

53. Domain: $(0, \infty)$

Vertical asymptote: $x = 0$

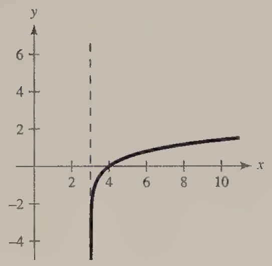
Intercept: $(1, 0)$



55. Domain: $(3, \infty)$

Vertical asymptote: $x = 3$

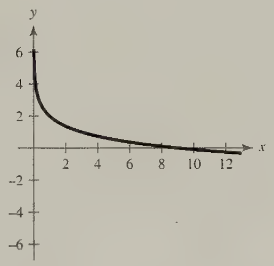
Intercept: $(4, 0)$



57. Domain: $(0, \infty)$

Vertical asymptote: $x = 0$

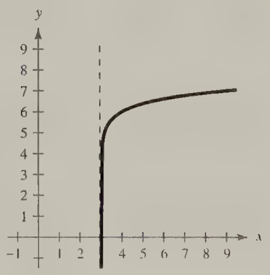
Intercept: $(9, 0)$



59. Domain: $(3, \infty)$

Vertical asymptote: $x = 3$

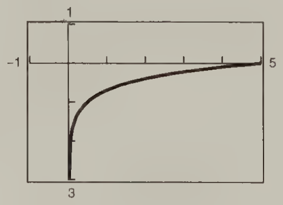
Intercept:
$(3 + 6^{-6}, 0) \approx (3, 0)$



61. Domain: $(0, \infty)$

Vertical asymptote: $x = 0$

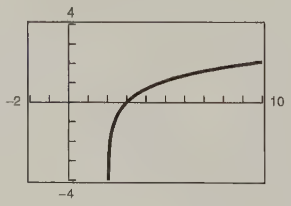
Intercept: $(5, 0)$



63. Domain: $(2, \infty)$

Vertical asymptote: $x = 2$

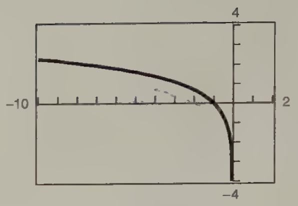
Intercept: $(3, 0)$



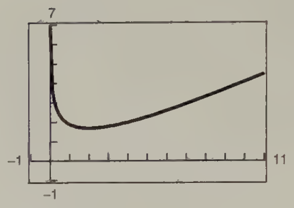
65. Domain: $(-\infty, 0)$

Vertical asymptote: $x = 0$

Intercept: $(-1, 0)$



67. (a)

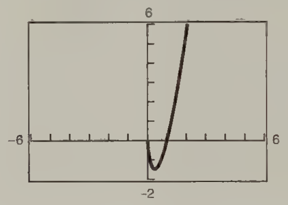


(b) Domain: $(0, \infty)$

(c) Decreasing: $(0, 2)$; Increasing: $(2, \infty)$

(d) Relative minimum: $(2, 1.693)$

69. (a)



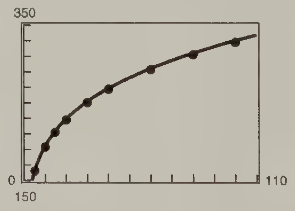
(b) Domain: $(0, \infty)$

(c) Decreasing: $(0, 0.37)$; Increasing: $(0.37, \infty)$

(d) Relative minimum: $(0.37, -1.47)$

71. 23.68 years **73.** (a) 80 (b) 68.1 (c) 62.3

75. (a)



The model fits the data well.

(b) 67.3 pounds per square inch (c) 306.5°F

77.

r	0.005	0.010	0.015
t	138.6	69.3	46.2

r	0.020	0.025	0.030
t	34.7	27.7	23.1

As the growth rate of the population increases, the time for the population to double decreases.

79. (a) 120 decibels (b) 100 decibels (c) No

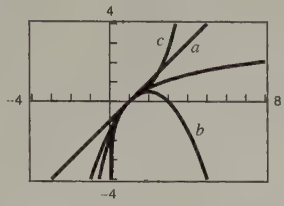
81. (a) 15 cubic feet per minute (b) 382 cubic feet

(c) 382 square feet

83. 10 years

85. Total amount: \$199,108.80; Interest: \$49,108.80

87. True **89.**

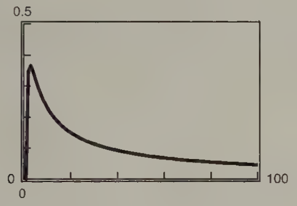


91. (a)

x	1	5	10	10^2	10^4	10^6
$f(x)$	0	0.322	0.230	0.046	0.00092	0.0000138

(b) 0

(c)

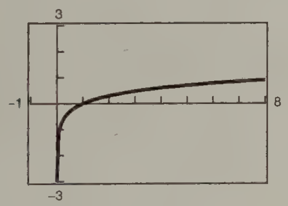


93. Vertical asymptote: $x = 0$

95. Vertical asymptote: $x = 7$ **97.** 33.115 **99.** 0.002

Section 4.3 *(page 326)*

1.



$f = g$

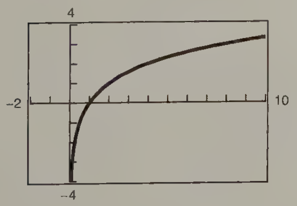
3. 1.771 **5.** -2 **7.** -0.102 **9.** 2.691

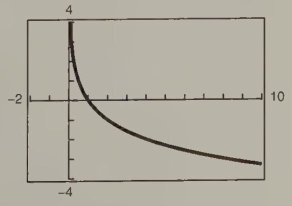
11. (a) $\dfrac{\log_{10} x}{\log_{10} 5}$ (b) $\dfrac{\ln x}{\ln 5}$ **13.** (a) $\dfrac{\log_{10} x}{\log_{10} \frac{1}{5}}$ (b) $\dfrac{\ln x}{\ln \frac{1}{5}}$

15. (a) $\dfrac{\log_{10} \frac{3}{10}}{\log_{10} x}$ (b) $\dfrac{\ln \frac{3}{10}}{\ln x}$

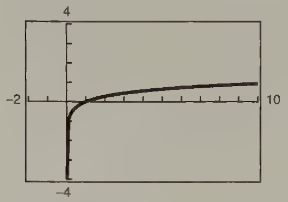
17. (a) $\dfrac{\log_{10} x}{\log_{10} 2.6}$ (b) $\dfrac{\ln x}{\ln 2.6}$

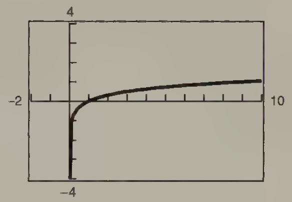
19. $\dfrac{\log_{10} x}{\log_{10} 2}$ **21.** $\dfrac{\log_{10} x}{\log_{10} \frac{1}{2}}$





23. $\dfrac{\ln x}{\ln 11.8}$ **25.** $\dfrac{\frac{1}{2} \ln x}{\ln 3}$





27. $\log_{10} 5 + \log_{10} x$ **29.** $\log_{10} 5 - \log_{10} x$

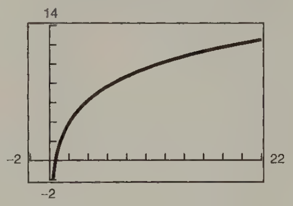
31. $4 \log_8 x$ **33.** $\frac{1}{2} \ln z$ **35.** $\ln x + \ln y + \ln z$

37. $\frac{1}{2} \ln(a - 1)$ **39.** $\ln z + 2 \ln(z - 1)$

41. $\frac{1}{3} \ln x - \frac{1}{3} \ln y$ **43.** $4 \ln x + \frac{1}{2} \ln y - 5 \ln z$

45. $2 \log_b x - 2 \log_b y - 3 \log_b z$

47.



$y_1 = y_2$ for positive values of x.

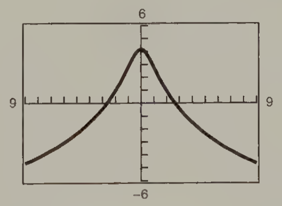
49. $\ln 4x$ **51.** $\log_4 \dfrac{z}{y}$ **53.** $\log_2(x + 3)^2$

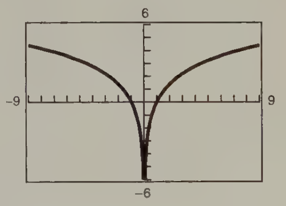
55. $\log_3 \sqrt[3]{7x}$ **57.** $\ln \dfrac{x}{(x + 1)^3}$ **59.** $\ln \dfrac{x - 2}{x + 2}$

61. $\ln \dfrac{x}{(x^2 - 4)^2}$ **63.** $\ln \sqrt[3]{\dfrac{x(x + 3)^2}{x^2 - 1}}$

65. $\ln \dfrac{\sqrt[3]{y(y + 4)^2}}{y - 1}$ **67.** $\ln \dfrac{9}{\sqrt{x^2 + 1}}$

69. **71.**





$y_1 = y_2$ No. The domains differ.

73. 2 **75.** 6.8 **77.** -4 is not in the domain of $\log_2 x$.

79. 2 **81.** -4 **83.** 0 is not in the domain of $\log_{10} x$.

85. 8.5 **87.** $\frac{3}{2}$ **89.** $\frac{1}{2}(1 + \log_7 10)$

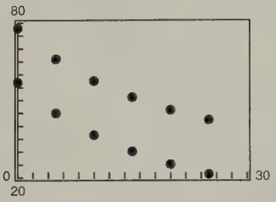
91. $-3 - \log_5 2$ **93.** $6 + \ln 5$

95. (a) $120 + 10\log_{10} I$

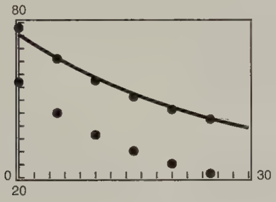
(b) and (c)

I	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}	10^{-14}
B	80	60	40	20	0	-20

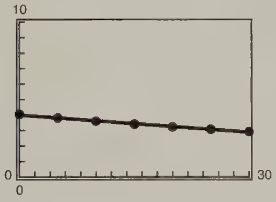
97. (a)



(b) $T = 54.438(0.964)^x + 21$

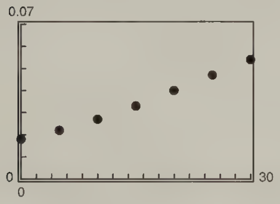


(c) $\ln(T - 21) = -0.037t + 3.997$

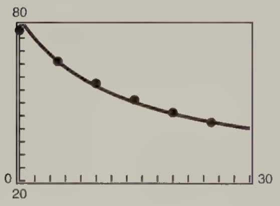


$T = e^{(-0.037t+3.997)} + 21$

(d)



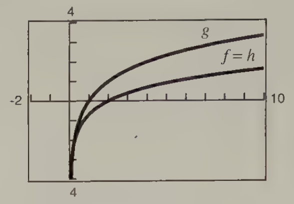
$$T = \frac{1}{0.0012t + 0.0162} + 21$$



99. False. $\ln 1 = 0$ **101.** False. $f(x) - f(2) = \ln \frac{x}{2}$

103. False. $u = v^2$

105.



$f(x) = h(x)$

The graphs are identical because for each positive value of x, Property 2 of logarithms holds.

107. Answers will vary. **109.** $\frac{3x^4}{2y^3}$

111. $1,\ x \neq 0, y \neq 0$ **113.** $3 \pm \sqrt{7}$ **115.** $\pm 4, \pm\sqrt{3}$

117. $\pm 2, 6$ **119.** 0.052 **121.** 15,235.494

123. 2.342 **125.** 0.697

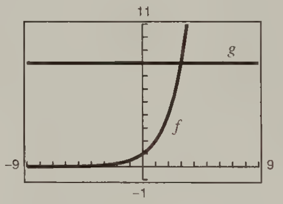
Section 4.4 *(page 336)*

1. (a) Yes (b) No **3.** (a) No (b) Yes (c) Yes

5. (a) No (b) No (c) Yes

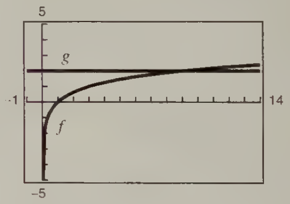
7. (a) Yes (b) Yes (c) No

9.



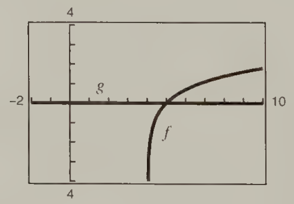
(3, 8)

11.



(9, 2)

13.



(5, 0)

15. 2 **17.** 4 **19.** $\frac{2}{3}$ **21.** -3 **23.** 4 **25.** 5

27. $\ln 4 \approx 1.386$ **29.** $e^{-7} \approx 0.00091$ **31.** 5

33. 0.1 **35.** 1 **37.** x^2 **39.** $5x + 2$ **41.** x^2

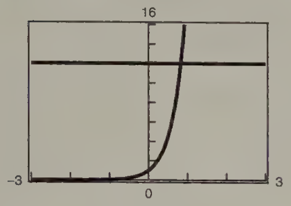
43. 2.756 **45.** $\ln 10 \approx 2.303$ **47.** 2 **49.** -6.142

51. 0.511 **53.** 0 **55.** $\ln 5 \approx 1.609$

57. $2 \ln 108 \approx 9.364$ **59.** 6.960

61.

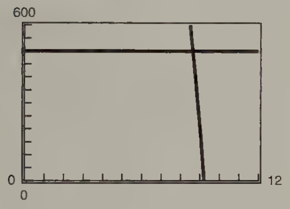
x	0.6	0.7	0.8	0.9	1.0
$f(x)$	6.05	8.17	11.02	14.88	20.09



0.828

63.

x	5	6	7	8	9
$f(x)$	1756	1598	1338	908	200

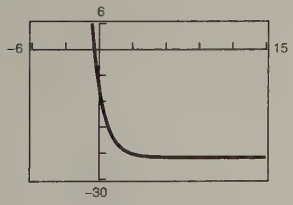


8.635

65. 1.881 **67.** 0.051 **69.** 6.146 **71.** 21.330

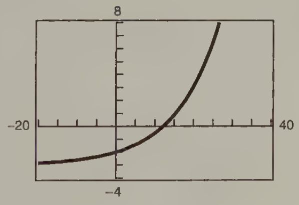
73. 3.656

75.



−0.427

77.



12.207

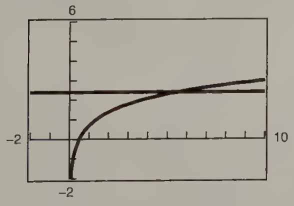
79. 0.050 **81.** 2.042 **83.** 4453.242 **85.** 103

87. 17.945 **89.** 5.389 **91.** 1.718, −3.718 **93.** 2

95. No real solution **97.** 180.384

99.

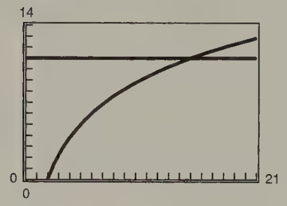
x	2	3	4	5	6
$f(x)$	1.39	1.79	2.08	2.30	2.48



5.512

101.

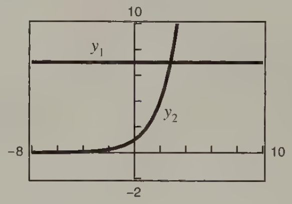
x	12	13	14	15	16
$f(x)$	9.79	10.22	10.63	11.00	11.36



14.988

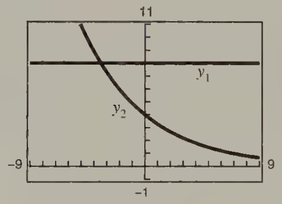
103. 14 **105.** 5.294 **107.** 3.729 **109.** 5.275

111.



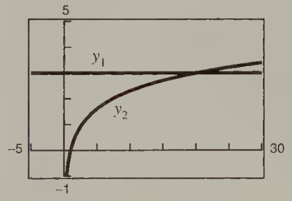
(2.807, 7)

113.



(−3.466, 8)

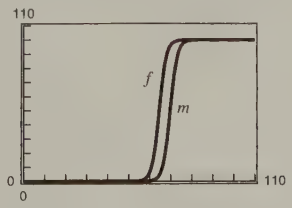
115.



(20.086, 3)

117. (a) 8.2 years (b) 12.9 years

119. (a)

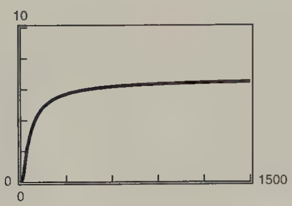


(b) $y = 100$ and $y = 0$

(c) Males: 69.71 inches; Females: 64.51 inches

121. (a) 1426 units (b) 1498 units

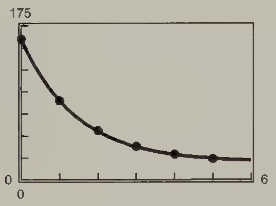
123. (a)



(b) $y = 6.7$. Yield will approach 6.7 million cubic feet per acre.

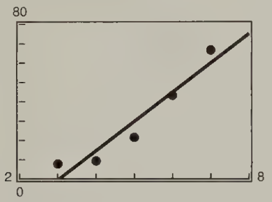
(c) 29.3 years

125. (a)



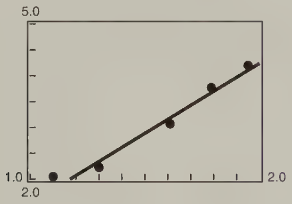
(b) $y = 20$. Room temperature (c) 0.81 hour

127. (a) $y = 15.17x - 46.15$



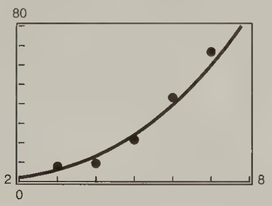
1999

(b) $\ln y = 2.706 \ln x - 1.175$



1998

(c) $y = e^{2.706 \ln x - 1.175}$

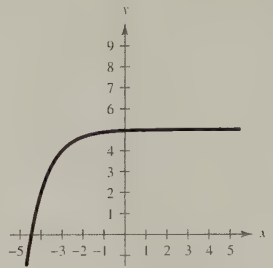


This model better fits the original data and will better predict future shipping levels.

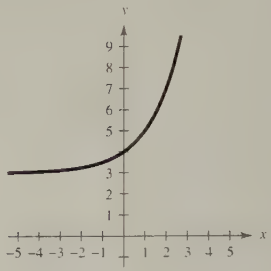
129. False. For example, $\ln(2x - 1) + \ln(x + 2) = \ln(x^2 - x + 1)$ has 2 extraneous solutions, $x = -1$ and $x = -3$.

131. No. It is dependent on the interest rate.

133.



135.



137. 2.814 **139.** 1.623

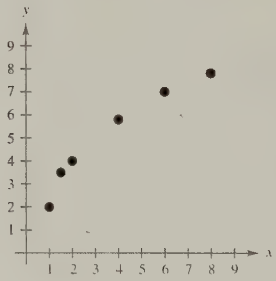
Section 4.5 *(page 348)*

1. (c) **3.** (b) **5.** (d) **7.** Logarithmic model

9. Gaussian model **11.** Exponential model

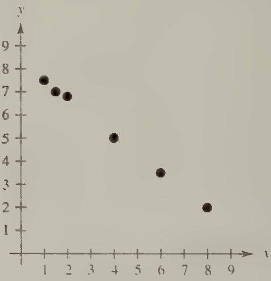
13. Gaussian model

15.



Logarithmic model

17.



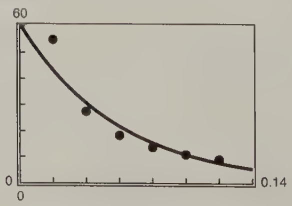
Linear model

	Initial Investment	*Annual % Rate*	*Time to Double*	*Amount After 10 Years*
19.	\$1000	12%	5.78 yr	\$3320.12
21.	\$750	8.94%	7.75 yr	\$1833.67
23.	\$500	9.5%	7.30 yr	\$1292.85
25.	\$6376.28	4.5%	15.4 yr	\$10,000.00

27. (a)

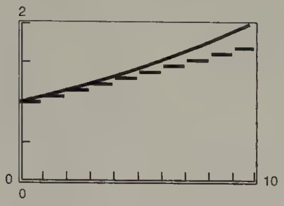
r	2%	4%	6%	8%	10%	12%
t	54.93	27.47	18.31	13.73	10.99	9.16

(b)



$t = 60.89(3.613 \times 10^{-8})^r$

29.



Continuous compounding

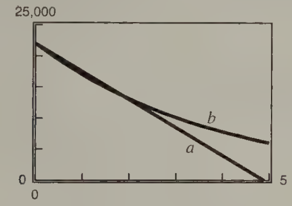
	Isotope	Half-Life (years)	Initial Quantity	Amount After 1000 Years
31.	^{226}Ra	1620	10 g	6.52 g
33.	^{14}C	5730	3 g	2.66 g

35. 2023 **37.** $k = 0.0112$; 2796 **39.** 3.15 hours

41. 95.8%

43. (a) $V = -4500t + 22{,}000$ (b) $V = 22{,}000e^{-0.263t}$

(c)



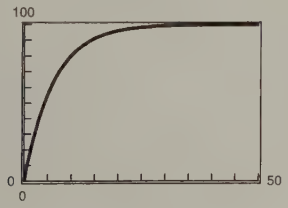
Exponential

(d) 1 year. Straight-line: $17,500; Exponential: $16,912
3 years. Straight-line: $8500; Exponential: $9995

(e) Value decreases $4500 per year.

45. (a) $S(t) = 100(1 - e^{-0.1625t})$

(b)



(c) 55,625

47. (a) $N = 30(1 - e^{-0.050t})$ (b) 36 days

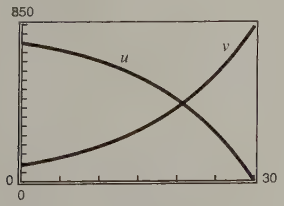
(c) No. It is not a linear function.

49. (a) 7.6 (b) 7.1

51. (a) 20 decibels (b) 70 decibels (c) 120 decibels

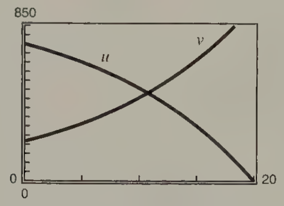
53. 95% **55.** 4.64 **57.** 10,000,000 times

59. (a)



(b) Interest. $t \approx 20.7$ years

(c)

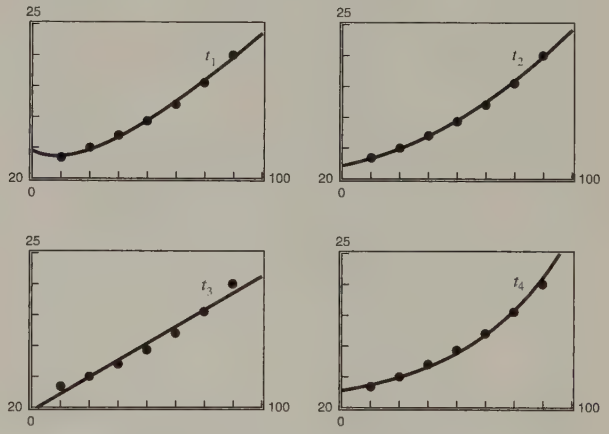


Interest. $t \approx 10.7$ years
Less interest is paid over the life of the loan.

61. $y = e^{0.768x}$ **63.** $y = \frac{1}{2}e^{0.576x}$

65. (a) $t_3 = 0.2729s - 6.0143$; $t_4 = 1.5385e^{0.0291s}$

(b)



(c)

s	30	40	50	60	70	80	90
t_1	3.6	4.7	6.7	9.4	12.5	15.9	19.6
t_2	3.3	4.9	7.0	9.5	12.5	15.9	19.9
t_3	2.2	4.9	7.6	10.4	13.1	15.8	18.5
t_4	3.7	4.9	6.6	8.8	11.8	15.8	21.1

(d) Model: t_1; Sum ≈ 1.9
Model: t_2; Sum ≈ 1.1
Model: t_3; Sum ≈ 5.6
Model: t_4; Sum ≈ 2.6
Quadratic model fits best.

67. 7:30 A.M.

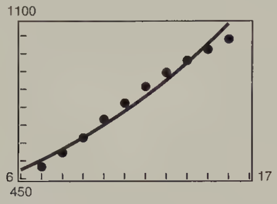
69. (a) $y = 0.082x + 4.45$

(b) $y = 4.536(1.015)^x$

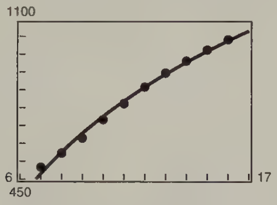
(c) The linear model fits the data better; Answers will vary.

(d) Linear: 6.50 billion; Exponential: 6.58 billion

71. (a) $y = 298.794(1.085)^x$



(b) $y = -837.7 + 673.619 \ln x$



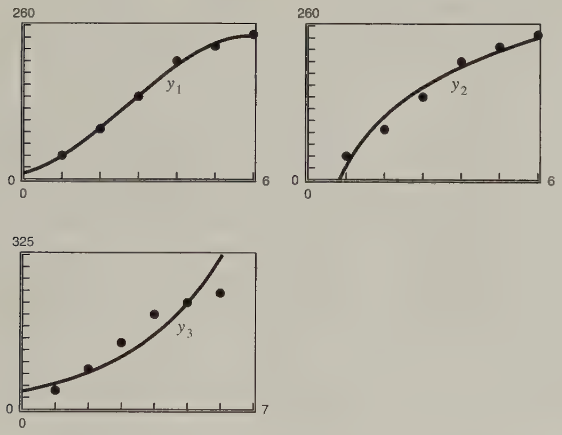
(c) The logarithmic model is better. It will continue to be better if growth of healthcare costs is slowed.

73. (a) $y_1 = -1.81x^3 + 14.58x^2 + 16.39x + 10.00$

$y_2 = 23.07 + 121.08 \ln x$

$y_3 = 38.38(1.4227)^x$

(b)



Cubic model

(c)

x	y	$y - y_1$	$(y - y_1)^2$	$y - y_2$	$(y - y_2)^2$	$y - y_3$	$(y - y_3)^2$
1	40	0.84	0.71	16.93	286.62	−14.60	213.25
2	85	−1.62	2.62	−22.00	483.84	7.32	53.52
3	140	−1.52	2.31	−16.09	258.89	29.48	869.01
4	200	7.00	49.00	9.08	82.40	42.76	1828.56
5	225	−5.20	27.04	7.06	49.83	1.30	1.68
6	245	2.74	7.51	4.98	24.84	−73.26	5367.34

(d) y_1: 89.19; y_2: 1186.42; y_3: 8333.36

Cubic model

(e) Sum of the squares of the errors

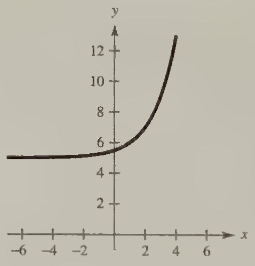
75. False. For example, $y = \dfrac{3}{1 + 2e^{-0.5x}}$ does not have an x-intercept.

77. True **79.** (b); $(0, -3)$, $\left(\frac{9}{4}, 0\right)$ **81.** (f); $(0, 25)$, $\left(\frac{100}{9}, 0\right)$

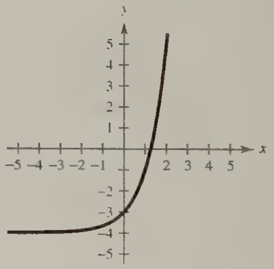
83. (d); $(0, 3)$ **85.** $4x^2 - 12x + 9,\ x \neq -4$

87. $2x^2 + 3 + \dfrac{3}{x - 4}$

89.



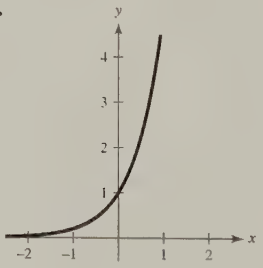
91.



Review Exercises *(page 357)*

1. 10.325 **3.** 0.201 **5.** (e) **7.** (b) **9.** (d)

11.



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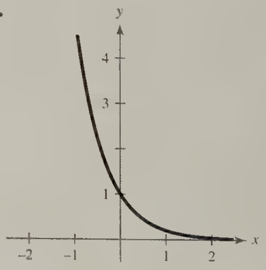

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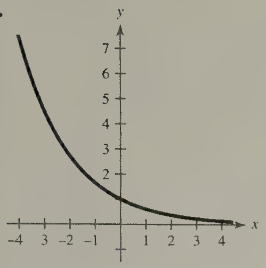

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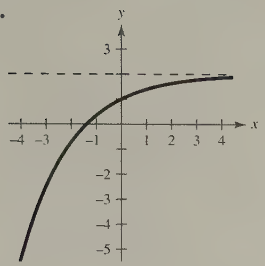

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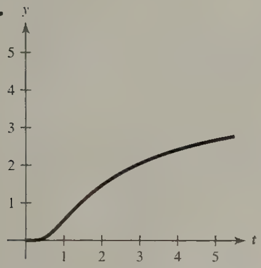
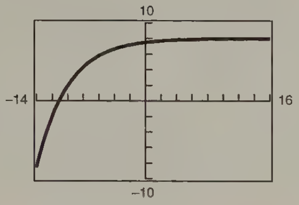
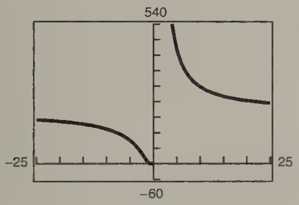


21.



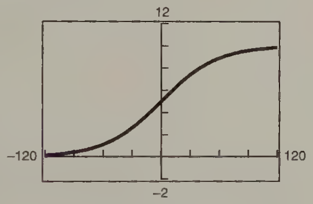
$y = 8$

23.



$y = 200, x = 0$

25.



$y = 0, y = 10$

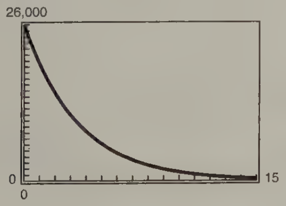
27. 0.069 **29.** 369.355

31.

t	2	10	20
P	\$184,623.27	\$89,865.79	\$40,379.30

t	30	40	50
P	\$18,143.59	\$8152.44	\$3663.13

33. (a)



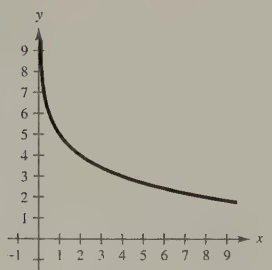
(b) \$14,625

(c) When it is first sold; Yes; Answers will vary.

35. $\log_4 64 = 3$ **37.** $\log_{25} 125 = \frac{3}{2}$ **39.** -0.34

41. 3 **43.** $-\frac{1}{2}$

45.



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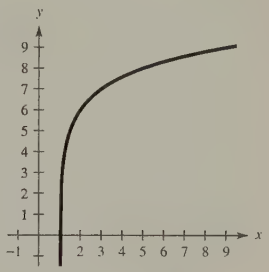

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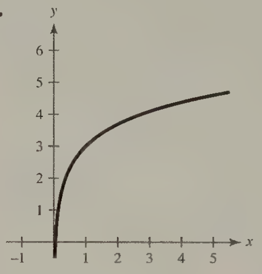
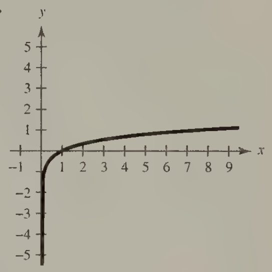
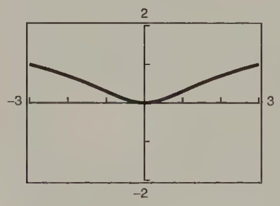


51.

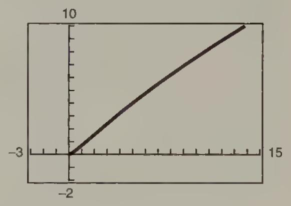


53.



$-\infty < x < \infty$

55.

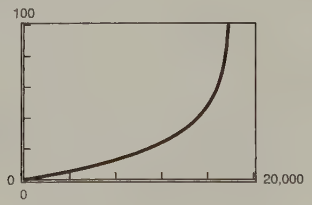


$x \geq 0$

57. 7 **59.** -18

61. (a) $0 \leq h < 18,000$

(b)



$h = 18,000$

(c) The time required to further increase its altitude increases.

(d) 5.46 minutes

63. 1.585 **65.** 2.132 **67.** $\ln 4 + \ln 5$

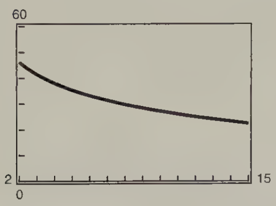
69. $\ln 5 - 3 \ln 4$ **71.** 1.6542 **73.** 0.2823

75. $1 + 2 \log_5 x$ **77.** $\log_{10} 5 + \frac{1}{2} \log_{10} y - 2 \log_{10} x$

79. $\ln(x^2 + 1) + \ln(x - 1)$ **81.** $\log_2 5x$

83. $\ln \frac{\sqrt{|2x - 1|}}{(x + 1)^2}$ **85.** $\ln \frac{3\sqrt[3]{4 - x^2}}{x}$

87. (a)



(b)

h	4	6	8	10	12	14
s	38	33	30	27	25	23

(c) The decrease in productivity starts to level off.

89. 3 **91.** −3 **93.** 2401 **95.** $\ln 12 \approx 2.485$

97. $-\frac{\ln 44}{5} \approx -0.757$ **99.** $\ln 22 \approx 3.091$

101. $\log_5 17 \approx 1.760$ **103.** $\ln 5 \approx 1.609, \ln 2 \approx 0.693$

105. $\frac{1}{3}e^{8.2} \approx 1213.650$ **107.** $\frac{1}{4}e^{15/2} \approx 452.011$

109. $3e^2 \approx 22.167$ **111.** $e^4 - 1 \approx 53.598$

113. No solution **115.** $\frac{9}{10}$ **117.** ≈15.2 years

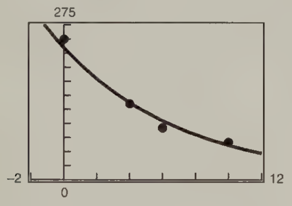
119. (e) **121.** (f) **123.** (a) **125.** 2025

127. (a) 5.78% (b) $10,595.03 (c) 5.95%

129. (a) 7.7 weeks (b) 13.3 weeks

131. $y = 2e^{0.1014x}$ **133.** $y = \frac{1}{2}e^{0.4605x}$

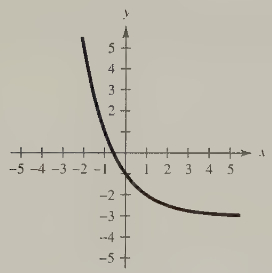
135. $y = 234.684(0.8746)^x$



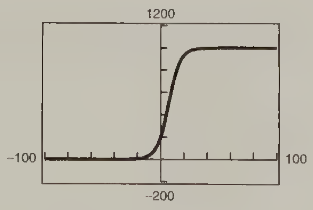
137. True **139.** False. $\ln(xy) = \ln x + \ln y$ **141.** True

Chapter Test *(page 362)*

1.



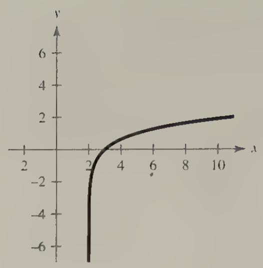
2.



$y = 0, y = 1000$

3. $40,386.38 **4.** $4^3 = 64$

5.



6. $\ln 6 + 2\ln x - \frac{1}{2}\ln(x^2 + 1)$ **7.** 2 **8.** −3

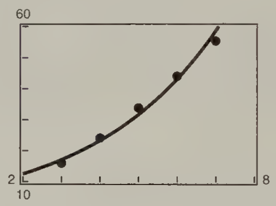
9. 14.955 **10.** 18.447 **11.** −3.975

12. $10,204 **13.** (a) 0.154 (b) 0.487 (c) 0.811

14. (a) 300 (b) 570 (c) At the end of the 8th year

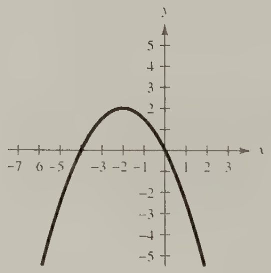
15. (c); it passes through the point (0, 0). Symmetric to the y-axis, and $y = 6$ is a horizontal asymptote.

16. $y = 6.775(1.361)^x$

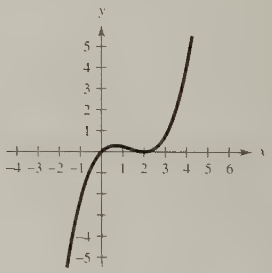


Cumulative Test for Chapters 3–4 *(page 363)*

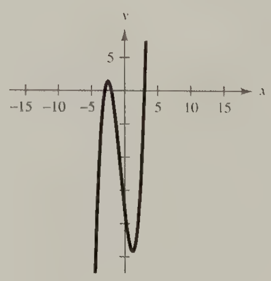
1.



2.



3.

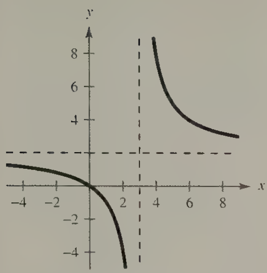


4. $-2, \pm 2i$ **5.** 1.42

6. $4x + 2 - \frac{15}{x + 3}$ **7.** $2x^2 + 7x + 48 + \frac{268}{x - 6}$

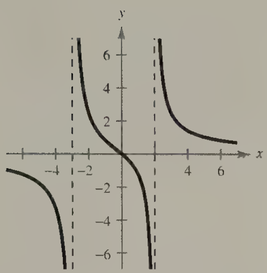
8. $f(x) = x^4 + x^3 + 18x$

9.



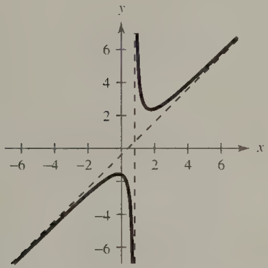
Asymptotes: $x = 3, y = 2$

10.



Asymptotes:
$x = -3, x = 2, y = 0$

11.



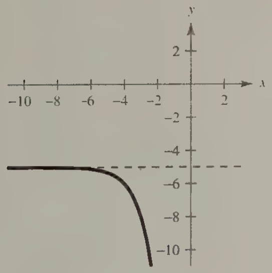
Asymptotes: $x = 2, y = x - 1$

12. $f(x) = \dfrac{4x^2}{x^2 + 1}$; Answers will vary.

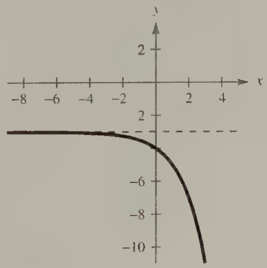
13. 6 **14.** −4 **15.** 10 **16.** −3 **17.** 6.733

18. 8772.934 **19.** 0.162 **20.** 51.743

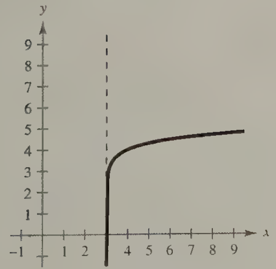
21.



22.



23.



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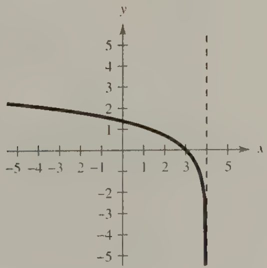


25. 1.892 **26.** 0.872 **27.** −7.484 **28.** −3.036

29. $\ln \dfrac{x^2}{\sqrt{x + 5}}$ **30.** $\ln x + \dfrac{1}{3}\ln(x - 5)$

31. $\frac{1}{2}\ln 12 \approx 1.242$ **32.** $\ln\left(\frac{9}{4}\right) + 3 \approx 3.811$

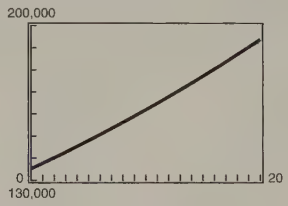
33. 12.8 **34.** $\dfrac{e^3}{4} \approx 5.021$ **35.** $e^6 - 2 \approx 401.429$

36. $\pm\dfrac{\sqrt{e^7}}{2} \approx 16.558$ **37.** $y = 2.5e^{0.2871x}$ **38.** $20,000

39. $50, $9.50, $5.45; $5.00

40. $16,302.05 **41.** $108.63

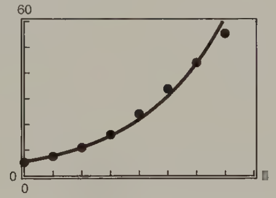
42.



2008

43. (a) 3.00 (b) 570 (c) after about 9 years

44. $y = 5.55e^{0.344x}$



Chapter 5

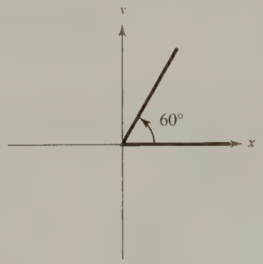
Section 5.1 *(page 373)*

1. 210° **3.** −45°

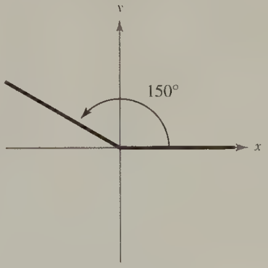
5. (a) Quadrant II (b) Quadrant IV

7. (a) Quadrant III (b) Quadrant I

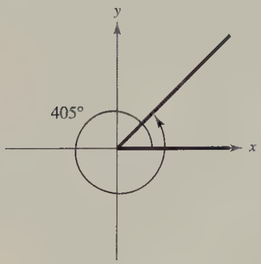
9. (a)



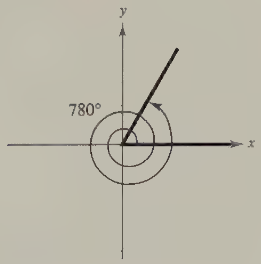
(b)



11. (a)



(b)



13. (a) 412°, −308° (b) 324°, −396°

15. (a) 660°, −60° (b) 590°, −130°

17. (a) 64.75° (b) −124.5°

19. (a) 85.308° (b) −408.274°

21. (a) $280°36'$ (b) $-115°48'$

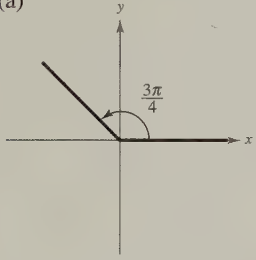
23. (a) $257°49'52''$ (b) $-205°7'8''$

25. 2 **27.** −3 **29.** (a) Quadrant I (b) Quadrant III

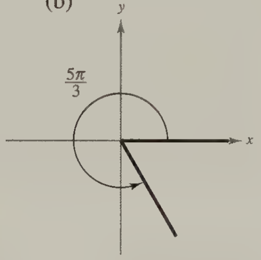
31. (a) Quadrant IV (b) Quadrant II

33. (a) Quadrant III (b) Quadrant II

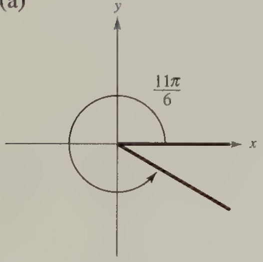
35. (a)



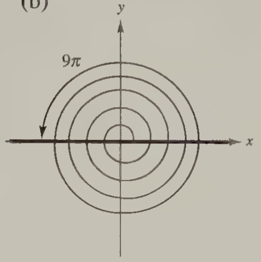
(b)



37. (a)



(b)



39. (a) $\frac{25\pi}{12}, -\frac{23\pi}{12}$ (b) $\frac{8\pi}{3}, -\frac{4\pi}{3}$

41. (a) $\frac{5\pi}{4}, -\frac{3\pi}{4}$ (b) $\frac{28\pi}{15}, -\frac{32\pi}{15}$

43. (a) Complement: 66°; Supplement: 156°

(b) Complement: none; Supplement: 54°

45. (a) Complement: $\frac{\pi}{6}$; Supplement: $\frac{2\pi}{3}$

(b) Complement: none; Supplement: $\frac{\pi}{4}$

47. (a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ **49.** (a) $-\frac{\pi}{9}$ (b) $-\frac{4\pi}{3}$

51. (a) 270° (b) −210° **53.** (a) 420° (b) −39°

55. 2.007 **57.** −3.776 **59.** 11.205 **61.** −0.014

63. 25.714° **65.** 562.5° **67.** −756°

69. −114.592° **71.** $\frac{6}{5}$ rad **73.** $4\frac{4}{7}$ rad **75.** $\frac{8}{15}$ rad

77. $2\frac{12}{29}$ rad **79.** 14π inches ≈ 43.98 inches

81. 4π meters ≈ 12.57 meters **83.** 1141.02 miles

85. 0.094 rad ≈ 5.39° **87.** $\frac{5}{12}$ rad ≈ 23.87°

89. (a) 540° ≈ 9.42 rad (b) 900° ≈ 15.71 rad

(c) 1260° ≈ 21.99 rad

91. (a) 448.2 revolutions per minute

(b) 2816 radians per minute

93. 20.16π inches per second

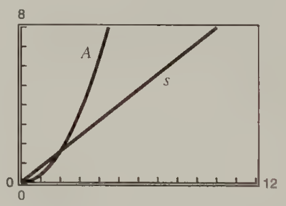
95. False. A radian is larger: 1 rad ≈ 57.3°.

97. True. The sum of the angles is π, or 180°.

99. Two angles in standard position are coterminal angles if they have the same initial and terminal sides.

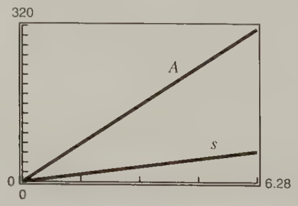
101. $\frac{50\pi}{3}$ square meters

103. (a) $A = 0.4r^2,\ r > 0;\ s = 0.8r,\ r > 0$



The area function changes more rapidly for $r > 1$ because it is quadratic and the arc length function is linear.

(b) $A = 50\theta,\ 0 < \theta < 2\pi;\ s = 10\theta,\ 0 < \theta < 2\pi$



105. $y = -6.59x + 38.06$ **107.** $y = -36.38x + 189.71$

109. $x = -3, 0, 0, 3$ **111.** $x = -8, -3, 0, 0$

Section 5.2 *(page 384)*

1. $\sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

$\tan\theta = \frac{3}{4}$

$\csc\theta = \frac{5}{3}$

$\sec\theta = \frac{5}{4}$

$\cot\theta = \frac{4}{3}$

3. $\sin\theta = \frac{8}{17}$

$\cos\theta = \frac{15}{17}$

$\tan\theta = \frac{8}{15}$

$\csc\theta = \frac{17}{8}$

$\sec\theta = \frac{17}{15}$

$\cot\theta = \frac{15}{8}$

5. $\sin\theta = \frac{3\sqrt{13}}{13}$

$\cos\theta = \frac{2\sqrt{13}}{13}$

$\tan\theta = \frac{3}{2}$

$\csc\theta = \frac{\sqrt{13}}{3}$

$\sec\theta = \frac{\sqrt{13}}{2}$

$\cot\theta = \frac{2}{3}$

7. $\sin\theta = \frac{1}{3}$

$\cos\theta = \frac{2\sqrt{2}}{3}$

$\tan\theta = \frac{\sqrt{2}}{4}$

$\csc\theta = 3$

$\sec\theta = \frac{3\sqrt{2}}{4}$

$\cot\theta = 2\sqrt{2}$

The triangles are similar and corresponding sides are proportional.

9. $\sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

$\tan\theta = \frac{3}{4}$

$\csc\theta = \frac{5}{3}$

$\sec\theta = \frac{5}{4}$

$\cot\theta = \frac{4}{3}$

The triangles are similar and corresponding sides are proportional.

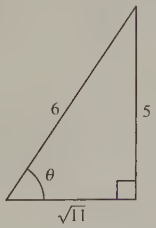
11. $\cos\theta = \frac{\sqrt{11}}{6}$

$\tan\theta = \frac{5\sqrt{11}}{11}$

$\csc\theta = \frac{6}{5}$

$\sec\theta = \frac{6\sqrt{11}}{11}$

$\cot\theta = \frac{\sqrt{11}}{5}$



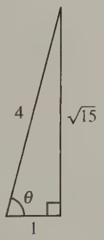
13. $\sin\theta = \frac{\sqrt{15}}{4}$

$\cos\theta = \frac{1}{4}$

$\tan\theta = \sqrt{15}$

$\csc\theta = \frac{4\sqrt{15}}{15}$

$\cot\theta = \frac{\sqrt{15}}{15}$



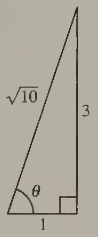
15. $\sin\theta = \frac{3\sqrt{10}}{10}$

$\cos\theta = \frac{\sqrt{10}}{10}$

$\csc\theta = \frac{\sqrt{10}}{3}$

$\sec\theta = \sqrt{10}$

$\cot\theta = \frac{1}{3}$



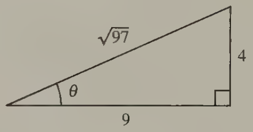
17. $\sin\theta = \frac{4\sqrt{97}}{97}$

$\cos\theta = \frac{9\sqrt{97}}{97}$

$\tan\theta = \frac{4}{9}$

$\csc\theta = \frac{\sqrt{97}}{4}$

$\sec\theta = \frac{\sqrt{97}}{9}$



19. (a) $\sqrt{3}$ (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}}{3}$

21. (a) $\frac{1}{3}$ (b) $\frac{2\sqrt{2}}{3}$ (c) $\frac{\sqrt{2}}{4}$ (d) 3

23. (a) 4 (b) $\pm\frac{\sqrt{15}}{4}$ (c) $\pm\frac{\sqrt{15}}{15}$ (d) $\frac{1}{4}$

25.–31. Answers will vary.

33. (a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{3}$ **35.** (a) 1 (b) $\frac{\sqrt{2}}{2}$

37. (a) $\frac{\sqrt{3}}{2}$ (b) 2 **39.** (a) 0.4226 (b) 0.4226

41. (a) 1.3499 (b) 1.3432

43. (a) 5.0273 (b) 0.1989

45. (a) 1.1884 (b) 0.5463

47. (a) $30° = \frac{\pi}{6}$ (b) $30° = \frac{\pi}{6}$

49. (a) $60° = \frac{\pi}{3}$ (b) $45° = \frac{\pi}{4}$

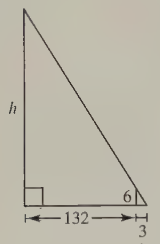
51. (a) $60° = \frac{\pi}{3}$ (b) $45° = \frac{\pi}{4}$

53. (a) $55° \approx 0.960$ (b) $89° \approx 1.553$

55. (a) $50° \approx 0.873$ (b) $25° \approx 0.436$

57. $35\sqrt{3}$ **59.** $\frac{38\sqrt{3}}{3}$ **61.** 11.5

63. (a)



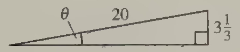
Not drawn to scale

(b) $\frac{6}{3} = \frac{h}{135}$

(c) 270 feet

65. 160.03 feet

67. (a)



(b) $\sin \theta = \frac{3\frac{1}{3}}{20} = \frac{1}{6}$

(c) 9.59°

(d) 9.59°

69. 425.3 meters; 431.9 meters

71. 6.57 centimeters

73. $\sin 75° \approx 0.97$
$\cos 75° \approx 0.26$
$\tan 75° \approx 3.73$
$\csc 75° \approx 1.04$
$\sec 75° \approx 3.86$
$\cot 75° \approx 0.27$

75. True. $\sec x = \csc(90° - x)$

77. True for all θ

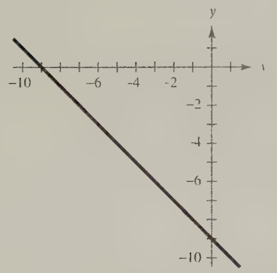
79.

θ	0°	20°	40°	60°	80°
$\cos \theta$	1	0.9397	0.7660	0.5000	0.1736
$\sin(90° - \theta)$	1	0.9397	0.7660	0.5000	0.1736

$\cos \theta = \sin(90° - \theta)$

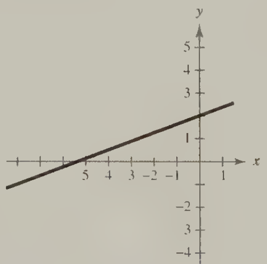
θ and $90° - \theta$ are complementary angles.

81.



x-intercept: -9
y-intercept: -9

83.



x-intercept: $-5\frac{1}{3}$
y-intercept: 2

85. 18.661

87. 17.420

89. Quadrant II

91. Quadrant I

Section 5.3 *(page 397)*

1. (a) $\sin \theta = \frac{3}{5}$
$\cos \theta = \frac{4}{5}$
$\tan \theta = \frac{3}{4}$
$\csc \theta = \frac{5}{3}$
$\sec \theta = \frac{5}{4}$
$\cot \theta = \frac{4}{3}$

(b) $\sin \theta = -\frac{15}{17}$
$\cos \theta = -\frac{8}{17}$
$\tan \theta = \frac{15}{8}$
$\csc \theta = -\frac{17}{15}$
$\sec \theta = -\frac{17}{8}$
$\cot \theta = \frac{8}{15}$

3. (a) $\sin \theta = -\frac{1}{2}$
$\cos \theta = -\frac{\sqrt{3}}{2}$
$\tan \theta = \frac{\sqrt{3}}{3}$
$\csc \theta = -2$
$\sec \theta = -\frac{2\sqrt{3}}{3}$
$\cot \theta = \sqrt{3}$

(b) $\sin \theta = \frac{\sqrt{2}}{2}$
$\cos \theta = -\frac{\sqrt{2}}{2}$
$\tan \theta = -1$
$\csc \theta = \sqrt{2}$
$\sec \theta = -\sqrt{2}$
$\cot \theta = -1$

5. $\sin \theta = \frac{24}{25}$
$\cos \theta = \frac{7}{25}$
$\tan \theta = \frac{24}{7}$
$\csc \theta = \frac{25}{24}$
$\sec \theta = \frac{25}{7}$
$\cot \theta = \frac{7}{24}$

7. $\sin \theta = -\frac{12}{13}$
$\cos \theta = \frac{5}{13}$
$\tan \theta = -\frac{12}{5}$
$\csc \theta = -\frac{13}{12}$
$\sec \theta = \frac{13}{5}$
$\cot \theta = -\frac{5}{12}$

9. $\sin \theta = \frac{5\sqrt{29}}{29}$
$\cos \theta = -\frac{2\sqrt{29}}{29}$
$\tan \theta = -\frac{5}{2}$
$\csc \theta = \frac{\sqrt{29}}{5}$
$\sec \theta = -\frac{\sqrt{29}}{2}$
$\cot \theta = -\frac{2}{5}$

11. $\sin \theta = \frac{9\sqrt{85}}{85}$
$\cos \theta = -\frac{2\sqrt{85}}{85}$
$\tan \theta = -\frac{9}{2}$
$\csc \theta = \frac{\sqrt{85}}{9}$
$\sec \theta = -\frac{\sqrt{85}}{2}$
$\cot \theta = -\frac{2}{9}$

13. Quadrant III

15. Quadrant II

17. Quadrant I

19. $\sin \theta = \frac{3}{5}$
$\cos \theta = -\frac{4}{5}$
$\tan \theta = -\frac{3}{4}$
$\csc \theta = \frac{5}{3}$
$\sec \theta = -\frac{5}{4}$
$\cot \theta = -\frac{4}{3}$

21. $\sin \theta = -\frac{15}{17}$
$\cos \theta = \frac{8}{17}$
$\tan \theta = -\frac{15}{8}$
$\csc \theta = -\frac{17}{15}$
$\sec \theta = \frac{17}{8}$
$\cot \theta = -\frac{8}{15}$

23. $\sin\theta = \frac{\sqrt{3}}{2}$

$\cos\theta = -\frac{1}{2}$

$\tan\theta = -\sqrt{3}$

$\csc\theta = \frac{2\sqrt{3}}{3}$

$\sec\theta = -2$

$\cot\theta = -\frac{\sqrt{3}}{3}$

25. $\sin\theta = 0$

$\cos\theta = -1$

$\tan\theta = 0$

$\csc\theta$ is undefined.

$\sec\theta = -1$

$\cot\theta$ is undefined.

27. $\sin\theta = \frac{\sqrt{2}}{2}$

$\cos\theta = -\frac{\sqrt{2}}{2}$

$\tan\theta = -1$

$\csc\theta = \sqrt{2}$

$\sec\theta = -\sqrt{2}$

$\cot\theta = -1$

29. $\sin\theta = -\frac{2\sqrt{5}}{5}$

$\cos\theta = -\frac{\sqrt{5}}{5}$

$\tan\theta = 2$

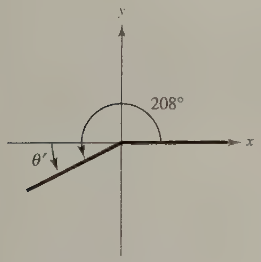
$\csc\theta = -\frac{\sqrt{5}}{2}$

$\sec\theta = -\sqrt{5}$

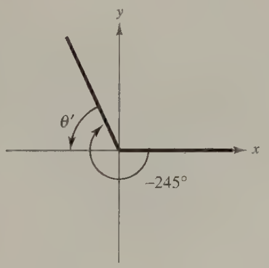
$\cot\theta = \frac{1}{2}$

31. -1 **33.** 0 **35.** 1 **37.** Undefined

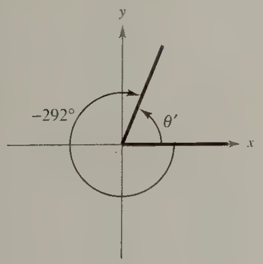
39. $\theta' = 28°$



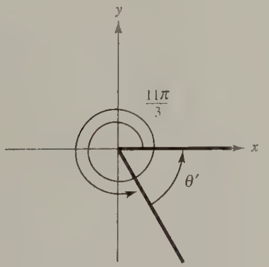
41. $\theta' = 65°$



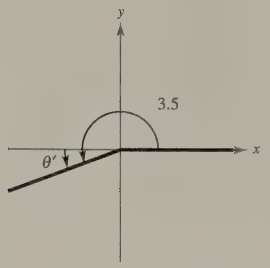
43. $\theta' = 68°$



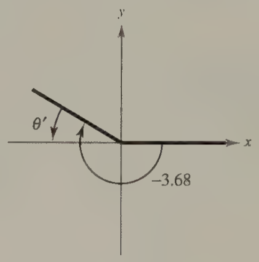
45. $\theta' = \frac{\pi}{3}$



47. $\theta' = 3.5 - \pi$



49. $\theta' = 3.68 - \pi \approx 0.5384$



51. $\sin(225°) = -\frac{\sqrt{2}}{2}$

$\cos(225°) = -\frac{\sqrt{2}}{2}$

$\tan(225°) = 1$

53. $\sin(-750°) = -\frac{1}{2}$

$\cos(-750°) = \frac{\sqrt{3}}{2}$

$\tan(-750°) = -\frac{\sqrt{3}}{3}$

55. $\sin(-240°) = \frac{\sqrt{3}}{2}$

$\cos(-240°) = -\frac{1}{2}$

$\tan(-240°) = -\sqrt{3}$

57. $\sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$

$\cos\frac{5\pi}{3} = \frac{1}{2}$

$\tan\frac{5\pi}{3} = -\sqrt{3}$

59. $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

$\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

61. $\sin\frac{11\pi}{4} = \frac{\sqrt{2}}{2}$

$\cos\frac{11\pi}{4} = -\frac{\sqrt{2}}{2}$

$\tan\frac{11\pi}{4} = -1$

63. $\sin\left(-\frac{7\pi}{6}\right) = \frac{1}{2}$

$\cos\left(-\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$\tan\left(-\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

65. 0.1736 **67.** 1.7321

69. -0.3420 **71.** 5.7588 **73.** 0.6052

75. 0.3640 **77.** -2.9238

79. (a) $30° = \frac{\pi}{6}, 150° = \frac{5\pi}{6}$

(b) $210° = \frac{7\pi}{6}, 330° = \frac{11\pi}{6}$

81. (a) $60° = \frac{\pi}{3}, 120° = \frac{2\pi}{3}$ (b) $135° = \frac{3\pi}{4}, 315° = \frac{7\pi}{4}$

83. (a) $150° = \frac{5\pi}{6}, 210° = \frac{7\pi}{6}$

(b) $120° = \frac{2\pi}{3}, 240° = \frac{4\pi}{3}$

85. 54.99°, 125.01° **87.** 33.12°, 213.12°

89. 144.78°, 215.22° **91.** 208.64°, 331.36°

93. 10°, 350° **95.** 50°, 230° **97.** $\frac{4}{5}$ **99.** $-\frac{\sqrt{13}}{2}$

101. $\frac{8}{5}$

103. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\tan\frac{\pi}{4} = 1$

105. $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$\sin\frac{5\pi}{6} = \frac{1}{2}$

$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

$\tan\frac{5\pi}{6} = -\frac{\sqrt{3}}{3}$

107. $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

$\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$

$\cos\frac{4\pi}{3} = -\frac{1}{2}$

$\tan\frac{4\pi}{3} = \sqrt{3}$

109. $(0, -1)$

$\sin\frac{3\pi}{2} = -1$

$\cos\frac{3\pi}{2} = 0$

$\tan\frac{3\pi}{2}$ is undefined.

111. (a) -1 (b) -0.4

113. (a) 0.25, 2.89 (b) 1.82, 4.46

115. (a) 25.2°F (b) 65.1°F (c) 50.8°F

117. (a) 2 centimeters

(b)

t	0.50	1.02	1.54	2.07	2.59
y	-1.2	0.71	-0.42	0.25	-0.15

(c) Friction within the system damps the oscillations and is modeled by the factor e^{-t}.

(d) 0.26 second, 0.79 second

119. 0.79 amperes **121.** True. $\theta' = 180° - \theta$

123. False. $\theta' \neq 180° - \theta$

125.

θ	0°	20°	40°	60°	80°
$\sec\theta$	1	1.0642	1.3054	2	5.7588
$\sec(-\theta)$	1	1.0642	1.3054	2	5.7588
$\tan\theta$	0	0.3640	0.8391	1.7321	5.6713
$\tan(-\theta)$	0	-0.3640	-0.8391	-1.7321	-5.6713

The secant is an even function and the tangent is an odd function.

127. (a)

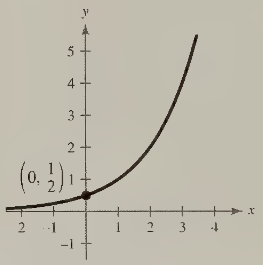
θ	0	0.3	0.6
$\cos\left(\frac{3\pi}{2} - \theta\right)$	0	-0.2955	-0.5646
$-\sin\theta$	0	-0.2955	-0.5646

θ	0.9	1.2	1.5
$\cos\left(\frac{3\pi}{2} - \theta\right)$	-0.7833	-0.9320	-0.9975
$-\sin\theta$	-0.7833	-0.9320	-0.9975

(b) $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$

129. It is an odd function.

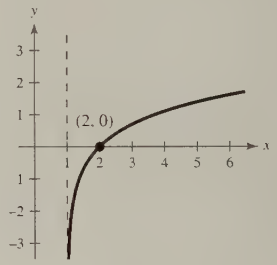
131.



Asymptote at $y = 0$

y-intercept at $y = \frac{1}{2}$

133.



Asymptote at $x = 1$

x-intercept at $x = 2$

135. $x = -1.752$ **137.** $x = 0.002$

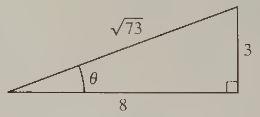
139. $\sin\theta = \frac{3\sqrt{73}}{73}$

$\cos\theta = \frac{8\sqrt{73}}{73}$

$\tan\theta = \frac{3}{8}$

$\csc\theta = \frac{\sqrt{73}}{3}$

$\sec\theta = \frac{\sqrt{73}}{8}$



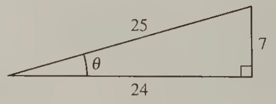
141. $\sin\theta = \frac{7}{25}$

$\cos\theta = \frac{24}{25}$

$\csc\theta = \frac{25}{7}$

$\sec\theta = \frac{25}{24}$

$\cot\theta = \frac{24}{7}$



Section 5.4 *(page 408)*

1. Period: π; Amplitude: 3

3. Period: 4π; Amplitude: $\frac{5}{2}$

5. Period: 2; Amplitude: $\frac{2}{3}$

7. Period: 2π; Amplitude: 2

9. Period: $\frac{\pi}{3}$; Amplitude: 3

11. Period: 3π; Amplitude: $\frac{1}{4}$

13. Period: $\frac{1}{2}$; Amplitude: 3

15. g is a shift of f π units to the right.

17. g is a reflection of f about the x-axis.

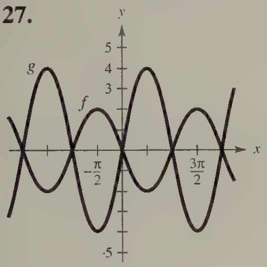
19. g is a reflection of f in the x-axis and has five times the amplitude of f.

21. g is the graph of f shifted 4 units up.

23. g has twice the amplitude of f.

25. g is a horizontal shift of f π units to the right.

27.



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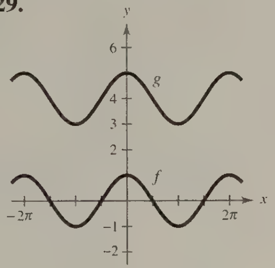
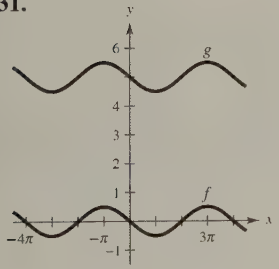
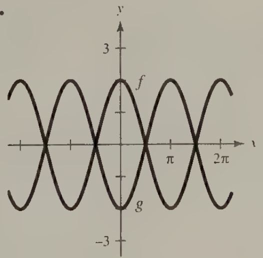


31.



33.



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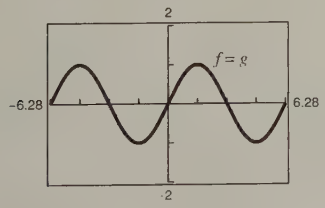
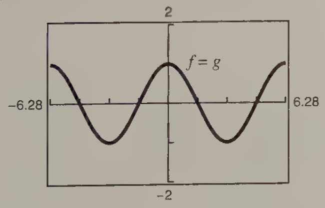


37.



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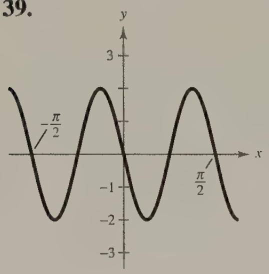

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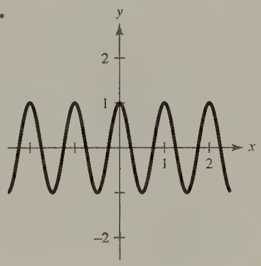

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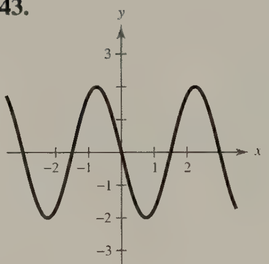

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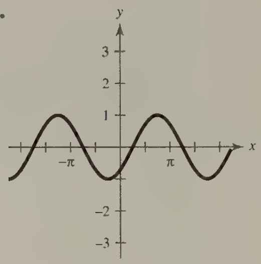

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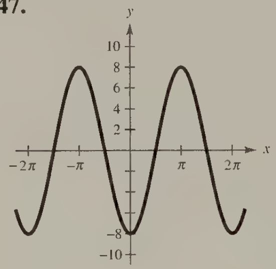

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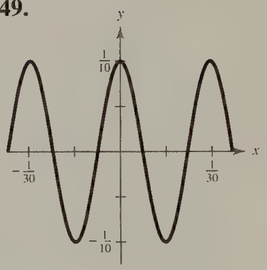

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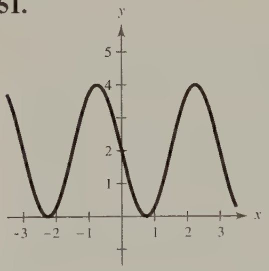

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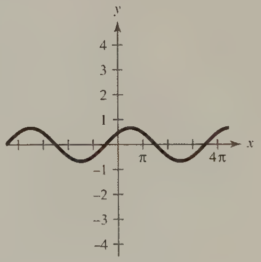

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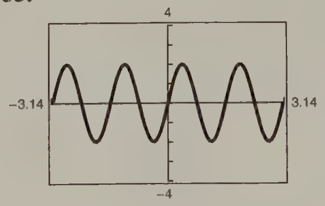
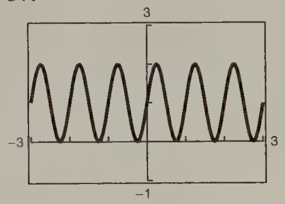


Amplitude: 2

Period: $\frac{\pi}{2}$

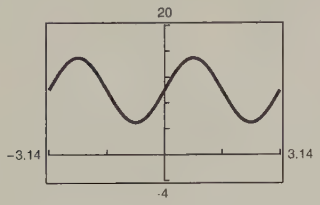
57.



Amplitude: 1

Period: 1

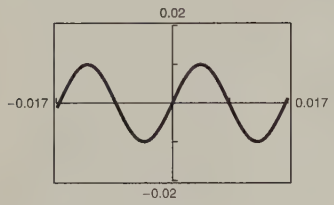
59.



Amplitude: 5

Period: π

61.



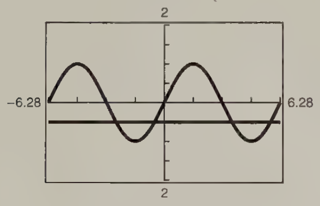
Amplitude: $\frac{1}{100}$

Period: $\frac{1}{60}$

63. $f(x) = 4 - 4\cos x$ **65.** $f(x) = 1 - 6\cos x$

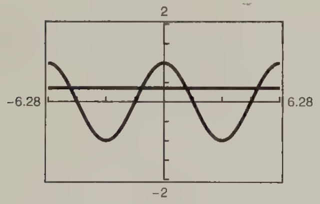
67. $f(x) = -3\sin(2x)$ **69.** $f(x) = \sin\left(x - \frac{\pi}{4}\right)$

71.



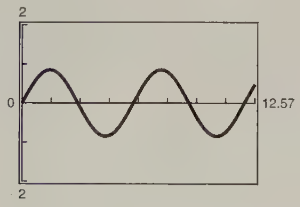
$x = -\frac{\pi}{6}, -\frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

73.



$x = \pm\frac{\pi}{4}, \pm\frac{7\pi}{4}$

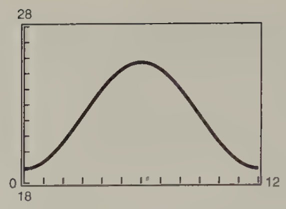
75. (a)



(b) 6 seconds

(c) 10 cycles per minute

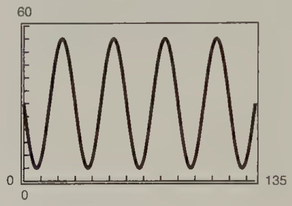
77.



Maximum sales: June

Minimum sales: December

79. (a)



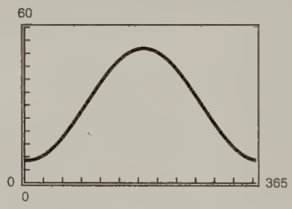
(b) Minimum height: 5 feet

Maximum height: 55 feet

81. (a) 365 days. The cycle is one year.

(b) 30.35 gallons per day; Both terms of the model; Average of the minimum and maximum of the model

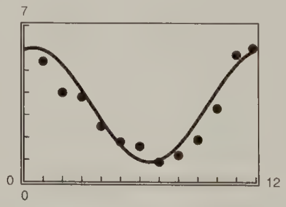
(c)



Consumption exceeds 40 gallons per day from the beginning of May through part of September.

83. (a) $p = 2.55\cos\left(\frac{\pi t}{6} - \frac{\pi}{12}\right) + 3.45$

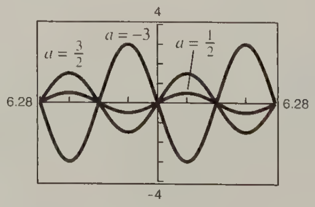
(b)



The model fits the data fairly well.

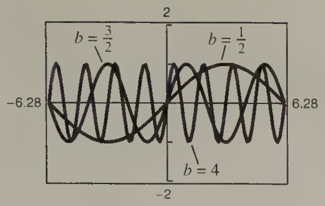
85. True

87.



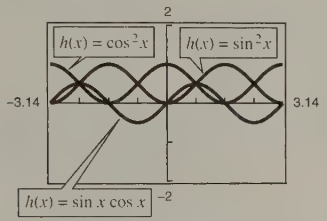
Amplitude changes

89.



Period changes

91.



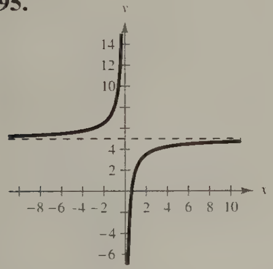
(a) Even

(b) Even

(c) Odd

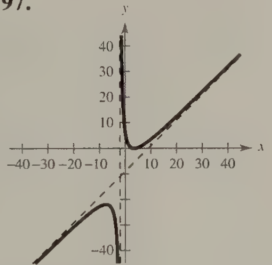
93. g is the periodic function and f is the parabola.

95.



Asymptotes: $x = 0, y = 5$

97.



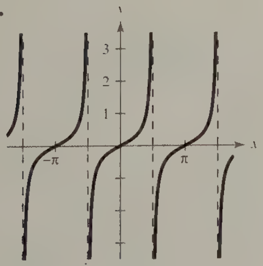
Asymptotes: $x = -2, y = x - 9$

99. $-20°$ **101.** $-27.502°$

Section 5.5 *(page 419)*

1. (g), 4π **3.** (f), $\frac{\pi}{2}$ **5.** (b), 2 **7.** (e), 2π

9.



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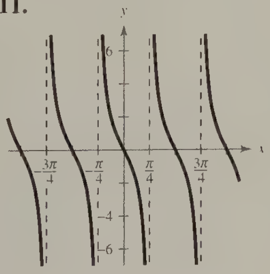

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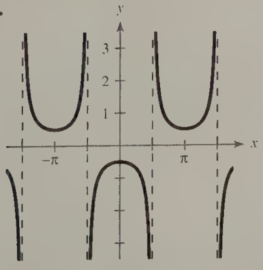

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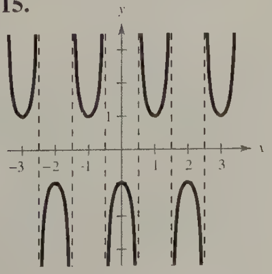

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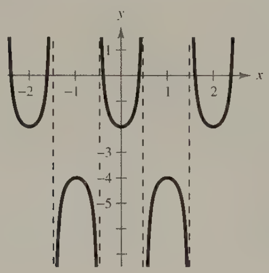

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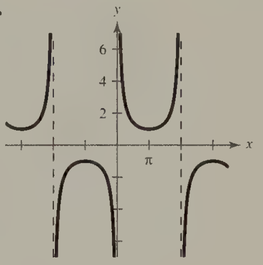

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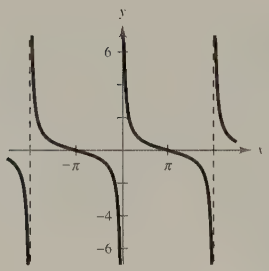

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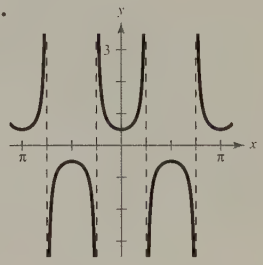

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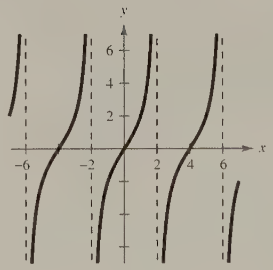

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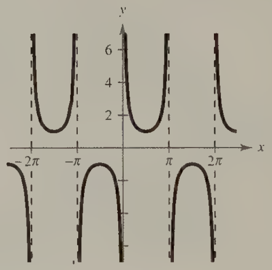

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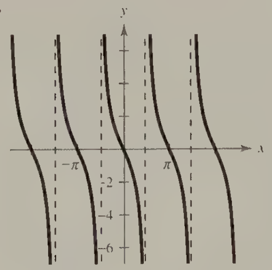

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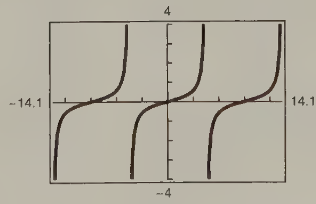

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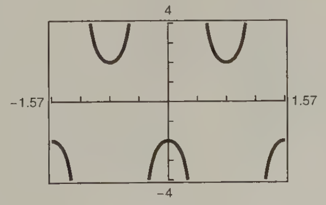

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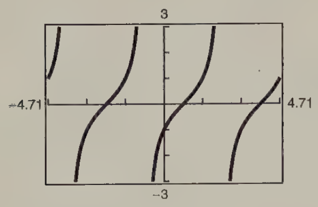

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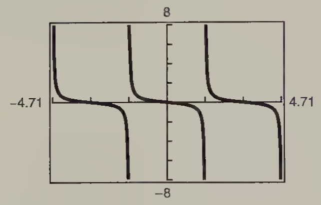

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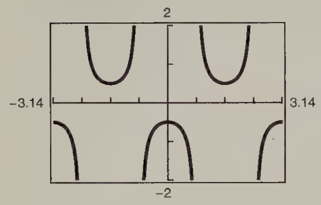
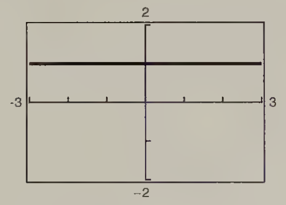


41. $-\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$ **43.** $-\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$

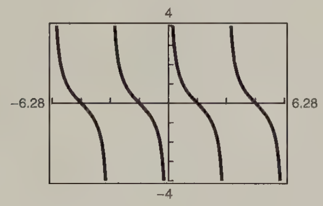
45. Even

47.



Not equivalent; y_1 is undefined at $x = 0$.

49.

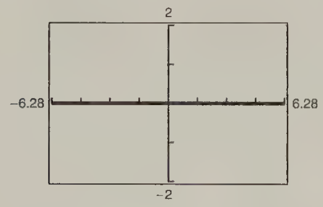


Equivalent

51. (d); as x approaches 0, $f(x)$ approaches 0.

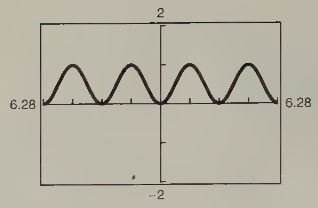
53. (b); as x approaches 0, $g(x)$ approaches 0.

55.



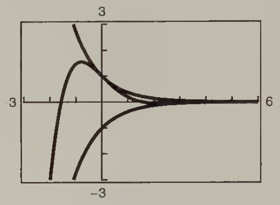
Equal

57.



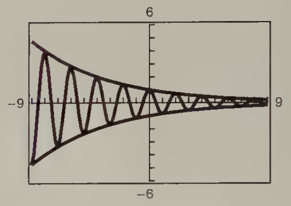
Equal

59.



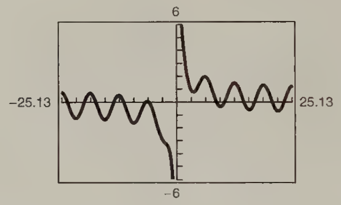
$f \to 0$ as $x \to \infty$

61.



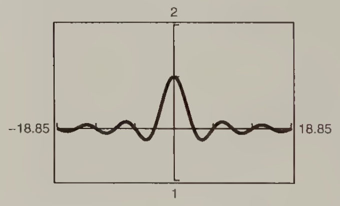
$f \to 0$ as $x \to \infty$

63.



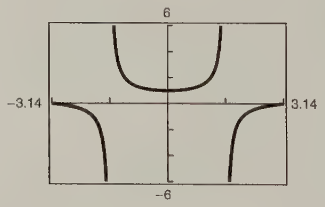
Approaches ±∞

65.



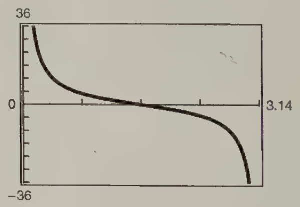
Approaches 1

67.



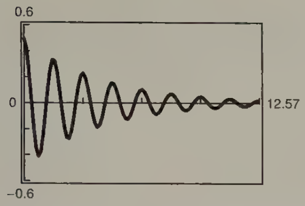
Approaches 1

69. $d = 5 \cot x$



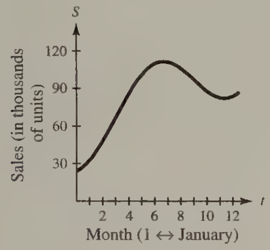
71. As the predator population increases, the number of prey decreases. When the number of prey is small, the number of predators decreases.

73. (a)



(b) Not periodic and damped; goes to 0 as t increases.

75. (a)



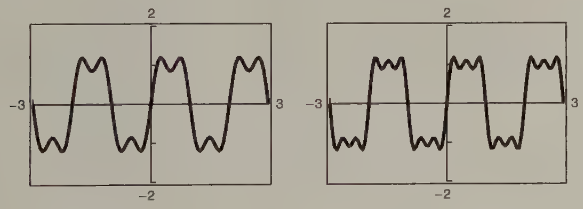
(b) According to the graph, the greatest number of sales occurs when $t = 6$, or in June. The least number of sales is in January.

77. (a) $0 < c < 8$ (b) $0.22 < b < 1$

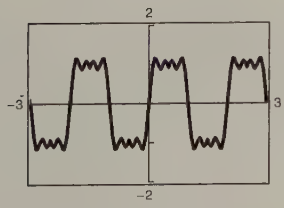
79. True **81.** True

83. As x approaches π from the left, f approaches ∞. As x approaches π from the right, f approaches $-\infty$.

85. (a)



(b) $y_3 = \dfrac{4}{\pi}\left[\sin(\pi x) + \dfrac{1}{3}\sin(3\pi x) + \dfrac{1}{5}\sin(5\pi x) + \dfrac{1}{7}\sin(7\pi x)\right]$



(c) $y_4 = \dfrac{4}{\pi}\left[\sin(\pi x) + \dfrac{1}{3}\sin(3\pi x) + \dfrac{1}{5}\sin(5\pi x) + \dfrac{1}{7}\sin(7\pi x) + \dfrac{1}{9}\sin(9\pi x)\right]$

87. Not one-to-one **89.** Yes. $f^{-1}(x) = x^3 + 5$

91. $\sin\theta = \dfrac{9\sqrt{277}}{277}$ $\csc\theta = \dfrac{\sqrt{277}}{9}$

$\cos\theta = \dfrac{14\sqrt{277}}{277}$ $\sec\theta = \dfrac{\sqrt{277}}{14}$

$\tan\theta = \dfrac{9}{14}$ $\cot\theta = \dfrac{14}{9}$

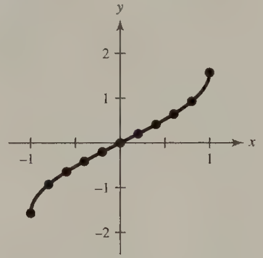
Section 5.6 *(page 429)*

1. (a)

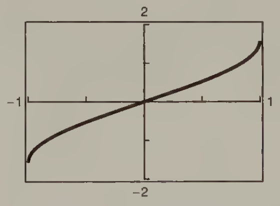
x	-1	-0.8	-0.6	-0.4	-0.2
y	-1.5708	-0.9273	-0.6435	-0.4115	-0.2014

x	0	0.2	0.4	0.6	0.8	1
y	0	0.2014	0.4115	0.6435	0.9273	1.5708

(b)



(c)



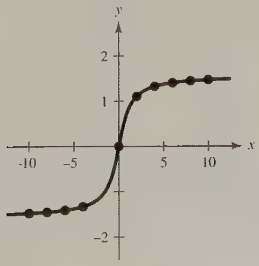
(d) Intercept: $(0, 0)$; Symmetric about the origin

3. (a)

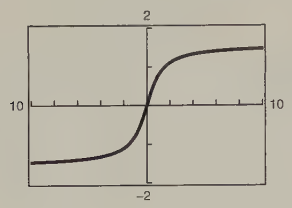
x	-10	-8	-6	-4	-2
y	-1.4711	-1.4464	-1.4056	-1.3258	-1.1071

x	0	2	4	6	8	10
y	0	1.1071	1.3258	1.4056	1.4464	1.4711

(b)



(c)



(d) $y = \pm\frac{\pi}{2}$

5. $\arctan(-1) = -\frac{\pi}{4}$ **7.** $\sin\left(-\frac{\pi}{2}\right) = -1$

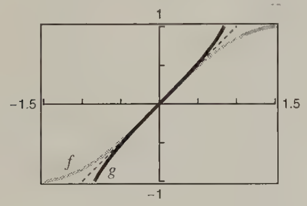
9. (a) 1.047 (b) 1.571 **11.** (a) 2.356 (b) −0.785

13. (a) 2.094 (b) 0.785 **15.** (a) −1.571 (b) 0

17. $\pi, \frac{2\pi}{3}, \frac{\sqrt{3}}{2}$ **19.** (a) −0.85 (b) 2.35

21. (a) 0.42 (b) 1.20 **23.** (a) 0.78 (b) 1.36

25.



27. $\theta = \arccos\frac{4}{x}$

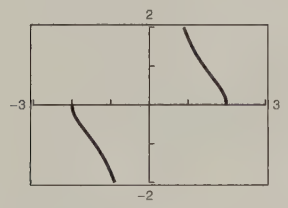
29. $\theta = \arctan\frac{x+1}{10}$ **31.** 35 **33.** −0.1 **35.** $\frac{\pi}{2}$

37. $-\frac{\pi}{4}$ **39.** $\frac{5}{4}$ **41.** $-\frac{13}{12}$ **43.** $-\frac{3\sqrt{55}}{55}$ **45.** $\frac{11}{6}$

47. $\frac{x}{\sqrt{x^2+1}}$ **49.** $\frac{1}{\sqrt{2x-x^2}}$ **51.** x

53. $\frac{\sqrt{r^2-(x-h)^2}}{r}$

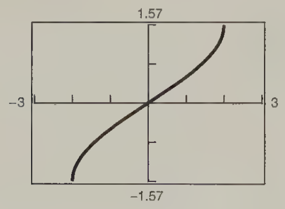
55.



$x = 0$

57. $\frac{x}{6}$ **59.** $\frac{\sqrt{4x-x^2}}{x-2}$

61.



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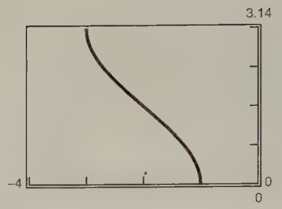

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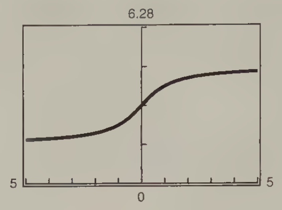

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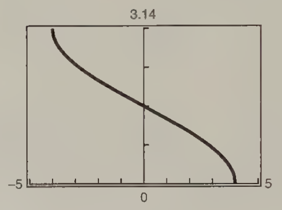
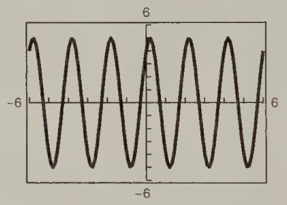


69. $5\sin\left(\pi t + \arctan\frac{4}{3}\right)$



The two forms are equivalent.

71. (a) $\theta = \arctan\frac{s}{750}$ (b) 0.49 rad, 1.13 rad

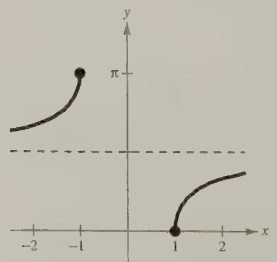
73. (a) 0.45 rad (b) 24.39 feet

75. (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) 1.25 (d) 2.03

77. (a) $\theta = \arctan\frac{x}{20}$ (b) 0.24 rad, 0.54 rad

79. False; Inverse trigonometric functions do not have the same relationships that trigonometric functions have.

81.



83. (a) $\frac{\pi}{4}$

(b) 0

(c) $\frac{5\pi}{6}$

(d) $\frac{\pi}{6}$

85.–89. Answers will vary.

91. $\sin(-585^\circ) = \frac{\sqrt{2}}{2}$

$\cos(-585^\circ) = -\frac{\sqrt{2}}{2}$

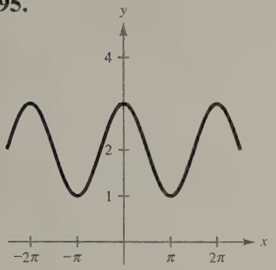
$\tan(-585^\circ) = -1$

93. $\sin\left(-\frac{19\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

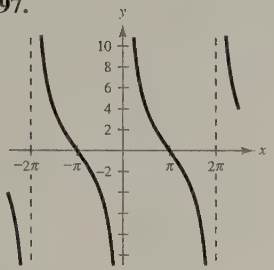
$\cos\left(-\frac{19\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$\tan\left(-\frac{19\pi}{4}\right) = 1$

95.



97.



Section 5.7 *(page 439)*

1. $a \approx 4.66$
$c \approx 11.03$
$B = 65^\circ$

3. $a \approx 8.26$
$c \approx 25.38$
$A = 19^\circ$

5. $c \approx 17.09$
$A \approx 20.56^\circ$
$B \approx 69.44^\circ$

7. $a \approx 37.95$
$A \approx 64.62^\circ$
$B \approx 25.38^\circ$

9. $a \approx 91.34$
$b \approx 420.70$
$B = 77^\circ 45'$

11. 2.56 inches **13.** 6.30 feet

15. (a) $L = 60 \cot\theta$

(b)

θ	10°	20°	30°	40°	50°
L	340	165	104	72	50

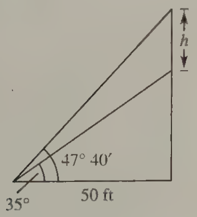
(c) No. Cotangent is not a linear function.

17. (a) $h = 20 \sin\theta$

(b)

θ	60°	65°	70°	75°	80°
h	17.3	18.1	18.8	19.3	19.7

19. (a)



(b) $h = 50(\tan 47^\circ 40' - \tan 35^\circ)$ (c) 19.9 feet

21. ≈2090 feet **23.** ≈38.3° **25.** ≈15.5°

27. ≈5099 feet **29.** ≈0.66 miles

31. ≈437 miles north; ≈700 miles east

33. (a) N 58° E (b) ≈68.82 meters **35.** N 55° W

37. ≈1933.3 feet **39.** 17,054 feet ≈ 3.23 miles

41. 78.69° **43.** ≈35.3° **45.** $y = \sqrt{3}r$

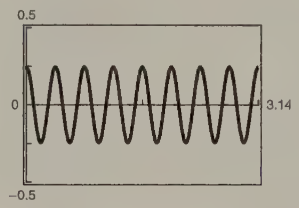
47. ≈28.2 inches **49.** $a \approx 12.2, b \approx 7$

51. (a) 4 (b) 4 (c) $\frac{1}{16}$

53. (a) $\frac{1}{16}$ (b) 70 (c) $\frac{1}{140}$ **55.** $y = 8 \sin \pi t$

57. $y = 3 \cos\left(\frac{10\pi t}{9}\right)$ **59.** $\omega = 528\pi$

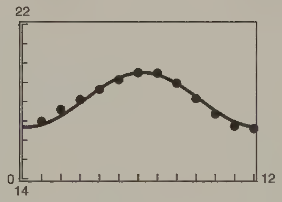
61. (a)



(b) $\frac{\pi}{8}$ seconds

(c) $\frac{\pi}{32}$ seconds

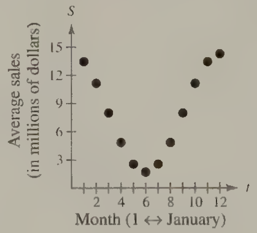
63. (a)



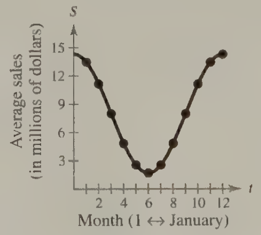
(b) 12 months. Yes, 1 period is 1 year.

(c) 1.41 hours. 1.41 represents the maximum change in time from the average time ($d = 18.09$) of sunset.

65. (a)



(b)



$S = 8 + 6.3 \cos\left(\frac{\pi t}{6}\right)$

The model is a good fit.

(c) 12 months. Sales of outerwear is seasonal.

(d) Maximum displacement of 6.3 from the average sales of 8

67. False. $a = \dfrac{22.56}{\tan(48.1^\circ)}$ **69.** $\dfrac{2x^3 + x^2 - x - 6}{2x^2 + 13x + 6}$

71. $\dfrac{(x-4)(x-3)}{4(x+2)}, \ x \neq -6, 0, \dfrac{1}{3}$ **73.** $x = 1.994$

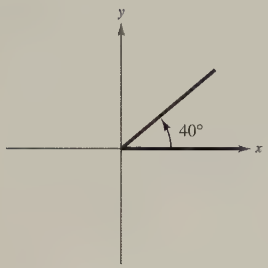
75. $x = \pm 4.851$ **77.** 1.44 radians or 82.53°

79. -0.11 radian or $-6.32°$

Review Exercises *(page 446)*

1. Negative

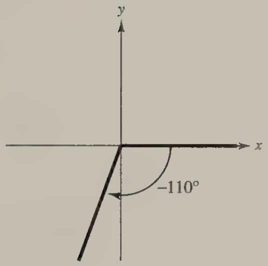
3. (a)



(b) Quadrant I

(c) 400°, −320°

5. (a)



(b) Quadrant III

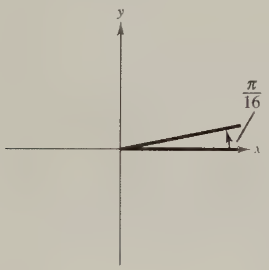
(c) 250°, −470°

7. Complement: 82°; Supplement: 172°

9. Complement: none; Supplement: 9° **11.** 135.28°

13. 5.38° **15.** 135° 17′24″ **17.** −85° 21′36″

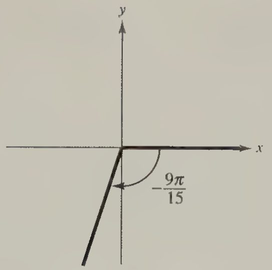
19. (a)



(b) Quadrant I

(c) $\dfrac{33\pi}{16}, -\dfrac{31\pi}{16}$

21. (a)



(b) Quadrant III

(c) $\dfrac{21\pi}{15}, -\dfrac{39\pi}{15}$

23. Complement: $\dfrac{3\pi}{8}$; Supplement: $\dfrac{7\pi}{8}$

25. Complement: $\dfrac{\pi}{5}$; Supplement: $\dfrac{7\pi}{10}$ **27.** 128.57°

29. −200.54° **31.** 8.3776 **33.** −0.5890

35. 2.083 rad **37.** 48.171 meters **39.** 0.094 rad ≈ 5.39°

41. $\sin\theta = \dfrac{5\sqrt{61}}{61}$

$\cos\theta = \dfrac{6\sqrt{61}}{61}$

$\tan\theta = \dfrac{5}{6}$

$\csc\theta = \dfrac{\sqrt{61}}{5}$

$\sec\theta = \dfrac{\sqrt{61}}{6}$

$\cot\theta = \dfrac{6}{5}$

43. $\sin\theta = \dfrac{\sqrt{65}}{9}$

$\cos\theta = \dfrac{4}{9}$

$\tan\theta = \dfrac{\sqrt{65}}{4}$

$\csc\theta = \dfrac{9\sqrt{65}}{65}$

$\sec\theta = \dfrac{9}{4}$

$\cot\theta = \dfrac{4\sqrt{65}}{65}$

45. Answers will vary. **47.** (a) 0.1045 (b) 0.1045

49. (a) 0.7071 (b) 1.4142 **51.** 9.2 meters

53. $\sin\theta = \dfrac{4}{5}$

$\cos\theta = \dfrac{3}{5}$

$\tan\theta = \dfrac{4}{3}$

$\csc\theta = \dfrac{5}{4}$

$\sec\theta = \dfrac{5}{3}$

$\cot\theta = \dfrac{3}{4}$

55. $\sin\theta = \dfrac{2\sqrt{53}}{53}$

$\cos\theta = -\dfrac{7\sqrt{53}}{53}$

$\tan\theta = -\dfrac{2}{7}$

$\csc\theta = \dfrac{\sqrt{53}}{2}$

$\sec\theta = -\dfrac{\sqrt{53}}{7}$

$\cot\theta = -\dfrac{7}{2}$

57. $\sin\theta = \dfrac{15\sqrt{241}}{241}$

$\cos\theta = \dfrac{4\sqrt{241}}{241}$

$\tan\theta = \dfrac{15}{4}$

$\csc\theta = \dfrac{\sqrt{241}}{15}$

$\sec\theta = \dfrac{\sqrt{241}}{4}$

$\cot\theta = \dfrac{4}{15}$

59. $\sin\theta = -\dfrac{\sqrt{11}}{6}$

$\cos\theta = \dfrac{5}{6}$

$\tan\theta = -\dfrac{\sqrt{11}}{5}$

$\csc\theta = -\dfrac{6\sqrt{11}}{11}$

$\cot\theta = -\dfrac{5\sqrt{11}}{11}$

61. $\cos\theta = -\frac{\sqrt{55}}{8}$

$\tan\theta = -\frac{3\sqrt{55}}{55}$

$\csc\theta = \frac{8}{3}$

$\sec\theta = -\frac{8\sqrt{55}}{55}$

$\cot\theta = -\frac{\sqrt{55}}{3}$

63. $\sqrt{3}$ **65.** $\frac{\sqrt{3}}{2}$ **67.** $-\frac{\sqrt{2}}{2}$ **69.** 0.65

71. 3.24 **73.** 84° **75.** $\frac{\pi}{5}$

77. $\sin 240^\circ = -\frac{\sqrt{3}}{2}$

$\cos 240^\circ = -\frac{1}{2}$

$\tan 240^\circ = \sqrt{3}$

79. $\sin(-210^\circ) = \frac{1}{2}$

$\cos(-210^\circ) = -\frac{\sqrt{3}}{2}$

$\tan(-210^\circ) = -\frac{\sqrt{3}}{3}$

81. $\sin\left(-\frac{9\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$\cos\left(-\frac{9\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\tan\left(-\frac{9\pi}{4}\right) = -1$

83. $\sin\left(-\frac{\pi}{2}\right) = -1$

$\cos\left(-\frac{\pi}{2}\right) = 0$

$\tan\left(-\frac{\pi}{2}\right)$ is undefined.

85. $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{2\pi}{3} = -\frac{1}{2}$

$\tan\frac{2\pi}{3} = -\sqrt{3}$

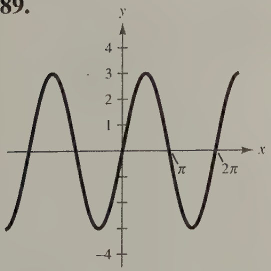
87. $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

$\sin\frac{7\pi}{6} = -\frac{1}{2}$

$\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

$\tan\frac{7\pi}{6} = \frac{\sqrt{3}}{3}$

89.



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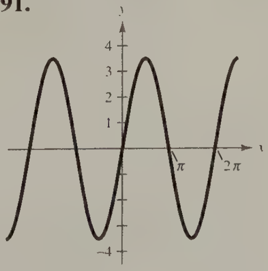
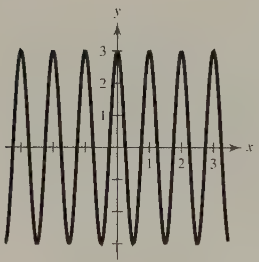


93. Period: 2; Amplitude: 5

95. Period: π; Amplitude: 3.4

97.



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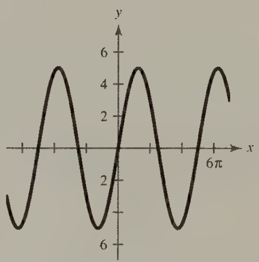

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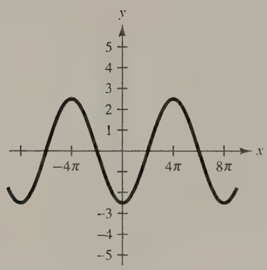

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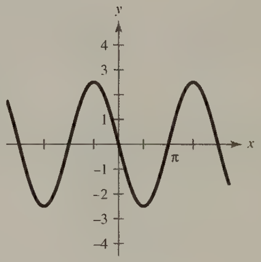

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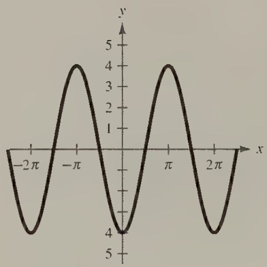

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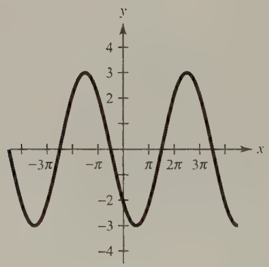
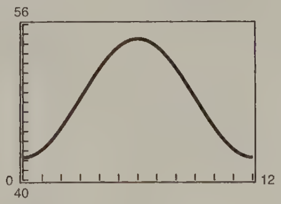


109. $f(x) = -2\cos\left(x - \frac{\pi}{4}\right)$

111. $f(x) = -4\cos\left(2x - \frac{\pi}{2}\right)$

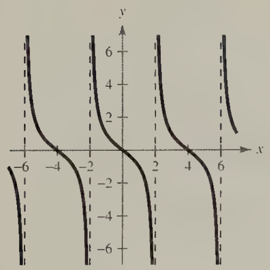
113.



Maximum sales: June; Minimum sales: December

115. $f(x) = \frac{1}{2}\tan\left(\frac{1}{2}x\right)$

117.



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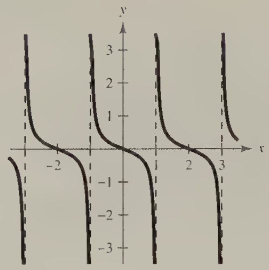

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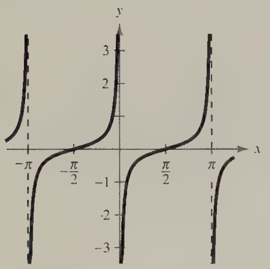

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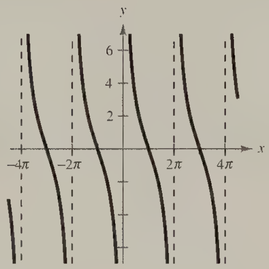

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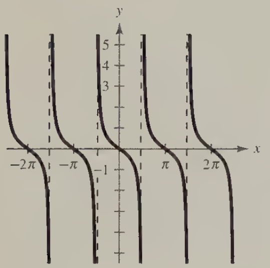

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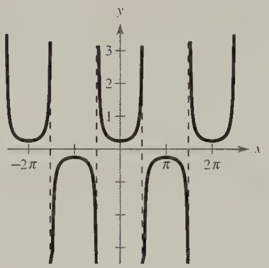

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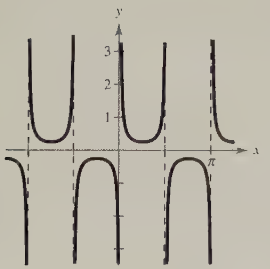

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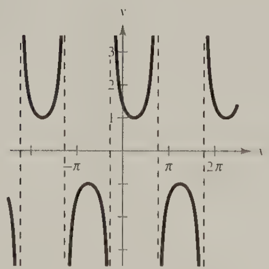

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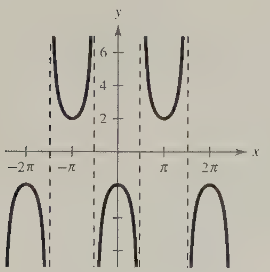

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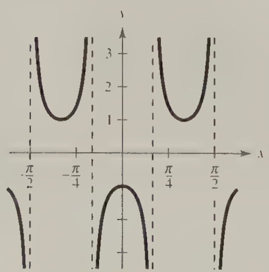

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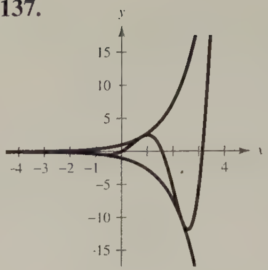
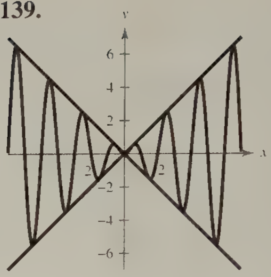


139.



141. (a) $\frac{\pi}{2}$ (b) Does not exist **143.** (a) $\frac{\pi}{4}$ (b) $\frac{5\pi}{6}$

145. (a) 1.137 (b) 0.682

147. (a) −1.488 (b) 1.523 **149.** $\theta = \arcsin\left(\frac{x+3}{16}\right)$

151. $\frac{\sqrt{-x^2+2x}}{-x^2+2x}$ **153.** $\frac{2\sqrt{4-2x^2}}{4-x^2}$ **155.** $\theta \approx 57.3°$

157. About 0.071 kilometer or 71 meters

159. ≈612 miles north and ≈634 miles west

161. $y = 3\cos\left(\frac{2\pi t}{15}\right)$

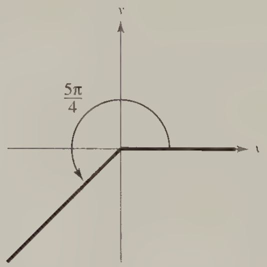
163. False. The expressions are not equal.

165. False. $3\pi/4$ is not in the range of the arctangent function.

167. Increases. The linear velocity is proportional to the radius.

Chapter Test *(page 452)*

1. (a)



(b) $\frac{13\pi}{4}, -\frac{3\pi}{4}$

(c) 225°

2. 3000 radians per minute

3. $\sin\theta = \frac{4\sqrt{17}}{17}$

$\cos\theta = -\frac{\sqrt{17}}{17}$

$\tan\theta = -4$

$\csc\theta = \frac{\sqrt{17}}{4}$

$\sec\theta = -\sqrt{17}$

$\cot\theta = -\frac{1}{4}$

4. $\sin\theta = \pm\frac{11\sqrt{157}}{157}$

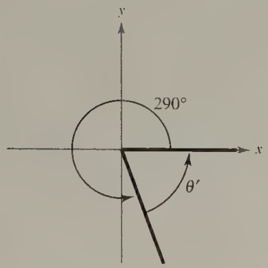
$\cos\theta = \pm\frac{6\sqrt{157}}{157}$

$\csc\theta = \pm\frac{\sqrt{157}}{11}$

$\sec\theta = \pm\frac{\sqrt{157}}{6}$

$\cot\theta = \frac{6}{11}$

5. $\theta' = 70°$



6. Quadrant III **7.** 150°, 210° **8.** 1.33, 1.81

9. $\sin\theta = -\dfrac{\sqrt{11}}{6}$

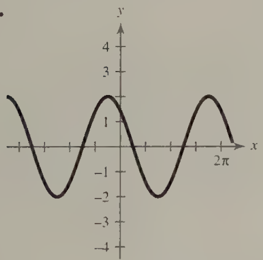
$\cos\theta = \dfrac{5}{6}$

$\tan\theta = -\dfrac{\sqrt{11}}{5}$

$\csc\theta = -\dfrac{6\sqrt{11}}{11}$

$\cot\theta = -\dfrac{5\sqrt{11}}{11}$

10.



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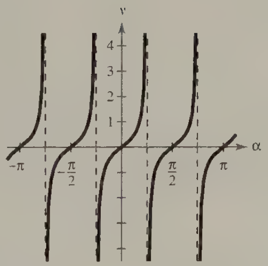

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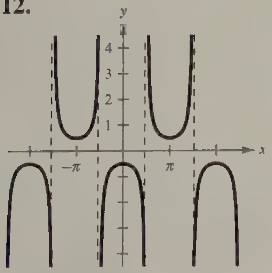

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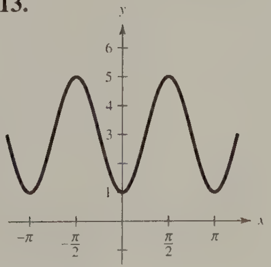

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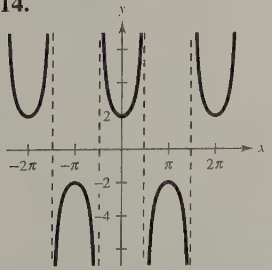

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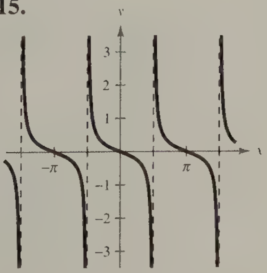

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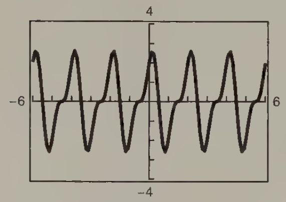
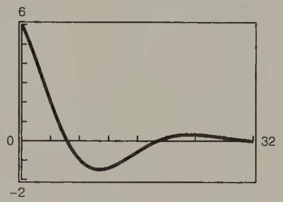


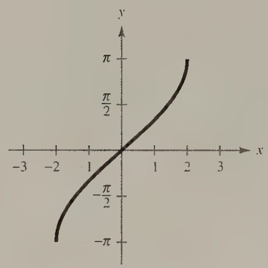
Period: 2

17.



18. $y = -2\sin\left(\dfrac{x}{2} - \dfrac{\pi}{4}\right)$ **19.** $\dfrac{\sqrt{5}}{2}$

20.



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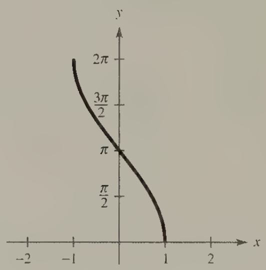
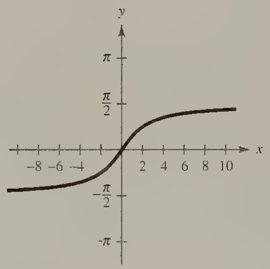


22.



23. S 34.5° W **24.** ≈ 2290 feet ≈ 0.434 mile

Chapter 6

Section 6.1 *(page 459)*

1. $\tan x = \sqrt{3}$

$\csc x = \dfrac{2\sqrt{3}}{3}$

$\sec x = 2$

$\cot x = \dfrac{\sqrt{3}}{3}$

3. $\cos\theta = \dfrac{\sqrt{2}}{2}$

$\tan\theta = -1$

$\csc\theta = -\sqrt{2}$

$\cot\theta = -1$

5. $\sin x = -\frac{7}{25}$

$\cos x = -\frac{24}{25}$

$\csc x = -\frac{25}{7}$

$\cot x = \frac{24}{7}$

7. $\cos\phi = -1$

$\tan\phi = 0$

$\csc\phi$ is undefined.

$\cot\phi$ is undefined.

9. $\sin x = \frac{2}{3}$

$\cos x = -\frac{\sqrt{5}}{3}$

$\csc x = \frac{3}{2}$

$\sec x = -\frac{3\sqrt{5}}{5}$

$\cot x = -\frac{\sqrt{5}}{2}$

11. $\sin \theta = -\frac{4\sqrt{17}}{17}$

$\cos \theta = -\frac{\sqrt{17}}{17}$

$\csc \theta = -\frac{\sqrt{17}}{4}$

$\sec \theta = -\sqrt{17}$

$\cot \theta = \frac{1}{4}$

13. $\cos \theta = 0$

$\tan \theta$ is undefined.

$\csc \theta = -1$

$\sec \theta$ is undefined.

15. 1, 1 **17.** $\infty, 0$ **19.** (d) **21.** (a)

23. (e) **25.** (b) **27.** (f) **29.** (e)

31. $\cos x$ **33.** $\cos^2 \phi$ **35.** $\cos x$

37. 1 **39.** $-\tan x$ **41.** $\cot x$ **43.** $1 + \sin y$

45.–61. Answers will vary.

63. $\cos^2 x$ **65.** $\sin^2 x \tan^2 x$ **67.** $\sec^4 x$

69. $\sin^2 x - \cos^2 x$ **71.** $1 + 2 \sin x \cos x$

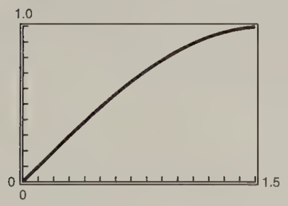
73. $\tan^2 x$ **75.** $2 \csc^2 x$ **77.** $2 \sec x$

79. $1 + \cos y$ **81.** $3(\sec x + \tan x)$

83.

x	0.2	0.4	0.6	0.8
y_1	0.1987	0.3894	0.5646	0.7174
y_2	0.1987	0.3894	0.5646	0.7174

x	1.0	1.2	1.4
y_1	0.8415	0.9320	0.9854
y_2	0.8415	0.9320	0.9854

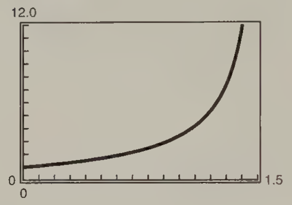


$y_1 = y_2$

85.

x	0.2	0.4	0.6	0.8
y_1	1.2230	1.5085	1.8958	2.4650
y_2	1.2230	1.5085	1.8958	2.4650

x	1.0	1.2	1.4
y_1	3.4082	5.3319	11.6814
y_2	3.4082	5.3319	11.6814



$y_1 = y_2$

87. $\csc x$ **89.** $\tan x$ **91.** $5 \cos \theta$ **93.** $3 \tan \theta$

95. $5 \sec \theta$ **97.** $0 \le \theta \le \pi$

99. $0 \le \theta < \frac{\pi}{2}, \frac{3\pi}{2} < \theta < 2\pi$ **101.** $\ln|\cot \theta|$

103. $\ln[(\cos x)(1 + \sin x)]$

105. (a) $\csc^2(132°) - \cot^2(132°) \approx 1.8107 - 0.8107 = 1$

(b) $\csc^2\left(\frac{2\pi}{7}\right) - \cot^2\left(\frac{2\pi}{7}\right) \approx 1.63596 - 0.63596 = 1$

107. (a) $\cos(90° - 80°) = \cos(10°) \approx 0.9848$;

$\sin(80°) \approx 0.9848$

(b) $\cos\left(\frac{\pi}{2} - 0.8\right) \approx 0.7174$; $\sin(0.8) \approx 0.7174$

109. Answers will vary.

111. False. $5 \sec \theta = \frac{5}{\cos \theta} \neq \frac{1}{5 \cos \theta}$

113. False. $\sin \theta \csc \phi = \frac{\sin \theta}{\sin \phi} \neq 1$ when $\theta \neq \phi$.

115. $\sin \theta = \pm\sqrt{1 - \cos^2 \theta}$

$\tan \theta = \pm\frac{\sqrt{1 - \cos^2 \theta}}{\cos \theta}$

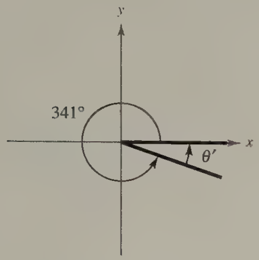
$\csc \theta = \pm\frac{1}{\sqrt{1 - \cos^2 \theta}}$

$\sec \theta = \frac{1}{\cos \theta}$

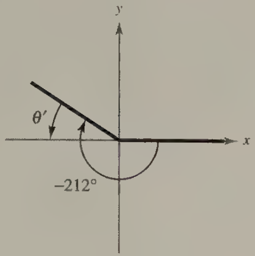
$\cot \theta = \pm\frac{\cos \theta}{\sqrt{1 - \cos^2 \theta}}$

The sign depends on the choice of θ.

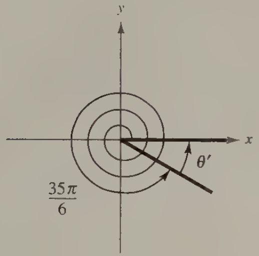
117. $\theta = 341°, \theta' = 19°$



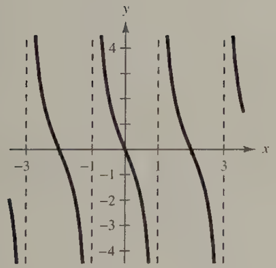
119. $\theta = -212°, \theta' = 32°$



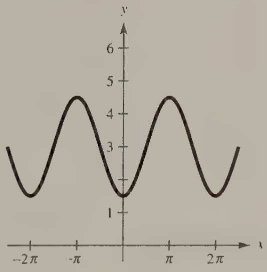
121. $\theta = \dfrac{35\pi}{6}, \theta' = \dfrac{\pi}{6}$



123.



125.



127. $B = 62°$

$C = 90°$

$a = 9.39$

$b = 17.66$

129. $A = 60.37°$

$B = 29.63°$

$C = 90°$

$a = 10.9$

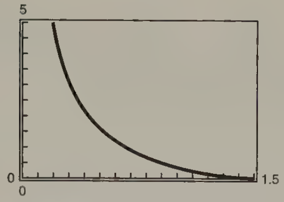
Section 6.2 *(page 467)*

1.–9. Answers will vary.

11.

x	0.2	0.4	0.6	0.8
y_1	4.835	2.1785	1.2064	0.6767
y_2	4.835	2.1785	1.2064	0.6767

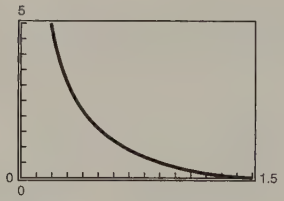
x	1.0	1.2	1.4
y_1	0.3469	0.1409	0.0293
y_2	0.3469	0.1409	0.0293



13.

x	0.2	0.4	0.6	0.8
y_1	4.835	2.1785	1.2064	0.6767
y_2	4.835	2.1785	1.2064	0.6767

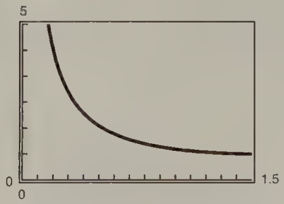
x	1.0	1.2	1.4
y_1	0.3469	0.1409	0.0293
y_2	0.3469	0.1409	0.0293



15.

x	0.2	0.4	0.6	0.8
y_1	5.0335	2.5679	1.771	1.394
y_2	5.0335	2.5679	1.771	1.394

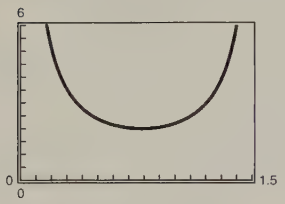
x	1.0	1.2	1.4
y_1	1.1884	1.0729	1.0148
y_2	1.1884	1.0729	1.0148



17.

x	0.2	0.4	0.6	0.8
y_1	5.1359	2.7880	2.1458	2.0009
y_2	5.1359	2.7880	2.1458	2.0009

x	1.0	1.2	1.4
y_1	2.1995	2.9609	5.9704
y_2	2.1995	2.9609	5.9704



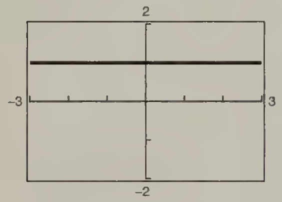
19. $\cot(-x) = -\cot(x)$ so,

$(1 + \tan x)[1 + \cos(-x)] = \tan x - \cot x.$

21. $(\tan^2 x)^2$

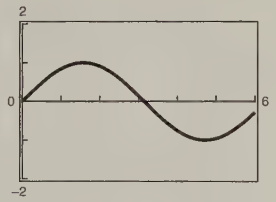
23.–51. Answers will vary.

53.



1

55.



$\sin x$

57. and 59. Answers will vary. **61.** 1 **63.** 2

65. and 67. Answers will vary.

69. $\mu = \tan\theta,\ W \neq 0$ **71.** Answers will vary.

73. True. Cosine and secant are even functions.

75. False. $\sin^2(x) = (\sin x)^2 \neq \sin(x^2)$

77. $|\sin\theta| = \sqrt{1 - \cos^2\theta};\ \frac{7\pi}{4}$

79. $\sqrt{\sin^2 x + \cos^2 x} \neq \sin x + \cos x$. The left side is 1 for any x, but the right side is not necessarily 1; $\pi/4$.

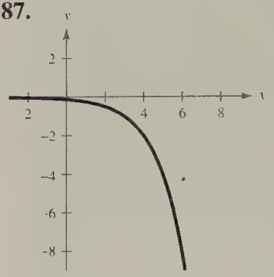
81. Answers will vary.

83. Answer is not unique. Sample answer: $y = x^4 + 17x + 16$

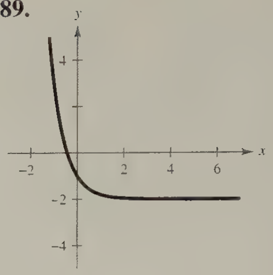
85. Answer is not unique. Sample answer:

$y = x^5 - 4x^4 + 6x^3 - 4x^2$

87.



89.



91. $\theta \approx 2.36$ rad **93.** Quadrant III **95.** Quadrant III

Section 6.3 *(page 478)*

1.–5. Answers will vary. **7.** $-1, 3$ **9.** ± 2

11. $\frac{2\pi}{3}, \frac{4\pi}{3}$ **13.** $\frac{\pi}{6}, \frac{11\pi}{6}$ **15.** $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

17. $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

19. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **21.** $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

23. $\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}$ **25.** $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

27. $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$ **29.** $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

31. No solution **33.** $\frac{\pi}{2}$ **35.** $\frac{\pi}{2}$ **37.** π

39. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **41.** $2.0344, 5.1760, \frac{\pi}{4}, \frac{5\pi}{4}$

43. 1.4595, 4.8237 **45.** 0.8614, 5.4218

47. 0.7854, 2.3562, 3.6652, 3.9270, 5.4978, 5.7596

49. 0.5236, 2.6180 **51.** 4.9172

53. 0, 2.6779, 3.1416, 5.8195

55. 0.3398, 0.8481, 2.2935, 2.8018

57. −1.154, 0.534 **59.** 1.110

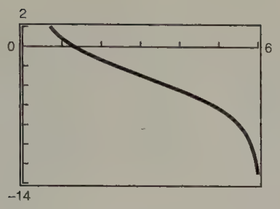
61. (a)

x	0	1	2	3
$f(x)$	Undef.	0.83	−1.36	−2.93

x	4	5	6
$f(x)$	−4.46	−6.34	−13.02

The zero is in the interval (1, 2) because f changes signs in that interval.

(b)



The interval is the same as in part (a).

(c) 1.3065

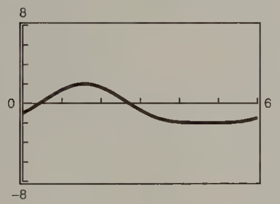
63. (a)

x	0	1	2	3
$f(x)$	-1	1.39	1.65	-0.70

x	4	5	6
$f(x)$	-1.94	-2.00	-1.48

The zeros are in the intervals $(0, 1)$ and $(2, 3)$ because f changes signs in those intervals.

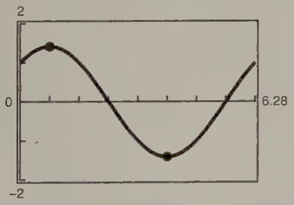
(b)



The intervals are the same as in part (a).

(c) 0.4271, 2.7145

65. (a)

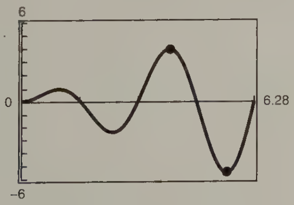


Maximum: $(0.785, 1.41)$

Minimum: $(3.93, -1.41)$

(b) $\frac{\pi}{4}, \frac{5\pi}{4}$

67. (a)



Maximum: $(3.99, 3.96)$

Minimum: $(5.54, -5.52)$

(b) 0, 1.01, 2.46, 3.99, 5.44

69. 1

71. (a) All real numbers x except $x = 0$

(b) y-axis symmetry; Horizontal asymptote: $y = 1$

(c) Oscillates (d) Infinite number of solutions

(e) Yes, 0.6366

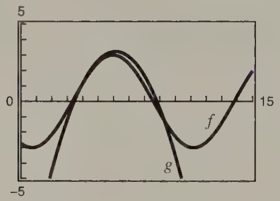
73. 0.04 second, 0.43 second, 0.83 second

75. 122 days $< t <$ 223 days

77. 1.91° **79.** 1.96 seconds

81. (a) 3.33

(b)



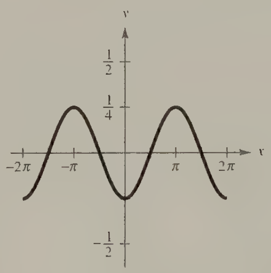
(c) 3.46, 8.81. The zero of $g(x)$ on $[0, 6]$ is 3.46, which is closest to the zero of $f(x)$.

83. False. Some trigonometric equations don't have any solutions in the interval $[0, 2\pi)$.

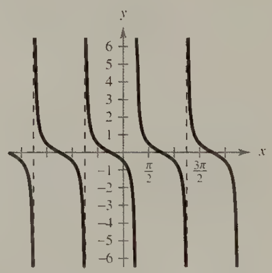
85. 2.164 rad **87.** -0.007 rad

89. 24.249 **91.** 24.892

93.



95.



Section 6.4 *(page 486)*

1. (a) 0 (b) $\frac{1 + \sqrt{3}}{2}$ **3.** (a) $\frac{1}{2}$ (b) $\frac{-1 - \sqrt{3}}{2}$

5. (a) $-\frac{\sqrt{2}}{2}$ (b) $\frac{2 - \sqrt{2}}{2}$

7. (a) $\frac{-\sqrt{2} - \sqrt{6}}{4}$ (b) $\frac{-\sqrt{2} - \sqrt{3}}{2}$

9. $\sin(75°) = \frac{\sqrt{2} + \sqrt{6}}{4}$

$\cos(75°) = \frac{\sqrt{6} - \sqrt{2}}{4}$

$\tan(75°) = 2 + \sqrt{3}$

11. $\sin(105°) = \frac{\sqrt{6} + \sqrt{2}}{4}$

$\cos(105°) = \frac{\sqrt{2} - \sqrt{6}}{4}$

$\tan(105°) = -2 - \sqrt{3}$

13. $\sin(195°) = \dfrac{\sqrt{2} - \sqrt{6}}{4}$

$\cos(195°) = \dfrac{-\sqrt{2} - \sqrt{6}}{4}$

$\tan(195°) = 2 - \sqrt{3}$

15. $\sin\left(\dfrac{11\pi}{12}\right) = \dfrac{\sqrt{6} - \sqrt{2}}{4}$

$\cos\left(\dfrac{11\pi}{12}\right) = \dfrac{-\sqrt{6} - \sqrt{2}}{4}$

$\tan\left(\dfrac{11\pi}{12}\right) = -2 + \sqrt{3}$

17. $\sin\left(-\dfrac{\pi}{12}\right) = \dfrac{\sqrt{2} - \sqrt{6}}{4}$

$\cos\left(-\dfrac{\pi}{12}\right) = \dfrac{\sqrt{6} + \sqrt{2}}{4}$

$\tan\left(-\dfrac{\pi}{12}\right) = -2 + \sqrt{3}$

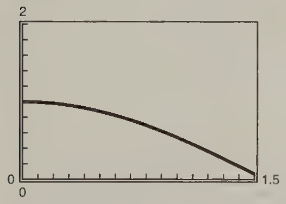
19. cos 55° **21.** sin 290° **23.** tan 239°

25. sin 1.8 **27.** $\cos \dfrac{12\pi}{35}$

29.

x	0.2	0.4	0.6	0.8
y_1	0.9801	0.9211	0.8253	0.6967
y_2	0.9801	0.9211	0.8253	0.6967

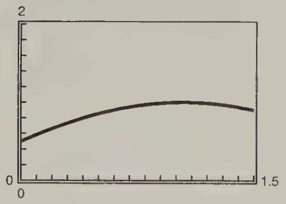
x	1.0	1.2	1.4
y_1	0.5403	0.3624	0.1700
y_2	0.5403	0.3624	0.1700



31.

x	0.2	0.4	0.6	0.8
y_1	0.6621	0.7978	0.9017	0.9696
y_2	0.6621	0.7978	0.9017	0.9696

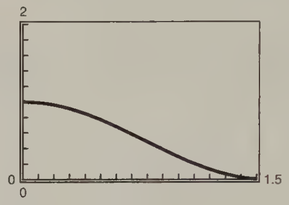
x	1.0	1.2	1.4
y_1	0.9989	0.9883	0.9384
y_2	0.9989	0.9883	0.9384



33.

x	0.2	0.4	0.6	0.8
y_1	0.9605	0.8484	0.6812	0.4854
y_2	0.9605	0.8484	0.6812	0.4854

x	1.0	1.2	1.4
y_1	0.2919	0.1313	0.0289
y_2	0.2919	0.1313	0.0289



35. $\frac{33}{65}$ **37.** $-\frac{56}{65}$ **39.** $-\frac{3}{5}$ **41.** $\frac{44}{125}$

43.–49. Answers will vary. **51.** 1

53. $\dfrac{2x^2 - \sqrt{1 - x^2}}{\sqrt{4x^2 + 1}}$ **55.** $\dfrac{\pi}{2}$ **57.** $\dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

59. $0, \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$ **61.** 0.7854, 5.4978 **63.** 0, 3.1416

65. Answers will vary.

67. False. $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$

69. False. $\sin(75°) = \dfrac{\sqrt{6} + \sqrt{2}}{4}$

71. and 73. Answers will vary.

75. (a) $\sqrt{2} \sin\left(\theta + \dfrac{\pi}{4}\right)$

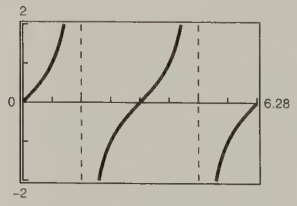
(b) $\sqrt{2} \cos\left(\theta - \dfrac{\pi}{4}\right)$

77. (a) $13 \sin(3\theta + 0.3948)$

(b) $13 \cos(3\theta - 1.1760)$

79. $2 \cos \theta$ **81.** $u + v = \omega$

83.



$h(\theta) = \tan \theta$

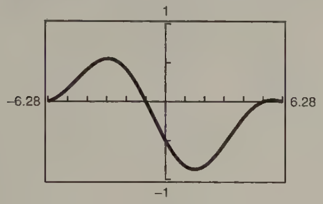
85. (a) $(-\infty, 0), (0, \infty)$

(b)

h	0.01	0.02	0.05
$f(h)$	-0.5043	-0.5086	-0.5214
$g(h)$	-0.5043	-0.5086	-0.5214

h	0.1	0.2	0.5
$f(h)$	-0.5424	-0.5830	-0.6915
$g(h)$	-0.5424	-0.5830	-0.6915

(c)



(d) $-\frac{1}{2}$

87. (38, 0), (0, 19) **89.** (0, 4), (2, 0), (7, 0)

91. $\frac{\pi}{6}$ **93.** $\frac{\pi}{2}$

Section 6.5 *(page 496)*

1. $\frac{3}{5}$ **3.** $\frac{7}{25}$ **5.** $\frac{24}{7}$ **7.** $\frac{25}{24}$ **9.** $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

11. 0.1263, 1.4445, 3.2679, 4.5860 **13.** $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

15. $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ **17.** $4 \sin 2x$ **19.** $5 \cos 2x$

21. $\sin 2u = \frac{24}{25}$
$\cos 2u = \frac{7}{25}$
$\tan 2u = \frac{24}{7}$

23. $\sin 2u = \frac{4}{5}$
$\cos 2u = \frac{3}{5}$
$\tan 2u = \frac{4}{3}$

25. $\frac{1}{8}(3 + 4 \cos 2x + \cos 4x)$ **27.** $\frac{1}{8}(1 - \cos 4x)$

29. $\frac{1}{32}(2 + \cos 2x - 2 \cos 4x - \cos 6x)$

31. $\frac{5\sqrt{26}}{26}$ **33.** $\frac{1}{5}$ **35.** $\sqrt{26}$ **37.** $\frac{5}{13}$

39. $\sin 15° = \frac{\sqrt{2 - \sqrt{3}}}{2}$
$\cos 15° = \frac{\sqrt{2 + \sqrt{3}}}{2}$
$\tan 15° = 2 - \sqrt{3}$

41. $\sin 112° 30' = \frac{\sqrt{2 + \sqrt{2}}}{2}$
$\cos 112° 30' = -\frac{\sqrt{2 - \sqrt{2}}}{2}$
$\tan 112° 30' = -1 - \sqrt{2}$

43. $\sin \frac{\pi}{8} = \frac{\sqrt{2 - \sqrt{2}}}{2}$
$\cos \frac{\pi}{8} = \frac{\sqrt{2 + \sqrt{2}}}{2}$
$\tan \frac{\pi}{8} = \sqrt{2} - 1$

45. $\sin \frac{3\pi}{8} = \frac{\sqrt{2 + \sqrt{2}}}{2}$
$\cos \frac{3\pi}{8} = \frac{\sqrt{2 - \sqrt{2}}}{2}$
$\tan \frac{3\pi}{8} = \sqrt{2} + 1$

47. $\sin \frac{u}{2} = \frac{5\sqrt{26}}{26}$
$\cos \frac{u}{2} = \frac{\sqrt{26}}{26}$
$\tan \frac{u}{2} = 5$

49. $\sin \frac{u}{2} = \sqrt{\frac{89 - 5\sqrt{89}}{178}}$
$\cos \frac{u}{2} = -\sqrt{\frac{89 + 5\sqrt{89}}{178}}$
$\tan \frac{u}{2} = \frac{5 - \sqrt{89}}{8}$

51. $|\sin 3x|$ **53.** $-|\tan 4x|$ **55.** $\frac{\pi}{3}, \frac{5\pi}{3}$

57. $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$ **59.** $3 \sin \frac{2\pi}{3}$ **61.** $\frac{1}{2}(\sin 8\theta + \sin 2\theta)$

63. $\frac{5}{2}(\cos 8\beta + \cos 2\beta)$ **65.** $2 \sin 45° \cos 15°$

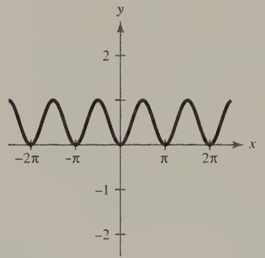
67. $2 \cos 3\theta \sin 2\theta$ **69.** $2 \cos \alpha \sin \beta$

71. $-2 \sin \theta \sin \frac{\pi}{2} = -2 \sin \theta$

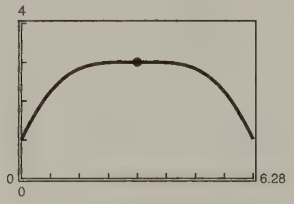
73. $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$ **75.** $\frac{\pi}{6}, \frac{5\pi}{6}$ **77.** $\frac{25}{169}$

79. $\frac{4}{13}$ **81.–91.** Answers will vary.

93.



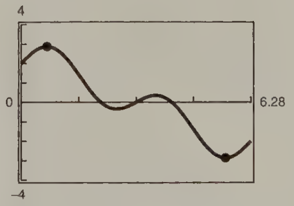
95. (a)



Maximum: $(\pi, 3)$

(b) π

97. (a)



Minimum: (5.5839, −2.8642)

Maximum: (0.6993, 2.8642)

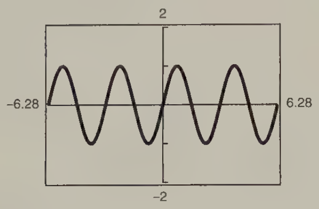
(b) 0.6993, 2.6078, 3.6754, 5.5839

99. $2x\sqrt{1 - x^2}$ **101.** $1 - 2x^2$ **103.** $\frac{1}{16} v_0 \sin \theta \cos \theta$

105. $\theta = 0.4482$

107. False. $\sin\frac{x}{2} = -\sqrt{\frac{1-\cos x}{2}}$ for $\pi \le \frac{x}{2} \le 2\pi$.

109. (a)



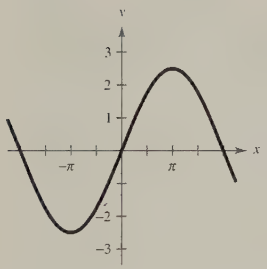
(b) $\sin 2x$

(c) Answers will vary.

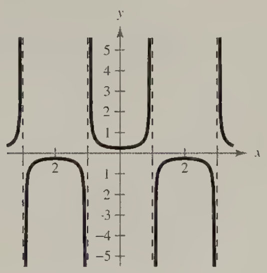
111. (a) Complement: $\frac{4\pi}{9}$; Supplement: $\frac{17\pi}{18}$

(b) Complement: $\frac{\pi}{20}$; Supplement: $\frac{11\pi}{20}$

113.



115.



Review Exercises *(page 500)*

1. $\sec x$ **3.** $\cos x$ **5.** $\cot x$ **7.** $\sec x$ **9.** $\sec x$

11. $\tan x = \frac{3}{4}$

$\csc x = \frac{5}{3}$

$\sec x = \frac{5}{4}$

$\cot x = \frac{4}{3}$

13. $\cos x = \frac{\sqrt{2}}{2}$

$\tan x = -1$

$\csc x = -\sqrt{2}$

$\sec x = \sqrt{2}$

$\cot x = -1$

15. $\sin^2 x$ **17.** $1 + \cot\alpha$ **19.** 1 **21.** $\csc x$

23.–37. Answers will vary.

39. $\frac{\pi}{6}, \frac{5\pi}{6}$ **41.** $\frac{\pi}{3}, \frac{2\pi}{3}$ **43.** $\frac{\pi}{6}, \frac{7\pi}{6}$

45. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **47.** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **49.** $0, \pi$

51. $0, \frac{2\pi}{3}, \frac{4\pi}{3}$ **53.** $0, \frac{\pi}{2}, \pi$ **55.** $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$

57. $0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$ **59.** $\frac{\pi}{2}, \frac{3\pi}{2}$

61. $0, \pi$ **63.** 1.2490, 1.8158, 4.3906, 4.9574

65. $\sin 285° = \frac{-\sqrt{2}-\sqrt{6}}{4}$

$\cos 285° = \frac{\sqrt{6}-\sqrt{2}}{4}$

$\tan 285° = -2 - \sqrt{3}$

67. $\sin\frac{5\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$

$\cos\frac{5\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$

$\tan\frac{5\pi}{12} = 2 + \sqrt{3}$

69. $\sin 190°$ **71.** $\tan 35°$ **73.** $\frac{-15-12\sqrt{7}}{52}$

75. $\frac{5\sqrt{7}+36}{52}$ **77.** $\frac{5\sqrt{7}-36}{52}$

79.–83. Answers will vary. **85.** $\frac{\pi}{6}, \frac{11\pi}{6}$

87.–91. Answers will vary.

93. $\sin 2u = \frac{20\sqrt{6}}{49}$

$\cos 2u = -\frac{1}{49}$

$\tan 2u = -20\sqrt{6}$

95. 15°, 75°

97. $\frac{1}{128}(3 - 4\cos 4x + \cos 8x)$

99. $\frac{1}{8}(3 - 4\cos 4x + \cos 8x)$

101. $\sin 67°30' = \frac{\sqrt{2+\sqrt{2}}}{2}$

$\cos 67°30' = \frac{\sqrt{2-\sqrt{2}}}{2}$

$\tan 67°30' = \sqrt{2} + 1$

103. $\sin\frac{11\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}$

$\cos\frac{11\pi}{12} = -\frac{\sqrt{2+\sqrt{3}}}{2}$

$\tan\frac{11\pi}{12} = -2 + \sqrt{3}$

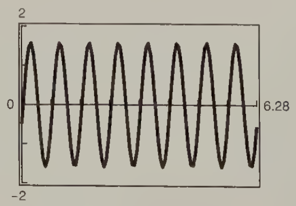
105. $\tan 3x$ **107.** Answers will vary.

109. $2\cos\frac{5\theta}{2}\cos\frac{\theta}{2}$ **111.** $-2\sin\frac{\pi}{2}\sin\frac{\pi}{4}$

113. $\frac{1}{2}(\cos\alpha - \cos 5\alpha)$ **115.** $3\sin\frac{\pi}{2}$

117. (a) $y = \frac{1}{2}\sqrt{10}\sin\left(8t - \arctan\frac{1}{3}\right)$

(b)



(c) $\frac{1}{2}\sqrt{10}$ (d) $\frac{4}{\pi}$

119. False. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

121. True **123.** No; answers will vary.

125. $y_3 = 1 - y_2$

Chapter Test *(page 504)*

1. $\sin\theta = -\dfrac{6\sqrt{61}}{61}$ **2.** 1 **3.** 1

$\cos\theta = -\dfrac{5\sqrt{61}}{61}$

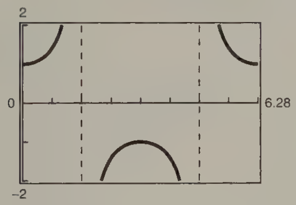
$\csc\theta = -\dfrac{\sqrt{61}}{6}$

$\sec\theta = -\dfrac{\sqrt{61}}{5}$

$\cot\theta = \dfrac{5}{6}$

4. $\csc\theta\sec\theta$ **5.** $\dfrac{\pi}{2} < \theta \le \pi, \dfrac{3\pi}{2} < \theta < 2\pi$

6.



$y_1 = y_2$

7.–12. Answers will vary.

13. $\sin\left(-\dfrac{7\pi}{12}\right) = \dfrac{-\sqrt{6} - \sqrt{2}}{4}$

$\cos\left(-\dfrac{7\pi}{12}\right) = \dfrac{-\sqrt{6} + \sqrt{2}}{4}$

$\tan\left(-\dfrac{7\pi}{12}\right) = 2 + \sqrt{3}$

14. $\frac{1}{8}(1 - \cos 4x)$ **15.** $\frac{3}{2}(\cos 4\theta - \cos 8\theta)$

16. $2\cos 4\theta\cos\theta$ **17.** $0, \dfrac{3\pi}{4}, \pi, \dfrac{7\pi}{4}$ **18.** $\dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}$

19. $\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$ **20.** $\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}$ **21.** 1.306

22. $\sin 2u = \frac{4}{5}$ **23.** Answers will vary.

$\tan 2u = -\frac{4}{3}$

24. $-2 - \sqrt{3}$ **25.** 76.52°

Chapter 7

Section 7.1 *(page 512)*

1. $C = 105°, b \approx 16.97, c \approx 23.18$

3. $C = 110°, b \approx 22.44, c \approx 24.35$

5. $B \approx 13.6°, C \approx 130.4°, c \approx 12.96$

7. $B = 10°, b \approx 69.46, c \approx 136.81$

9. $B = 42°4', a \approx 22.05, b \approx 14.88$

11. $A \approx 10°11', C \approx 154°19', c \approx 11.03$

13. $B \approx 18°13', C \approx 51°32', c \approx 40.05$

15. No solution

17. Two solutions

$B \approx 72.2°, C \approx 49.8°, c \approx 10.27$

$B \approx 107.8°, C \approx 14.2°, c \approx 3.3$

19. $B \approx 48.74°, C \approx 21.26°, c \approx 48.23$

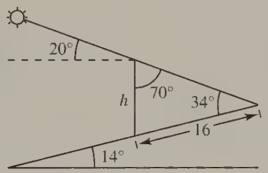
21. (a) $b \le 5, b = \dfrac{5}{\sin 36°}$ (b) $5 < b < \dfrac{5}{\sin 36°}$

(c) $b > \dfrac{5}{\sin 36°}$

23. ≈28.2 square units **25.** ≈1782.3 square units

27. ≈1057.1 square units

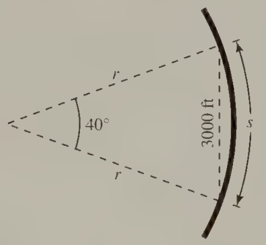
29. (a)



(b) $\dfrac{16}{\sin 70°} = \dfrac{h}{\sin 34°}$ (c) ≈9.5 meters

31. S 65° W

33. (a)



(b) 4385.71 feet

(c) 3061.80 feet

35. Distance from *A*: 42.3 kilometers

Distance from *B*: 25.8 kilometers

37. ≈4.55 miles

39. (a)

θ	0°	45°	90°	135°	180°
d	0	0.5338	1.6905	2.6552	3

(b) 0.0071 inches

41. $d = \dfrac{2\sin\theta}{\sin(\phi - \theta)}$

43. False. The triangle can't be solved if only three angles are known.

45. False. For the case SSA there could be two possible triangles, one triangle, or no triangle.

47. $\tan\theta = -\frac{12}{5}$; $\csc\theta = -\frac{13}{12}$; $\sec\theta = \frac{13}{5}$; $\cot\theta = -\frac{5}{12}$

49. $\sin\theta = \dfrac{\sqrt{122}}{122}$; $\cos\theta = -\dfrac{11\sqrt{122}}{122}$; $\tan\theta = -\dfrac{1}{11}$; $\csc\theta = \sqrt{122}$

51. Answers will vary. **53.** $3(\sin 11\theta + \sin 5\theta)$

55. $\dfrac{3}{2}\left[\sin\dfrac{11\pi}{6} - \sin\left(-\dfrac{9\pi}{6}\right)\right]$

Section 7.2 *(page 519)*

1. $A \approx 26.4°$, $B \approx 36.3°$, $C \approx 117.3°$

3. $B \approx 29.4°$, $C \approx 100.6°$, $a \approx 23.4$

5. $A \approx 36.9°$, $B \approx 53.1°$, $C = 90°$

7. $A \approx 103.5°$, $B \approx 38.2°$, $C \approx 38.2°$

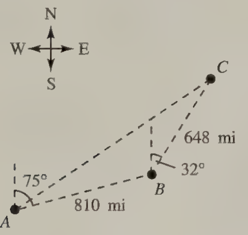
9. $b \approx 8.6$, $A \approx 153.9°$, $C \approx 17.7°$

	a	b	c	d	θ	ϕ
11.	4	8	11.6	5	30°	150°
13.	10	14	20	13.86	68.2°	111.8°
15.	10	11.58	18	12	67.1°	112.9°

17. ≈22.45 square units **19.** ≈15.52 square units

21. 96.82 square units

23.



1357.8 miles, S 56° W

25. 483.4 meters **27.** 43.3 miles

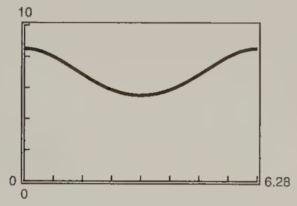
29. (a) N 58.4° W (b) S 81.5° W **31.** ≈63.7 feet

33. $PQ \approx 9.4$, $QS \approx 5.0$, $RS \approx 12.8$

35. (a) $49 = 2.25 + x^2 - 3x\cos\theta$

(b) $x = \frac{1}{2}\left(3\cos\theta + \sqrt{9\cos^2\theta + 187}\right)$

(c)



(d) 6 inches

37. ≈2.76 feet

39. False. A triangle cannot be formed with sides of length 10 cm, 16 cm, and 5 cm.

41. (a) and (b) Answers will vary.

43. $r \approx 64.5$ feet, track length ≈ 405 feet **45.** $\dfrac{\pi}{3}, \dfrac{5\pi}{3}$

47. $\dfrac{\pi}{6}, \dfrac{11\pi}{6}$ **49.** $-2\sin\left(\dfrac{7\pi}{12}\right)\sin\left(\dfrac{\pi}{4}\right)$

Section 7.3 *(page 531)*

1. $\|\mathbf{u}\| = \|\mathbf{v}\| = \sqrt{17}$, $\text{slope}_{\mathbf{u}} = \text{slope}_{\mathbf{v}} = \frac{1}{4}$

u and **v** have the same magnitude and direction so they are equal.

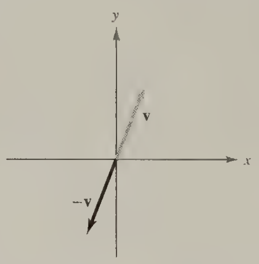
3. $\langle 4, 3\rangle$, $\|\mathbf{v}\| = 5$ **5.** $\langle -3, 2\rangle$, $\|\mathbf{v}\| = \sqrt{13}$

7. $\langle 0, 5\rangle$, $\|\mathbf{v}\| = 5$ **9.** $\left\langle -\frac{9}{2}, -\frac{5}{2}\right\rangle$, $\|\mathbf{v}\| = \frac{1}{2}\sqrt{106}$

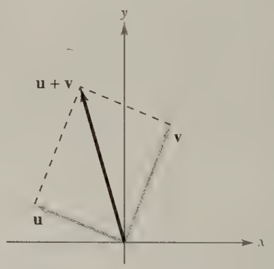
11. $\langle 8, 6\rangle$, $\|\mathbf{v}\| = 10$

13. $\langle 7.9, -17.9\rangle$, $\|\mathbf{v}\| = \sqrt{382.82} \approx 19.6$

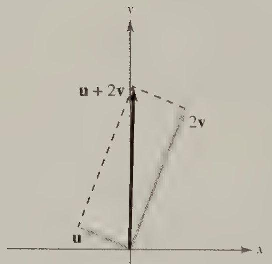
15.



17.



19.



21. (a) $\langle 11, 3\rangle$ (b) $\langle -3, 1\rangle$ (c) $\langle -13, 1\rangle$ (d) $\langle 23, 9\rangle$

23. (a) $\langle -4, -5\rangle$ (b) $\langle -6, 1\rangle$ (c) $\langle -13, 5\rangle$ (d) $\langle -19, -11\rangle$

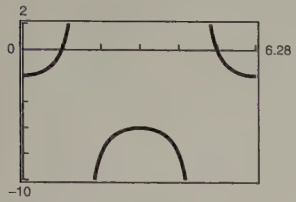
25. (a) $3\mathbf{i} - 2\mathbf{j}$ (b) $-\mathbf{i} + 4\mathbf{j}$ (c) $-4\mathbf{i} + 11\mathbf{j}$ (d) $6\mathbf{i} + \mathbf{j}$

27. $\langle 1, 0\rangle$ **29.** $\left\langle -\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right\rangle$ **31.** $\left\langle -\dfrac{24}{25}, -\dfrac{7}{25}\right\rangle$

33. $\dfrac{4}{5}\mathbf{i} - \dfrac{3}{5}\mathbf{j}$ **35.** $\mathbf{j}$ **37.** $\left\langle \dfrac{5\sqrt{2}}{2}, \dfrac{5\sqrt{2}}{2}\right\rangle$

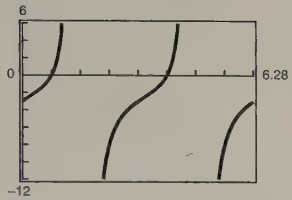
39. $\frac{21}{5}\mathbf{i} + \frac{28}{5}\mathbf{j}$ **41.** $-8\mathbf{i}$

18.



$\frac{\pi}{3}, \frac{5\pi}{3}$

19.



$\frac{\pi}{4}, \frac{5\pi}{4}$

20. $8x^2 - 1$ **21.** $\frac{2x}{x^2 + 1}$

22. $\sin 67°30' = \frac{\sqrt{2 + \sqrt{2}}}{2}$

$\cos 67°30' = \frac{\sqrt{2 - \sqrt{2}}}{2}$

$\tan 67°30' = \sqrt{2} + 1$

23. $2 \cos 6x \cos 2x$ **24.–27.** Answers will vary.

28. $B = 14.89°$
$C = 119.11°$
$c = 17$

29. $B = 52.48°$
$C = 97.52°$
$a = 5.04$

30. $B = 55°$
$b = 20.14$
$c = 24.13$

31. $A = 26.07°$
$B = 33.33°$
$C = 120.60°$

32. 131.71 square inches

33. 85.21 square inches **34.** $3\mathbf{i} + 5\mathbf{j}$

35. $\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$ **36.** -5

37. $\langle -\frac{1}{13}, -\frac{5}{13} \rangle; \langle \frac{105}{13}, -\frac{21}{13} \rangle$

38. $3\sqrt{2}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$ **39.** $-4\sqrt{3} + 4i$

40. $-12\sqrt{3} + 12i$

41. $\cos 0 + i \sin 0$

$\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$

$\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$

42. $3\left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}\right)$

$3\left(\cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5}\right)$

$3(\cos \pi + i \sin \pi)$

$3\left(\cos \frac{7\pi}{5} + i \sin \frac{7\pi}{5}\right)$

$3\left(\cos \frac{9\pi}{5} + i \sin \frac{9\pi}{5}\right)$

43. ≈ 4.79 feet **44.** $d = 4 \sin \frac{\pi}{4} t$

45. N 32.63° E; 543.88 kilometers per hour

46. $\approx 80.28°$

Chapter 8

Section 8.1 *(page 573)*

1. (a) No (b) No (c) No (d) Yes

3. (a) No (b) Yes (c) No (d) No

5. $(2, 2)$ **7.** $(2, 6), (-1, 3)$

9. $(0, 2), (\sqrt{3}, 2 - 3\sqrt{3}), (-\sqrt{3}, 2 + 3\sqrt{3})$

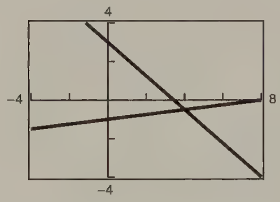
11. $(0, 0), (2, -4)$ **13.** $(4, 4)$ **15.** $(5, 5)$

17. $(\frac{1}{2}, 3)$ **19.** $(1, 1)$ **21.** $(\frac{20}{3}, \frac{40}{3})$ **23.** No solution

25. No solution **27.** $(0, 0)$ **29.** $(4, 3)$ **31.** $(\frac{5}{2}, \frac{3}{2})$

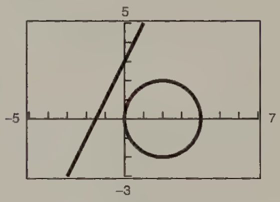
33. $(2, 2), (4, 0)$ **35.** $(1, 4), (4, 7)$

37.



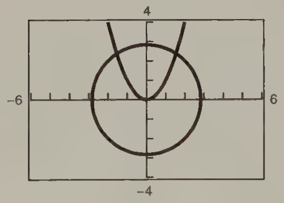
$(4, -0.5)$

39.



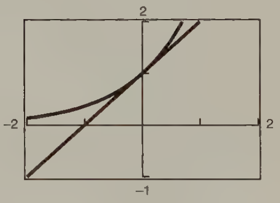
No points of intersection

41.



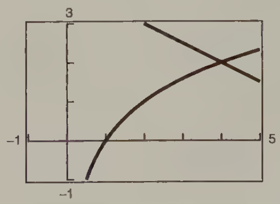
$(\pm 1.540, 2.372)$

43.



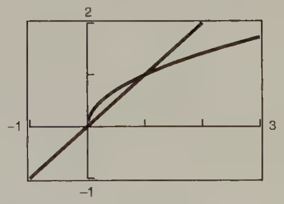
$(0, 1)$

45.



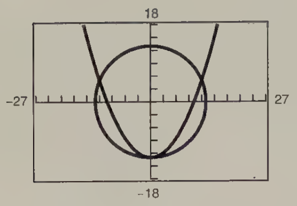
$(4, 2)$

47.



$(0, 0), (1, 1)$

49.



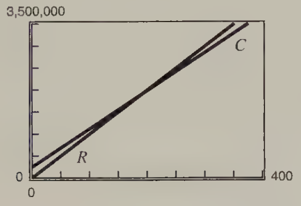
$(0, -13), (\pm 12, 5)$

51. $(1, 2)$

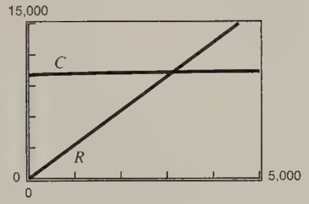
53. $(-2, 0), \left(\frac{29}{10}, \frac{21}{10}\right)$ **55.** No solution **57.** $(0.287, 1.751)$

59. $(0, 1), (1, 0)$ **61.** $\left(-4, -\frac{1}{4}\right), \left(\frac{1}{2}, 2\right)$

63.



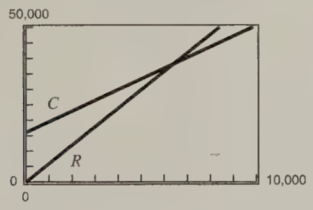
65.



192 units; \$1,910,400 3133 units; \$10,308

67. (a) $C = 3.45x + 16{,}000$; $R = 5.95x$

(b)



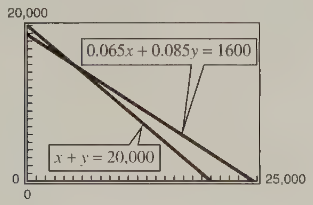
6400 units

(c) 6400 units

69. \$8333.33

71. (a) $\begin{cases} x + y = 20{,}000 \\ 0.065x + 0.085y = 1600 \end{cases}$

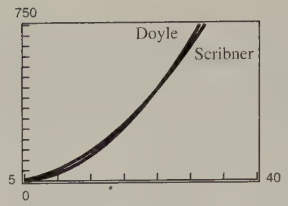
(b)



More invested at 6.5% means less invested at 8.5% and less interest.

(c) \$5000

73. (a)



(b) 24.7 inches

(c) Doyle Log Rule; for logs of large diameter, V is larger using the Doyle Log Rule.

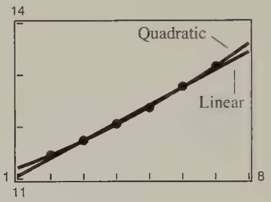
75. 60 × 80 centimeters **77.** 42 × 63 feet

79. $2 \times \sqrt{2} \times \sqrt{2}$ inches

81. (a) $E = 0.342t + 10.72$

$E = 0.0163t^2 + 0.196t + 11.00$

(b)



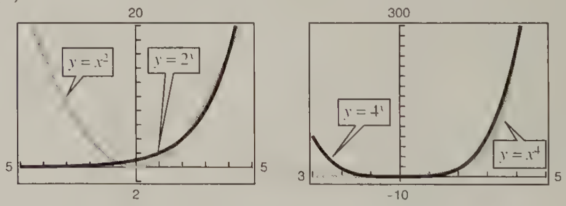
(c) $(6.18, 12.83), (2.78, 11.67)$

(d) Linear: \$14.14; Quadratic: \$14.59. Answers will vary.

83. False. Example: a parabola and a circle can have 4 intersection points.

85. Graphical solutions may be approximate.

87. (a)



(b) Three points of intersection when b is even

89. $4x - 13y + 38 = 0$ **91.** $x = 4$

93. $45x + 29y - 127 = 0$

95. Domain: $-\infty < x < -\frac{2}{3}, -\frac{2}{3} < x < \infty$

Asymptotes: $x = -\frac{2}{3}, y = \frac{2}{3}$

97. Domain: $-\infty < x < 0, 0 < x < \infty$

Asymptotes: $x = 0, y = 3$

99. 0.272 **101.** 0.405

Section 8.2 *(page 583)*

1. $(2, 1)$ **3.** $(1, -1)$ **5.** Inconsistent

7. All points on $3x - 2y - 5 = 0$ **9.** $\left(\frac{1}{3}, -\frac{2}{3}\right)$

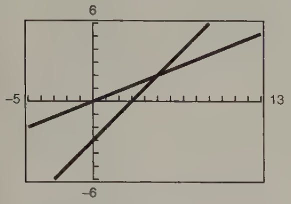
11. $\left(\frac{5}{2}, \frac{3}{4}\right)$ **13.** $(3, 4)$ **15.** $(4, -1)$ **17.** $\left(\frac{12}{7}, \frac{18}{7}\right)$

19. Inconsistent **21.** $\left(-\frac{6}{35}, \frac{43}{35}\right)$ **23.** $\left(\frac{38}{15}, \frac{11}{5}\right)$

25. All points on $6x + 8y - 1 = 0$ **27.** $(5, -2)$

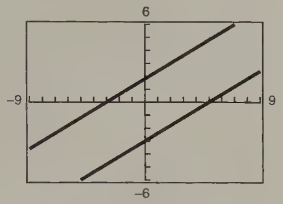
29. All points on $5x - 6y - 3 = 0$ **31.** $(8, 7)$

33.



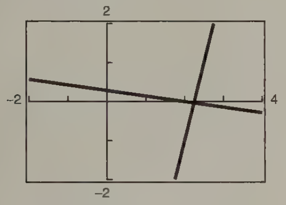
Consistent, (5, 2)

35.



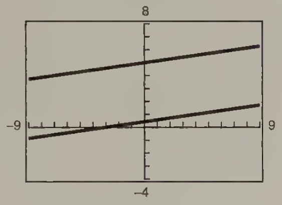
Inconsistent

37.



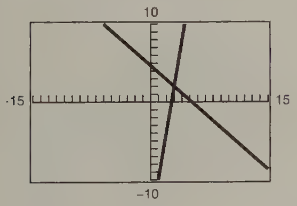
Consistent, $\left(\frac{65}{29}, -\frac{1}{29}\right)$

39.



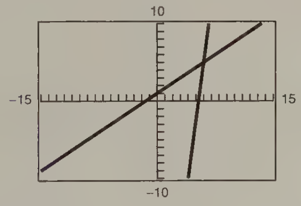
Inconsistent

41.



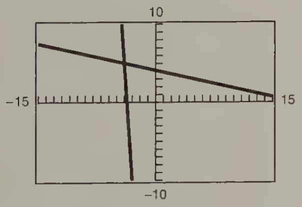
(3, 2)

43.



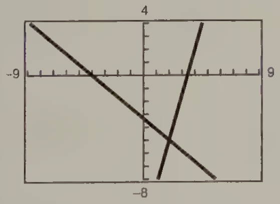
(6, 5)

45.



(−4, 5)

47.



(2, −5)

49. (4, 1) **51.** (2, −1) **53.** (6, −3) **55.** $\left(\frac{43}{6}, \frac{25}{6}\right)$

57. $\begin{cases} x - 2y = 0 \\ x - y - 3 = 0 \end{cases}$

Answer is not unique.

59. $\begin{cases} 2x + 2y = 11 \\ x - 4y = -7 \end{cases}$

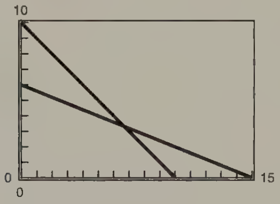
Answer is not unique.

61. (80, 10) **63.** (2,000,000, 100)

65. Plane: 550 miles per hour; Wind: 50 miles per hour

67. (a) $\begin{cases} x + y = 10 \\ 0.2x + 0.5y = 3 \end{cases}$

(b)



Decreases

(c) 20% solution: $6\frac{2}{3}$ liters; 50% solution: $3\frac{1}{3}$ liters

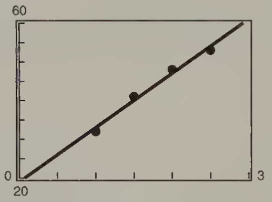
69. $6000 **71.** 375 adults, 125 children

73. 225 kilometers and 75 kilometers

75. $y = 0.97x + 2.10$ **77.** $y = 0.318x + 4.061$

79. (a) and (b) $y = 14x + 19$

(c)



(d) 41.4 bushels

81. True **83.** True

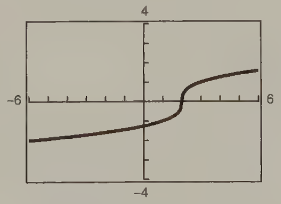
85. (300, 315). It is necessary to change the scale on the axes to see the point of intersection.

87. (a) $\begin{cases} x + y = 10 \\ x + y = 20 \end{cases}$ (b) $\begin{cases} x + y = 4 \\ 3x + 3y = 12 \end{cases}$

89. $k = -2$

91. $u = -\frac{1}{2}, v = \frac{\cot 2x}{2}$ **93.** $u = -1, v = \frac{1}{x}$

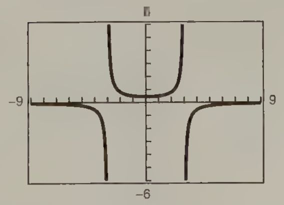
95.



Domain: All real numbers

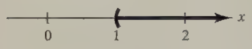
Range: All real numbers

97.

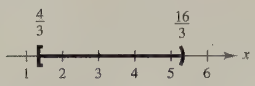


Domain: $-\infty < x < -3, -3 < x < 3, 3 < x < \infty$
Range: $-\infty < y < 0, 0 < y < \infty$

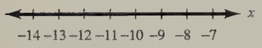
99. $x > 1$



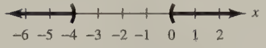
101. $\frac{4}{3} \le x < \frac{16}{3}$



103. $-\infty < x < \infty$



105. $x < -4, x > 0$



107. $\ln \dfrac{x}{(x+3)^5}$ **109.** $\log_6 \sqrt[4]{3x}$ **111.** $(-2, 3)$

Section 8.3 *(page 597)*

1. (a) No (b) No (c) No (d) Yes

3. (a) No (b) No (c) Yes (d) No

5. $(-7, -8, 6)$ **7.** $(-3, -2, -6)$ **9.** $\left(\frac{1}{2}, -2, 2\right)$

11.
$$\begin{cases} x - 2y + 3z = 5 \\ y - 2z = 9 \\ 2x - 3z = 0 \end{cases}$$

First step in putting the system in row-echelon form

13. $(-1, 3, 4)$ **15.** $(-2, 7, 5)$ **17.** $(6, -3, 0)$

19. Inconsistent **21.** $\left(1, -\frac{3}{2}, \frac{1}{2}\right)$

23. $(-3a + 10, 5a - 7, a)$ **25.** $(-a + 3, a + 1, a)$

27. Inconsistent **29.** Inconsistent **31.** $(0, 0, 0)$

33. $(2a, 21a - 1, 8a)$ **35.** $\left(\frac{1}{2} - \frac{3}{2}a, 1 - \frac{2}{3}a, a\right)$

37. $(9a, -35a, 67a)$ **39.** $(1, 1, 1, 1)$

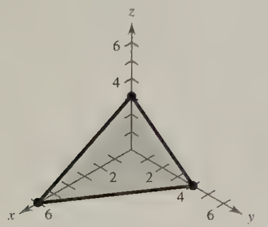
41.
$$\begin{cases} 3x + y - z = 9 \\ x + 2y - z = 0 \\ -x + y + 3z = 1 \end{cases}$$

Answer is not unique.

43.
$$\begin{cases} x + 6y + 4z = 7 \\ 2x - 2y - 4z = 0 \\ -x + y + z = -\frac{7}{4} \end{cases}$$

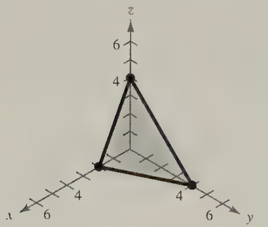
Answer is not unique.

45.



$(6, 0, 0), (0, 4, 0), (0, 0, 3), (4, 0, 1)$

47.



$(2, 0, 0), (0, 4, 0), (0, 0, 4), (0, 2, 2)$

49. $\dfrac{A}{x} + \dfrac{B}{x - 14}$ **51.** $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x - 10}$

53. $\dfrac{A}{x - 5} + \dfrac{B}{(x - 5)^2} + \dfrac{C}{(x - 5)^3}$

55. $\dfrac{1}{2}\left(\dfrac{1}{x - 1} - \dfrac{1}{x + 1}\right)$ **57.** $\dfrac{1}{x} - \dfrac{1}{x + 1}$

59. $\dfrac{1}{x} - \dfrac{2}{2x + 1}$ **61.** $\dfrac{1}{x - 1} - \dfrac{1}{x + 2}$

63. $-\dfrac{3}{x} - \dfrac{1}{x + 2} + \dfrac{5}{x - 2}$ **65.** $\dfrac{3}{x} - \dfrac{1}{x^2} + \dfrac{1}{x + 1}$

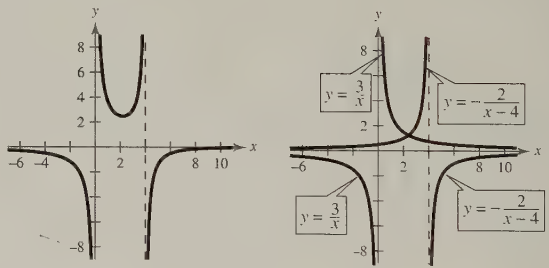
67. $\dfrac{3}{x - 3} + \dfrac{9}{(x - 3)^2}$ **69.** $2x - 7 + \dfrac{17}{x + 2} + \dfrac{1}{x + 1}$

71. $x + 3 + \dfrac{6}{x - 1} + \dfrac{4}{(x - 1)^2} + \dfrac{1}{(x - 1)^3}$

73. $\dfrac{3}{2x - 1} - \dfrac{2}{x + 1}$ **75.** $\dfrac{2}{x} - \dfrac{1}{x^2} - \dfrac{2}{x + 1}$

77. $2x + \dfrac{1}{2}\left(\dfrac{3}{x - 4} - \dfrac{1}{x + 2}\right)$

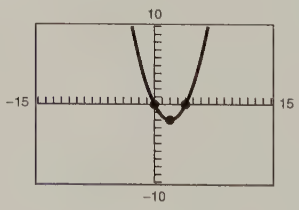
79. $\dfrac{3}{x} - \dfrac{2}{x - 4}$



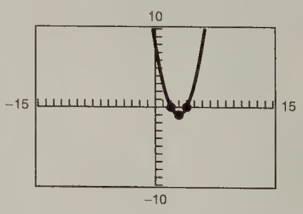
Vertical asymptotes are the same.

81. $s = -16t^2 + 144$ **83.** $s = 24t^2 - 264t + 692$

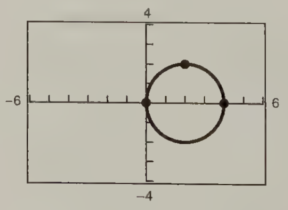
85. $y = \frac{1}{2}x^2 - 2x$



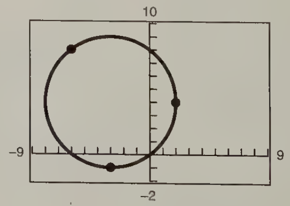
87. $y = x^2 - 6x + 8$



89. $x^2 + y^2 - 4x = 0$



91. $x^2 + y^2 + 6x - 8y = 0$



93. \$4000 at 5%, \$5000 at 6%, \$7000 at 7%

95. \$366,666.67 at 8%, \$316,666.67 at 9%, \$91,666.67 at 10%

97. $250{,}000 - \frac{1}{2}s$ in certificates of deposit

$125{,}000 + \frac{1}{2}s$ in municipal bonds

$125{,}000 - s$ in blue-chip stocks

s in growth stocks

99. 20 liters of spray X, 18 liters of spray Y, 16 liters of spray Z

101. Use four medium trucks or two large trucks, one medium truck, and two small trucks. (Other answers possible.)

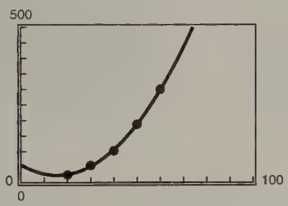
103. $t_1 = 96$ pounds

$t_2 = 48$ pounds

$a = -16$ feet per second squared

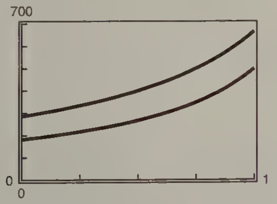
105. $y = -\frac{5}{24}x^2 - \frac{3}{10}x + \frac{41}{6}$ **107.** $y = x^2 - x$

109. (a) $y = 0.14x^2 - 4.4x + 58$ (b) 436 feet



111. (a) $\dfrac{2000}{7 - 4x} - \dfrac{2000}{11 - 7x}$, $0 \le x \le 1$

(b)



113. False. Leading coefficients are not all 1.

115. False. $\dfrac{A}{x + 10} + \dfrac{B}{x - 10} + \dfrac{C}{(x - 10)^2}$

117. $\dfrac{1}{2a}\left(\dfrac{1}{a + x} + \dfrac{1}{a - x}\right)$ **119.** $\dfrac{1}{a}\left(\dfrac{1}{y} + \dfrac{1}{a - y}\right)$

Answers will vary. Answers will vary.

121. No. There are two arithmetic errors. They are the constant in the second equation and the coefficient of z in the third equation.

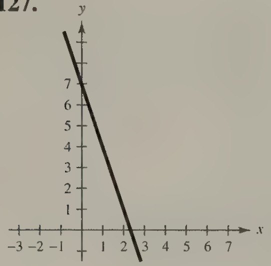
123. $x = 5, y = 5, \lambda = -5$

125. $x = \dfrac{\sqrt{2}}{2}, y = \dfrac{1}{2}, \lambda = 1$

$x = -\dfrac{\sqrt{2}}{2}, y = \dfrac{1}{2}, \lambda = 1$

$x = 0, y = 0, \lambda = 0$

127.



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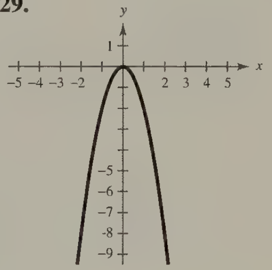

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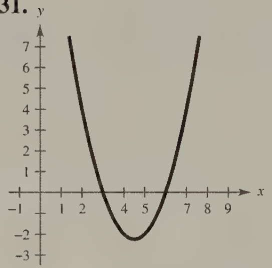

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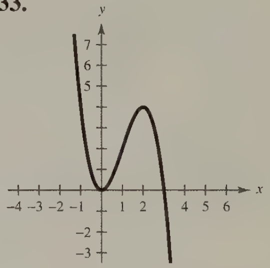
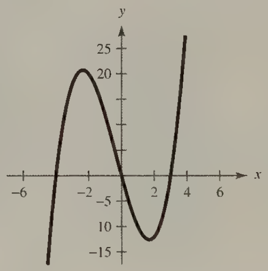
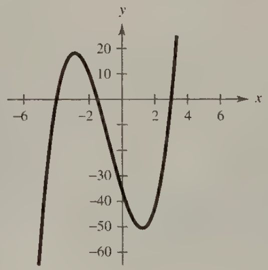


135. (a) $-4, 0, 3$ (b)

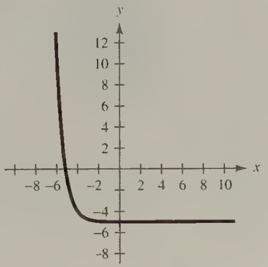


137. (a) $-4, -\frac{3}{2}, 3$ (b)



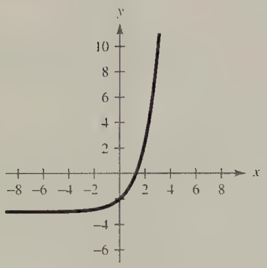
139.

x	-6	-5	-4	-3	0
y	11	-1	-4	-4.75	-4.996



141.

x	-3	-1	0	1	2	3
y	-2.9	-2.6	-2	-0.7	2.5	9.9

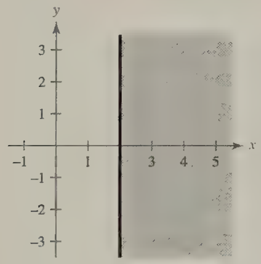


143. $\left(4\frac{1}{3}, -2\right)$ **145.** $(40, 40)$

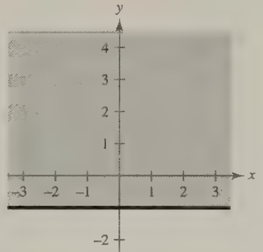
Section 8.4 *(page 610)*

1. (g) **3.** (a) **5.** (e) **7.** (f)

9.



11.



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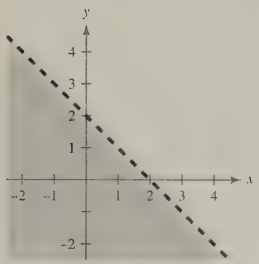

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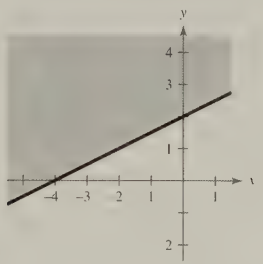

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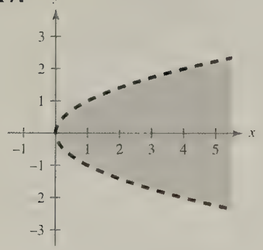

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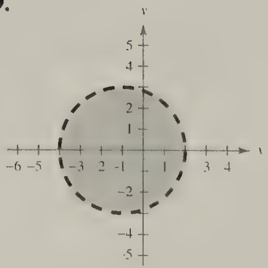

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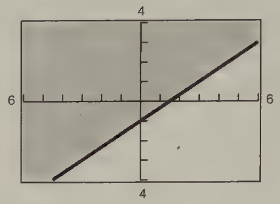

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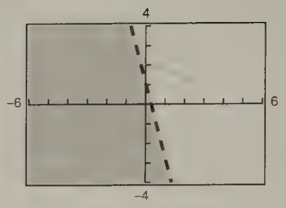
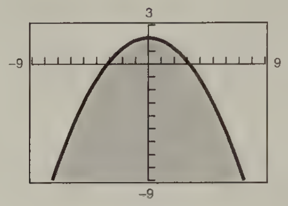
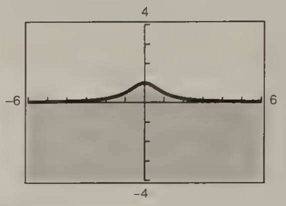


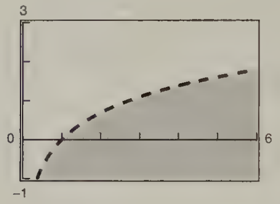
25.



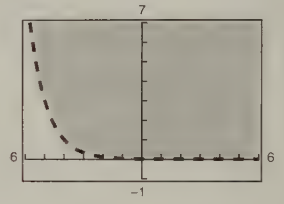
27.



29.



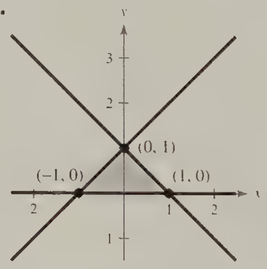
31.



33. $\frac{x}{3} + \frac{y}{2} \geq 1$ **35.** $x^2 + y^2 \leq 9$

37. (a) Yes (b) No (c) No (d) No

39.



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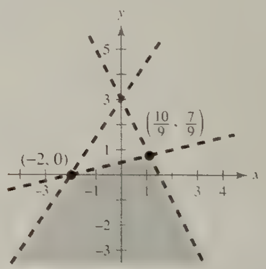
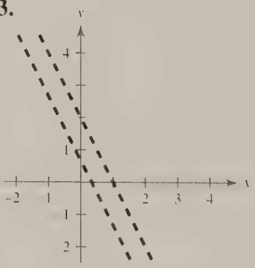
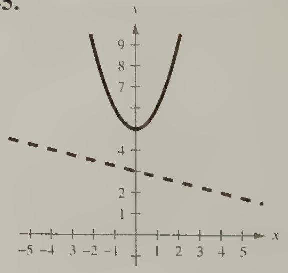


43.

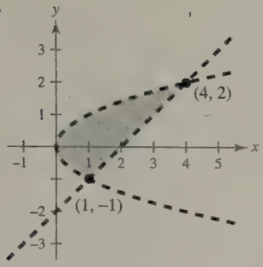


45.

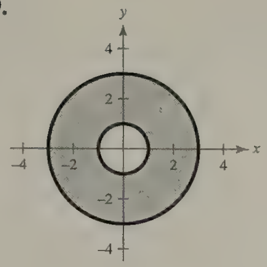


Solution set is empty.

47.



49.



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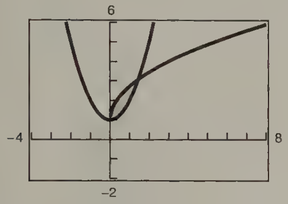

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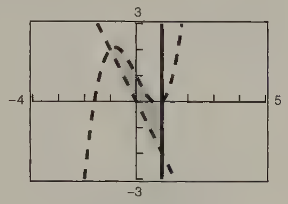
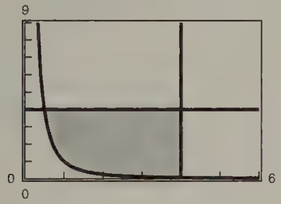


55.

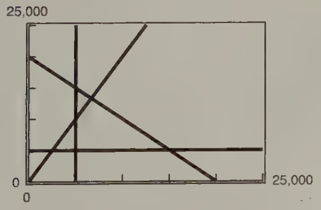


57. $\begin{cases} \frac{1}{4}x + \frac{1}{4}y \le 1 \\ x \ge 0 \\ y \ge 0 \end{cases}$

59. $\begin{cases} y \le 4 - x \\ y \le 2 - \frac{1}{4}x \\ x \ge 0 \\ y \ge 0 \end{cases}$ **61.** $\begin{cases} 2 \le x \le 5 \\ 1 \le y \le 7 \end{cases}$ **63.** $\begin{cases} y \le \frac{3}{2}x \\ y \le -x + 5 \\ y \ge 0 \end{cases}$

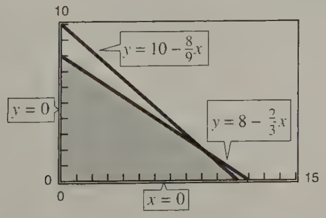
65. (a) $\begin{cases} x + y \le 20{,}000 \\ y \ge 2x \\ x \ge 5000 \\ y \ge 5000 \end{cases}$

(b)



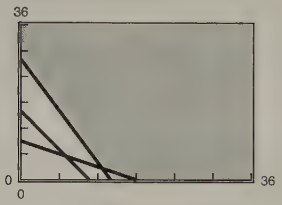
67. (a) $\begin{cases} x + \frac{3}{2}y \le 12 \\ \frac{4}{3}x + \frac{3}{2}y \le 15 \\ x \ge 0 \\ y \ge 0 \end{cases}$

(b)



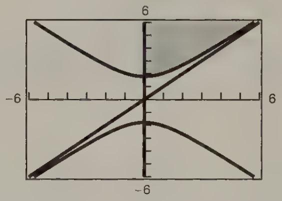
69. (a) $\begin{cases} 20x + 10y \ge 280 \\ 15x + 10y \ge 160 \\ 10x + 20y \ge 180 \\ x \ge 0 \\ y \ge 0 \end{cases}$

(b)



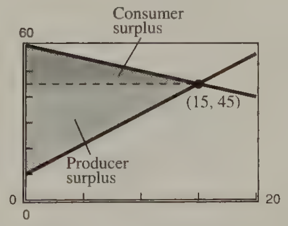
71. (a) $\begin{cases} \pi y^2 - \pi x^2 \ge 10 \\ x > 0 \\ y > x \end{cases}$

(b)



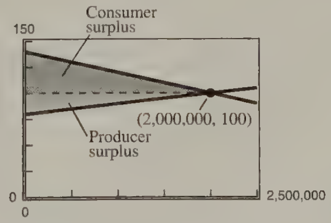
(c) The line is an asymptote to the boundary. The larger the circles, the closer the radii can be and the constraint still be satisfied.

73.



Consumer surplus: 112.5
Producer surplus: 262.5

75.



Consumer surplus: 40,000,000
Producer surplus: 20,000,000

77. False. $3x + y^2 \ge 2$ is outside the parabola.

79. Answers will vary. **81.** $x + 11y + 8 = 0$

83. $x + y + 1.8 = 0$

Section 8.5 *(page 619)*

1. Minimum at (0, 0): 0
Maximum at (0, 6): 30

3. Minimum at (0, 0): 0
Maximum at (6, 0): 60

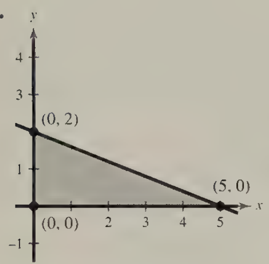
5. Minimum at (0, 0): 0
Maximum at (3, 4): 23

7. Minimum at (0, 0): 0
Maximum at (4, 0): 20

9. Minimum at (0, 0): 0
Maximum at (60, 20): 740

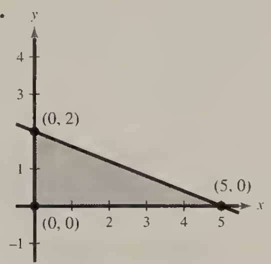
11. Minimum at (0, 0): 0
Maximum at any point on the line segment connecting (60, 20) and (30, 45): 2100

13.



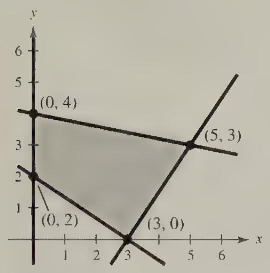
Minimum at (0, 0): 0
Maximum at (5, 0): 30

15.



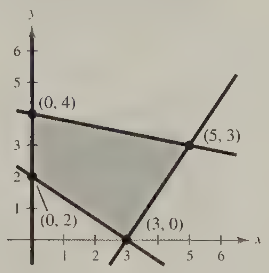
Minimum at (0, 0): 0
Maximum at (0, 2): 48

17.



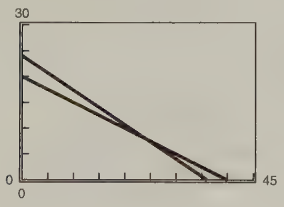
Minimum at (0, 2): 6
Maximum at (5, 3): 29

19.



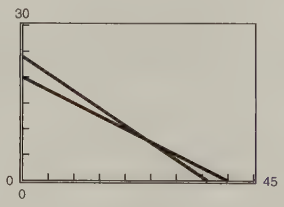
Minimum at (3, 0): 9
Maximum at (5, 3): 36

21.



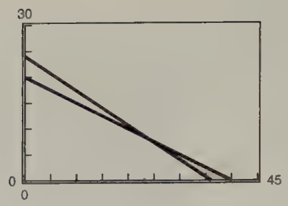
Minimum at (24, 8): 104
Maximum at (40, 0): 160

23.



Minimum at (36, 0): 36
Maximum at (24, 8): 56

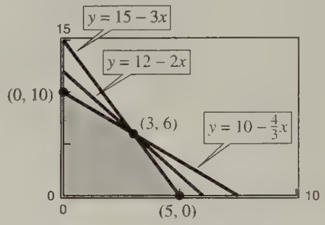
25.



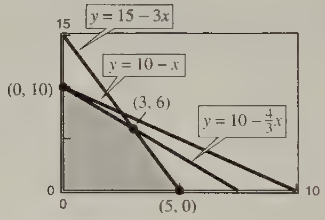
Minimum at any point on the line segment connecting (24, 8) and (36, 0): 72

Maximum at (40, 0): 80

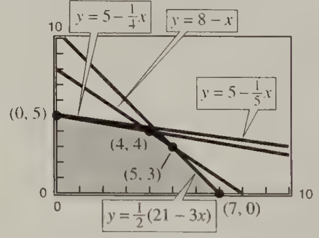
27. (a) and (b) (c) (3, 6)



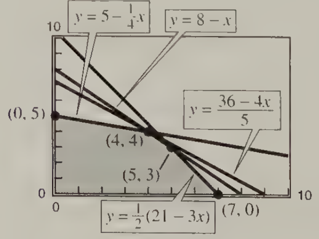
29. (a) and (b) (c) (0, 10)



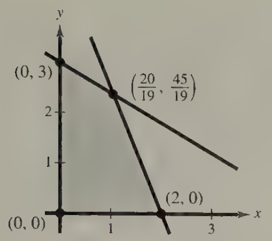
31. (a) and (b) (c) (0, 5)



33. (a) and (b) (c) (4, 4)

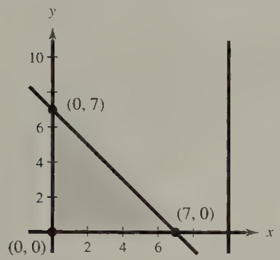


35.



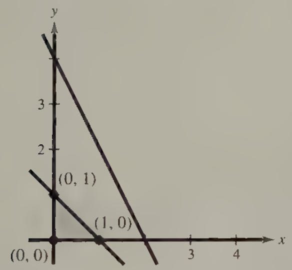
z is maximum at any point on the line segment connecting $(2, 0)$ and $\left(\frac{20}{19}, \frac{45}{19}\right)$.

37.



The constraint $x \le 10$ is extraneous.
Maximum at $(0, 7)$: 14

39.



The constraint $2x + y \le 4$ is extraneous.
Maximum at $(0, 1)$: 4

41. 200 units of the \$250 model; 50 units of the \$400 model
Maximum profit: \$11,500

43. Type A: $\frac{1}{6}$ gallon; Type B: $\frac{5}{6}$ gallon
Minimum cost: \$1.255 per gallon

45. No audits; 40 tax returns
Maximum revenue: \$12,000

47. 1000 units of Model A; 500 units of Model B
Maximum profit: \$76,000

49. True **51.** $z = x + 5y$ (Answer is not unique.)

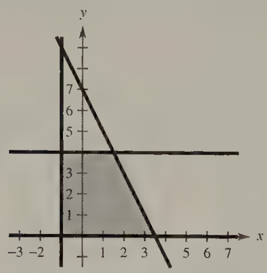
53. $z = 4x + y$ (Answer is not unique.)

55. (a) $t > 9$ (b) $\frac{3}{4} < t < 9$ **57.** $\frac{9}{2(x + 3)}$, $x \ne 0$

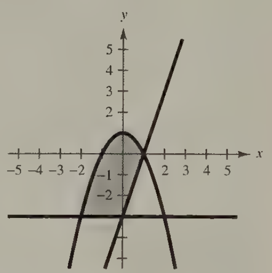
59. $\frac{x^2 + 2x - 13}{x(x - 2)}$, $x \ne \pm 3$ **61.** 1.099 **63.** 14.550

65. 1.851 **67.** $(-2, -3)$ **69.** $(-1, 2)$

71.



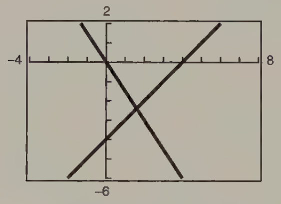
73.



Review Exercises *(page 624)*

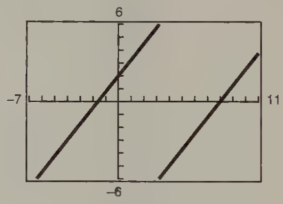
1. $(1, 1)$ **3.** $(5, 4)$ **5.** $(0, 0), (2, 8), (-2, 8)$
7. $\left(2, -\frac{1}{2}\right)$ **9.** $(0, 0), (-3, 3)$ **11.** $(4, 4)$
13. 4762 units **15.** 96 × 144 meters **17.** $\left(\frac{5}{2}, 3\right)$
19. $(-0.5, 0.8)$ **21.** $(0, 0)$ **23.** $\left(\frac{14}{5} + \frac{8}{5}a, a\right)$

25.



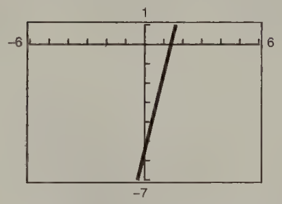
Consistent: $(1.6, -2.4)$

27.



Inconsistent

29.



Consistent: All points on the line $y = 4x - 5.5$

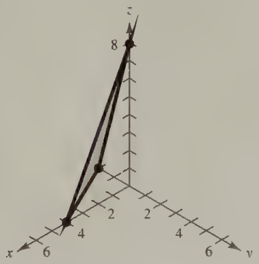
31. 218.75 miles per hour; 193.75 miles per hour

33. $\left(\frac{500{,}000}{7}, \frac{159}{7}\right)$ **35.** $(2, -4, -5)$

37. $\left(\frac{38}{17}, \frac{40}{17}, -\frac{63}{17}\right)$ **39.** $(3a + 4, 2a + 5, a)$

41. $y = 2x^2 + x - 5$ **43.** $(a - 4, a - 3, a)$

45.



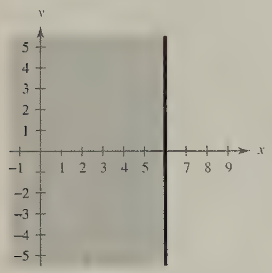
$(0, 0, 8), (0, -2, 0), (4, 0, 0), (1, -1, 2)$

47. $\dfrac{3}{x+2} - \dfrac{4}{x+4}$ **49.** $1 + \dfrac{1}{8}\left(\dfrac{9}{x-3} - \dfrac{25}{x+5}\right)$

51. $\dfrac{1}{2}\left(\dfrac{3}{x-1} - \dfrac{x-3}{x^2+1}\right)$

53. Spray X: 10 gallons; Spray Y: 5 gallons; Spray Z: 12 gallons

55.



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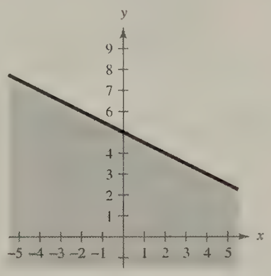

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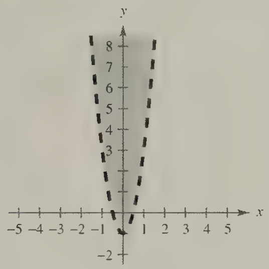
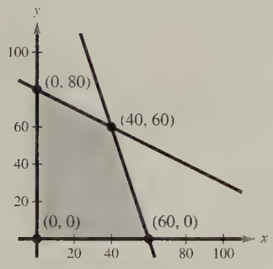
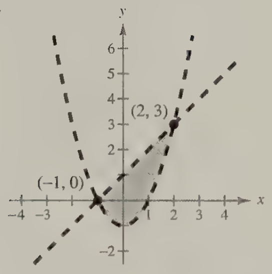


61. (d) **63.** (c)

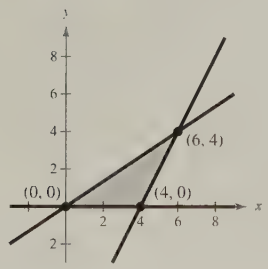
65.



67.

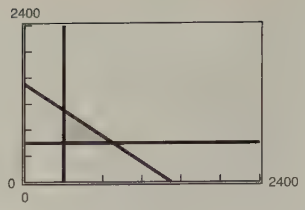


69.

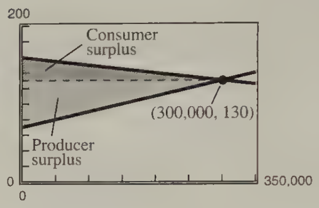


71. $\begin{cases} -x + y \le 4 \\ 2x + y \le 22 \\ -x + y \ge -2 \\ 2x + y \ge 7 \end{cases}$

73. $\begin{cases} x + y \le 1500 \\ x \ge 400 \\ y \ge 600 \end{cases}$



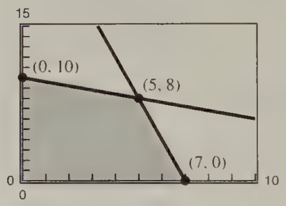
75.



Consumer surplus: 4,500,000

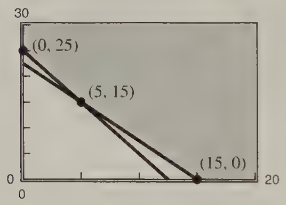
Producer surplus: 9,000,000

77.



Maximum at (5, 8): 47

79.



Minimum at (15, 0): 26.25

81. 20 perms, 2 haircuts; Maximum revenue: $1234

83. False. There is no solution for the system.

Chapter Test *(page 628)*

1. $(4, -2)$ **2.** $(0, -1), (1, 0), (2, 1)$

3. $(8, 5), (2, -1)$ **4.** $(-3, 1)$ **5.** $(3, 7), (-4, 0)$

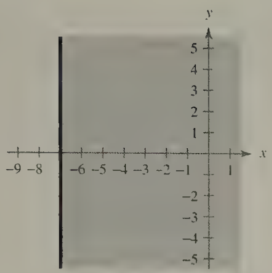
6. $(1, 8)$ **7.** $\left(\frac{28}{9}, -\frac{31}{9}\right)$ **8.** $(-1, 5, 2)$

9. $\left(-\frac{3}{5}a, \frac{4}{5}a, a\right)$ **10.** $\begin{cases} 9x + 2y = -4 \\ 3x + y = -4 \end{cases}$

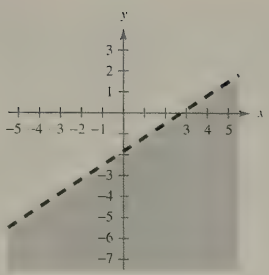
11. $\begin{cases} 6x + 2y + 4z = -2 \\ -2x + y + 8z = -12 \\ 2x + y + 4z = -5 \end{cases}$

12. $y = -\dfrac{1}{2}x^2 + x + 6$ **13.** $\dfrac{5}{x-1} + \dfrac{3}{(x-1)^2}$

14.



15.



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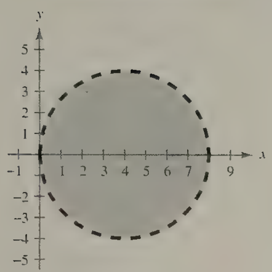

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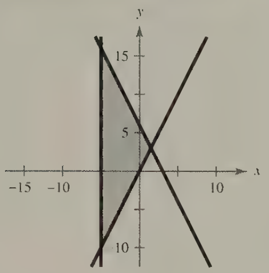

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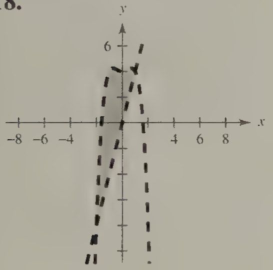
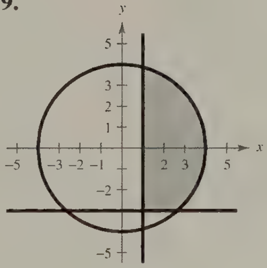


19.



20. $\begin{cases} y \le -\frac{1}{3}x + 15 \\ y \le -\frac{7}{3}x + 33 \\ x \le 12 \\ x \ge 0 \\ y \ge 0 \end{cases}$

21. Maximum at (12, 0): 240

22. Stock 160 units of the \$275 model and 140 units of the \$400 model.

Chapter 9

Section 9.1 *(page 640)*

1. 3×2 **3.** 3×1 **5.** 2×2

7. $\left[\begin{array}{rrcr} 4 & -5 & \vdots & 33 \\ -1 & 5 & \vdots & -27 \end{array}\right]$

9. $\left[\begin{array}{rrrcr} 1 & 10 & -2 & \vdots & 2 \\ 5 & -3 & 4 & \vdots & 0 \\ 2 & 1 & 0 & \vdots & 6 \end{array}\right]$

11. $\begin{cases} x + 2y = 7 \\ 2x - 3y = 4 \end{cases}$ **13.** $\begin{cases} 9x + 12y + 3z = 0 \\ -2x + 18y + 5z + 2w = 10 \\ x + 7y - 8z = -4 \end{cases}$

15. Reduced row-echelon form

17. Not in row-echelon form **19.** $\begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & -1 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 5 & -2 & 6 \\ 0 & 3 & 20 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 1 & -\frac{2}{5} & \frac{6}{5} \\ 0 & 3 & 20 & 4 \end{bmatrix}$

23. Add 5 times R_2 to R_1. **25.** Interchange R_1 and R_2.

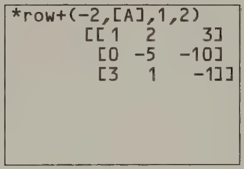
27. (a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -10 \\ 3 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -10 \\ 0 & -5 & -10 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -10 \\ 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

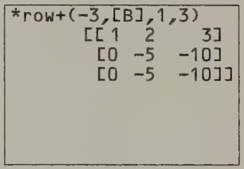
The matrix is in reduced row-echelon form.

29. (a)



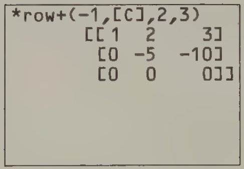
```
*row+(-2,[A],1,2)
        [[1  2    3]
         [0 -5  -10]
         [3  1   -1]]
```

(b)



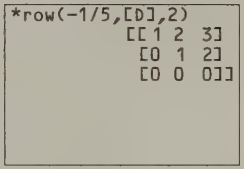
```
*row+(-3,[B],1,3)
        [[1  2    3]
         [0 -5  -10]
         [0 -5  -10]]
```

(c)



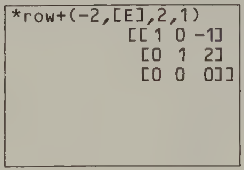
```
*row+(-1,[C],2,3)
        [[1  2    3]
         [0 -5  -10]
         [0  0    0]]
```

(d)



```
*row(-1/5,[D],2)
              [[1 2  3]
               [0 1  2]
               [0 0  0]]
```

(e)



```
*row+(-2,[E],2,1)
           [[1 0 -1]
            [0  1  2]
            [0  0  0]]
```

31. $\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ **33.** $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

35. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **37.** $\begin{bmatrix} 1 & 0 & 3 & 16 \\ 0 & 1 & 2 & 12 \end{bmatrix}$

39. $\begin{cases} x - 2y = 4 \\ \quad\;\; y = -3 \end{cases}$

$(-2, -3)$

41. $\begin{cases} x - y + 2z = 4 \\ \quad\;\; y - z = 2 \\ \qquad\quad z = -2 \end{cases}$

$(8, 0, -2)$

43. $(7, -5)$ **45.** $(-4, -8, 2)$ **47.** $(3, 2)$

49. $(6, -2)$ **51.** $\left(\frac{3}{2}, -\frac{1}{4}\right)$ **53.** Inconsistent

55. $(4, -3, 2)$ **57.** $(2a + 1, 3a + 2, a)$ **59.** $(0, 0)$

61. $(0, 2 - 4a, a)$ **63.** $(1, 0, 4, -2)$ **65.** $(-2a, a, a)$

67. Yes; $(-1, 1, -3)$ **69.** No **71.** $y = x^2 + 2x + 5$

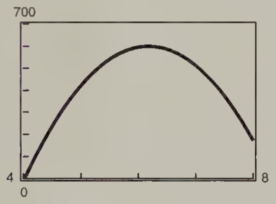
73. $\frac{1}{4}x^3 + x^2 - x - 2$ **75.** $y = -x^4 + 4x^2$

77. \$800,000 at 8%, \$500,000 at 9%, \$200,000 at 12%

79. $I_1 = \frac{13}{10}, I_2 = \frac{11}{5}, I_3 = \frac{9}{10}$

81. (a) $y = -128.5t^2 + 1587.5t - 4304$

(b)



(c) \$172 million

83. (a) $x_1 = s, x_2 = t, x_3 = 600 - s,$

$x_4 = s - t, x_5 = 500 - t, x_6 = s, x_7 = t$

(b) $x_1 = 0, x_2 = 0, x_3 = 600, x_4 = 0, x_5 = 500,$

$x_6 = 0, x_7 = 0$

(c) $x_1 = 0, x_2 = -500, x_3 = 600, x_4 = 500,$

$x_5 = 1000, x_6 = 0, x_7 = -500$

85. (a) $x_1 = 100 + t, x_2 = -100 + t, x_3 = 200 + t, x_4 = t$

(b) $x_1 = 100, x_2 = -100, x_3 = 200, x_4 = 0$

(c) $x_1 = 200, x_2 = 0, x_3 = 300, x_4 = 100$

87. False. Rows are not arranged.

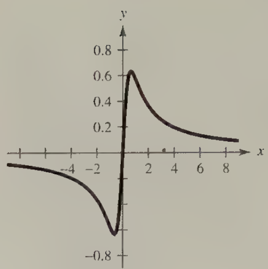
89. $\begin{cases} x + y + 7z = -1 \\ x + 2y + 11z = 0 \\ 2x + y + 10z = -3 \end{cases}$

(Answer is not unique.)

91. (a) There exists a row with all zeros except for the entry in the last column.

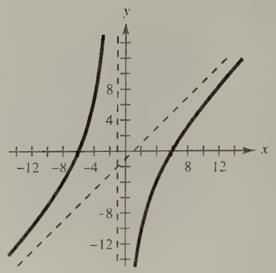
(b) There are fewer rows with nonzero entries than variables.

93.



Asymptote: $y = 0$

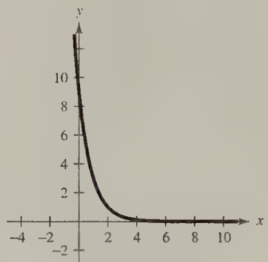
95.



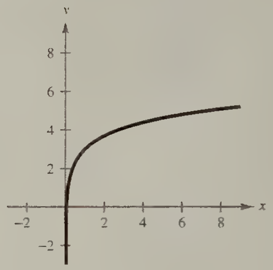
Asymptotes:

$x = -1, y = x - 1$

97.



99.



101. $(-3, -7)$ **103.** $(-2, 3, 1)$

Section 9.2 *(page 655)*

1. $x = -4, y = 22$ **3.** $x = -1, y = 4, z = 6$

5. (a) $\begin{bmatrix} 3 & -2 \\ 1 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 \\ 3 & -9 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -3 \\ 6 & -3 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & -1 \\ 8 & -19 \end{bmatrix}$

7. (a) $\begin{bmatrix} 9 & 5 \\ 1 & -2 \\ -3 & 15 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & -7 \\ 3 & 8 \\ -5 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} 24 & -3 \\ 6 & 9 \\ -12 & 15 \end{bmatrix}$ (d) $\begin{bmatrix} 22 & -15 \\ 8 & 19 \\ -14 & -5 \end{bmatrix}$

9. (a) $\begin{bmatrix} 3 & 3 & -2 & 1 & 1 \\ -2 & 5 & 7 & -6 & -8 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 0 & -1 & 1 \\ 4 & -3 & -11 & 6 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 6 & 6 & -3 & 0 & 3 \\ 3 & 3 & -6 & 0 & -3 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 4 & -1 & -2 & 3 \\ 9 & -5 & -24 & 12 & 11 \end{bmatrix}$

11. (a) Not possible (b) Not possible

(c) $\begin{bmatrix} 18 & 0 & 9 \\ -3 & -12 & 0 \end{bmatrix}$ (d) Not possible

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Calculus
with Analytic Geometry

Seventh Edition

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DERIVATIVES AND INTEGRALS

Basic Differentiation Rules

1. $\dfrac{d}{dx}[cu] = cu'$

2. $\dfrac{d}{dx}[u \pm v] = u' \pm v'$

3. $\dfrac{d}{dx}[uv] = uv' + vu'$

4. $\dfrac{d}{dx}\left[\dfrac{u}{v}\right] = \dfrac{vu' - uv'}{v^2}$

5. $\dfrac{d}{dx}[c] = 0$

6. $\dfrac{d}{dx}[u^n] = nu^{n-1}u'$

7. $\dfrac{d}{dx}[x] = 1$

8. $\dfrac{d}{dx}[|u|] = \dfrac{u}{|u|}(u'), \quad u \neq 0$

9. $\dfrac{d}{dx}[\ln u] = \dfrac{u'}{u}$

10. $\dfrac{d}{dx}[e^u] = e^u u'$

11. $\dfrac{d}{dx}[\sin u] = (\cos u)u'$

12. $\dfrac{d}{dx}[\cos u] = -(\sin u)u'$

13. $\dfrac{d}{dx}[\tan u] = (\sec^2 u)u'$

14. $\dfrac{d}{dx}[\cot u] = -(\csc^2 u)u'$

15. $\dfrac{d}{dx}[\sec u] = (\sec u \tan u)u'$

16. $\dfrac{d}{dx}[\csc u] = -(\csc u \cot u)u'$

17. $\dfrac{d}{dx}[\arcsin u] = \dfrac{u'}{\sqrt{1 - u^2}}$

18. $\dfrac{d}{dx}[\arccos u] = \dfrac{-u'}{\sqrt{1 - u^2}}$

19. $\dfrac{d}{dx}[\arctan u] = \dfrac{u'}{1 + u^2}$

20. $\dfrac{d}{dx}[\operatorname{arccot} u] = \dfrac{-u'}{1 + u^2}$

21. $\dfrac{d}{dx}[\operatorname{arcsec} u] = \dfrac{u'}{|u|\sqrt{u^2 - 1}}$

22. $\dfrac{d}{dx}[\operatorname{arccsc} u] = \dfrac{-u'}{|u|\sqrt{u^2 - 1}}$

Basic Integration Formulas

1. $\displaystyle\int kf(u)\,du = k\int f(u)\,du$

2. $\displaystyle\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$

3. $\displaystyle\int du = u + C$

4. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

5. $\displaystyle\int \frac{du}{u} = \ln|u| + C$

6. $\displaystyle\int e^u\,du = e^u + C$

7. $\displaystyle\int \sin u\,du = -\cos u + C$

8. $\displaystyle\int \cos u\,du = \sin u + C$

9. $\displaystyle\int \tan u\,du = -\ln|\cos u| + C$

10. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$

11. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$

12. $\displaystyle\int \csc u\,du = -\ln|\csc u + \cot u| + C$

13. $\displaystyle\int \sec^2 u\,du = \tan u + C$

14. $\displaystyle\int \csc^2 u\,du = -\cot u + C$

15. $\displaystyle\int \sec u \tan u\,du = \sec u + C$

16. $\displaystyle\int \csc u \cot u\,du = -\csc u + C$

17. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$

18. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

19. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

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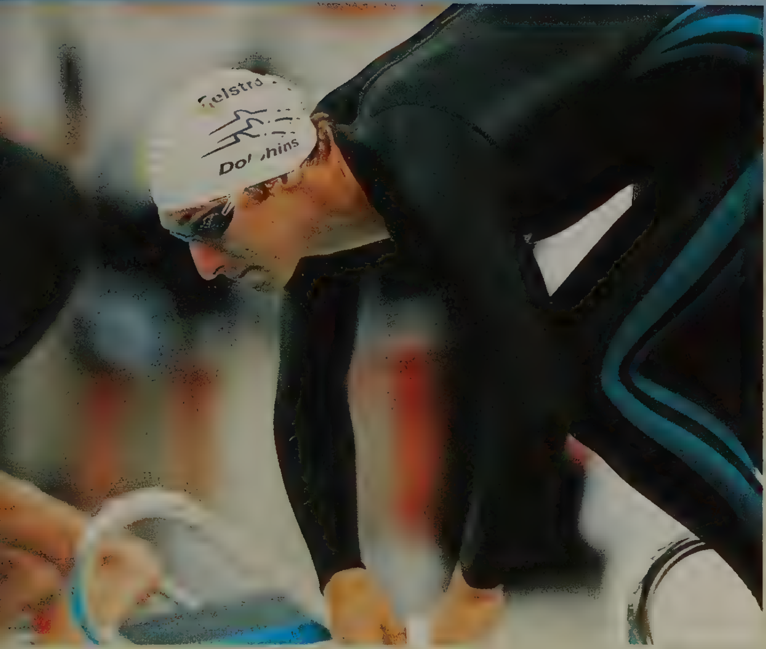
Limits and Their Properties

1



y the age of 17, Australian swimmer Ian Thorpe had set ten world cords. At the 2000 Summer Olympics in Sydney, Australia, he broke s own world record in the 400 meter freestyle.

How High? How Fast?

Excerpted from Sharon Begley and
Adam Rogers, "How High? How Fast?"
from *Newsweek*, July 22, 1996.

Look more closely at the march of winning times and record distances, of gold-medal weights and precedent-setting heights. The law of diminishing returns has set in. The world-record time in the women's 400-meter freestyle, for example, dropped more than two minutes —a full 33 percent—from 1921 (6:16.6) to 1976 (4:11.69). In the 20 years since, it has fallen just eight seconds, to Janet Evans's 4:03.85 at the 1988 Seoul Olympics. If you were to plot world records on graph paper, you would get curves that seem to approach a limit asymptotically, coming tantalizingly closer but never quite reaching it. It is as if the curves were little south-pole magnets and the limit an imposing bar of north polarity. But what is the limit?

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Excerpted from John MacDonald, "Carlile calls for hold on use of bodysuits" from Sports.com, September 4, 2000

Stager (Joel Stager, Indiana University's Councilman Centre for the Science of Swimming) did an analysis of times and the recent U.S. Swimming Trials, where 90 percent of the 1309 competitors wore Speedo suits.

He found there was only a 0.34 percent improvement compared with predictions made based on performances from the past 25 years.

This compared with manufacturers' claims of between 3 and 7 percent.

Al Bello/Allsport

The recent development of a swimming bodysuit proves to be a controversial issue.

Section 1.1 A Preview of Calculus

- Understand what calculus is and how it compares to precalculus.
- Understand that the tangent line problem is basic to calculus.
- Understand that the area problem is also basic to calculus.

What Is Calculus?

Calculus is the mathematics of change—velocities and accelerations. Calculus is al the mathematics of tangent lines, slopes, areas, volumes, arc lengths, centroid curvatures, and a variety of other concepts that have enabled scientists, engineers, ar economists to model real-life situations.

Although precalculus mathematics also deals with velocities, acceleration tangent lines, slopes, and so on, there is a fundamental difference between precalcul mathematics and calculus. Precalculus mathematics is more static, whereas calcul is more dynamic. Here are some examples.

- An object traveling at a constant velocity can be analyzed with precalculus mat ematics. To analyze the velocity of an accelerating object, you need calculus.
- The slope of a line can be analyzed with precalculus mathematics. To analyze tl slope of a curve, you need calculus.
- A tangent line to a circle can be analyzed with precalculus mathematics. T analyze a tangent line of a general graph, you need calculus.
- The area of a rectangle can be analyzed with precalculus mathematics. To analy the area under a general curve, you need calculus.

Each of these situations involves the same general strategy—the reformulation of pr calculus mathematics through the use of a limit process. So, one way to answer tl question "What is calculus?" is to say that calculus is a "limit machine" that involv three stages. The first stage is precalculus mathematics, such as the slope of a line the area of a rectangle. The second stage is the limit process, and the third stage is new calculus formulation, such as a derivative or integral.

Precalculus mathematics ⇨ Limit process ⇨ Calculus

Some students try to learn calculus as if it were simply a collection of ne formulas. This is unfortunate. If you reduce calculus to the memorization of differe tiation and integration formulas, you will miss a great deal of understanding, se confidence, and satisfaction.

On the following two pages we have listed some familiar precalculus concep coupled with their calculus counterparts. Throughout the text, your goal should be learn how precalculus formulas and techniques are used as building blocks to produ the more general calculus formulas and techniques. Don't worry if you are unfamili with some of the "old formulas" listed on the following two pages—we will reviewing all of them.

As you proceed through this text, we suggest that you come back to this discu sion repeatedly. Try to keep track of where you are relative to the three stages involv in the study of calculus. For example, the first three chapters break down as follow

Chapter P: Preparation for Calculus	Precalculus
Chapter 1: Limits and Their Properties	Limit process
Chapter 2: Differentiation	Calculus

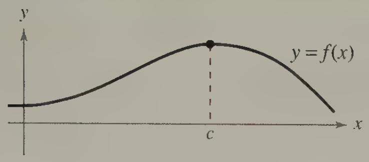
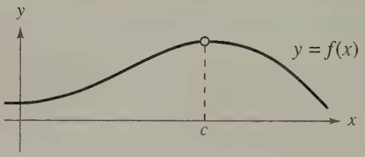
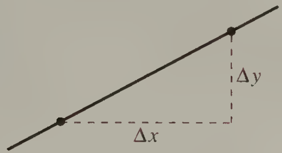
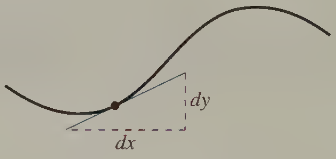
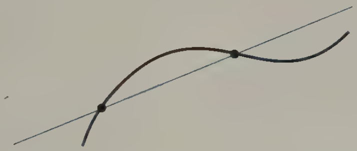
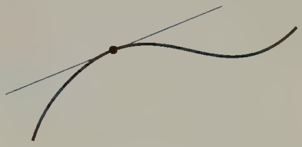
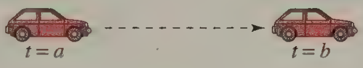
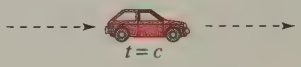
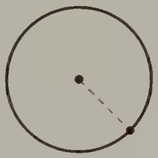
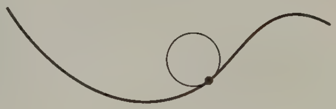
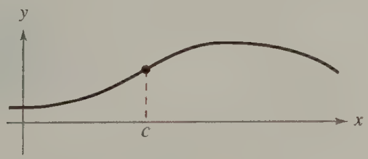
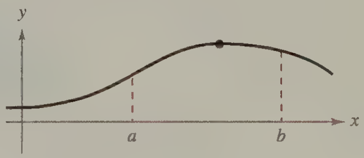
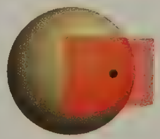
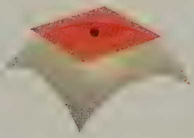
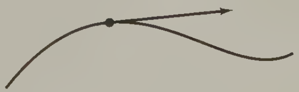
STUDY TIP As you progress through this course, remember that learning calculus is just one of your goals. Your most important goal is to learn how to use calculus to model and solve real-life problems. Here are a few problem-solving strategies that may help you.

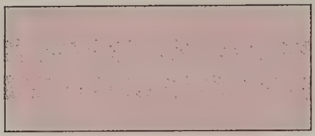
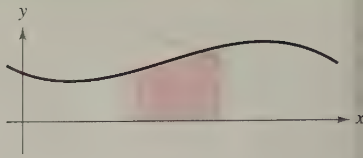
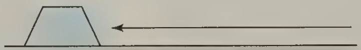
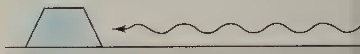
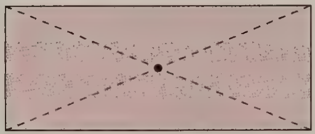
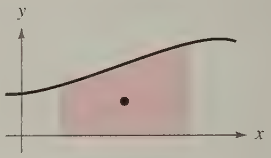
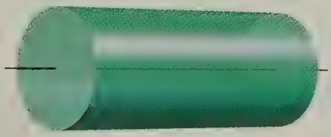
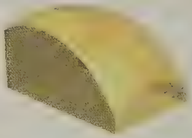
- Be sure you understand the question. What is given? What are you asked to find?
- Outline a plan. There are many approaches you could use: look for a pattern, solve a simpler problem, work backwards, draw a diagram, use technology, or any of many other approaches.
- Complete your plan. Be sure to answer the question. Verbalize your answer. For example, rather than writing the answer as $x = 4.6$, it would be better to write the answer as "The area of the region is 4.6 meters."
- Look back at your work. Does your answer make sense? Is there a way you can check the reasonableness of your answer?

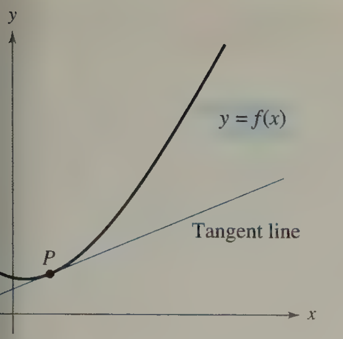
The Mistress Fellows, Girton College, Cambridge

GRACE CHISHOLM YOUNG (1868–1944)

Grace Chisholm Young received her degree in mathematics from Girton College in Cambridge, England. Her early work was published under the name of William Young, her husband. Between 1914 and 1916, Grace Young published work on the foundations of calculus that won her the Gamble Prize from Girton College.

Without Calculus		**With Differential Calculus**	
Value of $f(x)$ when $x = c$		Limit of $f(x)$ as x approaches c	
Slope of a line		Slope of a curve	
Secant line to a curve		Tangent line to a curve	
Average rate of change between $t = a$ and $t = b$		Instantaneous rate of change at $t = c$	
Curvature of a circle		Curvature of a curve	
Height of a curve when $x = c$		Maximum height of a curve on an interval	
Tangent plane to a sphere		Tangent plane to a surface	
Direction of motion along a straight line		Direction of motion along a curved line	

Without Calculus	**With Integral Calculus**
Area of a rectangle 	Area under a curve 
Work done by a constant force 	Work done by a variable force 
Center of a rectangle 	Centroid of a region 
Length of a line segment	Length of an arc
Surface area of a cylinder 	Surface area of a solid of revolution
Mass of a solid of constant density	Mass of a solid of variable density
Volume of a rectangular solid	Volume of a region under a surface 
Sum of a finite number of terms $a_1 + a_2 + \cdots + a_n = S$	Sum of an infinite number of terms $a_1 + a_2 + a_3 + \cdots = S$



The tangent line to the graph of f at P
Figure 1.1

The Tangent Line Problem

The notion of a limit is fundamental to the study of calculus. The following brief descriptions of two classic problems in calculus—*the tangent line problem* and *the area problem*—should give you some idea of the way limits are used in calculus.

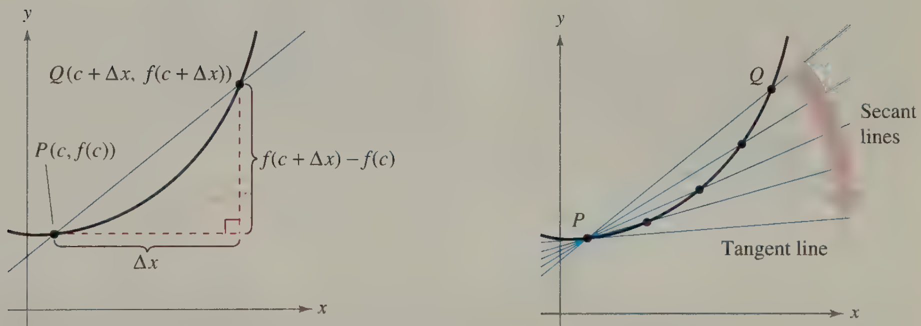
In the tangent line problem, you are given a function f and a point P on its graph and are asked to find an equation of the tangent line to the graph at point P, as shown in Figure 1.1.

Except for cases involving a vertical tangent line, the problem of finding the **tangent line** at a point P is equivalent to finding the *slope* of the tangent line at P. You can approximate this slope by using a line through the point of tangency and a second point on the curve, as shown in Figure 1.2(a). Such a line is called a **secant line.** If $P(c, f(c))$ is the point of tangency and

$$Q(c + \Delta x, f(c + \Delta x))$$

is a second point on the graph of f, the slope of the secant line through these two points is given by

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{c + \Delta x - c} = \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$



(a) The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$

(b) As Q approaches P, the secant lines approach the tangent line.

Figure 1.2

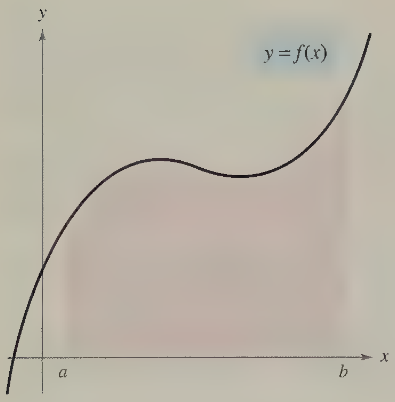
As point Q approaches point P, the slope of the secant line approaches the slope of the tangent line, as shown in Figure 1.2(b). When such a "limiting position" exists, the slope of the tangent line is said to be the **limit** of the slope of the secant line. (Much more will be said about this important problem in Chapter 2.)

EXPLORATION

The following points lie on the graph of $f(x) = x^2$.

$Q_1(1.5, f(1.5))$, $Q_2(1.1, f(1.1))$, $Q_3(1.01, f(1.01))$,
$Q_4(1.001, f(1.001))$, $Q_5(1.0001, f(1.0001))$

Each successive point gets closer to the point $P(1, 1)$. Find the slope of the secant line through Q_1 and P, Q_2 and P, and so on. Graph these secant lines on a graphing utility. Then use your results to estimate the slope of the tangent line to the graph of f at the point P.



Area under a curve
Figure 1.3

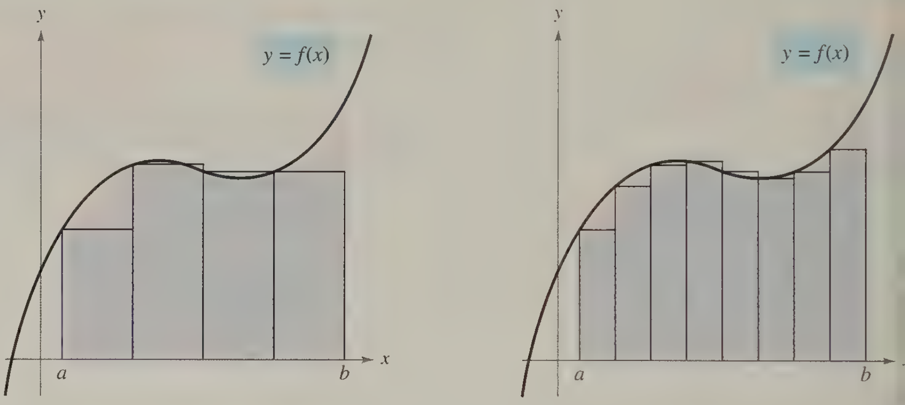
The Area Problem

In the tangent line problem, you saw how the limit process can be applied to the slop of a line to find the slope of a general curve. A second classic problem in calculus finding the area of a plane region that is bounded by the graphs of functions. Th problem can also be solved with a limit process. In this case, the limit process applied to the area of a rectangle to find the area of a general region.

As a simple example, consider the region bounded by the graph of the functio $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$, as shown in Figure 1. You can approximate the area of the region with several rectangular regions, as show in Figure 1.4. As you increase the number of rectangles, the approximation tends t become better and better because the amount of area missed by the rectangle decreases. Your goal is to determine the limit of the sum of the areas of the rectangle as the number of rectangles increases without bound.

Historical Note

In one of the most astounding events ever to occur in mathematics, it was discovered that the tangent line problem and the area problem are closely related. This discovery led to the birth of calculus. You will learn about the relationship between these two problems when you study the Fundamental Theorem of Calculus in Chapter 4.

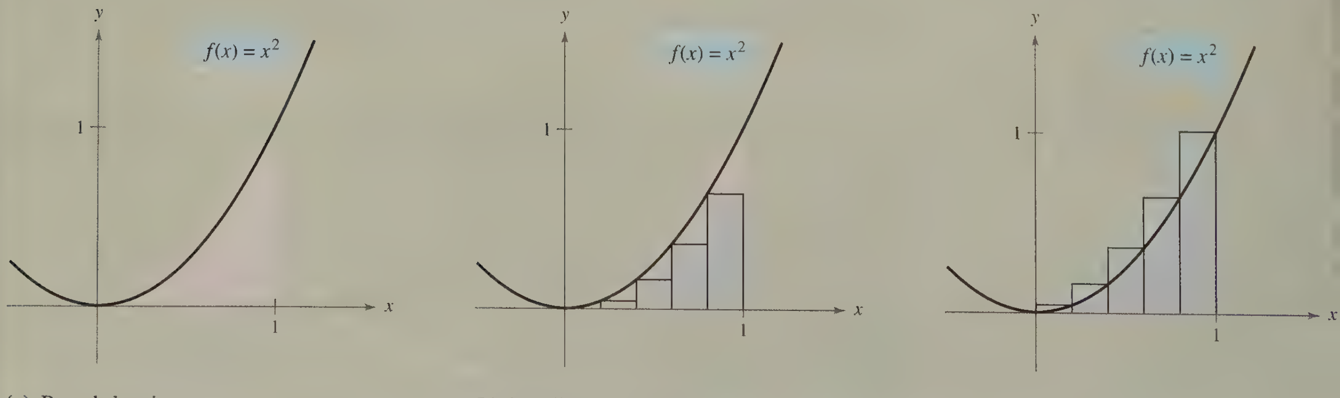


Approximation using four rectangles
Figure 1.4

Approximation using eight rectangles

EXPLORATION

Consider the region bounded by the graphs of $f(x) = x^2$, $y = 0$, and $x = 1$, as shown in part (a) of the figure. The area of the region can be approximated by two sets of rectangles—one set inscribed within the region and the other set circumscribed over the region, as shown in parts (b) and (c). Find the sum of the areas of each set of rectangles. Then use your results to approximate the area of the region.



(a) Bounded region

(b) Inscribed rectangles

(c) Circumscribed rectangles

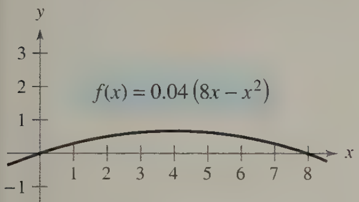
EXERCISES FOR SECTION 1.1

**n Exercises 1–4, decide whether the problem can be solved
sing precalculus, or whether calculus is required. If the prob-
m can be solved using precalculus, solve it. If the problem
eems to require calculus, explain your reasoning and use a
raphical or numerical approach to estimate the solution.**

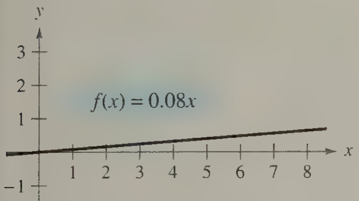
Find the distance traveled in 15 seconds by an object traveling at a constant velocity of 20 feet per second.

Find the distance traveled in 15 seconds by an object moving with a velocity of $v(t) = 20 + 7\cos t$ feet per second.

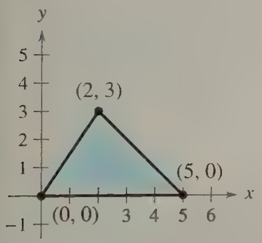
A bicyclist is riding on a path modeled by the function $f(x) = 0.04(8x - x^2)$, where x and $f(x)$ are measured in miles. Find the rate of change of elevation when $x = 2$.



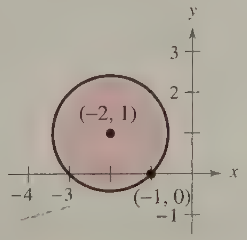
A bicyclist is riding on a path modeled by the function $f(x) = 0.08x$, where x and $f(x)$ are measured in miles. Find the rate of change of elevation when $x = 2$.



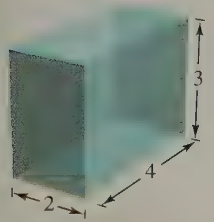
n Exercises 5 and 6, find the area of the shaded region.



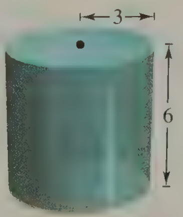
6.



n Exercises 7 and 8, find the volume of the solid shown.



8.



Getting at the Concept

9. (a) Use the *list* feature of a graphing utility to graph the following.

$$y_1 = 4x - x^2$$

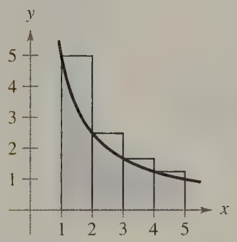
$$y_2 = \left[\frac{y_1(1 + \{2, 1.5, 1, 0.5\}) - y_1(1)}{\{2, 1.5, 1, 0.5\}}\right](x - 1) + y_1(1)$$

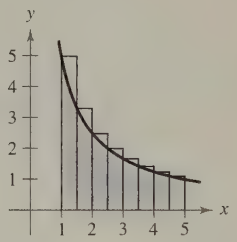
(*Note:* If you cannot use lists on your graphing utility, graph y_2 four times using 2, 1.5, 1, and 0.5.)

(b) Give a written description of the graphs of y_2 relative to the graph of y_1.

(c) Use the results in part (a) to estimate the slope of the tangent line to the graph of y_1 at (1, 3). If you want to improve your approximation of the slope, how could you change the list in the formula for y_2?

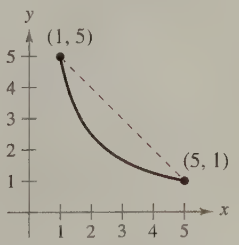
10. (a) Use the rectangles in each graph to approximate the area of the region bounded by $y = 5/x$, $y = 0$, $x = 1$, and $x = 5$.

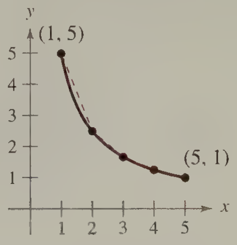




(b) Describe how you could continue this process to obtain a more accurate approximation of the area.

11. Consider the length of the graph of $f(x) = 5/x$ from (1, 5) to (5, 1).





(a) Approximate the length of the curve by finding the distance between its two endpoints, as shown in the first figure.

(b) Approximate the length of the curve by finding the lengths of four line segments, as shown in the second figure.

(c) Describe how you could continue this process to obtain a more accurate approximation of the length of the curve.

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ne symbol indicates an exercise in which you are instructed to use graphing technology a symbolic computer algebra system. The solutions of other exercises may also be facilitated use of appropriate technology.

Section 1.2 Finding Limits Graphically and Numerically

- Estimate a limit using a numerical or graphical approach.
- Learn different ways that a limit can fail to exist.
- Study and use a formal definition of a limit.

An Introduction to Limits

Suppose you are asked to sketch the graph of the function f given by

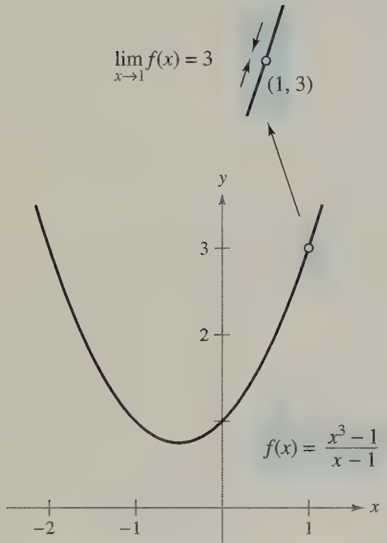
$$f(x) = \frac{x^3 - 1}{x - 1}, \qquad x \neq 1.$$

For all values other than $x = 1$, you can use standard curve-sketching techniqu However, at $x = 1$, it is not clear what to expect. To get an idea of the behavior of t graph of f near $x = 1$, you can use two sets of x-values—one set that approaches from the left and one that approaches 1 from the right, as shown in the table.

x approaches 1 from the left. → ← x approaches 1 from the right.

x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.710	2.970	2.997	?	3.003	3.030	3.310	3.813

$f(x)$ approaches 3. → ← $f(x)$ approaches 3.



The limit of $f(x)$ as x approaches 1 is 3.
Figure 1.5

The graph of f is a parabola that has a gap at the point (1, 3), as shown in Figu 1.5. Although x cannot equal 1, you can move arbitrarily close to 1, and as a res $f(x)$ moves arbitrarily close to 3. Using limit notation, you can write

$$\lim_{x\to 1} f(x) = 3.$$

This is read as "the limit of $f(x)$ as x approaches 1 is 3."

This discussion leads to an informal description of a limit. If $f(x)$ becomes arbitrar close to a single number L as x approaches c from either side, the **limit** of $f(x)$, a approaches c, is L. This limit is written as

$$\lim_{x\to c} f(x) = L.$$

EXPLORATION

The discussion above gives an example of how you can estimate a limit *numerically* by constructing a table and *graphically* by drawing a graph. Estimate the following limit numerically by completing the table.

$$\lim_{x\to 2} \frac{x^2 - 3x + 2}{x - 2}$$

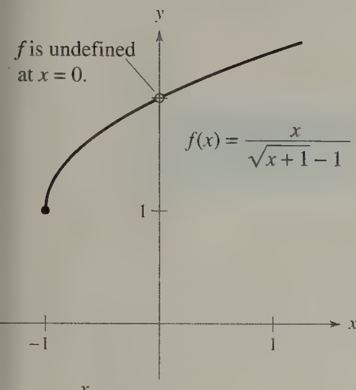
x	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25
$f(x)$	?	?	?	?	?	?	?	?	?

Then use a graphing utility to estimate the limit graphically.

Example 1 Estimating a Limit Numerically

Evaluate the function $f(x) = x/\left(\sqrt{x+1} - 1\right)$ at several points near $x = 0$ and use the result to estimate the limit

$$\lim_{x\to 0} \frac{x}{\sqrt{x+1} - 1}.$$



im $\frac{x}{\sqrt{x+1}-1} = 2$
$\to 0$

he limit of $f(x)$ as x approaches 0 is 2.
igure 1.6

Solution The table lists the values of $f(x)$ for several x-values near 0.

x approaches 0 from the left. x approaches 0 from the right.

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.99995	?	2.00005	2.00050	2.00499

$f(x)$ approaches 2. $f(x)$ approaches 2.

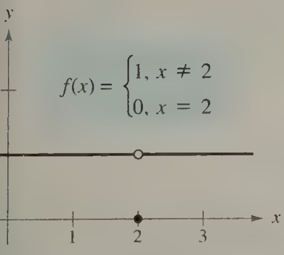
From the results shown in the table, you can estimate the limit to be 2. This limit is reinforced by the graph of f (see Figure 1.6).

In Example 1, note that the function is undefined at $x = 0$ and yet $f(x)$ appears to be approaching a limit as x approaches 0. This often happens, and it is important to realize that *the existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as x approaches c.*

Example 2 Finding a Limit

Find the limit of $f(x)$ as x approaches 2 where f is defined as

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2. \end{cases}$$



im $f(x) = 1$
$\to 2$

he limit of $f(x)$ as x approaches 2 is 1.
igure 1.7

Solution Because $f(x) = 1$ for all x other than $x = 2$, you can conclude that the limit is 1, as shown in Figure 1.7. So, you can write

$$\lim_{x\to 2} f(x) = 1.$$

The fact that $f(2) = 0$ has no bearing on the existence or value of the limit as x approaches 2. For instance, if the function were defined as

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 2, & x = 2 \end{cases}$$

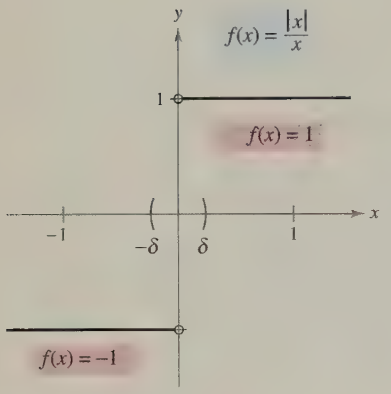
the limit would be the same.

So far in this section, you have been estimating limits numerically and graphically. Each of these approaches produces an estimate of the limit. In Section 1.3, you will study analytic techniques for evaluating limits. Throughout the course, try to develop a habit of using this three-pronged approach to problem solving.

1. Numerical approach	Construct a table of values.
2. Graphical approach	Draw a graph by hand or using technology.
3. Analytic approach	Use algebra or calculus.

Limits That Fail to Exist

In the next three examples you will examine some limits that fail to exist.



$\lim_{x\to 0} f(x)$ does not exist.
Figure 1.8

Example 3 Behavior That Differs from the Right and Left

Show that the limit does not exist.

$$\lim_{x\to 0} \frac{|x|}{x}$$

Solution Consider the graph of the function $f(x) = |x|/x$. From Figure 1.8, you ca see that for positive x-values

$$\frac{|x|}{x} = 1, \qquad x > 0$$

and for negative x-values

$$\frac{|x|}{x} = -1, \qquad x < 0.$$

This means that no matter how close x gets to 0, there will be both positive ar negative x-values that yield $f(x) = 1$ and $f(x) = -1$. Specifically, if δ (the lowercas Greek letter *delta*) is a positive number, then for x-values satisfying the inequali $0 < |x| < \delta$, you can classify the values of $|x|/x$ as follows.

$$\underset{\text{Negative } x\text{-values yield } |x|/x = -1.}{(-\delta, 0)} \qquad \underset{\text{Positive } x\text{-values yield } |x|/x = 1.}{(0, \delta)}$$

This implies that the limit does not exist.

Example 4 Unbounded Behavior

Discuss the existence of the limit

$$\lim_{x\to 0} \frac{1}{x^2}.$$

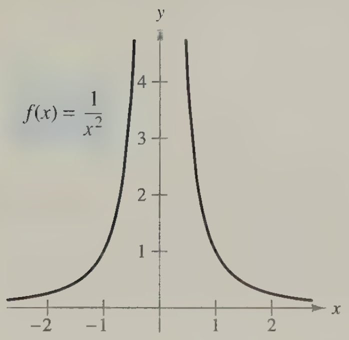
Solution Let $f(x) = 1/x^2$. In Figure 1.9, you can see that as x approaches 0 fro either the right or the left, $f(x)$ increases without bound. This means that by choosin x close enough to 0, you can force $f(x)$ to be as large as you want. For instance, $f($ will be larger than 100 if you choose x that is within $\frac{1}{10}$ of 0. That is,

$$0 < |x| < \frac{1}{10} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 100.$$

Similarly, you can force $f(x)$ to be larger than 1,000,000, as follows.

$$0 < |x| < \frac{1}{1000} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 1{,}000{,}000$$

Because $f(x)$ is not approaching a real number L as x approaches 0, you can conclu that the limit does not exist.

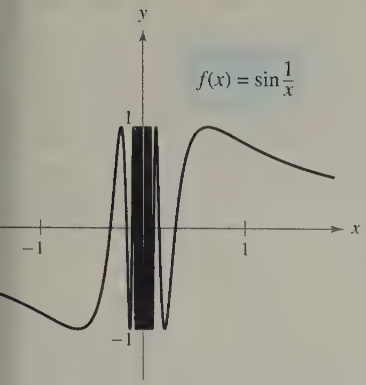


$\lim_{x\to 0} f(x)$ does not exist.
Figure 1.9

Example 5 Oscillating Behavior

Discuss the existence of the limit $\lim_{x \to 0} \sin \frac{1}{x}$.

Solution Let $f(x) = \sin(1/x)$. In Figure 1.10, you can see that as x approaches 0, $f(x)$ oscillates between -1 and 1. Therefore, the limit does not exist because no matter how small you choose δ, it is possible to choose x_1 and x_2 within δ units of 0 such that $\sin(1/x_1) = 1$ and $\sin(1/x_2) = -1$, as indicated in the table.



im $_{\to 0}$ $f(x)$ does not exist.
igure 1.10

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$x \to 0$
$\sin \frac{1}{x}$	1	-1	1	-1	1	-1	Limit does not exist.

Common Types of Behavior Associated with the Nonexistence of a Limit

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

There are many other interesting functions that have unusual limit behavior. An often cited one is the *Dirichlet function*

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational.} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

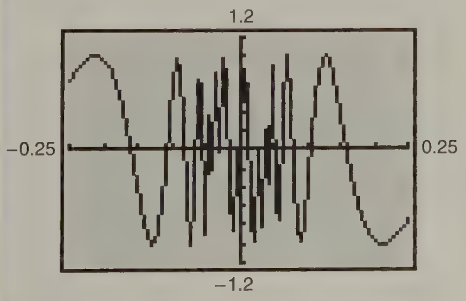
This function has *no limit* at any real number c.

The Granger Collection

PETER GUSTAV DIRICHLET (1805–1859)

In the early development of calculus, the definition of a function was much more restricted than it is today, and "functions" such as the Dirichlet function would not have been considered. The modern definition of a function was given by the German mathematician Peter Gustav Dirichlet.

TECHNOLOGY PITFALL When you use a graphing utility to investigate the behavior of a function near the x-value at which you are trying to evaluate a limit, remember that you can't always trust the pictures that graphing utilities draw. For instance, if you use a graphing utility to sketch the graph of the function in Example 5 over an interval containing 0, you will most likely obtain an incorrect graph such as that shown in Figure 1.11. The reason that a graphing utility can't show the correct graph is that the graph has infinitely many oscillations over any interval that contains 0.



Incorrect graph of $f(x) = \sin(1/x)$.
Figure 1.11

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indicates that in the Interactive 3.0 *CD-ROM and* Internet 3.0 *versions of this text (available at* college.hmco.com) *you will find an Open Exploration, which further explores this example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

A Formal Definition of a Limit

Let's take another look at the informal description of a limit. If $f(x)$ becomes arb trarily close to a single number L as x approaches c from either side, we say that tl limit of $f(x)$ as x approaches c is L, written as

$$\lim_{x \to c} f(x) = L.$$

At first glance, this description looks fairly technical. Even so, we call it inform because we have yet to give exact meanings to the two phrases

"$f(x)$ becomes arbitrarily close to L"

and

"x approaches c."

The first person to assign mathematically rigorous meanings to these two phrases wa Augustin-Louis Cauchy. His **ε-δ definition of a limit** is the standard used today.

In Figure 1.12, let ε (the lowercase Greek letter *epsilon*) represent a (smal positive number. Then the phrase "$f(x)$ becomes arbitrarily close to L" means th $f(x)$ lies in the interval $(L - \varepsilon, L + \varepsilon)$. Using absolute value, you can write this as

$$|f(x) - L| < \varepsilon.$$

Similarly, the phrase "x approaches c" means that there exists a positive number such that x lies in either the interval $(c - \delta, c)$ or the interval $(c, c + \delta)$. This fact ca be concisely expressed by the double inequality

$$0 < |x - c| < \delta.$$

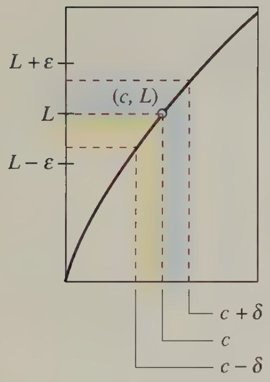
The first inequality

$0 < |x - c|$ The difference between x and c is more than 0.

expresses the fact that $x \neq c$. The second inequality

$|x - c| < \delta$ x is within δ units of c.

says that x is within a distance δ of c.



The ε-δ definition of the limit of $f(x)$ as x approaches c
Figure 1.12

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

NOTE Throughout this text, when we write

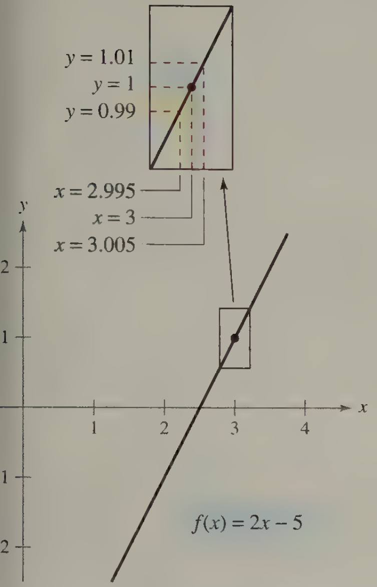
$$\lim_{x \to c} f(x) = L$$

we imply two statements—the limit **exists** *and* the limit is L.

Some functions do not have limits as $x \to c$, but those that do cannot have tw different limits as $x \to c$. That is, *if the limit of a function exists, it is unique* (se Exercise 55).

FOR FURTHER INFORMATION For more on the introduction of rigor to calculus, see "Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus" by Judith V. Grabiner in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

The next three examples should help you develop a better understanding of the ε-δ definition of a limit.



he limit of $f(x)$ as x approaches 3 is 1.
igure 1.13

Example 6 Finding a δ for a Given ε

Given the limit

$$\lim_{x\to 3} (2x - 5) = 1$$

find δ such that $|(2x - 5) - 1| < 0.01$ whenever $0 < |x - 3| < \delta$.

Solution In this problem, you are working with a given value of ε—namely, $\varepsilon = 0.01$. To find an appropriate δ, notice that

$$|(2x - 5) - 1| = |2x - 6| = 2|x - 3|.$$

Because the inequality $|(2x - 5) - 1| < 0.01$ is equivalent to $2|x - 3| < 0.01$, you can choose $\delta = \frac{1}{2}(0.01) = 0.005$. This choice works because

$$0 < |x - 3| < 0.005$$

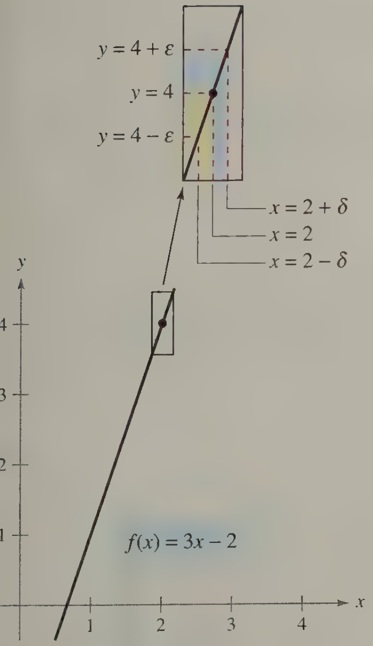
implies that

$$|(2x - 5) - 1| = 2|x - 3| < 2(0.005) = 0.01$$

as shown in Figure 1.13.

NOTE In Example 6, note that 0.005 is the *largest* value of δ that will guarantee $|(2x - 5) - 1| < 0.01$ whenever $0 < |x - 3| < \delta$. Any *smaller* positive value of δ would, of course, also work.

In Example 6, you found a δ-value for a *given* ε. This does not prove the existence of the limit. To do that, you must prove that you can find a δ for any ε, as demonstrated in the next example.



e limit of $f(x)$ as x approaches 2 is 4.
gure 1.14

Example 7 Using the ε-δ Definition of a Limit

Use the ε-δ definition of a limit to prove that

$$\lim_{x\to 2} (3x - 2) = 4.$$

Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|(3x - 2) - 4| < \varepsilon$ whenever $0 < |x - 2| < \delta$. Because your choice of δ depends on ε, you need to establish a connection between the absolute values $|(3x - 2) - 4|$ and $|x - 2|$.

$$|(3x - 2) - 4| = |3x - 6| = 3|x - 2|$$

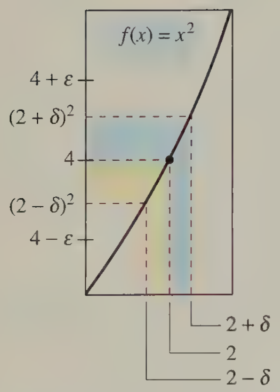
So, for a given $\varepsilon > 0$ you can choose $\delta = \varepsilon/3$. This choice works because

$$0 < |x - 2| < \delta = \frac{\varepsilon}{3}$$

implies that

$$|(3x - 2) - 4| = 3|x - 2| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

as shown in Figure 1.14.



The limit of $f(x)$ as x approaches 2 is 4.
Figure 1.15

Example 8 Using the ε-δ Definition of a Limit

Use the ε-δ definition of a limit to prove that

$$\lim_{x \to 2} x^2 = 4.$$

Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x^2 - 4| < \varepsilon \text{ when } 0 < |x - 2| < \delta.$$

To find an appropriate δ, begin by writing $|x^2 - 4| = |x - 2||x + 2|$. For all x in t interval $(1, 3)$, you know that $|x + 2| < 5$. So, letting δ be the minimum of $\varepsilon/5$ a 1, it follows that, whenever $0 < |x - 2| < \delta$, you have

$$|x^2 - 4| = |x - 2||x + 2| < \left(\frac{\varepsilon}{5}\right)(5) = \varepsilon$$

as shown in Figure 1.15.

Throughout this chapter you will use the ε-δ definition of a limit primarily prove theorems about limits and to establish the existence or nonexistence of partic lar types of limits. For *finding* limits, you will learn techniques that are easier to u than the ε-δ definition of a limit.

EXERCISES FOR SECTION 1.2

In Exercises 1–8, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

1. $\displaystyle\lim_{x \to 2} \frac{x - 2}{x^2 - x - 2}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

2. $\displaystyle\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

3. $\displaystyle\lim_{x \to 0} \frac{\sqrt{x + 3} - \sqrt{3}}{x}$

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$						

4. $\displaystyle\lim_{x \to -3} \frac{\sqrt{1 - x} - 2}{x + 3}$

x	−3.1	−3.01	−3.001	−2.999	−2.99	−2.9
$f(x)$						

5. $\displaystyle\lim_{x \to 3} \frac{[1/(x + 1)] - (1/4)}{x - 3}$

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$						

6. $\displaystyle\lim_{x \to 4} \frac{[x/(x + 1)] - (4/5)}{x - 4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$						

7. $\displaystyle\lim_{x \to 0} \frac{\sin x}{x}$

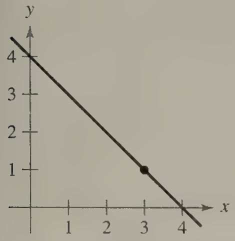
x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$						

8. $\displaystyle\lim_{x \to 0} \frac{\cos x - 1}{x}$

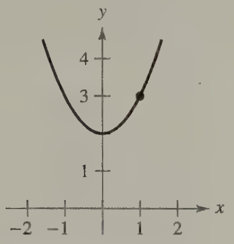
x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$						

n Exercises 9–18, use the graph to find the limit (if it exists). If he limit does not exist, explain why.

9. $\lim_{x\to3}(4-x)$

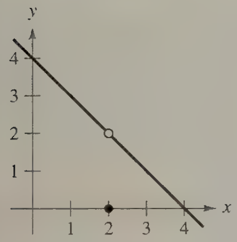


10. $\lim_{x\to1}(x^2+2)$



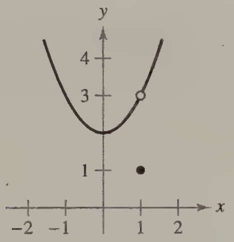
1. $\lim_{x\to2}f(x)$

$$f(x)=\begin{cases}4-x, & x\neq2\\ 0, & x=2\end{cases}$$

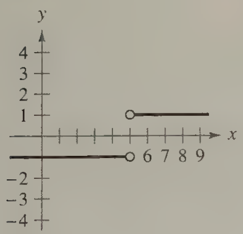


12. $\lim_{x\to1}f(x)$

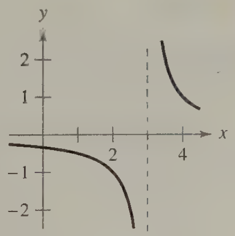
$$f(x)=\begin{cases}x^2+2, & x\neq1\\ 1, & x=1\end{cases}$$



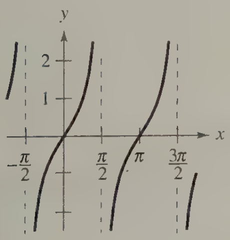
3. $\lim_{x\to5}\dfrac{|x-5|}{x-5}$



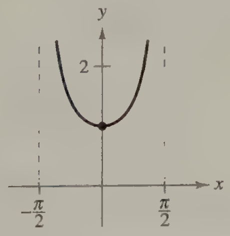
14. $\lim_{x\to3}\dfrac{1}{x-3}$



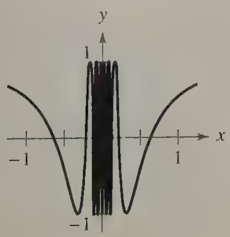
5. $\lim_{x\to\pi/2}\tan x$



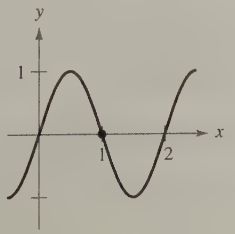
16. $\lim_{x\to0}\sec x$



7. $\lim_{x\to0}\cos\dfrac{1}{x}$



18. $\lim_{x\to1}\sin\pi x$



19. ***Modeling Data*** The cost of a telephone call between two cities is \$0.75 for the first minute and \$0.50 for each additional minute. A formula for the cost is given by

$$C(t)=0.75-0.50[\![-(t-1)]\!]$$

where t is the time in minutes.

(*Note:* $[\![x]\!]$ = greatest integer n such that $n\le x$. For example, $[\![3.2]\!]=3$ and $[\![-1.6]\!]=-2$.)

(a) Use a graphing utility to graph the cost function for $0<t\le5$.

(b) Use the graph to complete the table and observe the behavior of the function as t approaches 3.5. Use the graph and the table to find

$$\lim_{t\to3.5}C(t).$$

t	3	3.3	3.4	3.5	3.6	3.7	4
C				?			

(c) Use the graph to complete the table and observe the behavior of the function as t approaches 3.

t	2	2.5	2.9	3	3.1	3.5	4
C				?			

Does the limit of $C(t)$ as t approaches 3 exist? Explain.

20. Repeat Exercise 19 if $C(t)=0.35-0.12[\![-(t-1)]\!]$.

21. The graph of $f(x)=2-1/x$ is shown in the figure. Find δ such that if $0<|x-1|<\delta$ then $|f(x)-1|<0.1$.

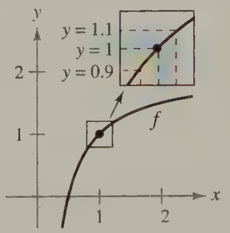


Figure for 21

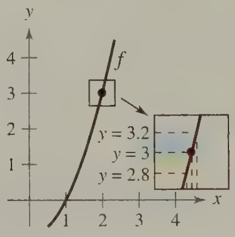


Figure for 22

22. The graph of $f(x)=x^2-1$ is shown in the figure. Find δ such that if $0<|x-2|<\delta$ then $|f(x)-3|<0.2$.

In Exercises 23–26, find the limit L. Then find $\delta>0$ such that $|f(x)-L|<0.01$ whenever $0<|x-c|<\delta$.

23. $\lim_{x\to2}(3x+2)$

24. $\lim_{x\to4}\left(4-\dfrac{x}{2}\right)$

25. $\lim_{x\to2}(x^2-3)$

26. $\lim_{x\to5}(x^2+4)$

In Exercises 27–38, find the limit L. Then use the ε-δ definition to prove that the limit is L.

27. $\lim_{x\to 2} (x + 3)$

28. $\lim_{x\to -3} (2x + 5)$

29. $\lim_{x\to -4} \left(\frac{1}{2}x - 1\right)$

30. $\lim_{x\to 1} \left(\frac{2}{3}x + 9\right)$

31. $\lim_{x\to 6} 3$

32. $\lim_{x\to 2} (-1)$

33. $\lim_{x\to 0} \sqrt[3]{x}$

34. $\lim_{x\to 4} \sqrt{x}$

35. $\lim_{x\to -2} |x - 2|$

36. $\lim_{x\to 3} |x - 3|$

37. $\lim_{x\to 1} (x^2 + 1)$

38. $\lim_{x\to -3} (x^2 + 3x)$

Writing **In Exercises 39–42, use a graphing utility to graph the function and estimate the limit (if it exists). What is the domain of the function? Can you detect a possible error in determining the domain of a function solely by analyzing the graph generated by a graphing utility? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

39. $f(x) = \dfrac{\sqrt{x+5} - 3}{x - 4}$

$\lim_{x\to 4} f(x)$

40. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

$\lim_{x\to 3} f(x)$

41. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$

$\lim_{x\to 9} f(x)$

42. $f(x) = \dfrac{x - 3}{x^2 - 9}$

$\lim_{x\to 3} f(x)$

Getting at the Concept

43. Write a brief description of the meaning of the notation $\lim_{x\to 8} f(x) = 25$.

44. (a) If $f(2) = 4$, can you conclude anything about the limit of $f(x)$ as x approaches 2? Explain your reasoning.

(b) If the limit of $f(x)$ as x approaches 2 is 4, can you conclude anything about $f(2)$? Explain your reasoning.

45. Identify three types of behavior associated with the nonexistence of a limit. Illustrate each type with a graph of a function.

46. Determine the limit of the function describing the atmospheric pressure on a plane as it descends from 32,000 feet to land at Honolulu, located at sea level. (The atmospheric pressure at sea level is 14.7 lb/in.^{2}.)

47. Consider the function $f(x) = (1 + x)^{1/x}$. Estimate the limit

$$\lim_{x\to 0} (1 + x)^{1/x}$$

by evaluating f at x-values near 0. Sketch the graph of f.

48. ***Graphical Analysis*** The statement

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = 4$$

means that for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such th if $0 < |x - 2| < \delta$, then

$$\left|\frac{x^2 - 4}{x - 2} - 4\right| < \varepsilon.$$

If $\varepsilon = 0.001$, then

$$\left|\frac{x^2 - 4}{x - 2} - 4\right| < 0.001.$$

Use a graphing utility to graph each side of this inequality. Us the *zoom* feature to find an interval $(2 - \delta, 2 + \delta)$ such th the graph of the left side is below the graph of the right side the inequality.

True or False? **In Exercises 49–52, determine whether th statement is true or false. If it is false, explain why or give a example that shows it is false.**

49. If f is undefined at $x = c$, then the limit of $f(x)$ as approaches c does not exist.

50. If the limit of $f(x)$ as x approaches c is 0, then there must exi a number k such that $f(k) < 0.001$.

51. If $f(c) = L$, then $\lim_{x\to c} f(x) = L$.

52. If $\lim_{x\to c} f(x) = L$, then $f(c) = L$.

53. ***Programming*** Use the programming capabilities of a grap ing utility to write a program for approximating $\lim_{x\to c} f(x)$.

Assume the program will be applied only to functions who limits exist as x approaches c. Let $y_1 = f(x)$ and generate tw lists whose entries form the ordered pairs

$$(c \pm [0.1]^n, f(c \pm [0.1]^n))$$

for $n = 0, 1, 2, 3,$ and 4.

54. Use the program you created in Exercise 53 to approximate t limit

$$\lim_{x\to 4} \frac{x^2 - x - 12}{x - 4}.$$

55. Prove that if the limit of $f(x)$ as $x \to c$ exists, then the limit mu be unique. [*Hint:* Let

$$\lim_{x\to c} f(x) = L_1 \quad \text{and} \quad \lim_{x\to c} f(x) = L_2$$

and prove that $L_1 = L_2$.]

56. Consider the line $f(x) = mx + b$, where $m \neq 0$. Use the ε definition of a limit to prove that $\lim_{x\to c} f(x) = mc + b$.

57. Prove that $\lim_{x\to c} f(x) = L$ is equivalent to $\lim_{x\to c} [f(x) - L] = 0$

58. Given that $\lim_{x\to c} g(x) = L$, where $L > 0$, prove that there exis an open interval (a, b) containing c such that $g(x) > 0$ for $x \neq c$ in (a, b).

Section 1.3 Evaluating Limits Analytically

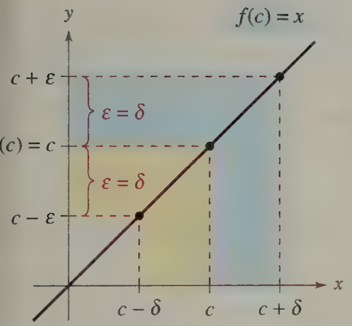
- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.
- Evaluate a limit using dividing out and rationalizing techniques.
- Evaluate a limit using the Squeeze Theorem.

Properties of Limits

In Section 1.2, you learned that the limit of $f(x)$ as x approaches c does not depend on the value of f at $x = c$. It may happen, however, that the limit is precisely $f(c)$. In such cases, the limit can be evaluated by **direct substitution.** That is,

$$\lim_{x \to c} f(x) = f(c). \qquad \text{Substitute } c \text{ for } x.$$

Such *well-behaved* functions are **continuous at c.** You will examine this concept more closely in Section 1.4.



igure 1.16

THEOREM 1.1 Some Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b = b$ **2.** $\lim_{x \to c} x = c$ **3.** $\lim_{x \to c} x^n = c^n$

Proof To prove Property 2 of Theorem 1.1, you need to show that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - c| < \varepsilon$ whenever $0 < |x - c| < \delta$. Because the second inequality is a stricter version of the first, you can simply choose $\delta = \varepsilon$, as shown in Figure 1.16. This completes the proof. (Proofs of the other properties of limits in this section are listed in Appendix B or are discussed in the exercises.) ◿

OTE When you encounter new nota-ons or symbols in mathematics, be sure ou know how the notations are read. or instance, the limit in Example 1c is ead as "the limit of x^2 as x approaches 2 4."

Example 1 Evaluating Basic Limits

a. $\lim_{x \to 2} 3 = 3$ **b.** $\lim_{x \to -4} x = -4$ **c.** $\lim_{x \to 2} x^2 = 2^2 = 4$ ◿

THEOREM 1.2 Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. Scalar multiple: $\lim_{x \to c} [b f(x)] = bL$ $\lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$

3. Product: $\lim_{x \to c} [f(x)g(x)] = LK$

4. Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$

5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

Example 2 The Limit of a Polynomial

$$\begin{aligned}\lim_{x\to 2}(4x^2+3) &= \lim_{x\to 2}4x^2 + \lim_{x\to 2}3 && \text{Property 2}\\ &= 4\left(\lim_{x\to 2}x^2\right) + \lim_{x\to 2}3 && \text{Property 1}\\ &= 4(2^2)+3 && \text{Example 1}\\ &= 19 && \text{Simplify.}\end{aligned}$$

In Example 2, note that the limit (as $x \to 2$) of the *polynomial function* $p(x)$ $4x^2 + 3$ is simply the value of p at $x = 2$.

$$\lim_{x\to 2} p(x) = p(2) = 4(2^2) + 3 = 19$$

This *direct substitution* property is valid for all polynomial and rational functions wi nonzero denominators.

The Square Root Symbol

The first use of a symbol to denote the square root can be traced to the sixteenth century. Mathematicians first used the symbol $\surd$, which had only two strokes. This symbol was chosen because it resembled a lowercase r, to stand for the Latin word *radix*, meaning root.

THEOREM 1.3 Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Example 3 The Limit of a Rational Function

Find the limit: $\lim_{x\to 1} \dfrac{x^2+x+2}{x+1}$.

Solution Because the denominator is not 0 when $x = 1$, you can apply Theorem 1 to obtain

$$\lim_{x\to 1} \frac{x^2+x+2}{x+1} = \frac{1^2+1+2}{1+1} = \frac{4}{2} = 2.$$

Polynomial functions and rational functions are two of the three basic types algebraic functions. The following theorem deals with the limit of the third type algebraic function—one that involves a radical. See Appendix B for a proof of th theorem.

THEOREM 1.4 The Limit of a Function Involving a Radical

Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$$

The following theorem greatly expands your ability to evaluate limits because it shows how to analyze the limit of a composite function. See Appendix B for a proof of this theorem.

THEOREM 1.5 The Limit of a Composite Function

If f and g are functions such that $\lim_{x\to c} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$, then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(L).$$

Example 4 The Limit of a Composite Function

a. Because

$$\lim_{x\to 0} (x^2 + 4) = 0^2 + 4 = 4 \quad \text{and} \quad \lim_{x\to 4} \sqrt{x} = 2$$

it follows that

$$\lim_{x\to 0} \sqrt{x^2 + 4} = \sqrt{4} = 2.$$

b. Because

$$\lim_{x\to 3} (2x^2 - 10) = 2(3^2) - 10 = 8 \quad \text{and} \quad \lim_{x\to 8} \sqrt[3]{x} = 2$$

it follows that

$$\lim_{x\to 3} \sqrt[3]{2x^2 - 10} = \sqrt[3]{8} = 2.$$

You have seen that the limits of many algebraic functions can be evaluated by direct substitution. Each of the six basic trigonometric functions also possesses this desirable quality, as shown in the next theorem (presented without proof).

THEOREM 1.6 Limits of Trigonometric Functions

Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x\to c} \sin x = \sin c$

2. $\lim_{x\to c} \cos x = \cos c$

3. $\lim_{x\to c} \tan x = \tan c$

4. $\lim_{x\to c} \cot x = \cot c$

5. $\lim_{x\to c} \sec x = \sec c$

6. $\lim_{x\to c} \csc x = \csc c$

Example 5 Limits of Trigonometric Functions

a. $\lim_{x\to 0} \tan x = \tan(0) = 0$

b. $\lim_{x\to \pi} (x \cos x) = \left(\lim_{x\to \pi} x\right)\left(\lim_{x\to \pi} \cos x\right) = \pi \cos(\pi) = -\pi$

c. $\lim_{x\to 0} \sin^2 x = \lim_{x\to 0} (\sin x)^2 = 0^2 = 0$

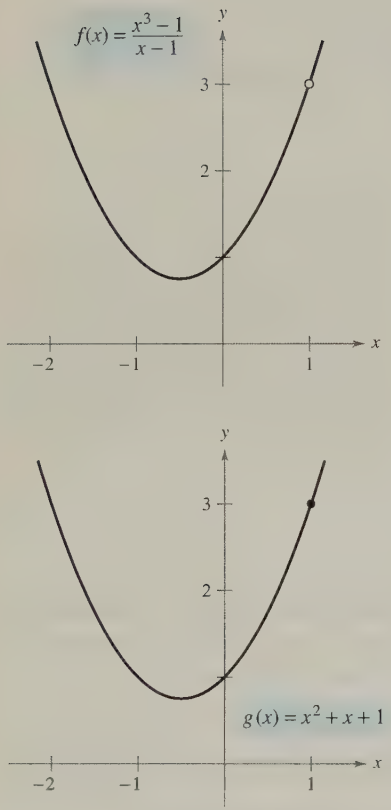
A Strategy for Finding Limits

On the previous three pages, you studied several types of functions whose limits c be evaluated by direct substitution. This knowledge, together with the followi theorem, can be used to develop a strategy for finding limits. A proof of this theor is given in Appendix B.

THEOREM 1.7 Functions That Agree at All But One Point

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$



f and *g* agree at all but one point.
Figure 1.17

Example 6 Finding the Limit of a Function

Find the limit: $\lim_{x \to 1} \dfrac{x^3 - 1}{x - 1}$.

Solution Let $f(x) = (x^3 - 1)/(x - 1)$. By factoring and dividing out like facto you can rewrite f as

$$f(x) = \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{(x-1)}} = x^2 + x + 1 = g(x), \qquad x \neq 1.$$

So, for all x-values other than $x = 1$, the functions f and g agree, as shown in Figu 1.17. Because $\lim_{x \to 1} g(x)$ exists, you can apply Theorem 1.7 to conclude that f an have the same limit at $x = 1$.

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1} \qquad \text{Factor.}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{x-1}} \qquad \text{Divide out like factors.}$$

$$= \lim_{x \to 1} (x^2 + x + 1) \qquad \text{Apply Theorem 1.7.}$$

$$= 1^2 + 1 + 1 \qquad \text{Use direct substitution.}$$

$$= 3 \qquad \text{Simplify.}$$

STUDY TIP When applying this strategy for finding a limit, remember that some functions do not have a limit (as x approaches c). For instance, the following limit does not exist.

$$\lim_{x \to 1} \frac{x^3 + 1}{x - 1}$$

A Strategy for Finding Limits

1. Learn to recognize which limits can be evaluated by direct substitution. (These limits are listed in Theorems 1.1 through 1.6.)
2. If the limit of $f(x)$ as x approaches c *cannot* be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$. [Choose g such that the limit of $g(x)$ *can* be evaluated by direct substitution.]
3. Apply Theorem 1.7 to conclude *analytically* that
$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c).$$
4. Use a *graph* or *table* to reinforce your conclusion.

Dividing Out and Rationalizing Techniques

Two techniques for finding limits analytically are shown in Examples 7 and 8. The first technique involves dividing out common factors, and the second technique involves rationalizing the numerator of a fractional expression.

Example 7 Dividing Out Technique

Find the limit: $\lim\limits_{x\to-3} \dfrac{x^2+x-6}{x+3}$.

Solution Although you are taking the limit of a rational function, you *cannot* apply Theorem 1.3 because the limit of the denominator is 0.

$$\lim_{x\to-3} \frac{x^2+x-6}{x+3} \quad \begin{array}{l} \nearrow \ \lim\limits_{x\to-3} (x^2+x-6) = 0 \\ \qquad\qquad\qquad \text{Direct substitution fails.} \\ \searrow \ \lim\limits_{x\to-3} (x+3) = 0 \end{array}$$

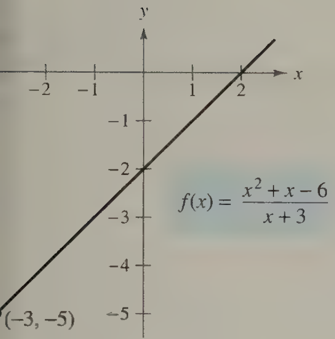
Because the limit of the numerator is also 0, the numerator and denominator have a *common factor* of $(x + 3)$. Thus, for all $x \neq -3$, you can divide out this factor to obtain

$$f(x) = \frac{x^2+x-6}{x+3} = \frac{\cancel{(x+3)}(x-2)}{\cancel{x+3}} = x - 2 = g(x), \qquad x \neq -3.$$

Using Theorem 1.7, it follows that

$$\lim_{x\to-3} \frac{x^2+x-6}{x+3} = \lim_{x\to-3} (x-2) \qquad \text{Apply Theorem 1.7.}$$
$$= -5. \qquad \text{Use direct substitution.}$$

This result is shown graphically in Figure 1.18. Note that the graph of the function f coincides with the graph of the function $g(x) = x - 2$, except that the graph of f has a gap at the point $(-3, -5)$.



s undefined when $x = -3$.
gure 1.18

In Example 7, direct substitution produced the meaningless fractional form 0/0. An expression such as 0/0 is called an **indeterminate form** because you cannot (from the form alone) determine the limit. When you try to evaluate a limit and encounter this form, remember that you must rewrite the fraction so that the new denominator does not have 0 as its limit. One way to do this is to *divide out like factors*, as shown in Example 7. A second way is to *rationalize the numerator*, as shown in Example 8.

TE In the solution of Example 7,
sure you see the usefulness of the
tor Theorem of Algebra. This
orem states that if c is a zero of a
ynomial function, $(x - c)$ is a factor
the polynomial. Thus, if you apply
ect substitution to a rational function
l obtain

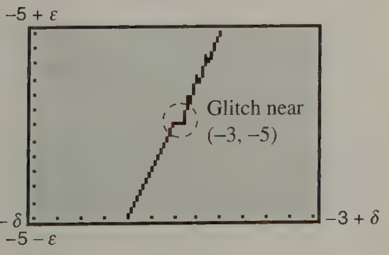
$$r(c) = \frac{p(c)}{q(c)} = \frac{0}{0},$$

u can conclude that $(x - c)$ must be a
nmon factor to both $p(x)$ and $q(x)$.

TECHNOLOGY PITFALL Because the graphs of

$$f(x) = \frac{x^2+x-6}{x+3} \qquad \text{and} \qquad g(x) = x - 2$$

differ only at the point $(-3, -5)$, a standard graphing utility setting may not distinguish clearly between these graphs. However, because of the pixel configuration and rounding error of a graphing utility, it may be possible to find screen settings that distinguish between the graphs. Specifically, by repeatedly zooming in near the point $(-3, -5)$ on the graph of f, your graphing utility may show glitches or irregularities that do not exist on the actual graph. (See Figure 1.19.) By changing the screen settings on your graphing utility you may obtain the correct graph of f.



rrect graph of f
ure 1.19

Example 8 Rationalizing Technique

Find the limit: $\lim\limits_{x\to 0} \dfrac{\sqrt{x+1}-1}{x}$.

Solution By direct substitution, you obtain the indeterminate form 0/0.

$$\lim_{x\to 0} (\sqrt{x+1}-1) = 0$$

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$$

Direct substitution fails.

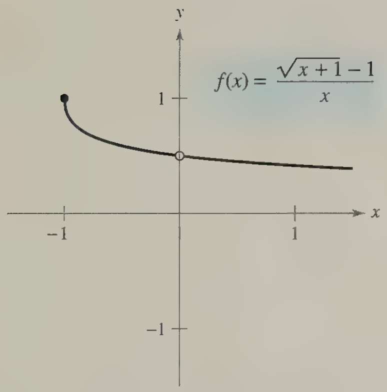
$$\lim_{x\to 0} x = 0$$

In this case, you can rewrite the fraction by rationalizing the numerator.

$$\begin{aligned}
\frac{\sqrt{x+1}-1}{x} &= \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)\\
&= \frac{(x+1)-1}{x(\sqrt{x+1}+1)}\\
&= \frac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)}\\
&= \frac{1}{\sqrt{x+1}+1}, \qquad x \neq 0
\end{aligned}$$

Now, using Theorem 1.7, you can evaluate the limit as follows.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1}\\
&= \frac{1}{1+1}\\
&= \frac{1}{2}
\end{aligned}$$



The limit of $f(x)$ as x approaches 0 is $\frac{1}{2}$.
Figure 1.20

A table or a graph can reinforce your conclusion that the limit is $\frac{1}{2}$. (See Figure 1.2

x approaches 0 from the left. x approaches 0 from the right.

x	−0.25	−0.1	−0.01	−0.001	0	0.001	0.01	0.1	0.25
$f(x)$	0.5359	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881	0.472

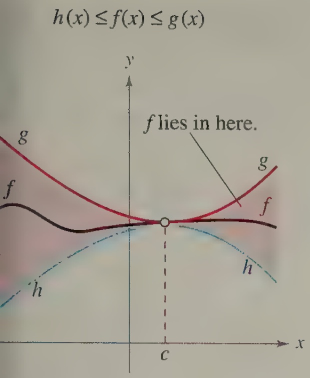
$f(x)$ approaches 0.5. $f(x)$ approaches 0.5.

NOTE The rationalizing technique for evaluating limits is based on multiplication b convenient form of 1. In Example 8, the convenient form is

$$1 = \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}.$$

The Squeeze Theorem

The next theorem concerns the limit of a function that is squeezed between two other functions, each of which has the same limit at a given x-value, as shown in Figure 1.21. (The proof of this theorem is given in Appendix B.)



ne Squeeze Theorem
gure 1.21

THEOREM 1.8 The Squeeze Theorem

If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

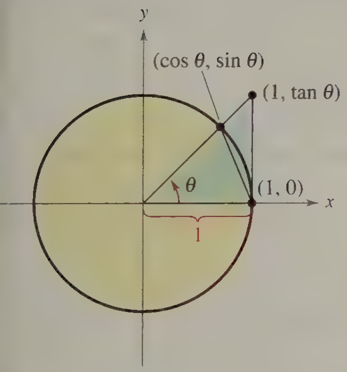
then $\lim_{x \to c} f(x)$ exists and is equal to L.

You can see the usefulness of the Squeeze Theorem in the proof of Theorem 1.9.

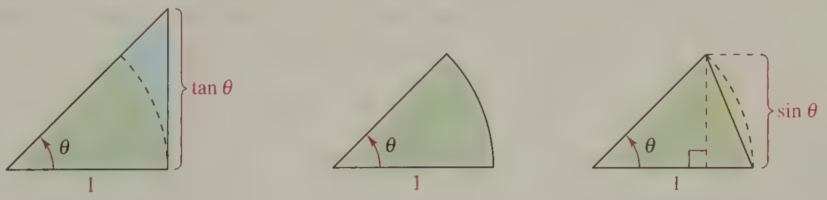
THEOREM 1.9 Two Special Trigonometric Limits

1. $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$ **2.** $\lim_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

Proof To avoid the confusion of two different uses of x, the proof is presented using the variable θ, where θ is an acute positive angle *measured in radians*. Figure 1.22 shows a circular sector that is squeezed between two triangles.



circular sector is used to prove Theorem 1.9.
gure 1.22



$$\begin{array}{ccccc} \text{Area of triangle} & \geq & \text{Area of sector} & \geq & \text{Area of triangle} \\ \dfrac{\tan \theta}{2} & \geq & \dfrac{\theta}{2} & \geq & \dfrac{\sin \theta}{2} \end{array}$$

Multiplying each expression by $2/\sin \theta$ produces

$$\frac{1}{\cos \theta} \geq \frac{\theta}{\sin \theta} \geq 1$$

and taking reciprocals and reversing the inequalities yields

$$\cos \theta \leq \frac{\sin \theta}{\theta} \leq 1.$$

Because $\cos \theta = \cos(-\theta)$ and $(\sin \theta)/\theta = [\sin(-\theta)]/(-\theta)$, we can conclude that this inequality is valid for *all* nonzero θ in the open interval $(-\pi/2, \pi/2)$. Finally, because $\lim_{\theta \to 0} \cos \theta = 1$ and $\lim_{\theta \to 0} 1 = 1$, you can apply the Squeeze Theorem to conclude that $\lim_{\theta \to 0} (\sin \theta)/\theta = 1$. The proof of the second limit is left as an exercise (see Exercise 120).

OR FURTHER INFORMATION For ore information on the function x) = $(\sin x)/x$, see the article "The nction $(\sin x)/x$" by William B. earhart and Harris S. Shultz in *The llege Mathematics Journal*. To view s article, go to the website *vw.matharticles.com*.

Example 9 A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x\to 0} \frac{\tan x}{x}$.

Solution Direct substitution yields the indeterminate form 0/0. To solve th problem, you can write $\tan x$ as $(\sin x)/(\cos x)$ and obtain

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right).$$

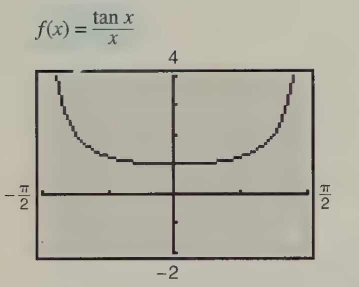
Now, because

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{1}{\cos x} = 1$$

you can obtain

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x}{x} &= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{\cos x}\right)\\ &= (1)(1)\\ &= 1.\end{aligned}$$

(See Figure 1.23.)



The limit of $f(x)$ as x approaches 0 is 1.
Figure 1.23

Example 10 A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x\to 0} \frac{\sin 4x}{x}$.

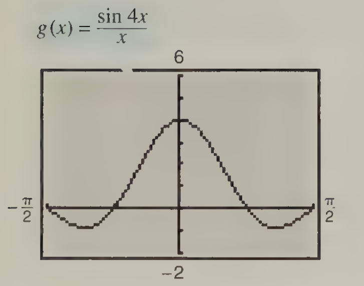
Solution Direct substitution yields the indeterminate form 0/0. To solve this pro lem, you can rewrite the limit as

$$\lim_{x\to 0} \frac{\sin 4x}{x} = 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right).$$

Now, by letting $y = 4x$ and observing that $x \to 0$ if and only if $y \to 0$, you can wr

$$\begin{aligned}\lim_{x\to 0} \frac{\sin 4x}{x} &= 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right)\\ &= 4\left(\lim_{y\to 0} \frac{\sin y}{y}\right)\\ &= 4(1)\\ &= 4.\end{aligned}$$

(See Figure 1.24.)



The limit of $g(x)$ as x approaches 0 is 4.
Figure 1.24

TECHNOLOGY Try using a graphing utility to confirm the limits in the examp and exercise set. For instance, Figures 1.23 and 1.24 show the graphs of

$$f(x) = \frac{\tan x}{x} \qquad \text{and} \qquad g(x) = \frac{\sin 4x}{x}.$$

Note that the first graph appears to contain the point (0, 1) and the second gra appears to contain the point (0, 4), which lends support to the conclusions obtain in Examples 9 and 10.

XERCISES FOR SECTION 1.3

Exercises 1–4, use a graphing utility to graph the function d visually estimate the limits.

. $h(x) = x^2 - 5x$
(a) $\lim_{x\to 5} h(x)$
(b) $\lim_{x\to -1} h(x)$

2. $g(x) = \dfrac{12(\sqrt{x} - 3)}{x - 9}$
(a) $\lim_{x\to 4} g(x)$
(b) $\lim_{x\to 0} g(x)$

. $f(x) = x \cos x$
(a) $\lim_{x\to 0} f(x)$
(b) $\lim_{x\to \pi/3} f(x)$

4. $f(t) = t|t - 4|$
(a) $\lim_{t\to 4} f(t)$
(b) $\lim_{t\to -1} f(t)$

Exercises 5–22, find the limit.

. $\lim_{x\to 2} x^4$

6. $\lim_{x\to -2} x^3$

. $\lim_{x\to 0} (2x - 1)$

8. $\lim_{x\to -3} (3x + 2)$

. $\lim_{x\to -3} (x^2 + 3x)$

10. $\lim_{x\to 1} (-x^2 + 1)$

. $\lim_{x\to -3} (2x^2 + 4x + 1)$

12. $\lim_{x\to 1} (3x^3 - 2x^2 + 4)$

. $\lim_{x\to 2} \dfrac{1}{x}$

14. $\lim_{x\to -3} \dfrac{2}{x + 2}$

. $\lim_{x\to 1} \dfrac{x - 3}{x^2 + 4}$

16. $\lim_{x\to 3} \dfrac{2x - 3}{x + 5}$

. $\lim_{x\to 7} \dfrac{5x}{\sqrt{x + 2}}$

18. $\lim_{x\to 3} \dfrac{\sqrt{x + 1}}{x - 4}$

. $\lim_{x\to 3} \sqrt{x + 1}$

20. $\lim_{x\to 4} \sqrt[3]{x + 4}$

. $\lim_{x\to -4} (x + 3)^2$

22. $\lim_{x\to 0} (2x - 1)^3$

Exercises 23–26, find the limits.

. $f(x) = 5 - x,\ g(x) = x^3$
(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

. $f(x) = x + 7,\ g(x) = x^2$
(a) $\lim_{x\to -3} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to -3} g(f(x))$

. $f(x) = 4 - x^2,\ g(x) = \sqrt{x + 1}$
(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 3} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

. $f(x) = 2x^2 - 3x + 1,\ g(x) = \sqrt[3]{x + 6}$
(a) $\lim_{x\to 4} f(x)$ (b) $\lim_{x\to 21} g(x)$ (c) $\lim_{x\to 4} g(f(x))$

Exercises 27–36, find the limit of the trigonometric function.

. $\lim_{x\to \pi/2} \sin x$

28. $\lim_{x\to \pi} \tan x$

. $\lim_{x\to 2} \cos \dfrac{\pi x}{3}$

30. $\lim_{x\to 1} \sin \dfrac{\pi x}{2}$

. $\lim_{x\to 0} \sec 2x$

32. $\lim_{x\to \pi} \cos 3x$

. $\lim_{x\to 5\pi/6} \sin x$

34. $\lim_{x\to 5\pi/3} \cos x$

35. $\lim_{x\to 3} \tan\left(\dfrac{\pi x}{4}\right)$

36. $\lim_{x\to 7} \sec\left(\dfrac{\pi x}{6}\right)$

In Exercises 37–40, use the information to evaluate the limits.

37. $\lim_{x\to c} f(x) = 2$
$\lim_{x\to c} g(x) = 3$
(a) $\lim_{x\to c} [5g(x)]$
(b) $\lim_{x\to c} [f(x) + g(x)]$
(c) $\lim_{x\to c} [f(x)g(x)]$
(d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

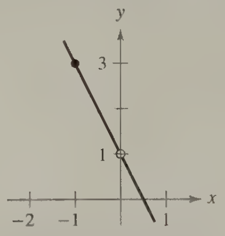
38. $\lim_{x\to c} f(x) = \frac{3}{2}$
$\lim_{x\to c} g(x) = \frac{1}{2}$
(a) $\lim_{x\to c} [4f(x)]$
(b) $\lim_{x\to c} [f(x) + g(x)]$
(c) $\lim_{x\to c} [f(x)g(x)]$
(d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

39. $\lim_{x\to c} f(x) = 4$
(a) $\lim_{x\to c} [f(x)]^3$
(b) $\lim_{x\to c} \sqrt{f(x)}$
(c) $\lim_{x\to c} [3f(x)]$
(d) $\lim_{x\to c} [f(x)]^{3/2}$

40. $\lim_{x\to c} f(x) = 27$
(a) $\lim_{x\to c} \sqrt[3]{f(x)}$
(b) $\lim_{x\to c} \dfrac{f(x)}{18}$
(c) $\lim_{x\to c} [f(x)]^2$
(d) $\lim_{x\to c} [f(x)]^{2/3}$

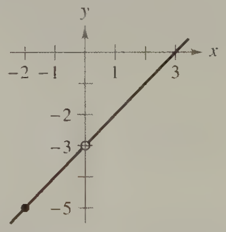
In Exercises 41–44, use the graph to determine the limit visually (if it exists). Write a simpler function that agrees with the given function at all but one point.

41. $g(x) = \dfrac{-2x^2 + x}{x}$



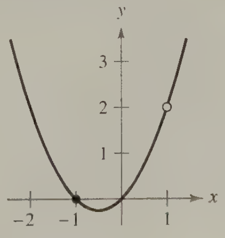
(a) $\lim_{x\to 0} g(x)$
(b) $\lim_{x\to -1} g(x)$

42. $h(x) = \dfrac{x^2 - 3x}{x}$



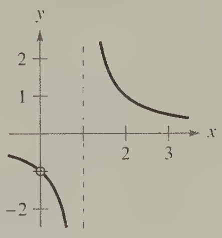
(a) $\lim_{x\to -2} h(x)$
(b) $\lim_{x\to 0} h(x)$

43. $g(x) = \dfrac{x^3 - x}{x - 1}$



(a) $\lim_{x\to 1} g(x)$
(b) $\lim_{x\to -1} g(x)$

44. $f(x) = \dfrac{x}{x^2 - x}$



(a) $\lim_{x\to 1} f(x)$
(b) $\lim_{x\to 0} f(x)$

In Exercises 45–48, find the limit of the function (if it exists). Write a simpler function that agrees with the given function at all but one point. Use a graphing utility to confirm your result.

45. $\lim_{x\to -1}\frac{x^2-1}{x+1}$

46. $\lim_{x\to -1}\frac{2x^2-x-3}{x+1}$

47. $\lim_{x\to 2}\frac{x^3-8}{x-2}$

48. $\lim_{x\to -1}\frac{x^3+1}{x+1}$

In Exercises 49–62, find the limit (if it exists).

49. $\lim_{x\to 5}\frac{x-5}{x^2-25}$

✓ **50.** $\lim_{x\to 2}\frac{2-x}{x^2-4}$

51. $\lim_{x\to -3}\frac{x^2+x-6}{x^2-9}$

52. $\lim_{x\to 4}\frac{x^2-5x+4}{x^2-2x-8}$

53. $\lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x}$

54. $\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$

55. $\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}$

56. $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}$

✓ **57.** $\lim_{x\to 0}\frac{[1/(3+x)]-(1/3)}{x}$

58. $\lim_{x\to 0}\frac{[1/(x+4)]-(1/4)}{x}$

59. $\lim_{\Delta x\to 0}\frac{2(x+\Delta x)-2x}{\Delta x}$

60. $\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}$

61. $\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x}$

62. $\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x}$

Graphical, Numerical, and Analytic Analysis **In Exercises 63–66, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.**

63. $\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}$

64. $\lim_{x\to 16}\frac{4-\sqrt{x}}{x-16}$

65. $\lim_{x\to 0}\frac{[1/(2+x)]-(1/2)}{x}$

66. $\lim_{x\to 2}\frac{x^5-32}{x-2}$

In Exercises 67–78, determine the limit of the trigonometric function (if it exists).

67. $\lim_{x\to 0}\frac{\sin x}{5x}$

68. $\lim_{x\to 0}\frac{3(1-\cos x)}{x}$

69. $\lim_{x\to 0}\frac{\sin x(1-\cos x)}{2x^2}$

70. $\lim_{\theta\to 0}\frac{\cos\theta\tan\theta}{\theta}$

71. $\lim_{x\to 0}\frac{\sin^2 x}{x}$

72. $\lim_{x\to 0}\frac{\tan^2 x}{x}$

73. $\lim_{h\to 0}\frac{(1-\cos h)^2}{h}$

74. $\lim_{\phi\to\pi}\phi\sec\phi$

75. $\lim_{x\to\pi/2}\frac{\cos x}{\cot x}$

76. $\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x}$

77. $\lim_{t\to 0}\frac{\sin 3t}{2t}$

78. $\lim_{x\to 0}\frac{\sin 2x}{\sin 3x}$ $\left[\textit{Hint: } \text{Find } \lim_{x\to 0}\left(\frac{2\sin 2x}{2x}\right)\left(\frac{3x}{3\sin 3x}\right).\right]$

Graphical, Numerical, and Analytic Analysis **In Exerci 79–82, use a graphing utility to graph the function and estim the limit. Use a table to reinforce your conclusion. Then find limit by analytic methods.**

79. $\lim_{t\to 0}\frac{\sin 3t}{t}$

80. $\lim_{h\to 0}(1+\cos 2h)$

81. $\lim_{x\to 0}\frac{\sin x^2}{x}$

82. $\lim_{x\to 0}\frac{\sin x}{\sqrt[3]{x}}$

In Exercises 83–86, find $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$**.**

83. $f(x)=2x+3$

84. $f(x)=\sqrt{x}$

85. $f(x)=\frac{4}{x}$

86. $f(x)=x^2-4x$

In Exercises 87 and 88, use the Squeeze Theorem to f $\lim_{x\to c}f(x)$**.**

87. $c=0$

$4-x^2\le f(x)\le 4+x^2$

88. $c=a$

$b-|x-a|\le f(x)\le b+|x-a|$

In Exercises 89–94, use a graphing utility to graph the gi function and the equations $y=|x|$ **and** $y=-|x|$ **in the sa viewing window. Using the graphs to visually observe Squeeze Theorem, find** $\lim_{x\to 0}f(x)$**.**

89. $f(x)=x\cos x$

90. $f(x)=|x\sin x|$

91. $f(x)=|x|\sin x$

92. $f(x)=|x|\cos x$

93. $f(x)=x\sin\frac{1}{x}$

94. $h(x)=x\cos\frac{1}{x}$

Getting at the Concept

95. In the context of finding limits, discuss what is meant b two functions that agree at all but one point.

96. Give an example of two functions that agree at all but or point.

97. What is meant by an indeterminate form?

98. In your own words, explain the Squeeze Theorem.

99. ***Writing*** Use a graphing utility to graph

$$f(x)=x,\quad g(x)=\sin x,\quad \text{and}\quad h(x)=\frac{\sin x}{x}$$

in the same viewing window. Compare the magnitudes of j and $g(x)$ when x is "close to" 0. Use the comparison to wri short paragraph explaining why

$$\lim_{x\to 0}h(x)=1.$$

0. ***Writing*** Use a graphing utility to graph

$$f(x) = x, \; g(x) = \sin^2 x, \text{ and } h(x) = \frac{\sin^2 x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is "close to" 0. Use the comparison to write a short paragraph explaining why

$$\lim_{x\to 0} h(x) = 0.$$

ee-Falling Object **In Exercises 101 and 102, use the position ction $s(t) = -16t^2 + 1000$, which gives the height (in feet) of object that has fallen for t seconds from a height of 1000 feet. e velocity at time $t = a$ seconds is given by**

$$\frac{s(a) - s(t)}{a - t}.$$

1. If a construction worker drops a wrench from a height of 1000 feet, how fast will the wrench be falling after 5 seconds?

2. If a construction worker drops a wrench from a height of 1000 feet, when will the wrench hit the ground? At what velocity will the wrench impact the ground?

ee-Falling Object **In Exercises 103 and 104, use the position ction $s(t) = -4.9t^2 + 150$, which gives the height (in meters) an object that has fallen from a height of 150 meters. The ocity at time $t = a$ seconds is given by**

$$\frac{s(a) - s(t)}{a - t}.$$

3. Find the velocity of the object when $t = 3$.

4. At what velocity will the object impact the ground?

5. Find two functions f and g such that $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist, but $\lim_{x\to 0} [f(x) + g(x)]$ does exist.

6. Prove that if $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} [f(x) + g(x)]$ does not exist, then $\lim_{x\to c} g(x)$ does not exist.

7. Prove Property 1 of Theorem 1.1.

8. Prove Property 3 of Theorem 1.1. (You may use Property 3 of Theorem 1.2.)

9. Prove Property 1 of Theorem 1.2.

0. Prove that if $\lim_{x\to c} f(x) = 0$, then $\lim_{x\to c} |f(x)| = 0$.

1. Prove that if $\lim_{x\to c} f(x) = 0$ and $|g(x)| \le M$ for a fixed number M and all $x \ne c$, then $\lim_{x\to c} f(x)g(x) = 0$.

2. (a) Prove that if $\lim_{x\to c} |f(x)| = 0$, then $\lim_{x\to c} f(x) = 0$.

   (*Note:* This is the converse of Exercise 110.)

   (b) Prove that if $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c} |f(x)| = |L|$.

   [*Hint:* Use the inequality $||f(x)| - |L|| \le |f(x) - L|$.]

True or False? **In Exercises 113–118, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

113. $\lim_{x\to 0} \frac{|x|}{x} = 1$

114. $\lim_{x\to 0} x^3 = 0$

115. If $f(x) = g(x)$ for all real numbers other than $x = 0$, and

$$\lim_{x\to 0} f(x) = L$$

then

$$\lim_{x\to 0} g(x) = L.$$

116. If $\lim_{x\to c} f(x) = L$, then $f(c) = L$.

117. $\lim_{x\to 2} f(x) = 3$, where $f(x) = \begin{cases} 3, & x \le 2 \\ 0, & x > 2 \end{cases}$

118. If $f(x) < g(x)$ for all $x \ne a$, then

$$\lim_{x\to a} f(x) < \lim_{x\to a} g(x).$$

119. ***Think About It*** Find a function f to show that the converse of Exercise 112(b) is not true. [*Hint:* Find a function f such that $\lim_{x\to c} |f(x)| = |L|$ but $\lim_{x\to c} f(x)$ does not exist.]

120. Prove the second part of Theorem 1.9 by proving that

$$\lim_{x\to 0} \frac{1 - \cos x}{x} = 0.$$

121. Let $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

and

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational.} \end{cases}$$

Find (if possible) $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$.

122. ***Graphical Reasoning*** Consider $f(x) = \frac{\sec x - 1}{x^2}$.

(a) Find the domain of f.

(b) Use a graphing utility to graph f. Is the domain of f obvious from the graph? If not, explain.

(c) Use the graph of f to approximate $\lim_{x\to 0} f(x)$.

(d) Confirm the answer in part (c) analytically.

123. ***Approximation***

(a) Find $\lim_{x\to 0} \frac{1 - \cos x}{x^2}$.

(b) Use the result in part (a) to derive the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ for x near 0.

(c) Use the result in part (b) to approximate $\cos(0.1)$.

(d) Use a calculator to approximate $\cos(0.1)$ to four decimal places. Compare the result with part (c).

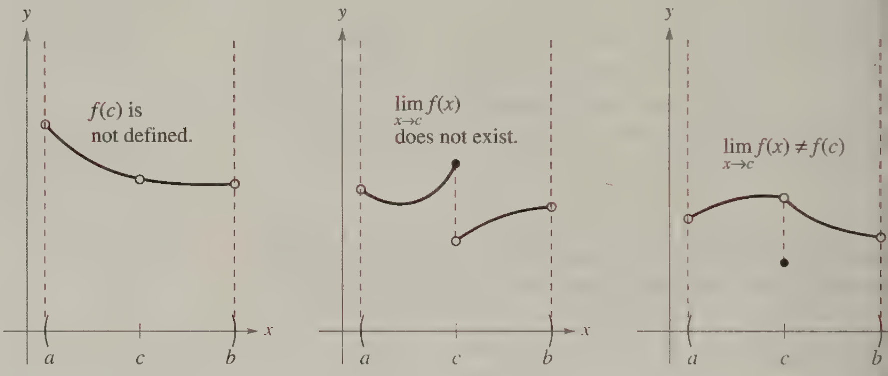
124. ***Think About It*** When using a graphing utility to generate a table to approximate $\lim_{x\to 0} [(\sin x)/x]$, a student concluded that the limit was 0.01745 rather than 1. Determine the probable cause of the error.

Section 1.4 Continuity and One-Sided Limits

- Determine continuity at a point and continuity on an open interval.
- Determine one-sided limits and continuity on a closed interval.
- Use properties of continuity.
- Understand and use the Intermediate Value Theorem.

Continuity at a Point and on an Open Interval

In mathematics, the term *continuous* has much the same meaning as it has in everyc usage. To say that a function f is continuous at $x = c$ means that there is no interr tion in the graph of f at c. That is, its graph is unbroken at c and there are no ho jumps, or gaps. Figure 1.25 identifies three values of x at which the graph of f is continuous. At all other points in the interval (a, b), the graph of f is uninterrupted a **continuous.**



y
$f(c)$ is not defined.
a
c
b
x
y
$\lim_{x\to c} f(x)$ does not exist.
a
c
b
x
y
$\lim_{x\to c} f(x) \neq f(c)$
a
c
b

Three conditions exist for which the graph of f is not continuous at $x = c$.
Figure 1.25

EXPLORATION

Informally, you might say that a function is *continuous* on an open interval if its graph can be drawn with a pencil without lifting the pencil from the paper. Use a graphing calculator to graph each of the following functions on the indicated interval. From the graphs, which functions would you say are continuous on the interval? Do you think you can trust the results you obtained graphically? Explain your reasoning.

Function	*Interval*
a. $y = x^2 + 1$	$(-3, 3)$
b. $y = \dfrac{1}{x - 2}$	$(-3, 3)$
c. $y = \dfrac{\sin x}{x}$	$(-\pi, \pi)$
d. $y = \dfrac{x^2 - 4}{x + 2}$	$(-3, 3)$
e. $y = \begin{cases} 2x - 4, & x \le 0 \\ x + 1, & x > 0 \end{cases}$	$(-3, 3)$

In Figure 1.25, it appears that continuity at $x = c$ can be destroyed by any one the following conditions.

1. The function is not defined at $x = c$.
2. The limit of $f(x)$ does not exist at $x = c$.
3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.

If *none* of the above three conditions is true, the function f is called **continuous a** as indicated in the following important definition.

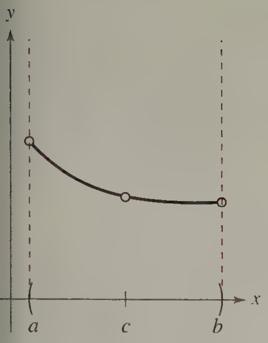
Definition of Continuity

Continuity at a Point: A function f is **continuous at c** if the following three conditions are met.

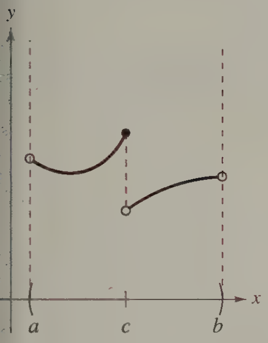
1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists.
3. $\lim_{x\to c} f(x) = f(c)$.

Continuity on an Open Interval: A function is **continuous on an open interva (a, b)** if it is continuous at each point in the interval. A function that is continuou on the entire real line $(-\infty, \infty)$ is **everywhere continuous.**

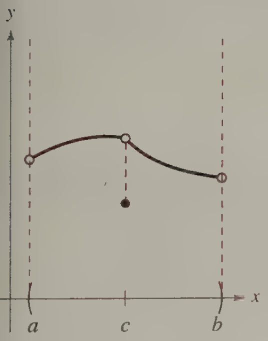
FOR FURTHER INFORMATION For more information on the concept of continuity, see the article "Leibniz and the Spell of the Continuous" by Hardy Grant in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.



Removable discontinuity



Nonremovable discontinuity



Removable discontinuity

gure 1.26

Consider an open interval I that contains a real number c. If a function f is defined on I (except possibly at c), and f is not continuous at c, then f is said to have a **discontinuity** at c. Discontinuities fall into two categories: **removable** and **nonremovable.** A discontinuity at c is called removable if f can be made continuous by appropriately defining (or redefining) $f(c)$. For instance, the functions shown in Figure 1.26(a) and (c) have removable discontinuities at c and the function shown in Figure 1.26(b) has a nonremovable discontinuity at c.

Example 1 Continuity of a Function

Discuss the continuity of each function.

a. $f(x) = \dfrac{1}{x}$ **b.** $g(x) = \dfrac{x^2 - 1}{x - 1}$ **c.** $h(x) = \begin{cases} x + 1, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$ **d.** $y = \sin x$

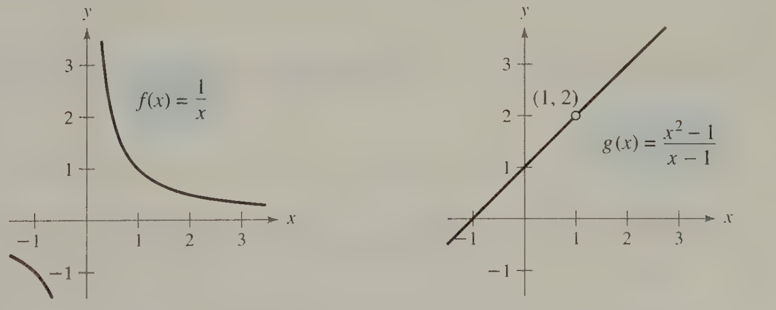
Solution

a. The domain of f is all nonzero real numbers. From Theorem 1.3, you can conclude that f is continuous at every x-value in its domain. At $x = 0$, f has a nonremovable discontinuity, as shown in Figure 1.27(a). In other words, there is no way to define $f(0)$ so as to make the function continuous at $x = 0$.

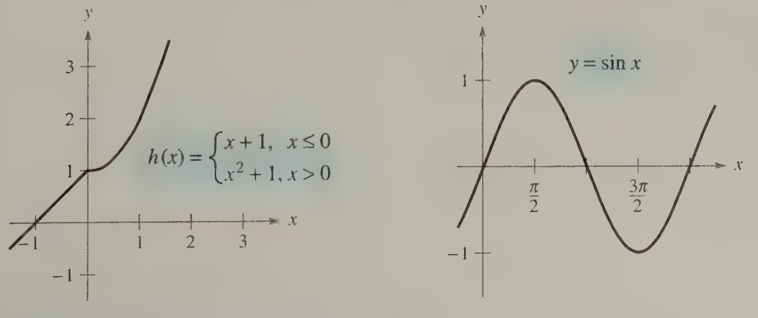
b. The domain of g is all real numbers except $x = 1$. From Theorem 1.3, you can conclude that g is continuous at every x-value in its domain. At $x = 1$, the function has a removable discontinuity, as shown in Figure 1.27(b). If $g(1)$ is defined as 2, the "newly defined" function is continuous for all real numbers.

c. The domain of h is all real numbers. The function h is continuous on $(-\infty, 0)$ and $(0, \infty)$, and, because $\lim\limits_{x\to 0} h(x) = 1$, h is continuous on the entire real line, as shown in Figure 1.27(c).

d. The domain of y is all real numbers. From Theorem 1.6, you can conclude that the function is continuous on its entire domain, $(-\infty, \infty)$, as shown in Figure 1.27(d).



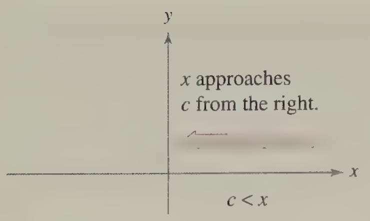
(a) Nonremovable discontinuity at $x = 0$ **(b)** Removable discontinuity at $x = 1$



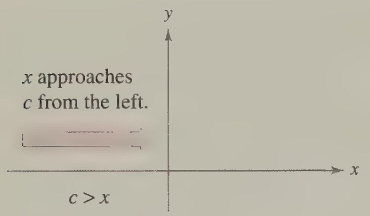
(c) Continuous on entire real line **(d)** Continuous on entire real line

Figure 1.27

TUDY TIP Some people may refer to
e function in Example 1a as "discon-
uous." We have found that this termi-
logy can be confusing. Rather than
ying the function is discontinuous, we
efer to say that it has a discontinuity
$x = 0$.



(a) Limit from right



(b) Limit from left

Figure 1.28

One-Sided Limits and Continuity on a Closed Interval

To understand continuity on a closed interval, you first need to look at a different ty of limit called a **one-sided limit.** For example, the **limit from the right** means tha approaches c from values greater than c [see Figure 1.28(a)]. This limit is denoted

$$\lim_{x\to c^+} f(x) = L.$$

Limit from the right

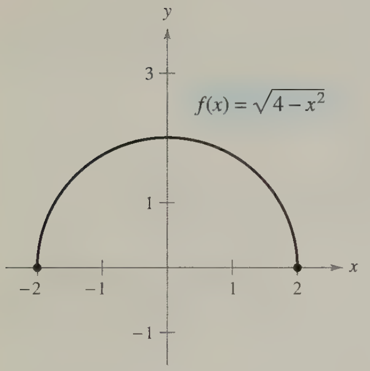
Similarly, the **limit from the left** means that x approaches c from values less tha [see Figure 1.28(b)]. This limit is denoted as

$$\lim_{x\to c^-} f(x) = L.$$

Limit from the left

One-sided limits are useful in taking limits of functions involving radicals. instance, if n is an even integer,

$$\lim_{x\to 0^+} \sqrt[n]{x} = 0.$$



The limit of $f(x)$ as x approaches -2 from the right is 0.

Figure 1.29

Example 2 A One-Sided Limit

Find the limit of $f(x) = \sqrt{4-x^2}$ as x approaches -2 from the right.

Solution As indicated in Figure 1.29, the limit as x approaches -2 from the righ

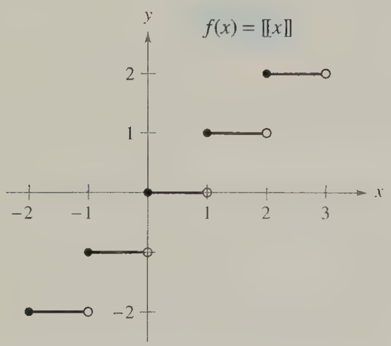
$$\lim_{x\to -2^+} \sqrt{4-x^2} = 0.$$

One-sided limits can be used to investigate the behavior of **step functions.** common type of step function is the **greatest integer function** $[\![x]\!]$, defined by

$$[\![x]\!] = \text{greatest integer } n \text{ such that } n \le x.$$

Greatest integer function

For instance, $[\![2.5]\!] = 2$ and $[\![-2.5]\!] = -3$.



Greatest integer function

Figure 1.30

Example 3 The Greatest Integer Function

Find the limit of the greatest integer function $f(x) = [\![x]\!]$ as x approaches 0 from left and from the right.

Solution As shown in Figure 1.30, the limit as x approaches 0 *from the left* is given

$$\lim_{x\to 0^-} [\![x]\!] = -1$$

and the limit as x approaches 0 *from the right* is given by

$$\lim_{x\to 0^+} [\![x]\!] = 0.$$

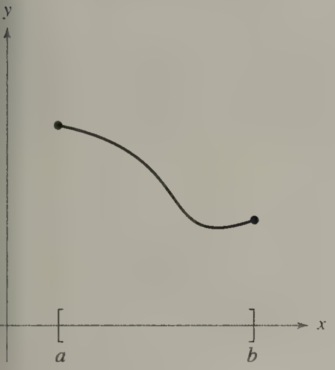
The greatest integer function has a discontinuity at 0 because the left and right lin at zero are different. By similar reasoning, you can see that the greatest inte function has a discontinuity at any integer n.

When the limit from the left is not equal to the limit from the right, the (two-sided) limit *does not exist*. The next theorem makes this more explicit. The proof of this theorem follows directly from the definition of a one-sided limit.

THEOREM 1.10 The Existence of a Limit

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$



ntinuous function on a closed interval
ure 1.31

The concept of a one-sided limit allows you to extend the definition of continuity to closed intervals. Basically, a function is continuous on a closed interval if it is continuous in the interior of the interval and possesses one-sided continuity at the endpoints. We state this formally as follows.

Definition of Continuity on a Closed Interval

A function f is **continuous on the closed interval** $[a, b]$ if it is continuous on the open interval (a, b) and

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} f(x) = f(b).$$

The function f is **continuous from the right** at a and **continuous from the left** at b (see Figure 1.31).

Similar definitions can be made to cover continuity on intervals of the form $(a, b]$ and $[a, b)$ that are neither open nor closed, or on infinite intervals. For example, the function

$$f(x) = \sqrt{x}$$

is continuous on the infinite interval $[0, \infty)$, and the function

$$g(x) = \sqrt{2 - x}$$

is continuous on the infinite interval $(-\infty, 2]$.

Example 4 Continuity on a Closed Interval

Discuss the continuity of $f(x) = \sqrt{1 - x^2}$.

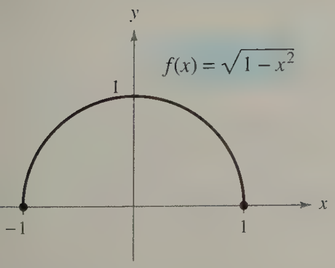
Solution The domain of f is the closed interval $[-1, 1]$. At all points in the open interval $(-1, 1)$, the continuity of f follows from Theorems 1.4 and 1.5. Moreover, because

$$\lim_{x\to -1^+} \sqrt{1 - x^2} = 0 = f(-1) \qquad \text{Continuous from the right}$$

and

$$\lim_{x\to 1^-} \sqrt{1 - x^2} = 0 = f(1) \qquad \text{Continuous from the left}$$

you can conclude that f is continuous on the closed interval $[-1, 1]$, as shown in Figure 1.32.



continuous on $[-1, 1]$.
ure 1.32

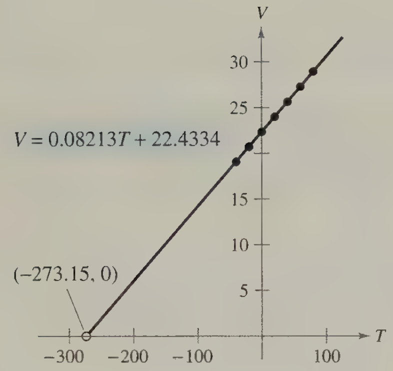
The next example shows how a one-sided limit can be used to determine the va of absolute zero on the Kelvin scale.

Example 5 Charles's Law and Absolute Zero

On the Kelvin scale, *absolute zero* is the temperature 0 K. Although temperature approximately 0.0001 K have been produced in laboratories, absolute zero has ne been attained. In fact, evidence suggests that absolute zero *cannot* be attained. H did scientists determine that 0 K is the "lower limit" of the temperature of mat What is absolute zero on the Celsius scale?

Solution The determination of absolute zero stems from the work of the Fre physicist Jacques Charles (1746–1823). Charles discovered that the volume of ga a constant pressure increases linearly with the temperature of the gas. The table il trates this relationship between volume and temperature. In the table, one mole hydrogen is held at a constant pressure of one atmosphere. The volume V is measu in liters and the temperature T is measured in degrees Celsius.

T	-40	-20	0	20	40	60	80
V	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038



The volume of hydrogen gas depends on its temperature.
Figure 1.33

The points represented by the table are shown in Figure 1.33. Moreover, by using points in the table, you can determine that T and V are related by the linear equati

$$V = 0.08213T + 22.4334 \qquad \text{or} \qquad T = \frac{V - 22.4334}{0.08213}.$$

By reasoning that the volume of the gas can approach 0 (but never equal or go belov you can determine that the "least possible temperature" is given by

$$\begin{aligned}\lim_{V\to 0^+} T &= \lim_{V\to 0^+} \frac{V - 22.4334}{0.08213} \\ &= \frac{0 - 22.4334}{0.08213} \qquad \text{Use direct substitution.} \\ &\approx -273.15.\end{aligned}$$

So, absolute zero on the Kelvin scale (0 K) is approximately −273.15° on the Cel scale.

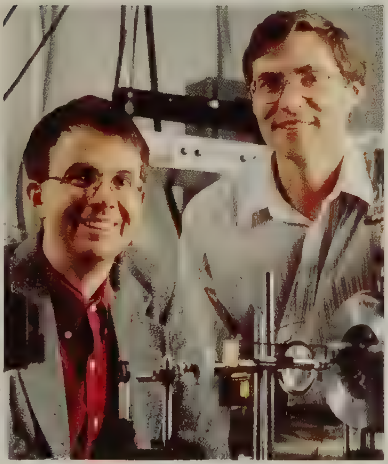


Photo courtesy of Ken Abbott/ University of Colorado at Boulder

In 1995, physicists Carl Wieman and Eric Cornell of the University of Colorado at Boulder used lasers and evaporation to produce a supercold gas in which atoms overlap. This gas is called a Bose-Einstein condensate. "We get to within a billionth of a degree of absolute zero," reported Wieman. *(Source:* Time *magazine, April 10, 2000)*

The following table shows the temperatures in Example 5, converted to Fahrenheit scale. Try repeating the solution shown in Example 5 using these temperat and volumes. Use the result to find the value of absolute zero on the Fahrenheit scale

T	-40	-4	32	68	104	140	176
V	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.003

NOTE Charles's Law for gases (assuming constant pressure) can be stated as

$$V = RT \qquad \text{Charles's Law}$$

where V is volume, R is constant, and T is temperature. In the statement of this law, property must the temperature scale have?

Properties of Continuity

In Section 1.3, you studied several properties of limits. Each of those properties yields a corresponding property pertaining to the continuity of a function. For instance, Theorem 1.11 follows directly from Theorem 1.2.

Bettmann/Corbis

Augustin-Louis Cauchy (1789–1857)

he concept of a continuous function was rst introduced by Augustin-Louis Cauchy in 821. The definition given in his text *Cours* '*Analyse* stated that indefinite small changes ı y were the result of indefinite small changes ı x. "...$f(x)$ will be called a *continuous* nction if ... the numerical values of the ifference $f(x + \alpha) - f(x)$ decrease ndefinitely with those of α ...."

THEOREM 1.11 Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. Scalar multiple: bf
2. Sum and difference: $f \pm g$
3. Product: fg
4. Quotient: $\dfrac{f}{g}$, if $g(c) \neq 0$

The following types of functions are continuous at every point in their domains.

1. Polynomial functions: $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1x + a_0$
2. Rational functions: $r(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$
3. Radical functions: $f(x) = \sqrt[n]{x}$
4. Trigonometric functions: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$

By combining Theorem 1.11 with this summary, you can conclude that a wide variety of elementary functions are continuous at every point in their domains.

Example 6 Applying Properties of Continuity

By Theorem 1.11, it follows that each of the following functions is continuous at every point in its domain.

$$f(x) = x + \sin x, \qquad f(x) = 3 \tan x, \qquad f(x) = \frac{x^2 + 1}{\cos x}$$

The next theorem, which is a consequence of Theorem 1.5, allows you to determine the continuity of *composite* functions such as

$$f(x) = \sin 3x, \qquad f(x) = \sqrt{x^2 + 1}, \qquad f(x) = \tan \frac{1}{x}.$$

THEOREM 1.12 Continuity of a Composite Function

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

One consequence of Theorem 1.12 is that if f and g satisfy the given conditions, you can determine the limit of $f(g(x))$ as x approaches c to be

$$\lim_{x \to c} f(g(x)) = f(g(c)).$$

Example 7 Testing for Continuity

Describe the interval(s) on which each function is continuous.

a. $f(x) = \tan x$ **b.** $g(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ **c.** $h(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution

a. The tangent function $f(x) = \tan x$ is undefined at

$$x = \frac{\pi}{2} + n\pi, \qquad n \text{ is an integer.}$$

At all other points it is continuous. So, $f(x) = \tan x$ is continuous on the o intervals

$$\ldots, \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$$

as shown in Figure 1.34(a).

b. Because $y = 1/x$ is continuous except at $x = 0$ and the sine function is continu for all real values of x, it follows that $y = \sin(1/x)$ is continuous at all real val except $x = 0$. At $x = 0$, the limit of $g(x)$ does not exist (see Example 5, Sect 1.2). So, g is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$, as indicatec Figure 1.34(b).

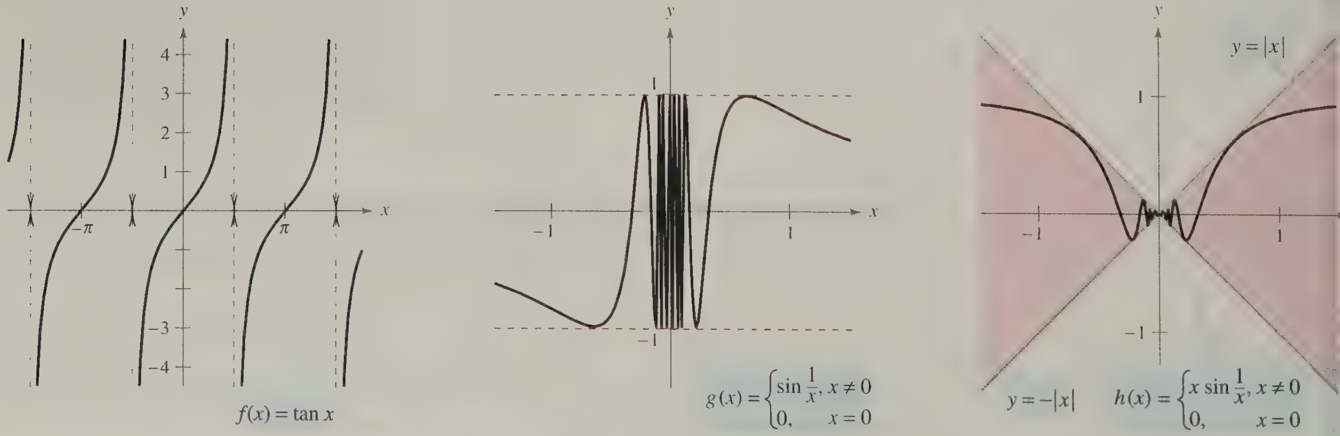
c. This function is similar to that in part (b) except that the oscillations are dam by the factor x. Using the Squeeze Theorem, you obtain

$$-|x| \leq x \sin \frac{1}{x} \leq |x|, \qquad x \neq 0$$

and you can conclude that

$$\lim_{x \to 0} h(x) = 0.$$

So, h is continuous on the entire real line, as indicated in Figure 1.34(c).



(a) f is continuous on each open interval in its domain.

(b) g is continuous on $(-\infty, 0)$ and $(0, \infty)$.

(c) h is continuous on the entire real line

Figure 1.34

The Intermediate Value Theorem

We conclude this section with an important theorem concerning the behavior of functions that are continuous on a closed interval.

THEOREM 1.13 Intermediate Value Theorem

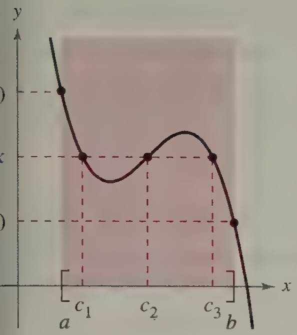
If f is continuous on the closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

NOTE The Intermediate Value Theorem tells you that at least one c exists, but it does not give a method for finding c. Such theorems are called **existence theorems.**

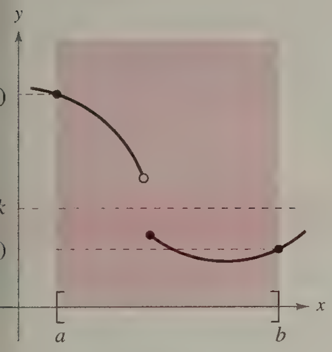
By referring to a text on advanced calculus, you will find that a proof of this theorem is based on a property of real numbers called *completeness*. The Intermediate Value Theorem states that for a continuous function f, if x takes on all values between a and b, $f(x)$ must take on all values between $f(a)$ and $f(b)$.

As a simple example of this theorem, consider a person's height. Suppose that a girl is 5 feet tall on her thirteenth birthday and 5 feet 7 inches tall on her fourteenth birthday. Then, for any height h between 5 feet and 5 feet 7 inches, there must have been a time t when her height was exactly h. This seems reasonable because human growth is continuous and a person's height does not abruptly change from one value to another.

The Intermediate Value Theorem guarantees the existence of *at least one* number c in the closed interval $[a, b]$. There may, of course, be more than one number c such that $f(c) = k$, as shown in Figure 1.35. A function that is not continuous does not necessarily possess the intermediate value property. For example, the graph of the function shown in Figure 1.36 jumps over the horizontal line given by $y = k$, and for this function there is no value of c in $[a, b]$ such that $f(c) = k$.



continuous on $[a, b]$.
ere exist three c's such that $f(c) = k$.]
gure 1.35



not continuous on $[a, b]$.
ere are no c's such that $f(c) = k$.]
gure 1.36

The Intermediate Value Theorem often can be used to locate the zeros of a function that is continuous on a closed interval. Specifically, if f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, the Intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval $[a, b]$.

Example 8 An Application of the Intermediate Value Theorem

Use the Intermediate Value Theorem to show that the polynomial function

$$f(x) = x^3 + 2x - 1$$

has a zero in the interval $[0, 1]$.

Solution Note that f is continuous on the closed interval $[0, 1]$. Because

$$f(0) = 0^3 + 2(0) - 1 = -1$$

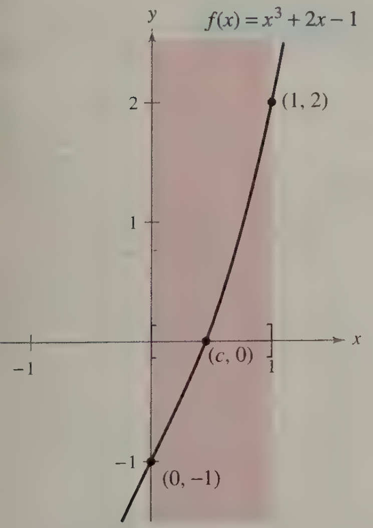
and

$$f(1) = 1^3 + 2(1) - 1 = 2$$

it follows that $f(0) < 0$ and $f(1) > 0$. You can therefore apply the Intermediate Value Theorem to conclude that there must be some c in $[0, 1]$ such that

$$f(c) = 0 \qquad f \text{ has a zero in the closed interval } [0, 1].$$

as shown in Figure 1.37.



continuous on $[0, 1]$ with $f(0) < 0$ and
$) > 0$.
gure 1.37

The **bisection method** for approximating the real zeros of a continuous funct is similar to the method used in Example 8. If you know that a zero exists in the clo interval $[a, b]$, the zero must lie in the interval $[a, (a + b)/2]$ or $[(a + b)/2, b]$. Fr the sign of $f([a + b]/2)$, you can determine which interval contains the zero. repeatedly bisecting the interval, you can "close in" on the zero of the function.

TECHNOLOGY You can also use the *zoom* feature of a graphing utility to appr imate the real zeros of a continuous function. By repeatedly zooming in on the po where the graph crosses the x-axis, and adjusting the x-axis scale, you can appr imate the zero of the function to any desired accuracy. The zero of $x^3 + 2x - 1$ approximately 0.453, as shown in Figure 1.38.

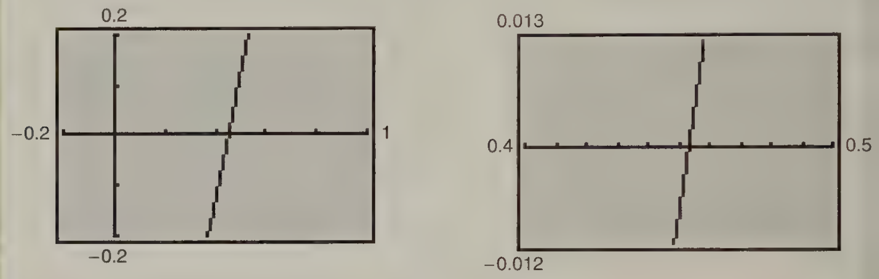


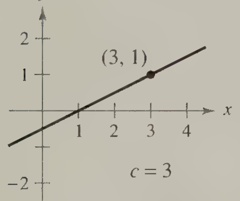
Figure 1.38 Zooming in on the zero of $f(x) = x^3 + 2x - 1$

EXERCISES FOR SECTION 1.4

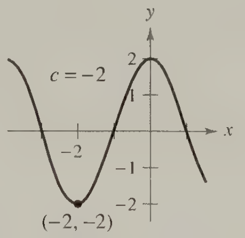
In Exercises 1–6, use the graph to determine the limit, and discuss the continuity of the function.

(a) $\lim_{x \to c^+} f(x)$ **(b)** $\lim_{x \to c^-} f(x)$ **(c)** $\lim_{x \to c} f(x)$

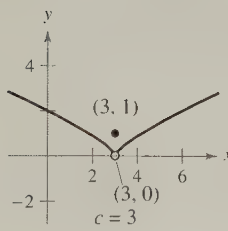
1.



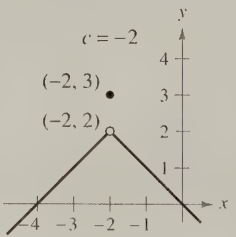
2.



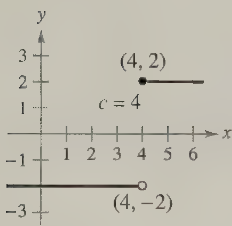
3.



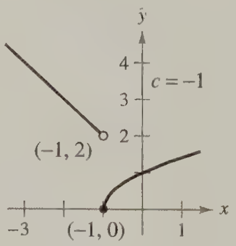
4.



5.



6.



In Exercises 7–24, find the limit (if it exists). If it does not ex explain why.

7. $\lim_{x \to 5^+} \dfrac{x - 5}{x^2 - 25}$

8. $\lim_{x \to 2^+} \dfrac{2 - x}{x^2 - 4}$

9. $\lim_{x \to -3^-} \dfrac{x}{\sqrt{x^2 - 9}}$

10. $\lim_{x \to 4} \dfrac{\sqrt{x} - 2}{x - 4}$

11. $\lim_{x \to 0} \dfrac{|x|}{x}$

12. $\lim_{x \to 2^+} \dfrac{|x - 2|}{x - 2}$

13. $\lim_{\Delta x \to 0} \dfrac{\dfrac{1}{x + \Delta x} - \dfrac{1}{x}}{\Delta x}$

14. $\lim_{\Delta x \to 0^+} \dfrac{(x + \Delta x)^2 + x + \Delta x - (x^2 + x)}{\Delta x}$

15. $\lim_{x \to 3^-} f(x)$, where $f(x) = \begin{cases} \dfrac{x + 2}{2}, & x \le 3 \\ \dfrac{12 - 2x}{3}, & x > 3 \end{cases}$

16. $\lim_{x \to 2} f(x)$, where $f(x) = \begin{cases} x^2 - 4x + 6, & x < 2 \\ -x^2 + 4x - 2, & x \ge 2 \end{cases}$

17. $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} x^3 + 1, & x < 1 \\ x + 1, & x \ge 1 \end{cases}$

18. $\lim_{x \to 1^+} f(x)$, where $f(x) = \begin{cases} x, & x \le 1 \\ 1 - x, & x > 1 \end{cases}$

19. $\lim_{x \to \pi} \cot x$

20. $\lim_{x \to \pi/2} \sec x$

21. $\lim_{x \to 4^-} (3[\![x]\!] - 5)$

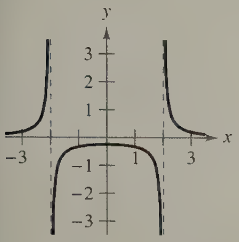
22. $\lim_{x \to 2^+} (2x - [\![x]\!])$

23. $\lim_{x \to 3} (2 - [\![-x]\!])$

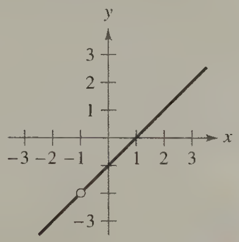
24. $\lim_{x \to 1} \left(1 - \left[\!\!\left[-\dfrac{x}{2}\right]\!\!\right]\right)$

n Exercises 25–28, discuss the continuity of each function.

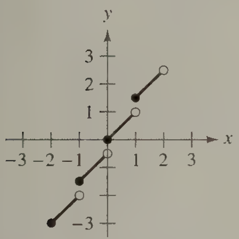
5. $f(x) = \dfrac{1}{x^2 - 4}$



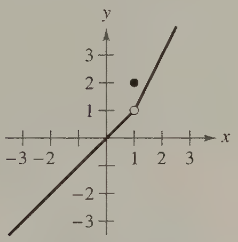
26. $f(x) = \dfrac{x^2 - 1}{x + 1}$



7. $f(x) = \frac{1}{2}[\![x]\!] + x$



28. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$



n Exercises 29–32, discuss the continuity of the function on the losed interval.

9. $g(x) = \sqrt{25 - x^2}$, $[-5, 5]$

0. $f(t) = 3 - \sqrt{9 - t^2}$, $[-3, 3]$

1. $f(x) = \begin{cases} 3 - x, & x \le 0 \\ 3 + \frac{1}{2}x, & x > 0 \end{cases}$, $[-1, 4]$

2. $g(x) = \dfrac{1}{x^2 - 4}$, $[-1, 2]$

n Exercises 33–54, find the x-values (if any) at which f is not ontinuous. Which of the discontinuities are removable?

3. $f(x) = x^2 - 2x + 1$

34. $f(x) = \dfrac{1}{x^2 + 1}$

5. $f(x) = 3x - \cos x$

36. $f(x) = \cos \dfrac{\pi x}{2}$

7. $f(x) = \dfrac{x}{x^2 - x}$

38. $f(x) = \dfrac{x}{x^2 - 1}$

9. $f(x) = \dfrac{x}{x^2 + 1}$

40. $f(x) = \dfrac{x - 3}{x^2 - 9}$

1. $f(x) = \dfrac{x + 2}{x^2 - 3x - 10}$

42. $f(x) = \dfrac{x - 1}{x^2 + x - 2}$

3. $f(x) = \dfrac{|x + 2|}{x + 2}$

44. $f(x) = \dfrac{|x - 3|}{x - 3}$

5. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

46. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

7. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

48. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

49. $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

50. $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

51. $f(x) = \csc 2x$

52. $f(x) = \tan \dfrac{\pi x}{2}$

53. $f(x) = [\![x - 1]\!]$

54. $f(x) = 3 - [\![x]\!]$

In Exercises 55 and 56, use a graphing utility to graph the function. From the graph, estimate

$$\lim_{x \to 0^+} f(x) \quad \text{and} \quad \lim_{x \to 0^-} f(x).$$

Is the function continuous on the entire real line? Explain.

55. $f(x) = \dfrac{|x^2 - 4|x}{x + 2}$

56. $f(x) = \dfrac{|x^2 + 4x|(x + 2)}{x + 4}$

In Exercises 57–60, find the constants a and b such that the function is continuous on the entire real line.

57. $f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$

58. $g(x) = \begin{cases} \dfrac{4 \sin x}{x}, & x < 0 \\ a - 2x, & x \ge 0 \end{cases}$

59. $f(x) = \begin{cases} 2, & x \le -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

60. $g(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a}, & x \ne a \\ 8, & x = a \end{cases}$

In Exercises 61–64, discuss the continuity of the composite function $h(x) = f(g(x))$.

61. $f(x) = x^2$

$g(x) = x - 1$

62. $f(x) = \dfrac{1}{\sqrt{x}}$

$g(x) = x - 1$

63. $f(x) = \dfrac{1}{x - 6}$

$g(x) = x^2 + 5$

64. $f(x) = \sin x$

$g(x) = x^2$

In Exercises 65–68, use a graphing utility to graph the function. Use the graph to determine any x-values at which the function is not continuous.

65. $f(x) = [\![x]\!] - x$

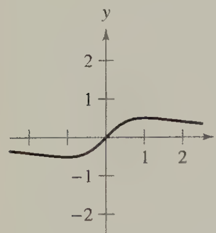
66. $h(x) = \dfrac{1}{x^2 - x - 2}$

67. $g(x) = \begin{cases} 2x - 4, & x \le 3 \\ x^2 - 2x, & x > 3 \end{cases}$

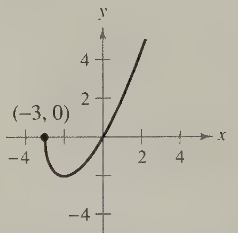
68. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

In Exercises 69–72, describe the interval(s) on which the function is continuous.

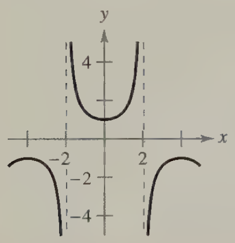
69. $f(x) = \dfrac{x}{x^2 + 1}$



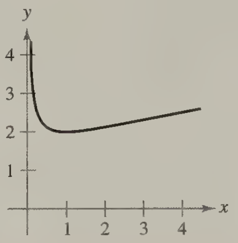
70. $f(x) = x\sqrt{x + 3}$



71. $f(x) = \sec \dfrac{\pi x}{4}$



72. $f(x) = \dfrac{x + 1}{\sqrt{x}}$



Writing **In Exercises 73 and 74, use a graphing utility to graph the function on the interval $[-4, 4]$. Does the graph of the function appear continuous on this interval? Is the function continuous on $[-4, 4]$? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

73. $f(x) = \dfrac{\sin x}{x}$

74. $f(x) = \dfrac{x^3 - 8}{x - 2}$

Writing **In Exercises 75–78, explain why the function has a zero in the specified interval.**

75. $f(x) = \frac{1}{16}x^4 - x^3 + 3, \quad [1, 2]$

76. $f(x) = x^3 + 3x - 2, \quad [0, 1]$

77. $f(x) = x^2 - 2 - \cos x, \quad [0, \pi]$

78. $f(x) = -\dfrac{4}{x} + \tan\left(\dfrac{\pi x}{8}\right), \quad [1, 3]$

In Exercises 79–82, use the Intermediate Value Theorem and a graphing utility to approximate the zero of the function in the interval [0, 1]. Repeatedly "zoom in" on the graph of the function to approximate the zero accurate to two decimal places. Use the root-finding capabilities of the graphing utility to approximate the zero accurate to four decimal places.

79. $f(x) = x^3 + x - 1$

80. $f(x) = x^3 + 3x - 2$

81. $g(t) = 2\cos t - 3t$

82. $h(\theta) = 1 + \theta - 3\tan\theta$

In Exercises 83–86, verify that the Intermediate Value Theor applies to the indicated interval and find the value of c guar teed by the theorem.

83. $f(x) = x^2 + x - 1, \quad [0, 5], \quad f(c) = 11$

84. $f(x) = x^2 - 6x + 8, \quad [0, 3], \quad f(c) = 0$

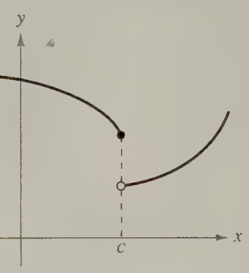
85. $f(x) = x^3 - x^2 + x - 2, \quad [0, 3], \quad f(c) = 4$

86. $f(x) = \dfrac{x^2 + x}{x - 1}, \quad \left[\dfrac{5}{2}, 4\right], \quad f(c) = 6$

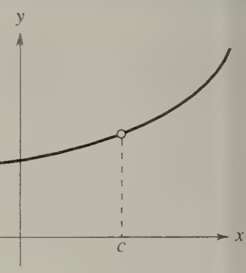
Getting at the Concept

87. State how continuity is destroyed at $x = c$ for each of th following.

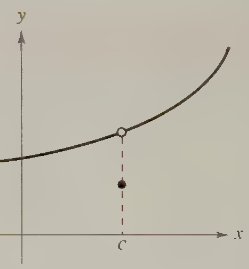
(a)



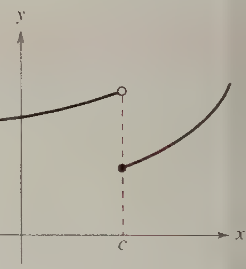
(b)



(c)



(d)



88. Describe the difference between a discontinuity that removable and one that is nonremovable. In your explana tion, give examples of the following.

(a) A function with a nonremovable discontinuity at $x =$

(b) A function with a removable discontinuity at $x = -$

(c) A function that has both of the characteristics describe in parts (a) and (b).

89. Sketch the graph of any function f such that

$$\lim_{x\to 3^+} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to 3} f(x) = 0.$$

Is the function continuous at $x = 3$? Explain.

90. If the functions f and g are continuous for all real x, is $f +$ always continuous for all real x? Is f/g always continuou for all real x? If either is not continuous, give an example verify your conclusion.

91. ***Think About It*** Describe how the functions $f(x) = 3 +$ and $g(x) = 3 - [\![-x]\!]$ differ.

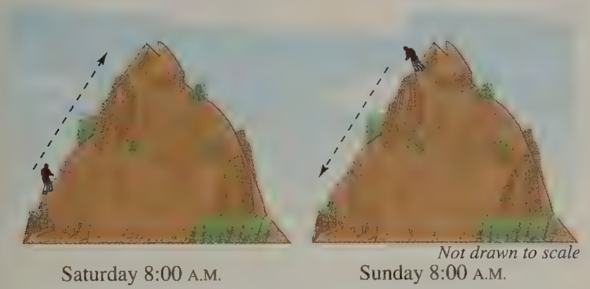
. ***Telephone Charges*** A dial-direct long distance call between two cities costs \$1.04 for the first 2 minutes and \$0.36 for each additional minute or fraction thereof. Use the greatest integer function to write the cost C of a call in terms of the time t (in minutes). Sketch a graph of this function and discuss its continuity.

. ***Inventory Management*** The number of units in inventory in a small company is given by

$$N(t) = 25\left(2\left[\!\left[\frac{t+2}{2}\right]\!\right] - t\right)$$

where t is the time in months. Sketch the graph of this function and discuss its continuity. How often must this company replenish its inventory?

. ***Déjà Vu*** At 8:00 A.M. on Saturday a man begins running up the side of a mountain to his weekend campsite (see figure). On Sunday morning at 8:00 A.M. he runs back down the mountain. It takes him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down, he realizes that he passed the same place at exactly the same time on Saturday. Prove that he is correct. [*Hint:* Let $s(t)$ and $r(t)$ be the position functions for the runs up and down, and apply the Intermediate Value Theorem to the function $f(t) = s(t) - r(t)$.]



Saturday 8:00 A.M. Sunday 8:00 A.M.

. ***Volume*** Use the Intermediate Value Theorem to show that for all spheres with radii in the interval $[1, 5]$, there is one with a volume of 275 cubic centimeters.

. Prove that if f is continuous and has no zeros on $[a, b]$, then either $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

. Show that the Dirichlet function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

is not continuous at any real number.

. Show that the function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ kx, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x = 0$. (Assume that k is any nonzero real number.)

. The **signum function** is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

Sketch a graph of $\operatorname{sgn}(x)$ and find the following (if possible).

(a) $\lim\limits_{x\to 0^-} \operatorname{sgn}(x)$ (b) $\lim\limits_{x\to 0^+} \operatorname{sgn}(x)$ (c) $\lim\limits_{x\to 0} \operatorname{sgn}(x)$

True or False? **In Exercises 100–103, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

100. If $\lim\limits_{x\to c} f(x) = L$ and $f(c) = L$, then f is continuous at c.

101. If $f(x) = g(x)$ for $x \neq c$ and $f(c) \neq g(c)$, then either f or g is not continuous at c.

102. A rational function can have infinitely many x-values at which it is not continuous.

103. The function $f(x) = |x - 1|/(x - 1)$ is continuous on $(-\infty, \infty)$.

104. ***Modeling Data*** After an object falls for t seconds, the speed S (in feet per second) of the object is recorded in the table.

t	0	5	10	15	20	25	30
S	0	48.2	53.5	55.2	55.9	56.2	56.3

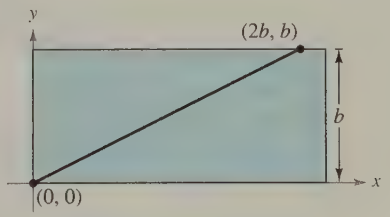
(a) Create a line graph of the data.

(b) Does there appear to be a limiting speed of the object? If there is a limiting speed, identify a possible cause.

105. ***Creating Models*** A swimmer crosses a pool of width b by swimming in a straight line from $(0, 0)$ to $(2b, b)$. (See figure.)

(a) Let f be a function defined as the y-coordinate of the point on the long side of the pool that is nearest the swimmer at any given time during the swimmer's path across the pool. Determine the function f and sketch its graph. Is it continuous? Explain.

(b) Let g be the minimum distance between the swimmer and the long sides of the pool. Determine the function g and sketch its graph. Is it continuous? Explain.



106. Prove that for any real number y there exists x in $(-\pi/2, \pi/2)$ such that $\tan x = y$.

107. Let $f(x) = \left(\sqrt{x + c^2} - c\right)/x$, $c > 0$. What is the domain of f? How can you define f at $x = 0$ in order for f to be continuous there?

108. Prove that if $\lim\limits_{\Delta x\to 0} f(c + \Delta x) = f(c)$, then f is continuous at c.

109. Discuss the continuity of the function $h(x) = x[\![x]\!]$.

110. Let $f_1(x)$ and $f_2(x)$ be continuous on the closed interval $[a, b]$. If $f_1(a) < f_2(a)$ and $f_1(b) > f_2(b)$, prove that there exists c between a and b such that $f_1(c) = f_2(c)$.

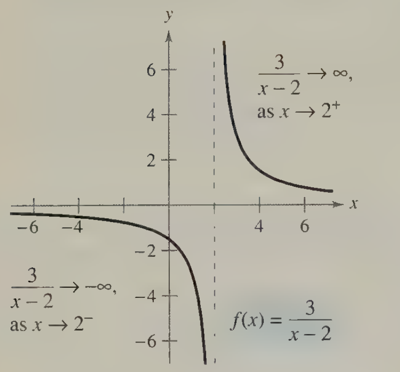
Section 1.5 Infinite Limits

- Determine infinite limits from the left and from the right.
- Find and sketch the vertical asymptotes of the graph of a function.

Infinite Limits

Let f be the function given by

$$f(x) = \frac{3}{x-2}.$$



$f(x)$ increases and decreases without bound as x approaches 2.
Figure 1.39

From Figure 1.39 and the table, you can see that $f(x)$ *decreases without bound* a approaches 2 from the left, and $f(x)$ *increases without bound* as x approaches 2 fr the right. This behavior is denoted as

$$\lim_{x\to 2^-} \frac{3}{x-2} = -\infty \qquad f(x) \text{ decreases without bound as } x \text{ approaches 2 from the left.}$$

and

$$\lim_{x\to 2^+} \frac{3}{x-2} = \infty \qquad f(x) \text{ increases without bound as } x \text{ approaches 2 from the right.}$$

x approaches 2 from the left. → ← x approaches 2 from the right.

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.
$f(x)$	−6	−30	−300	−3000	?	3000	300	30	6

$f(x)$ decreases without bound. → ← $f(x)$ increases without bound.

A limit in which $f(x)$ increases or decreases without bound as x approaches c is ca an **infinite limit.**

Definition of Infinite Limits

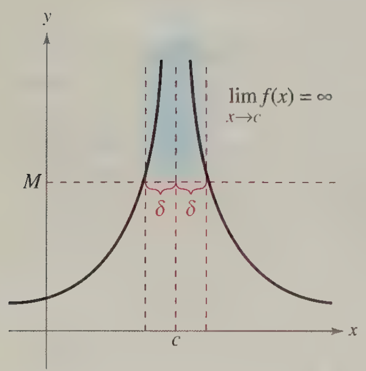
Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

$$\lim_{x\to c} f(x) = \infty$$

means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$ (see Figure 1.40). Similarly, the statement

$$\lim_{x\to c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - c| < \delta$. To define the **infinite limit from the left,** replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the **infinite limit from the right,** replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.



Infinite limits
Figure 1.40

Be sure you see that the equal sign in the statement $\lim f(x) = \infty$ does not m that the limit exists! On the contrary, it tells you how the limit *fails to exist* by de ing the unbounded behavior of $f(x)$ as x approaches c.

EXPLORATION

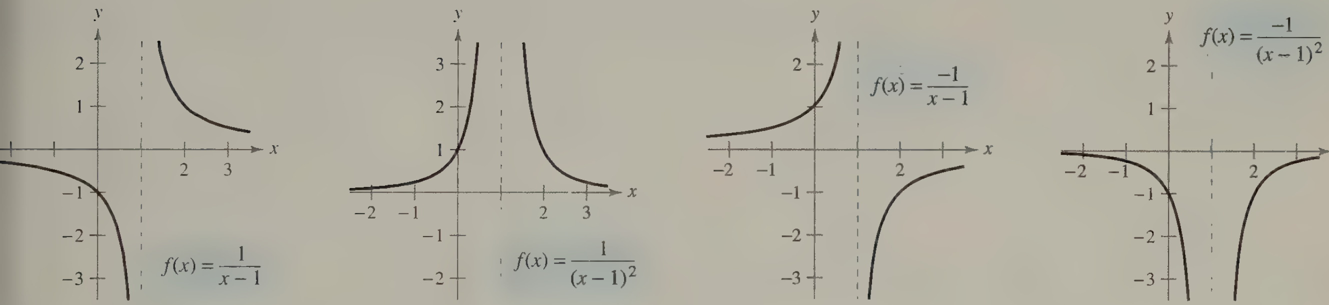
Use a graphing utility to graph each function. For each function, analytically find the single real number c that is not in the domain. Then graphically find the limit of $f(x)$ as x approaches c from the left and from the right.

a. $f(x) = \dfrac{3}{x - 4}$ **b.** $f(x) = \dfrac{1}{2 - x}$

c. $f(x) = \dfrac{2}{(x - 3)^2}$ **d.** $f(x) = \dfrac{-3}{(x + 2)^2}$

Example 1 Determining Infinite Limits from a Graph

Use Figure 1.41 to determine the limit of each function as x approaches 1 from the left and from the right.



a) (b) (c) (d)

igure 1.41 Each graph has an asymptote at $x = 1$.

Solution

a. $\displaystyle\lim_{x\to1^-} \frac{1}{x - 1} = -\infty$ and $\displaystyle\lim_{x\to1^+} \frac{1}{x - 1} = \infty$

b. $\displaystyle\lim_{x\to1} \frac{1}{(x - 1)^2} = \infty$ Limit from each side is ∞.

c. $\displaystyle\lim_{x\to1^-} \frac{-1}{x - 1} = \infty$ and $\displaystyle\lim_{x\to1^+} \frac{-1}{x - 1} = -\infty$

d. $\displaystyle\lim_{x\to1} \frac{-1}{(x - 1)^2} = -\infty$ Limit from each side is $-\infty$.

Vertical Asymptotes

If it were possible to extend the graphs in Figure 1.41 toward positive and negative infinity, you would see that each graph becomes arbitrarily close to the vertical line $x = 1$. This line is a **vertical asymptote** of the graph of f. (You will study other types of asymptotes in Sections 3.5 and 3.6.)

Definition of a Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of f.

OTE If a function f has a vertical symptote at $x = c$, then f is *not ontinuous* at c.

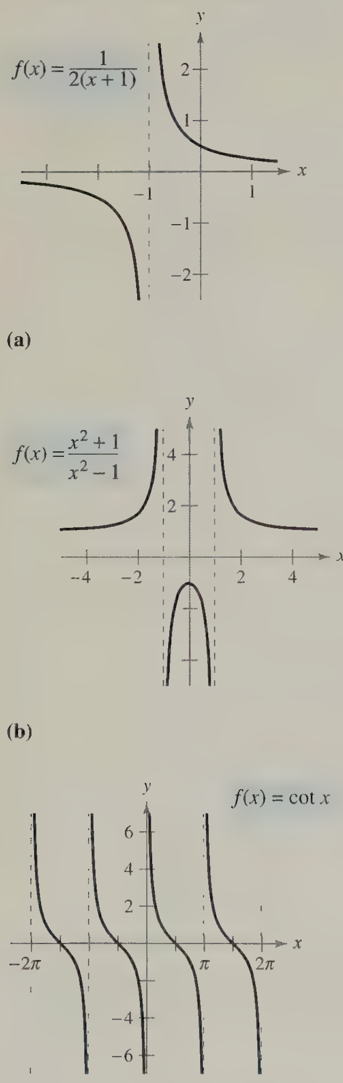
In Example 1, note that each of the functions is a *quotient* and that the verti asymptote occurs at a number where the denominator is 0 (and the numerator is 0). The next theorem generalizes this observation. (A proof of this theorem is given Appendix B.)

THEOREM 1.14 Vertical Asymptotes

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function given by

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.



(a)

(b)

(c)

Functions with vertical asymptotes

Figure 1.42

Example 2 Finding Vertical Asymptotes

Determine all vertical asymptotes of the graph of each function.

a. $f(x) = \dfrac{1}{2(x+1)}$ **b.** $f(x) = \dfrac{x^2+1}{x^2-1}$ **c.** $f(x) = \cot x$

Solution

a. When $x = -1$, the denominator of

$$f(x) = \frac{1}{2(x+1)}$$

is 0 and the numerator is not 0. Hence, by Theorem 1.14, you can conclude t $x = -1$ is a vertical asymptote, as shown in Figure 1.42(a).

b. By factoring the denominator as

$$f(x) = \frac{x^2+1}{x^2-1} = \frac{x^2+1}{(x-1)(x+1)}$$

you can see that the denominator is 0 at $x = -1$ and $x = 1$. Moreover, because numerator is not 0 at these two points, you can apply Theorem 1.14 to conclu that the graph of f has two vertical asymptotes, as shown in Figure 1.42(b).

c. By writing the cotangent function in the form

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

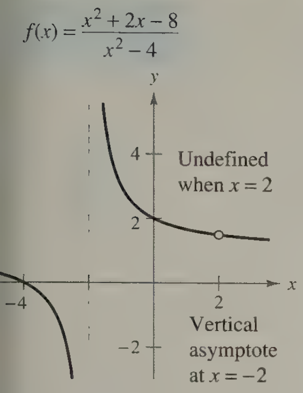
you can apply Theorem 1.14 to conclude that vertical asymptotes occur at values of x such that $\sin x = 0$ and $\cos x \neq 0$, as shown in Figure 1.42(c). Hen the graph of this function has infinitely many vertical asymptotes. These asym totes occur when $x = n\pi$, where n is an integer.

Theorem 1.14 requires that the value of the numerator at $x = c$ be nonzero both the numerator and the denominator are 0 at $x = c$, you obtain the *indetermin form* 0/0, and you cannot determine the limit behavior at $x = c$ without furt investigation, as illustrated in Example 3.

Example 3 A Rational Function with Common Factors

Determine all vertical asymptotes of the graph of

$$f(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$



(x) increases and decreases without bound s x approaches -2.
igure 1.43

Solution Begin by simplifying the expression, as follows.

$$\begin{aligned} f(x) &= \frac{x^2 + 2x - 8}{x^2 - 4} \\ &= \frac{(x + 4)\cancel{(x - 2)}}{(x + 2)\cancel{(x - 2)}} \\ &= \frac{x + 4}{x + 2}, \quad x \neq 2 \end{aligned}$$

At all x-values other than $x = 2$, the graph of f coincides with the graph of $g(x) = (x + 4)/(x + 2)$. So, you can apply Theorem 1.14 to g to conclude that there is a vertical asymptote at $x = -2$, as shown in Figure 1.43. From the graph, you can see that

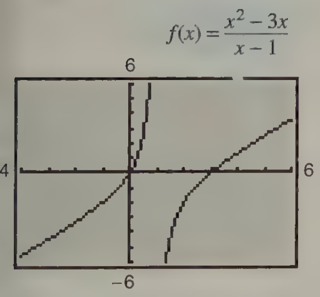
$$\lim_{x \to -2^-} \frac{x^2 + 2x - 8}{x^2 - 4} = -\infty \quad \text{and} \quad \lim_{x \to -2^+} \frac{x^2 + 2x - 8}{x^2 - 4} = \infty.$$

Note that $x = 2$ is *not* a vertical asymptote.

Example 4 Determining Infinite Limits

Find each limit.

$$\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1} \quad \text{and} \quad \lim_{x \to 1^+} \frac{x^2 - 3x}{x - 1}$$



has a vertical asymptote at $x = 1$.
igure 1.44

Solution Because the denominator is 0 when $x = 1$ (and the numerator is not zero), you know that the graph of

$$f(x) = \frac{x^2 - 3x}{x - 1}$$

has a vertical asymptote at $x = 1$. This means that each of the given limits is either ∞ or $-\infty$. A graphing utility can help determine the result. From the graph of f shown in Figure 1.44, you can see that the graph approaches ∞ from the left of $x = 1$ and approaches $-\infty$ from the right of $x = 1$. So, you can conclude that

$$\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1} = \infty$$ The limit from the left is infinity.

and

$$\lim_{x \to 1^+} \frac{x^2 - 3x}{x - 1} = -\infty.$$ The limit from the right is negative infinity.

TECHNOLOGY PITFALL When using a graphing calculator or graphing software, be careful to correctly interpret the graph of a function with a vertical asymptote—graphing utilities often have difficulty drawing this type of graph.

THEOREM 1.15 Properties of Infinite Limits

Let c and L be real numbers and let f and g be functions such that

$$\lim_{x\to c} f(x) = \infty \quad \text{and} \quad \lim_{x\to c} g(x) = L.$$

1. Sum or difference: $\displaystyle\lim_{x\to c} [f(x) \pm g(x)] = \infty$
2. Product: $\displaystyle\lim_{x\to c} [f(x)g(x)] = \infty, \quad L > 0$
   $\displaystyle\lim_{x\to c} [f(x)g(x)] = -\infty, \quad L < 0$
3. Quotient: $\displaystyle\lim_{x\to c} \frac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$.

Proof To show that the limit of $f(x) + g(x)$ is infinite, choose $M > 0$. You then ne
to find $\delta > 0$ such that

$$[f(x) + g(x)] > M$$

whenever $0 < |x - c| < \delta$. For simplicity's sake, you can assume L is positive a
let $M_1 = M + 1$. Because the limit of $f(x)$ is infinite, there exists δ_1 such th
$f(x) > M_1$ whenever $0 < |x - c| < \delta_1$. Also, because the limit of $g(x)$ is L, the
exists δ_2 such that $|g(x) - L| < 1$ whenever $0 < |x - c| < \delta_2$. By letting δ be t
smaller of δ_1 and δ_2, you can conclude that $0 < |x - c| < \delta$ implies $f(x) > M +$
and $|g(x) - L| < 1$. The second of these two inequalities implies that $g(x) > L -$
and, adding this to the first inequality, you can write

$$f(x) + g(x) > (M + 1) + (L - 1) = M + L > M.$$

So, you can conclude that

$$\lim_{x\to c} [f(x) + g(x)] = \infty.$$

We leave the proofs of the remaining properties as exercises (see Exercise 73).

Example 5 Determining Limits

a. Because $\displaystyle\lim_{x\to 0} 1 = 1$ and $\displaystyle\lim_{x\to 0} \frac{1}{x^2} = \infty$, you can write

$$\lim_{x\to 0}\left(1 + \frac{1}{x^2}\right) = \infty.$$

Property 1, Theorem 1.15

b. Because $\displaystyle\lim_{x\to 1^-} (x^2 + 1) = 2$ and $\displaystyle\lim_{x\to 1} (\cot \pi x) = -\infty$, you can write

$$\lim_{x\to 1^-} \frac{x^2 + 1}{\cot \pi x} = 0.$$

Property 3, Theorem 1.15

c. Because $\displaystyle\lim_{x\to 0^+} 3 = 3$ and $\displaystyle\lim_{x\to 0^+} \cot x = \infty$, you can write

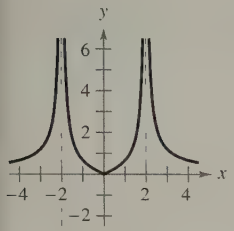
$$\lim_{x\to 0^+} 3 \cot x = \infty.$$

Property 2, Theorem 1.15

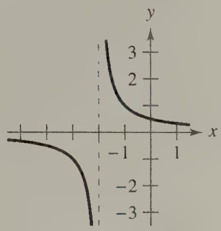
EXERCISES FOR SECTION 1.5

Exercises 1–4, determine whether $f(x)$ approaches ∞ or ∞ as x approaches -2 from the left and from the right.

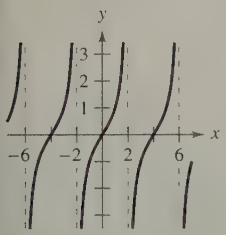
. $f(x) = 2\left|\dfrac{x}{x^2-4}\right|$



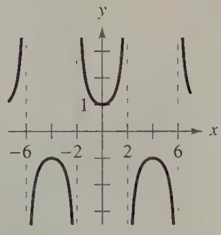
2. $f(x) = \dfrac{1}{x+2}$



. $f(x) = \tan \dfrac{\pi x}{4}$



4. $f(x) = \sec \dfrac{\pi x}{4}$



umerical and Graphical Analysis **In Exercises 5–8, determine hether $f(x)$ approaches ∞ or $-\infty$ as x approaches -3 from e left and from the right by completing the table. Use a graph- g utility to graph the function and confirm your answer.**

x	-3.5	-3.1	-3.01	-3.001
$f(x)$				

x	-2.999	-2.99	-2.9	-2.5
$f(x)$				

. $f(x) = \dfrac{1}{x^2-9}$

6. $f(x) = \dfrac{x}{x^2-9}$

. $f(x) = \dfrac{x^2}{x^2-9}$

8. $f(x) = \sec \dfrac{\pi x}{6}$

Exercises 9–28, find the vertical asymptotes (if any) of the nction.

. $f(x) = \dfrac{1}{x^2}$

10. $f(x) = \dfrac{4}{(x-2)^3}$

. $h(x) = \dfrac{x^2-2}{x^2-x-2}$

12. $g(x) = \dfrac{2+x}{x^2(1-x)}$

. $f(x) = \dfrac{x^2}{x^2-4}$

14. $f(x) = \dfrac{-4x}{x^2+4}$

. $g(t) = \dfrac{t-1}{t^2+1}$

16. $h(s) = \dfrac{2s-3}{s^2-25}$

17. $f(x) = \tan 2x$

18. $f(x) = \sec \pi x$

19. $T(t) = 1 - \dfrac{4}{t^2}$

20. $g(x) = \dfrac{\frac{1}{2}x^3 - x^2 - 4x}{3x^2 - 6x - 24}$

21. $f(x) = \dfrac{x}{x^2+x-2}$

22. $f(x) = \dfrac{4x^2+4x-24}{x^4-2x^3-9x^2+18x}$

23. $g(x) = \dfrac{x^3+1}{x+1}$

24. $h(x) = \dfrac{x^2-4}{x^3+2x^2+x+2}$

25. $f(x) = \dfrac{x^2-2x-15}{x^3-5x^2+x-5}$

26. $h(t) = \dfrac{t^2-2t}{t^4-16}$

27. $s(t) = \dfrac{t}{\sin t}$

28. $g(\theta) = \dfrac{\tan \theta}{\theta}$

In Exercises 29–32, determine whether the function has a vertical asymptote or a removable discontinuity at $x = -1$. Graph the function using a graphing utility to confirm your answer.

29. $f(x) = \dfrac{x^2-1}{x+1}$

30. $f(x) = \dfrac{x^2-6x-7}{x+1}$

31. $f(x) = \dfrac{x^2+1}{x+1}$

32. $f(x) = \dfrac{\sin(x+1)}{x+1}$

In Exercises 33–48, find the limit.

33. $\displaystyle\lim_{x\to 2^+} \frac{x-3}{x-2}$

34. $\displaystyle\lim_{x\to 1^+} \frac{2+x}{1-x}$

35. $\displaystyle\lim_{x\to 3^+} \frac{x^2}{x^2-9}$

36. $\displaystyle\lim_{x\to 4^-} \frac{x^2}{x^2+16}$

37. $\displaystyle\lim_{x\to -3^-} \frac{x^2+2x-3}{x^2+x-6}$

38. $\displaystyle\lim_{x\to (-1/2)^+} \frac{6x^2+x-1}{4x^2-4x-3}$

39. $\displaystyle\lim_{x\to 1} \frac{x^2-x}{(x^2+1)(x-1)}$

40. $\displaystyle\lim_{x\to 3} \frac{x-2}{x^2}$

41. $\displaystyle\lim_{x\to 0^-} \left(1 + \frac{1}{x}\right)$

42. $\displaystyle\lim_{x\to 0^-} \left(x^2 - \frac{1}{x}\right)$

43. $\displaystyle\lim_{x\to 0^+} \frac{2}{\sin x}$

44. $\displaystyle\lim_{x\to (\pi/2)^+} \frac{-2}{\cos x}$

45. $\displaystyle\lim_{x\to \pi} \frac{\sqrt{x}}{\csc x}$

46. $\displaystyle\lim_{x\to 0} \frac{x+2}{\cot x}$

47. $\displaystyle\lim_{x\to 1/2} x \sec \pi x$

48. $\displaystyle\lim_{x\to 1/2} x^2 \tan \pi x$

In Exercises 49–52, use a graphing utility to graph the function and determine the one-sided limit.

49. $f(x) = \dfrac{x^2+x+1}{x^3-1}$

$\displaystyle\lim_{x\to 1^+} f(x)$

50. $f(x) = \dfrac{x^3-1}{x^2+x+1}$

$\displaystyle\lim_{x\to 1^-} f(x)$

51. $f(x) = \dfrac{1}{x^2-25}$

$\displaystyle\lim_{x\to 5} f(x)$

52. $f(x) = \sec \dfrac{\pi x}{6}$

$\displaystyle\lim_{x\to 3^+} f(x)$

Getting at the Concept

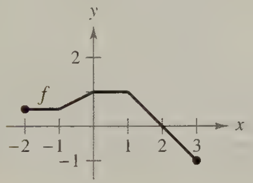
53. In your own words, describe the meaning of an infinite limit. Is ∞ a real number?

54. In your own words, describe what is meant by an asymptote of a graph.

55. Write a rational function with vertical asymptotes at $x = 6$ and $x = -2$, and with a zero at $x = 3$.

56. Does every rational function have a vertical asymptote? Explain.

57. Use the graph of the function f (see figure) to sketch the graph of $g(x) = 1/f(x)$ on the interval $[-2, 3]$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



58. ***Boyle's Law*** For a quantity of gas at a constant temperature, the pressure P is inversely proportional to the volume V. Find the limit of P as $V \to 0^+$.

59. A given sum S is inversely proportional to $1 - r$, where $0 < |r| < 1$. Find the limit of S as $r \to 1^-$.

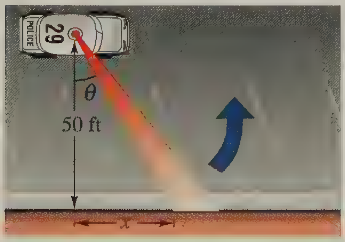
60. ***Rate of Change*** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of $\frac{1}{2}$ revolution per second. The rate at which the light beam moves along the wall is

$$r = 50\pi \sec^2 \theta \text{ ft/sec}.$$

(a) Find the rate r when θ is $\pi/6$.

(b) Find the rate r when θ is $\pi/3$.

(c) Find the limit of r as $\theta \to (\pi/2)^-$.



61. ***Illegal Drugs*** The cost in millions of dollars for a governmental agency to seize $x\%$ of an illegal drug is

$$C = \frac{528x}{100 - x}, \quad 0 \le x < 100.$$

(a) Find the cost of seizing 25% of the drug.

(b) Find the cost of seizing 50% of the drug.

(c) Find the cost of seizing 75% of the drug.

(d) Find the limit of C as $x \to 100^-$ and interpret its meaning.

62. ***Relativity*** According to the theory of relativity, the mass a particle depends on its velocity v. That is,

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$$

where m_0 is the mass when the particle is at rest and c is speed of light. Find the limit of the mass as v approaches

63. ***Rate of Change*** A 25-foot ladder is leaning against a ho (see figure). If the base of the ladder is pulled away from house at a rate of 2 feet per second, the top will move down wall at a rate of

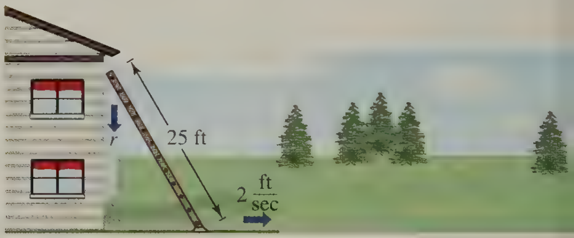
$$r = \frac{2x}{\sqrt{625 - x^2}} \text{ ft/sec}$$

where x is the distance between the base of the ladder and house.

(a) Find the rate r when x is 7 feet.

(b) Find the rate r when x is 15 feet.

(c) Find the limit of r as $x \to 25^-$.



64. ***Average Speed*** On a trip of d miles to another city, a tr driver's average speed was x miles per hour. On the return the average speed was y miles per hour. The average speed the round trip was 50 miles per hour.

(a) Verify that $y = \dfrac{25x}{x - 25}$. What is the domain?

(b) Complete the table.

x	30	40	50	60
y				

Are the values of y different than you expected? Explai

(c) Find the limit of y as $x \to 25^+$ and interpret its meanin

65. ***Numerical and Graphical Analysis*** Use a graphing utilit complete the table for each function and graph each functio estimate the limit. What is the value of the limit when the po on x in the denominator is greater than 3?

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$							

(a) $\displaystyle\lim_{x \to 0^+} \frac{x - \sin x}{x}$

(b) $\displaystyle\lim_{x \to 0^+} \frac{x - \sin x}{x^2}$

(c) $\displaystyle\lim_{x \to 0^-} \frac{x - \sin x}{x^3}$

(d) $\displaystyle\lim_{x \to 0^+} \frac{x - \sin x}{x^4}$

Numerical and Graphical Analysis Consider the shaded region outside the sector of a circle of radius 10 meters and inside a right triangle (see figure).

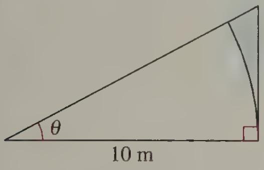
(a) Write the area $A = f(\theta)$ of the region as a function of θ. Determine the domain of the function.

(b) Use a graphing utility to complete the table.

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$					

(c) Use a graphing utility to graph the function over the appropriate domain.

(d) Find the limit of A as $\theta \to (\pi/2)^-$.



Numerical and Graphical Reasoning A crossed belt connects a 20-centimeter pulley (10-cm radius) on an electric motor with a 40-centimeter pulley (20-cm radius) on a saw arbor (see figure). The electric motor runs at 1700 revolutions per minute.

(a) Determine the number of revolutions per minute of the saw.

(b) How does crossing the belt affect the saw in relation to the motor?

(c) Let L be the total length of the belt. Write L as a function of ϕ, where ϕ is measured in radians. What is the domain of the function? (*Hint:* Add the lengths of the straight sections of the belt and the length of the belt around each pulley.)

(d) Use a graphing utility to complete the table.

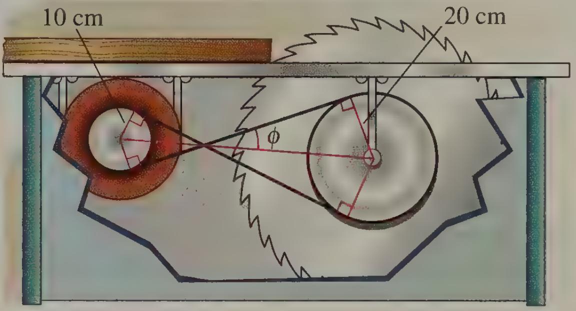
ϕ	0.3	0.6	0.9	1.2	1.5
L					

(e) Use a graphing utility to graph the function over the appropriate domain.

(f) Find $\lim\limits_{\phi \to (\pi/2)} L$.

Use a geometric argument as the basis of a second method of finding this limit.

(g) Find $\lim\limits_{\phi \to 0^+} L$.



True or False? **In Exercises 68–71, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

68. If $p(x)$ is a polynomial, then the function given by

$$f(x) = \frac{p(x)}{x-1}$$

has a vertical asymptote at $x = 1$.

69. A rational function has at least one vertical asymptote.

70. Polynomial functions have no vertical asymptotes.

71. If f has a vertical asymptote at $x = 0$, then f is undefined at $x = 0$.

72. Find functions f and g such that

$$\lim_{x \to c} f(x) = \infty \quad \text{and} \quad \lim_{x \to c} g(x) = \infty$$

but $\lim\limits_{x \to c} [f(x) - g(x)] \neq 0$.

73. Prove the remaining properties of Theorem 1.15.

74. Prove that if $\lim\limits_{x \to c} f(x) = \infty$ then $\lim\limits_{x \to c} \dfrac{1}{f(x)} = 0$.

75. Prove that if $\lim\limits_{x \to c} \dfrac{1}{f(x)} = 0$ then $\lim\limits_{x \to c} f(x)$ does not exist.

SECTION PROJECT GRAPHS AND LIMITS OF TRIGONOMETRIC FUNCTIONS

Recall from Theorem 1.9 that the limit of $f(x) = (\sin x)/x$ as x pproaches 0 is 1.

a) Use a graphing utility to graph the function f on the interval $-\pi \leq 0 \leq \pi$. Explain how this graph helps confirm that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

b) Explain how you could use a table of values to confirm the value of this limit numerically.

c) Graph $g(x) = \sin x$ by hand. Sketch a tangent line at the point $(0, 0)$ and visually estimate the slope of this tangent line.

(d) Let $(x, \sin x)$ be a point on the graph of g near $(0, 0)$, and write a formula for the slope of the secant line joining $(x, \sin x)$ and $(0, 0)$. Evaluate this formula for $x = 0.1$ and $x = 0.01$. Then find the exact slope of the tangent line to g at the point $(0, 0)$.

(e) Sketch the graph of the cosine function $h(x) = \cos x$. What is the slope of the tangent line at the point $(0, 1)$? Use limits to find this slope analytically.

(f) Find the slope of the tangent line to $k(x) = \tan x$ at $(0, 0)$.

REVIEW EXERCISES FOR CHAPTER 1

1.1 **In Exercises 1 and 2, determine whether the problem can be solved using precalculus or if calculus is required. If the problem can be solved using precalculus, solve it. If the problem seems to require calculus, explain your reasoning. Use a graphical or numerical approach to estimate the solution.**

1. Find the distance between the points (1, 1) and (3, 9) along the curve $y = x^2$.

2. Find the distance between the points (1, 1) and (3, 9) along the line $y = 4x - 3$.

1.2 **In Exercises 3 and 4, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.**

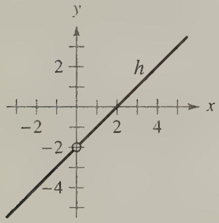
x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

3. $\displaystyle\lim_{x\to 0} \frac{[4/(x+2)] - 2}{x}$

4. $\displaystyle\lim_{x\to 0} \frac{4\left(\sqrt{x+2} - \sqrt{2}\right)}{x}$

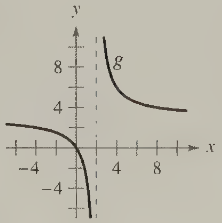
In Exercises 5 and 6, use the graph to determine each limit.

5. $h(x) = \dfrac{x^2 - 2x}{x}$



(a) $\displaystyle\lim_{x\to 0} h(x)$ (b) $\displaystyle\lim_{x\to -1} h(x)$

6. $g(x) = \dfrac{3x}{x - 2}$



(a) $\displaystyle\lim_{x\to 2} g(x)$ (b) $\displaystyle\lim_{x\to 0} g(x)$

In Exercises 7–10, find the limit L. Then use the ε-δ definition to prove that the limit is L.

7. $\displaystyle\lim_{x\to 1} (3 - x)$

8. $\displaystyle\lim_{x\to 9} \sqrt{x}$

9. $\displaystyle\lim_{x\to 2} (x^2 - 3)$

10. $\displaystyle\lim_{x\to 5} 9$

1.3 **In Exercises 11–24, find the limit (if it exists).**

11. $\displaystyle\lim_{t\to 4} \sqrt{t+2}$

12. $\displaystyle\lim_{y\to 4} 3|y - 1|$

13. $\displaystyle\lim_{t\to -2} \frac{t+2}{t^2 - 4}$

14. $\displaystyle\lim_{t\to 3} \frac{t^2 - 9}{t - 3}$

15. $\displaystyle\lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4}$

16. $\displaystyle\lim_{x\to 0} \frac{\sqrt{4+x} - 2}{x}$

17. $\displaystyle\lim_{x\to 0} \frac{[1/(x+1)] - 1}{x}$

18. $\displaystyle\lim_{s\to 0} \frac{\left(1/\sqrt{1+s}\right) - 1}{s}$

19. $\displaystyle\lim_{x\to -5} \frac{x^3 + 125}{x + 5}$

20. $\displaystyle\lim_{x\to -2} \frac{x^2 - 4}{x^3 + 8}$

21. $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{\sin x}$

22. $\displaystyle\lim_{x\to \pi/4} \frac{4x}{\tan x}$

23. $\displaystyle\lim_{\Delta x\to 0} \frac{\sin[(\pi/6) + \Delta x] - (1/2)}{\Delta x}$

[*Hint:* $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$]

24. $\displaystyle\lim_{\Delta x\to 0} \frac{\cos(\pi + \Delta x) + 1}{\Delta x}$

[*Hint:* $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$]

In Exercises 25 and 26, evaluate the limit given $\displaystyle\lim_{x\to c} f(x) =$ and $\displaystyle\lim_{x\to c} g(x) = \tfrac{2}{3}$.

25. $\displaystyle\lim_{x\to c} [f(x)g(x)]$

26. $\displaystyle\lim_{x\to c} [f(x) + 2g(x)]$

Numerical, Graphical, and Analytic Analysis **In Exercises and 28, consider**

$$\lim_{x\to 1^+} f(x).$$

(a) Complete the table to estimate the limit.

(b) Use a graphing utility to graph the function and use graph to estimate the limit.

(c) Rationalize the numerator to find the exact value of limit analytically.

x	1.1	1.01	1.001	1.0001
$f(x)$				

27. $f(x) = \dfrac{\sqrt{2x+1} - \sqrt{3}}{x - 1}$

28. $f(x) = \dfrac{1 - \sqrt[3]{x}}{x - 1}$

[*Hint:* $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$]

Free-Falling Object **In Exercises 29 and 30, use the posit function**

$$s(t) = -4.9t^2 + 200$$

which gives the height (in meters) of an object that has fal from a height of 200 meters. The velocity at time $t = a$ seco is given by

$$\lim_{t\to a} \frac{s(a) - s(t)}{a - t}.$$

29. Find the velocity of the object when $t = 4$.

30. At what velocity will the object impact the ground?

.4 **In Exercises 31–36, find the limit (if it exists). If the nit does not exist, explain why.**

. $\lim\limits_{x\to3} \dfrac{|x-3|}{x-3}$

32. $\lim\limits_{x\to4} [\![x-1]\!]$

. $\lim\limits_{x\to2} f(x)$, where $f(x) = \begin{cases} (x-2)^2, & x \le 2 \\ 2-x, & x > 2 \end{cases}$

. $\lim\limits_{x\to1^+} g(x)$, where $g(x) = \begin{cases} \sqrt{1-x}, & x \le 1 \\ x+1, & x > 1 \end{cases}$

. $\lim\limits_{t\to1} h(t)$, where $h(t) = \begin{cases} t^3+1, & t < 1 \\ \frac{1}{2}(t+1), & t \ge 1 \end{cases}$

. $\lim\limits_{s\to-2} f(s)$, where $f(s) = \begin{cases} -s^2-4s-2, & s \le -2 \\ s^2+4s+6, & s > -2 \end{cases}$

**Exercises 37–46, determine the intervals on which the func-
on is continuous.**

. $f(x) = [\![x+3]\!]$

38. $f(x) = \dfrac{3x^2-x-2}{x-1}$

. $f(x) = \begin{cases} \dfrac{3x^2-x-2}{x-1}, & x \ne 1 \\ 0, & x = 1 \end{cases}$

40. $f(x) = \begin{cases} 5-x, & x \le 2 \\ 2x-3, & x > 2 \end{cases}$

. $f(x) = \dfrac{1}{(x-2)^2}$

42. $f(x) = \sqrt{\dfrac{x+1}{x}}$

. $f(x) = \dfrac{3}{x+1}$

44. $f(x) = \dfrac{x+1}{2x+2}$

. $f(x) = \csc \dfrac{\pi x}{2}$

46. $f(x) = \tan 2x$

. Determine the value of c such that the function is continuous on the entire real line.

$$f(x) = \begin{cases} x+3, & x \le 2 \\ cx+6, & x > 2 \end{cases}$$

. Determine the values of b and c such that the function is continuous on the entire real line.

$$f(x) = \begin{cases} x+1, & 1 < x < 3 \\ x^2+bx+c, & |x-2| \ge 1 \end{cases}$$

. Use the Intermediate Value Theorem to show that $f(x) = 2x^3 - 3$ has a zero in the interval $[1, 2]$.

. *Cost of Overnight Delivery* The cost of sending an overnight package from New York to Atlanta is \$9.80 for the first pound and \$2.50 for each additional pound. Use the greatest integer function to create a model for the cost C of overnight delivery of a package weighing x pounds. Use a graphing utility to graph the function and discuss its continuity.

. Let $f(x) = \dfrac{x^2-4}{|x-2|}$. Find each limit (if possible).

(a) $\lim\limits_{x\to2^-} f(x)$

(b) $\lim\limits_{x\to2^+} f(x)$

(c) $\lim\limits_{x\to2} f(x)$

52. Let $f(x) = \sqrt{x(x-1)}$.

(a) Find the domain of f.

(b) Find $\lim\limits_{x\to0^-} f(x)$.

(c) Find $\lim\limits_{x\to1^+} f(x)$.

1.5 **In Exercises 53–56, find the vertical asymptotes (if any) of the function.**

53. $g(x) = 1 + \dfrac{2}{x}$

54. $h(x) = \dfrac{4x}{4-x^2}$

55. $f(x) = \dfrac{8}{(x-10)^2}$

56. $f(x) = \csc \pi x$

In Exercises 57–68, find the one-sided limit.

57. $\lim\limits_{x\to-2^-} \dfrac{2x^2+x+1}{x+2}$

58. $\lim\limits_{x\to(1/2)^+} \dfrac{x}{2x-1}$

59. $\lim\limits_{x\to-1^+} \dfrac{x+1}{x^3+1}$

60. $\lim\limits_{x\to-1^-} \dfrac{x+1}{x^4-1}$

61. $\lim\limits_{x\to1^-} \dfrac{x^2+2x+1}{x-1}$

62. $\lim\limits_{x\to-1^+} \dfrac{x^2-2x+1}{x+1}$

63. $\lim\limits_{x\to0^+} \left(x - \dfrac{1}{x^3}\right)$

64. $\lim\limits_{x\to2^-} \dfrac{1}{\sqrt[3]{x^2-4}}$

65. $\lim\limits_{x\to0^-} \dfrac{\sin 4x}{5x}$

66. $\lim\limits_{x\to0^+} \dfrac{\sec x}{x}$

67. $\lim\limits_{x\to0^+} \dfrac{\csc 2x}{x}$

68. $\lim\limits_{x\to0^-} \dfrac{\cos^2 x}{x}$

69. *Cost of Clean Air* A utility company burns coal to generate electricity. The cost C in dollars of removing $p\%$ of the air pollutants in the stack emissions is

$$C = \frac{80{,}000p}{100-p}, \qquad 0 \le p < 100.$$

Find the cost of removing (a) 15%, (b) 50%, and (c) 90% of the pollutants. (d) Find the limit of C as $p \to 100^-$.

70. The function f is defined as follows.

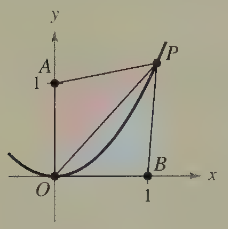
$$f(x) = \frac{\tan 2x}{x}, \qquad x \ne 0$$

(a) Find $\lim\limits_{x\to0} \dfrac{\tan 2x}{x}$ (if it exists).

(b) Can the function f be defined at $x = 0$ such that it is continuous at $x = 0$?

P.S. Problem Solving

1. Let $P(x, y)$ be a point on the parabola $y = x^2$ in the first quadrant. Consider the triangle $\triangle PAO$ formed by P, $A(0, 1)$, and the origin $O(0, 0)$, and the triangle $\triangle PBO$ formed by P, $B(1, 0)$, and the origin.



(a) Express the perimeter of each triangle in terms of x.

(b) Let $r(x)$ be the ratio of the perimeters of the two triangles,

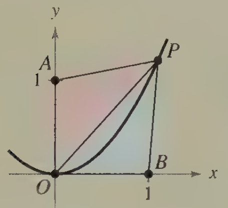
$$r(x) = \frac{\text{Perimeter } \triangle PAO}{\text{Perimeter } \triangle PBO}.$$

Complete the table.

x	4	2	1	0.1	0.01
Perimeter $\triangle PAO$					
Perimeter $\triangle PBO$					
$r(x)$					

(c) Calculate $\lim_{x\to 0^+} r(x)$.

2. Let $P(x, y)$ be a point on the parabola $y = x^2$ in the first quadrant. Consider the triangle $\triangle PAO$ formed by P, $A(0, 1)$, and the origin $O(0, 0)$, and the triangle $\triangle PBO$ formed by P, $B(1, 0)$, and the origin.



(a) Express the area of each triangle in terms of x.

(b) Let $a(x)$ be the ratio of the areas of the two triangles,

$$a(x) = \frac{\text{Area } \triangle PBO}{\text{Area } \triangle PAO}.$$

Complete the table.

x	4	2	1	0.1	0.01
Area $\triangle PAO$					
Area $\triangle PBO$					
$a(x)$					

(c) Calculate $\lim_{x\to 0^+} a(x)$.

3. (a) Find the area of a regular hexagon inscribed in a circle o radius 1. How close is this area to that of the circle?

(b) Find the area A_n of an n-sided regular polygon inscribec a circle of radius 1. Express your answer as a function o

(c) Complete the table.

n	6	12	24	48	96
A_n					

(d) What number does A_n approach as n gets larger and larg

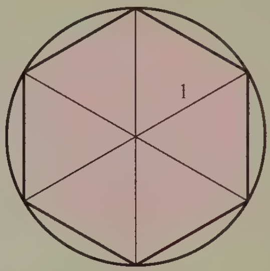


Figure for 3

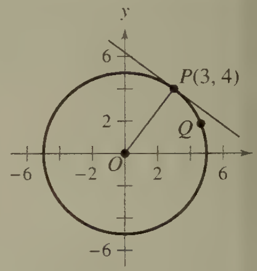


Figure for 4

4. Let $P(3, 4)$ be a point on the circle $x^2 + y^2 = 25$.

(a) What is the slope of the line joining P and $O(0, 0)$?

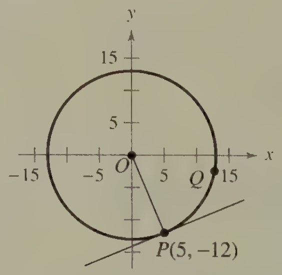
(b) Find an equation of the tangent line to the circle at P.

(c) Let $Q(x, y)$ be another point on the circle in the first quadr Find the slope m_x of the line joining P and Q in terms of

(d) Calculate $\lim_{x\to 3} m_x$.

How does this number relate to your answer in part (b)?

5. Let $P(5, -12)$ be a point on the circle $x^2 + y^2 = 169$.



(a) What is the slope of the line joining P and $O(0, 0)$?

(b) Find an equation of the tangent line to the circle at P.

(c) Let $Q(x, y)$ be another point on the circle in the fourth qu rant. Find the slope m_x of the line joining P and Q in te of x.

(d) Calculate $\lim_{x\to 5} m_x$.

How does this number relate to your answer in part (b)

6. Find the values for the constants a and b such that

$$\lim_{x\to 0} \frac{\sqrt{a + bx} - \sqrt{3}}{x} = \sqrt{3}.$$

7. Consider the function $f(x) = \dfrac{\sqrt{3 + x^{1/3}} - 2}{x - 1}$.

(a) Find the domain of f.

(b) Use a graphing utility to graph the function.

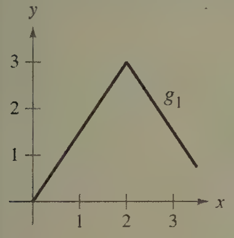
(c) Calculate $\lim_{x \to -27^+} f(x)$.

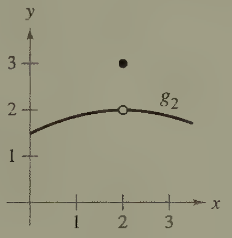
(d) Calculate $\lim_{x \to 1} f(x)$.

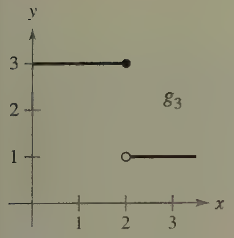
8. Determine all values of the constant a such that the following function is continuous for all real numbers.

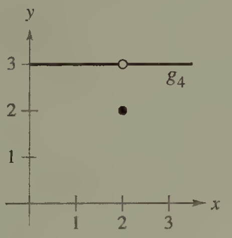
$$f(x) = \begin{cases} \dfrac{ax}{\tan x}, & x \geq 0 \\ a^2 - 2, & x < 0 \end{cases}$$

9. Consider the graphs of the four functions g_1, g_2, g_3, and g_4.









For the given condition of the function f, which of the graphs could be the graph of f?

(a) $\lim_{x \to 2} f(x) = 3$

(b) f is continuous at 2.

(c) $\lim_{x \to 2^-} f(x) = 3$

0. Sketch the graph of the function $f(x) = \left[\!\left[\dfrac{1}{x} \right]\!\right]$.

(a) Evaluate $f\left(\frac{1}{4}\right)$, $f(3)$, and $f(1)$.

(b) Evaluate the limits $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0^+} f(x)$.

(c) Discuss the continuity of the function.

1. Sketch the graph of the function $f(x) = [\![x]\!] + [\![-x]\!]$.

(a) Evaluate $f(1)$, $f(0)$, $f\left(\frac{1}{2}\right)$, and $f(-2.7)$.

(b) Evaluate the limits $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$, and $\lim_{x \to \frac{1}{2}} f(x)$.

(c) Discuss the continuity of the function.

12. To escape earth's gravitational field, a rocket must be launched with an initial velocity called the **escape velocity.** A rocket launched from the surface of earth has velocity v (in miles per second) given by

$$v = \sqrt{\frac{2GM}{r} + v_0^2 - \frac{2GM}{R}} \approx \sqrt{\frac{192{,}000}{r} + v_0^2 - 48}$$

where v_0 is the initial velocity, r is the distance from the rocket to the center of earth, G is the gravitational constant, M is the mass of earth, and R is the radius of earth (approximately 4000 miles).

(a) Find the value of v_0 for which you obtain an infinite limit for r as v tends to zero. This value of v_0 is the escape velocity for earth.

(b) A rocket launched from the surface of the moon has velocity v (in miles per second) given by

$$v = \sqrt{\frac{1920}{r} + v_0^2 - 2.17}.$$

Find the escape velocity for the moon.

(c) A rocket launched from the surface of a certain planet has velocity v (in miles per second) given by

$$v = \sqrt{\frac{10{,}600}{r} + v_0^2 - 6.99}.$$

Find the escape velocity for this planet. Is the mass of this planet larger or smaller than that of earth? (Assume that the mean density of this planet is the same as that of earth.)

13. For positive numbers $a < b$, the **pulse function** is defined as

$$P_{a,b}(x) = H(x - a) - H(x - b) = \begin{cases} 0, & x < a \\ 1, & a \leq x < b \\ 0, & x \geq b \end{cases}$$

where $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is the Heaviside function.

(a) Sketch the graph of the pulse function.

(b) Find the following limits:

(i) $\lim_{x \to a^+} P_{a,b}(x)$ (ii) $\lim_{x \to a^-} P_{a,b}(x)$

(iii) $\lim_{x \to b^+} P_{a,b}(x)$ (iv) $\lim_{x \to b^-} P_{a,b}(x)$

(c) Discuss the continuity of the pulse function.

(d) Why is $U(x) = \dfrac{1}{b - a} P_{a,b}(x)$ called the **unit** pulse function?

14. Let a be a nonzero constant. Prove that if

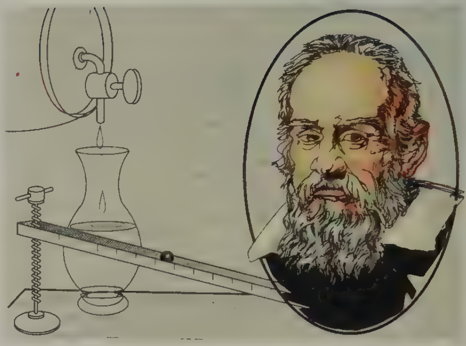
$\lim_{x \to 0} f(x) = L$

then

$\lim_{x \to 0} f(ax) = L.$

Show by means of an example that a must be nonzero.

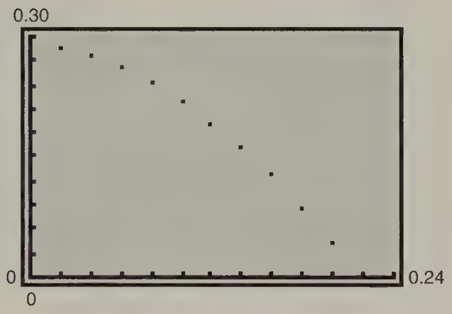
Gravity: Finding It Experimentally



The study of dynamics dates back to the sixteenth century. As the Dark Ages gave way to the Renaissance, Galileo Galilei (1564–1642) was one of the first to take steps toward understanding the motion of objects under the influence of gravity.

Up until Galileo's time, it was recognized that a falling object moved faster and faster as it fell, but what mathematical law governed this accelerating motion was unknown. Free-falling objects move too fast to have been measured with any of the equipment available at that time. Galileo solved this problem with a rather ingenious setup. He reasoned that gravity could be "diluted" by rolling a ball down an inclined plane. He used a water clock, which kept track of time by measuring the amount of water that poured through a small opening at the bottom.

We now have relatively inexpensive instruments, such as the *Texas Instruments Calculator-Based Laboratory (CBL) System*, that allow accurate position data to be gathered on a free-falling object. A CBL System was used to track the positions of a falling ball at time intervals of 0.02 second. The results are shown below.



Time (sec)	*Height (meters)*	*Velocity (meters/sec)*
0.00	0.290864	−0.16405
0.02	0.284279	−0.32857
0.04	0.274400	−0.49403
0.06	0.260131	−0.71322
0.08	0.241472	−0.93309
0.10	0.219520	−1.09409
0.12	0.189885	−1.47655
0.14	0.160250	−1.47891
0.16	0.126224	−1.69994
0.18	0.086711	−1.96997
0.20	0.045002	−2.07747
0.22	0.000000	−2.25010

QUESTIONS

1. Use a graphing utility to sketch a scatter plot of the positions of the falling ball. What type of model seems to be the best fit? Use the regression features of the graphing utility to find the best-fitting model.
2. Repeat the procedure in Question 1 for the velocities of the falling ball. Describe any relationships between the two models.
3. In theory, the position of a free-falling object in a vacuum is given by $s = \frac{1}{2}gt^2 + v_0^t + s_0$, where g is the acceleration due to gravity (meters per second per second), t is the time (seconds), v_0 is the initial velocity (meters per second), and s_0 is the initial height (meters). From this experiment, estimate the value of g. Do you think your estimate is too great or too small? Explain your reasoning.

The concepts presented here will be explored further in this chapter. For an extension of this application, see Lab 3 in the lab series that accompanies this text at college.hmco.com.

Differentiation

2

erial cinematographers nust have a thorough nderstanding of gravity's ffect on a falling object n order to control the amera mounted on their elmets.

Bruno Brokkens/Allsport

Excerpted from "Into the Stratosphere: Skysurfing Over Mission Bay" from wildca.com

Who would dare jump out of a plane with a bulky, 75-pound IMAX camera strapped to their chest? The answer turned out to be Joe Jennings, who is not only a skysurfer but also an innovative aerial cinematographer in his own right. Jennings designed a special harness to hold the camera, as well as a massive wing-suit—with fabric spanning from his knees to his wrists—to slow his rate of descent.

The pitfalls were enormous. Explains Krenzien: (Mark Krenzien, writer/producer) "One of the major problems is how do you balance the fall-rate of a photographer with the fall-rate of the surfer. Obviously, they have to be at fairly close levels to one another in the sky. In this case, Joe's winged suit and extraordinary skill made the difference."

ne work of Joe Jennings, a renowned aerial cameraman, can be en in many films, television shows, and commercials.

Section 2.1

The Derivative and the Tangent Line Problem

- Find the slope of the tangent line to a curve at a point.
- Use the limit definition to find the derivative of a function.
- Understand the relationship between differentiability and continuity.

Mary Evans Picture Library

Isaac Newton (1642–1727)

In addition to his work in calculus, Newton made revolutionary contributions to physics, including the Universal Law of Gravitation and his three laws of motion.

The Tangent Line Problem

Calculus grew out of four major problems that European mathematicians were wor ing on during the seventeenth century.

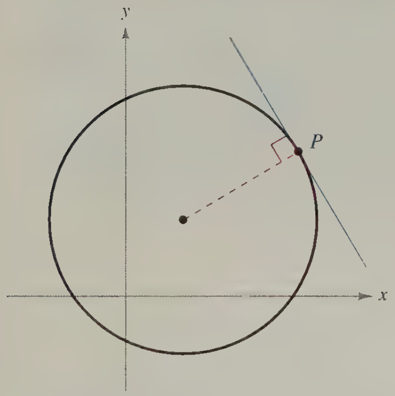
1. The tangent line problem (Section 1.1 and this section)
2. The velocity and acceleration problem (Sections 2.2 and 2.3)
3. The minimum and maximum problem (Section 3.1)
4. The area problem (Sections 1.1 and 4.2)

Each problem involves the notion of a limit, and we could introduce calculus with a of the four problems.

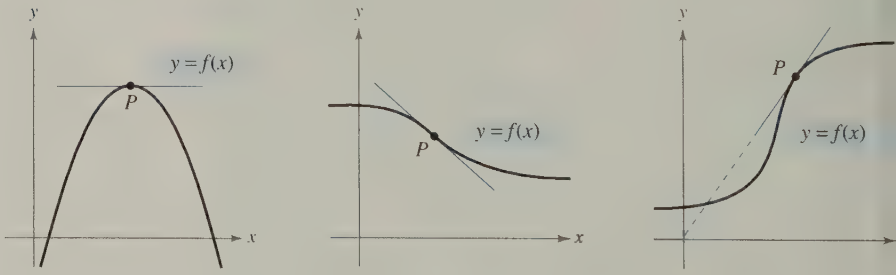
We gave a brief introduction to the tangent line problem in Section 1.1. Althou partial solutions to this problem were given by Pierre de Fermat (1601–166 René Descartes (1596–1650), Christian Huygens (1629–1695), and Isaac Barr (1630–1677), credit for the first general solution is usually given to Isaac Newt (1642–1727) and Gottfried Leibniz (1646–1716). Newton's work on this proble stemmed from his interest in optics and light refraction.

What does it mean to say that a line is tangent to a curve at a point? For a circ the tangent line at a point P is the line that is perpendicular to the radial line at po P, as shown in Figure 2.1.

For a general curve, however, the problem is more difficult. For example, ho would you define the tangent lines shown in Figure 2.2? You might say that a line tangent to a curve at a point P if it touches, but does not cross, the curve at point This definition would work for the first curve shown in Figure 2.2, but not for t second. *Or* you might say that a line is tangent to a curve if the line touches or int sects the curve at exactly one point. This definition would work for a circle but not f more general curves, as the third curve in Figure 2.2 shows.



Tangent line to a circle
Figure 2.1



Tangent line to a curve at a point
Figure 2.2

FOR FURTHER INFORMATION For more information on the crediting of mathematical discoveries to the first "discoverer," see the article "Mathematical Firsts—Who Done It?" by Richard H. Williams and Roy D. Mazzagatti in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

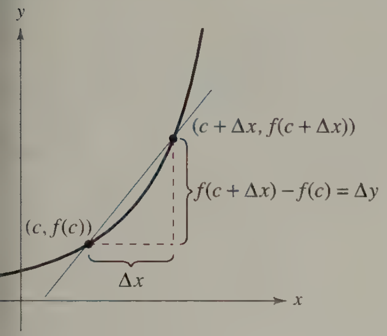
EXPLORATION

Identifying a Tangent Line Use a graphing utility to sketch the graph of $f(x) = 2x^3 - 4x^2 + 3x - 5$. On the same screen, sketch the graphs of $y = x - 5$, $y = 2x - 5$, and $y = 3x - 5$. Which of these lines, if any, appears to be tangent to the graph of f at the point $(0, -5)$? Explain your reasoning.

THE TANGENT LINE PROBLEM

In 1637, mathematician René Descartes stated this about the tangent line problem:

"And I dare say that this is not only the most useful and general problem in geometry that I know, but even that I ever desire to know."



he secant line through $(c, f(c))$ and
$+ \Delta x, f(c + \Delta x))$
igure 2.3

Essentially, the problem of finding the tangent line at a point P boils down to the problem of finding the *slope* of the tangent line at point P. You can approximate this slope using a **secant line*** through the point of tangency and a second point on the curve, as shown in Figure 2.3. If $(c, f(c))$ is the point of tangency and $(c + \Delta x, f(c + \Delta x))$ is a second point on the graph of f, the slope of the secant line through the two points is given by substitution into the slope formula

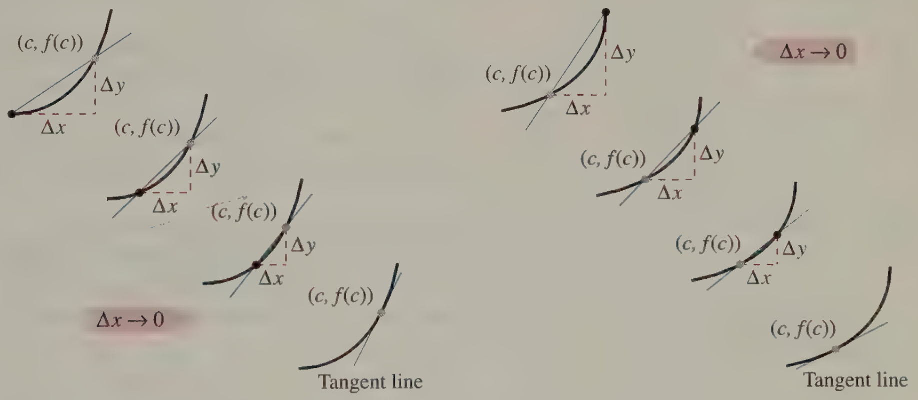
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c} \qquad \frac{\text{Change in } y}{\text{Change in } x}$$

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{\Delta x}. \qquad \text{Slope of secant line}$$

The right-hand side of this equation is a **difference quotient.** The denominator Δx is the **change in x,** and the numerator $\Delta y = f(c + \Delta x) - f(c)$ is the **change in y.**

The beauty of this procedure is that you can obtain more and more accurate approximations to the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in Figure 2.4.



Tangent line approximations
Figure 2.4

Definition of Tangent Line with Slope m

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the tangent line to the graph of f at the point $(c, f(c))$.

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the **slope of the graph of f at $x = c$.**

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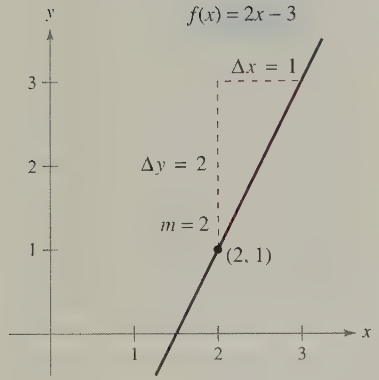
* *This use of the word* secant *comes from the Latin* secare, *meaning to cut, and is not a reference to the trigonometric function of the same name.*

Example 1 The Slope of the Graph of a Linear Function

Find the slope of the graph of

$$f(x) = 2x - 3$$

at the point (2, 1).



The slope of f at (2, 1) is $m = 2$.
Figure 2.5

Solution To find the slope of the graph of f when $c = 2$, you can apply the defin tion of the slope of a tangent line, as follows.

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{f(2+\Delta x)-f(2)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{[2(2+\Delta x)-3]-[2(2)-3]}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{4+2\Delta x-3-4+3}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{2\cancel{\Delta x}}{\cancel{\Delta x}}\\ &= \lim_{\Delta x\to 0} 2\\ &= 2\end{aligned}$$

The slope of f at $(c, f(c)) = (2, 1)$ is $m = 2$, as shown in Figure 2.5.

NOTE In Example 1, the limit definition of the slope of f agrees with the definition of t slope of a line as discussed in Section P.2.

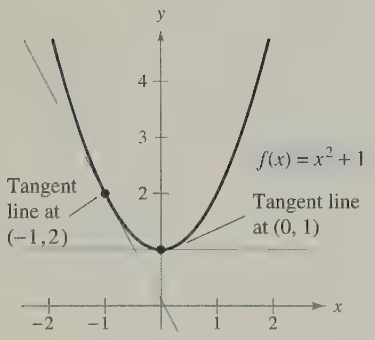
The graph of a linear function has the same slope at any point. This is not true nonlinear functions, as can be seen in the following example.

Example 2 Tangent Lines to the Graph of a Nonlinear Function

Find the slopes of the tangent lines to the graph of

$$f(x) = x^2 + 1$$

at the points (0, 1) and (−1, 2), as shown in Figure 2.6.



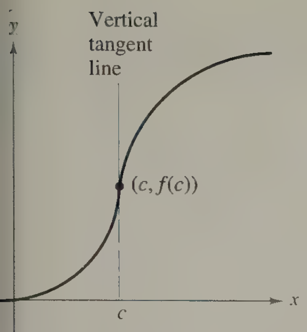
The slope of f at any point $(c, f(c))$ is $m = 2c$.
Figure 2.6

Solution Let $(c, f(c))$ represent an arbitrary point on the graph of f. Then the slo of the tangent line at $(c, f(c))$ is given by

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{f(c+\Delta x)-f(c)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{[(c+\Delta x)^2+1]-(c^2+1)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{c^2+2c(\Delta x)+(\Delta x)^2+1-c^2-1}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{2c(\Delta x)+(\Delta x)^2}{\Delta x}\\ &= \lim_{\Delta x\to 0}(2c+\Delta x)\\ &= 2c.\end{aligned}$$

So, the slope at *any* point $(c, f(c))$ on the graph of f is $m = 2c$. At the point (0, 1), t slope is $m = 2(0) = 0$, and at (−1, 2), the slope is $m = 2(-1) = -2$.

NOTE In Example 2, note that c is held constant in the limit process (as $\Delta x \to 0$).



e graph of f has a vertical tangent line at
$f(c))$.
gure 2.7

The definition of a tangent line to a curve does not cover the possibility of a vertical tangent line. For vertical tangent lines, you can use the following definition. If f is continuous at c and

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = -\infty$$

the vertical line $x = c$ passing through $(c, f(c))$ is a **vertical tangent line** to the graph of f. For example, the function shown in Figure 2.7 has a vertical tangent line at $(c, f(c))$. If the domain of f is the closed interval $[a, b]$, you can extend the definition of a vertical tangent line to include the endpoints by considering continuity and limits from the right (for $x = a$) and from the left (for $x = b$).

The Derivative of a Function

You have now arrived at a crucial point in the study of calculus. The limit used to define the slope of a tangent line is also used to define one of the two fundamental operations of calculus—**differentiation.**

Definition of the Derivative of a Function

The **derivative** of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

Be sure you see that the derivative of a function of x is also a function of x. This "new" function gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, provided that the graph has a tangent line at this point.

The process of finding the derivative of a function is called **differentiation.** A function is **differentiable** at x if its derivative exists at x and **differentiable on an open interval (a, b)** if it is differentiable at every point in the interval.

In addition to $f'(x)$, which is read as "f prime of x," other notations are used to denote the derivative of $y = f(x)$. The most common are

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y].$$

Notation for derivatives

The notation dy/dx is read as "the derivative of y *with respect to* x." Using limit notation, you can write

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= f'(x). \end{aligned}$$

Example 3 Finding the Derivative by the Limit Process

Find the derivative of $f(x) = x^3 + 2x$.

Solution

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \text{Definition of derivative} \\ &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 + 2(x+\Delta x) - (x^3 + 2x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}[3x^2 + 3x\Delta x + (\Delta x)^2 + 2]}{\cancel{\Delta x}} \\ &= \lim_{\Delta x \to 0} [3x^2 + 3x\Delta x + (\Delta x)^2 + 2] \\ &= 3x^2 + 2 \end{aligned}$$

STUDY TIP The key to finding the derivative of a function is to rewrite the difference quotient so that Δx does not occur as a factor of the denominator.

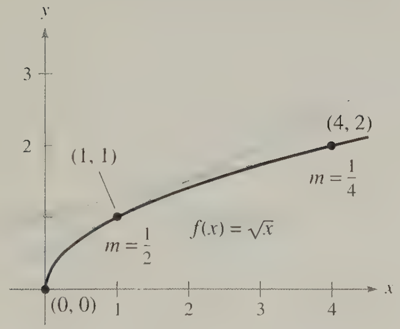
Remember that the derivative of a function f is itself a function, which can be u to find the slope of the tangent line at the point $(x, f(x))$ on the graph of f.

Example 4 Using the Derivative to Find the Slope at a Point

Find $f'(x)$ for $f(x) = \sqrt{x}$. Then find the slope of the graph of f at the points $(1, 1)$ a $(4, 2)$. Discuss the behavior of f at $(0, 0)$.

Solution Use the procedure for rationalizing numerators, as discussed in Section

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \text{Definition of derivative} \\ &= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left(\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}\right)\left(\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}\right) \\ &= \lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)} \\ &= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x}\left(\sqrt{x+\Delta x} + \sqrt{x}\right)} \\ &= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$



The slope of f at $(x, f(x))$, $x > 0$, is $m = 1/(2\sqrt{x})$.
Figure 2.8

At the point $(1, 1)$, the slope is $f'(1) = \frac{1}{2}$. At the point $(4, 2)$, the slope is $f'(4) =$ (See Figure 2.8.) At the point $(0, 0)$ the slope is undefined. Moreover, because limit of $f'(x)$ as $x \to 0$ from the right is infinite, the graph of f has a vertical tang line at $(0, 0)$.

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indicates that in the Interactive 3.0 *CD-ROM and* Internet 3.0 *versions of this (available at* college.hmco.com) *you will find an Open Exploration, which further explores example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

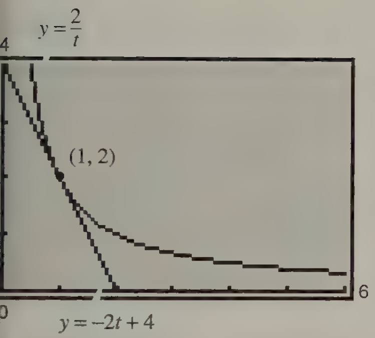
In many applications, it is convenient to use a variable other than x as the independent variable, as shown in Example 5.

Example 5 Finding the Derivative of a Function

Find the derivative with respect to t for the function $y = 2/t$.

Solution Considering $y = f(t)$, you obtain the following.

$$\begin{aligned}
\frac{dy}{dt} &= \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} && \text{Definition of derivative} \\
&= \lim_{\Delta t \to 0} \frac{\dfrac{2}{t+\Delta t} - \dfrac{2}{t}}{\Delta t} && f(t+\Delta t) = 2/(t+\Delta t) \text{ and } f(t) = 2/t \\
&= \lim_{\Delta t \to 0} \frac{\dfrac{2t - 2(t+\Delta t)}{t(t+\Delta t)}}{\Delta t} && \text{Combine fractions in numerator.} \\
&= \lim_{\Delta t \to 0} \frac{-2\cancel{\Delta t}}{\cancel{\Delta t}(t)(t+\Delta t)} && \text{Divide out common factor of } \Delta t. \\
&= \lim_{\Delta t \to 0} \frac{-2}{t(t+\Delta t)} && \text{Simplify.} \\
&= -\frac{2}{t^2} && \text{Evaluate limit as } \Delta t \to 0.
\end{aligned}$$



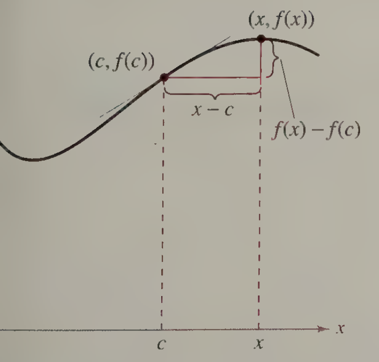
the point $(1, 2)$ the line $y = -2t + 4$ is
ngent to the graph of $y = 2/t$.
gure 2.9

TECHNOLOGY A graphing utility can be used to reinforce the result given in Example 5. For instance, using the formula $dy/dt = -2/t^2$, you know that the slope of the graph of $y = 2/t$ at the point $(1, 2)$ is $m = -2$. This implies that an equation of the tangent line to the graph at $(1, 2)$ is $y - 2 = -2(t - 1)$ or $y = -2t + 4$, as shown in Figure 2.9.

Differentiability and Continuity

The following alternative limit form of the derivative is useful in investigating the relationship between differentiability and continuity. The derivative of f at c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \text{Alternative form of derivative}$$



x approaches c, the secant line approaches
tangent line.
gure 2.10

provided this limit exists (see Figure 2.10). (A proof of the equivalence of this form is given in Appendix B.) Note that the existence of the limit in this alternative form requires that the one-sided limits

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

exist and are equal. These one-sided limits are called the **derivatives from the left and from the right,** respectively. We say that f is **differentiable on the closed interval $[a, b]$** if it is differentiable on (a, b) and if the derivative from the right at a and the derivative from the left at b both exist.

If a function is not continuous at $x = c$, it is also not differentiable at $x = c$. instance, the greatest integer function

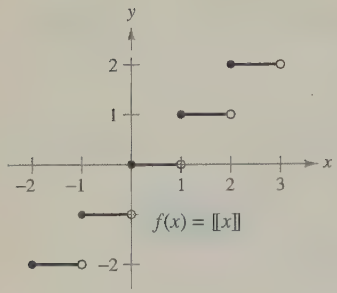
$$f(x) = [\![x]\!]$$

is not continuous at $x = 0$—hence, it is not differentiable at $x = 0$ (see Figure 2.1 You can verify this by observing that

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{[\![x]\!] - 0}{x} = \infty \qquad \text{Derivative from the left}$$

and

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{[\![x]\!] - 0}{x} = 0. \qquad \text{Derivative from the right}$$



The greatest integer function is not differentiable at $x = 0$, because it is not continuous at $x = 0$.
Figure 2.11

Although it is true that differentiability implies continuity (as we will show Theorem 2.1), the converse is not true. That is, it is possible for a function to continuous at $x = c$ and *not* differentiable at $x = c$. Examples 6 and 7 illustrate t possibility.

Example 6 A Graph with a Sharp Turn

The function

$$f(x) = |x - 2|$$

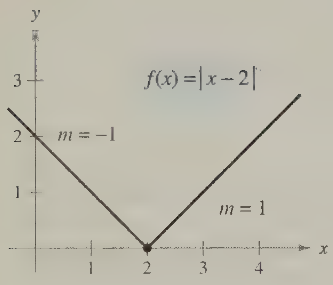
shown in Figure 2.12 is continuous at $x = 2$. However, the one-sided limits

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{|x - 2| - 0}{x - 2} = -1 \qquad \text{Derivative from the left}$$

and

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{|x - 2| - 0}{x - 2} = 1 \qquad \text{Derivative from the right}$$

are not equal. So, f is not differentiable at $x = 2$ and the graph of f does not have tangent line at the point $(2, 0)$.



f is not differentiable at $x = 2$, because the derivatives from the left and from the right are not equal.
Figure 2.12

Example 7 A Graph with a Vertical Tangent Line

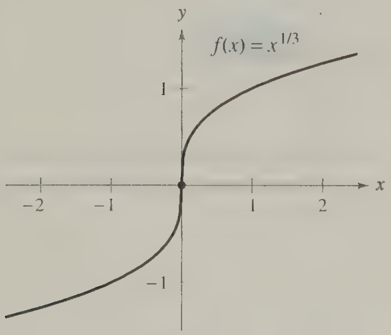
The function

$$f(x) = x^{1/3}$$

is continuous at $x = 0$, as shown in Figure 2.13. However, because the limit

$$\begin{aligned}\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0} \frac{x^{1/3} - 0}{x} \\ &= \lim_{x \to 0} \frac{1}{x^{2/3}} \\ &= \infty\end{aligned}$$

is infinite, you can conclude that the tangent line is vertical at $x = 0$. So, f is differentiable at $x = 0$.



f is not differentiable at $x = 0$, because f has a vertical tangent at $x = 0$.
Figure 2.13

From Examples 6 and 7, you can see that a function is not differentiable at a p at which its graph has a sharp turn *or* a vertical tangent.

TECHNOLOGY Some graphing utilities, such as *Derive, Maple, Mathcad, Mathematica,* and the *TI-89*, perform symbolic differentiation. Others perform *numerical differentiation* by finding values of derivatives using the formula

$$f'(x) \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$

where Δx is a small number such as 0.001. Can you see any problems with this definition? For instance, using this definition, what is the value of the derivative of $f(x) = |x|$ when $x = 0$?

THEOREM 2.1 Differentiability Implies Continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

Proof You can prove that f is continuous at $x = c$ by showing that $f(x)$ approaches $f(c)$ as $x \to c$. To do this, use the differentiability of f at $x = c$ and consider the following limit.

$$\begin{aligned}\lim_{x\to c}[f(x) - f(c)] &= \lim_{x\to c}\left[(x-c)\left(\frac{f(x)-f(c)}{x-c}\right)\right]\\ &= \left[\lim_{x\to c}(x-c)\right]\left[\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right]\\ &= (0)[f'(c)]\\ &= 0\end{aligned}$$

Because the difference $f(x) - f(c)$ approaches zero as $x \to c$, you can conclude that $\lim_{x\to c} f(x) = f(c)$. So, f is continuous at $x = c$.

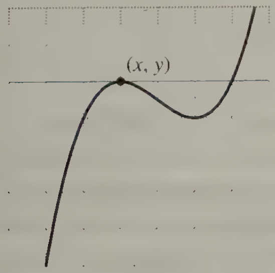
You can summarize the relationship between continuity and differentiability as follows.

1. If a function is differentiable at $x = c$, then it is continuous at $x = c$. So, differentiability implies continuity.
2. It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, continuity does not imply differentiability.

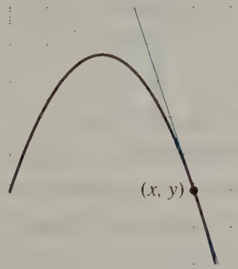
EXERCISES FOR SECTION 2.1

n Exercises 1 and 2, estimate the slope of the graph at the point *x*, *y*).

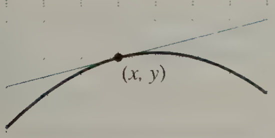
. (a)



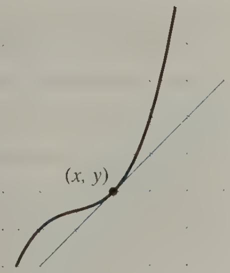
(b)



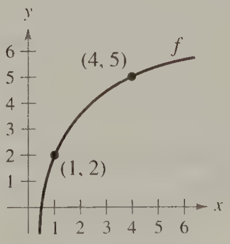
. (a)



(b)



In Exercises 3 and 4, use the graph shown in the figure. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



3. Identify or sketch each of the quantities on the figure.

(a) $f(1)$ and $f(4)$ (b) $f(4) - f(1)$

(c) $y = \dfrac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$

4. Insert the proper inequality symbol ($<$ or $>$) between the given quantities.

(a) $\dfrac{f(4) - f(1)}{4 - 1} \quad \dfrac{f(4) - f(3)}{4 - 3}$

(b) $\dfrac{f(4) - f(1)}{4 - 1} \quad f'(1)$

In Exercises 5–10, find the slope of the tangent line to the graph of the function at the specified point.

5. $f(x) = 3 - 2x, \quad (-1, 5)$

6. $g(x) = \frac{3}{2}x + 1, \quad (-2, -2)$

7. $g(x) = x^2 - 4, \quad (1, -3)$

8. $g(x) = 5 - x^2, \quad (2, 1)$

9. $f(t) = 3t - t^2, \quad (0, 0)$

10. $h(t) = t^2 + 3, \quad (-2, 7)$

In Exercises 11–24, find the derivative by the limit process.

11. $f(x) = 3$

12. $g(x) = -5$

13. $f(x) = -5x$

14. $f(x) = 3x + 2$

15. $h(s) = 3 + \frac{2}{3}s$

16. $f(x) = 9 - \frac{1}{2}x$

17. $f(x) = 2x^2 + x - 1$

18. $f(x) = 1 - x^2$

19. $f(x) = x^3 - 12x$

20. $f(x) = x^3 + x^2$

21. $f(x) = \dfrac{1}{x - 1}$

22. $f(x) = \dfrac{1}{x^2}$

23. $f(x) = \sqrt{x + 1}$

24. $f(x) = \dfrac{4}{\sqrt{x}}$

In Exercises 25–32, (a) find an equation of the tangent line to the graph of f at the indicated point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

25. $f(x) = x^2 + 1, \quad (2, 5)$

26. $f(x) = x^2 + 2x + 1, \quad (-3, 4)$

27. $f(x) = x^3, \quad (2, 8)$

28. $f(x) = x^3 + 1, \quad (1, 2)$

29. $f(x) = \sqrt{x}, \quad (1, 1)$

30. $f(x) = \sqrt{x - 1}, \quad (5, 2)$

31. $f(x) = x + \dfrac{4}{x}, \quad (4, 5)$

32. $f(x) = \dfrac{1}{x + 1}, \quad (0, 1)$

In Exercises 33–36, find an equation of the line that is tangent to the graph of f *and* parallel to the given line.

	Function	*Line*
33.	$f(x) = x^3$	$3x - y + 1 = 0$
34.	$f(x) = x^3 + 2$	$3x - y - 4 = 0$
35.	$f(x) = \dfrac{1}{\sqrt{x}}$	$x + 2y - 6 = 0$
36.	$f(x) = \dfrac{1}{\sqrt{x - 1}}$	$x + 2y + 7 = 0$

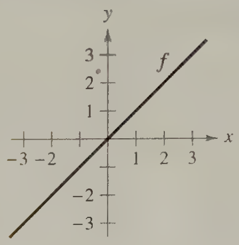
37. The tangent line to the graph of $y = g(x)$ at the point $(5, 2)$ passes through the point $(9, 0)$. Find $g(5)$ and $g'(5)$.

38. The tangent line to the graph of $y = h(x)$ at the point $(-1, 4)$ passes through the point $(3, 6)$. Find $h(-1)$ and $h'(-1)$.

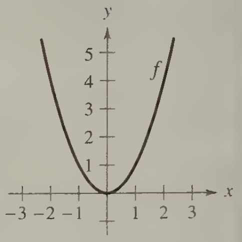
Getting at the Concept

In Exercises 39–42, the graph of f is given. Select the grap of f'.

39.



40.



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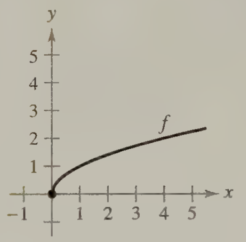
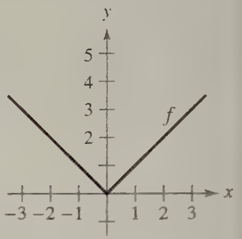
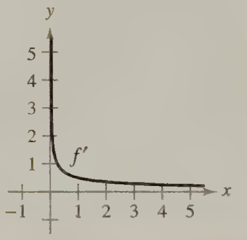


42.



(a)



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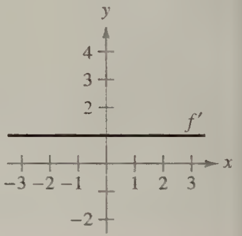
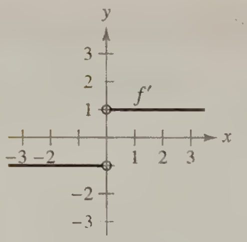
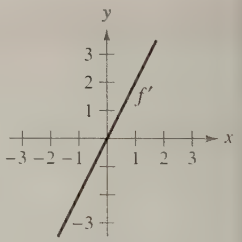


(c)



(d)



43. Sketch a graph of a function whose derivative is alway negative.

44. Sketch a graph of a function whose derivative is alway positive.

45. Assume that $f'(c) = 3$. Find $f'(-c)$ if (a) f is an odd func tion and if (b) f is an even function.

46. Determine whether the limit yields the derivative of differentiable function f. Explain.

(a) $\displaystyle\lim_{\Delta x \to 0} \frac{f(x + 2\Delta x) - f(x)}{2\Delta x}$

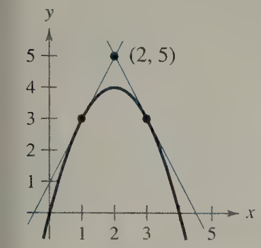
(b) $\displaystyle\lim_{\Delta x \to 0} \frac{f(x + 2) - f(x)}{\Delta x}$

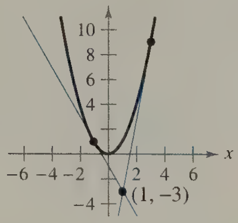
(c) $\displaystyle\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$

(d) $\displaystyle\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

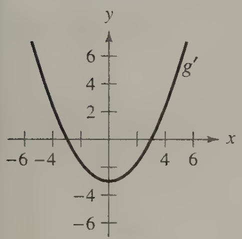
Exercises 47 and 48, find equations of the two tangent lines the graph of f that pass through the indicated point.

. $f(x) = 4x - x^2$ **48.** $f(x) = x^2$





. *Graphical Reasoning* The figure shows the graph of g'.



(a) $g'(0) =$

(b) $g'(3) =$

(c) What can you conclude about the graph of g knowing that $g'(1) = -\frac{8}{3}$?

(d) What can you conclude about the graph of g knowing that $g'(-4) = \frac{7}{3}$?

(e) Is $g(6) - g(4)$ positive or negative? Explain.

(f) Is it possible to find $g(2)$ from the graph? Explain.

. *Graphical Reasoning* Use a graphing utility to graph each function and its tangent lines when $x = -1$, $x = 0$, and $x = 1$. Based on the results, determine whether the slope of a tangent line to the graph of a function is always distinct for different values of x.

(a) $f(x) = x^2$ (b) $g(x) = x^3$

raphical, Numerical, and Analytic Analysis **In Exercises 51 nd 52, use a graphing utility to graph f on the interval $[-2, 2]$. omplete the table by graphically estimating the slopes of the aph at the indicated points. Then evaluate the slopes analyti- lly and compare your results with those obtained graphically.**

x	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
f(x)									
f′(x)									

. $f(x) = \frac{1}{4}x^3$ **52.** $f(x) = \frac{1}{2}x^2$

Graphical Reasoning **In Exercises 53 and 54, use a graphing utility to graph the functions f and g in the same viewing window where**

$$g(x) = \frac{f(x + 0.01) - f(x)}{0.01}.$$

Label the graphs and describe the relationship between them.

53. $f(x) = 2x - x^2$ **54.** $f(x) = 3\sqrt{x}$

In Exercises 55 and 56, evaluate $f(2)$ and $f(2.1)$ and use the results to approximate $f'(2)$.

55. $f(x) = x(4 - x)$ **56.** $f(x) = \frac{1}{4}x^3$

Graphical Reasoning **In Exercises 57 and 58, use a graphing utility to graph the function and its derivative in the same viewing window. Label the graphs and describe the relationship between them.**

57. $f(x) = \dfrac{1}{\sqrt{x}}$ **58.** $f(x) = \dfrac{x^3}{4} - 3x$

Writing **In Exercises 59 and 60, consider the functions f and $S_{\Delta x}$ where**

$$S_{\Delta x}(x) = \frac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2).$$

(a) Use a graphing utility to graph f and $S_{\Delta x}$ in the same viewing window for $\Delta x = 1$, 0.5, and 0.1.

(b) Give a written description of the graphs of S for the different values of Δx in part (a).

59. $f(x) = 4 - (x - 3)^2$ **60.** $f(x) = x + \dfrac{1}{x}$

In Exercises 61–70, use the alternative form of the derivative to find the derivative at $x = c$ (if it exists).

61. $f(x) = x^2 - 1, \quad c = 2$ **62.** $g(x) = x(x - 1), \quad c = 1$

63. $f(x) = x^3 + 2x^2 + 1, \quad c = -2$

64. $f(x) = x^3 + 2x, \quad c = 1$ **65.** $g(x) = \sqrt{|x|}, \quad c = 0$

66. $f(x) = 1/x, \quad c = 3$ **67.** $f(x) = (x - 6)^{2/3}, \quad c = 6$

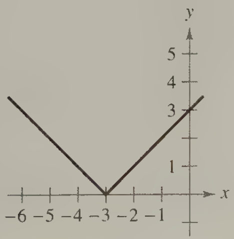
68. $g(x) = (x + 3)^{1/3}, \quad c = -3$

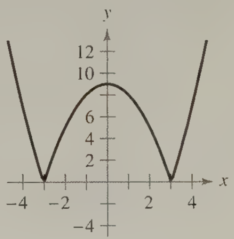
69. $h(x) = |x + 5|, \quad c = -5$

70. $f(x) = |x - 4|, \quad c = 4$

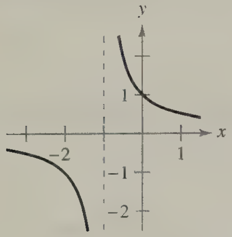
In Exercises 71–80, describe the x-values at which f is differentiable.

71. $f(x) = |x + 3|$ **72.** $f(x) = |x^2 - 9|$

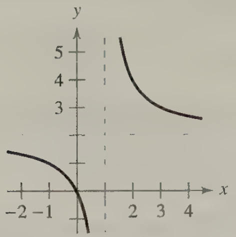




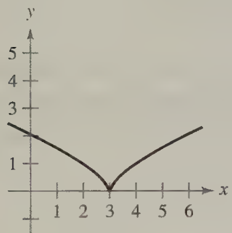
73. $f(x) = \dfrac{1}{x+1}$



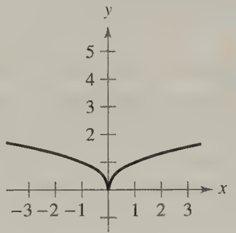
74. $f(x) = \dfrac{2x}{x-1}$



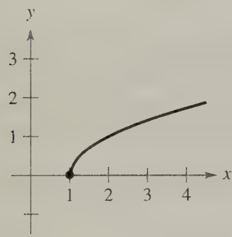
75. $f(x) = (x-3)^{2/3}$



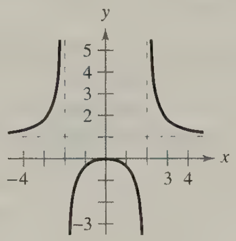
76. $f(x) = x^{2/5}$



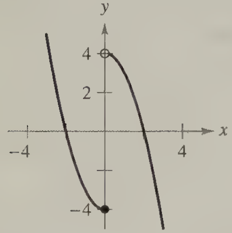
77. $f(x) = \sqrt{x-1}$



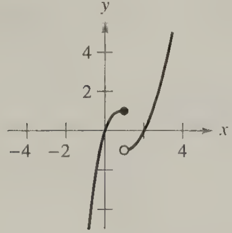
78. $f(x) = \dfrac{x^2}{x^2-4}$



79. $f(x) = \begin{cases} 4 - x^2, & x > 0 \\ x^2 - 4, & x \le 0 \end{cases}$



80. $f(x) = \begin{cases} x^2 - 2x, & x > 1 \\ x^3 - 3x^2 + 3x, & x \le 1 \end{cases}$



In Exercises 81–84, find the derivatives from the left and fr the right at $x = 1$ (if they exist). Is the function differentiabl $x = 1$?

81. $f(x) = |x - 1|$

82. $f(x) = \sqrt{1 - x^2}$

83. $f(x) = \begin{cases} (x-1)^3, & x \le 1 \\ (x-1)^2, & x > 1 \end{cases}$

84. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

In Exercises 85 and 86, determine whether the function differentiable at $x = 2$.

85. $f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 4x - 3, & x > 2 \end{cases}$

86. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x \\ \sqrt{2x}, & x \end{cases}$

87. ***Graphical Reasoning*** A line with slope m passes through point $(0, 4)$ and has the equation $y = mx + 4$.

(a) Write the distance d between the line and the point (3, 1 a function of m.

(b) Use a graphing utility to graph the function d in part Based on the graph, is the function differentiable at ev value of m? If not, where is it not differentiable?

88. ***Conjecture*** Consider the functions $f(x) = x^2$ and $g(x) =$

(a) Graph f and f' on the same set of axes.

(b) Graph g and g' on the same set of axes.

(c) Identify any pattern between the functions f and g and t respective derivatives. Use the pattern to make a conject about $h'(x)$ if $h(x) = x^n$, where n is an integer and $n \ge$

(d) Find $f'(x)$ if $f(x) = x^4$. Compare the result with the con ture in part (c). Is this a proof of your conjecture? Expl

True or False? **In Exercises 89–92, determine whether statement is true or false. If it is false, explain why or give example that shows it is false.**

89. The slope of the tangent line to the differentiable function the point $(2, f(2))$ is

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

90. If a function is continuous at a point, then it is differentiabl that point.

91. If a function has derivatives from both the right and the le a point, then it is differentiable at that point.

92. If a function is differentiable at a point, then it is continuou that point.

93. Let $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ and $g(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq \\ 0, & x = \end{cases}$

Show that f is continuous, but not differentiable, at x Show that g is differentiable at 0, and find $g'(0)$.

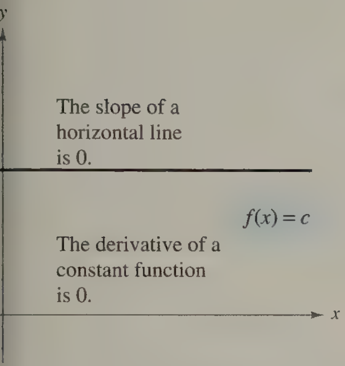
94. ***Writing*** Use a graphing utility to graph the two funct $f(x) = x^2 + 1$ and $g(x) = |x| + 1$ in the same viewing win Use the *zoom* and *trace* features to analyze the graphs nea point $(0, 1)$. What do you observe? Which functio differentiable at this point? Write a short paragraph descri the geometric significance of differentiability at a point.

Section 2.2 Basic Differentiation Rules and Rates of Change

- Find the derivative of a function using the Constant Rule.
- Find the derivative of a function using the Power Rule.
- Find the derivative of a function using the Constant Multiple Rule.
- Find the derivative of a function using the Sum and Difference Rules.
- Find the derivative of the sine function and of the cosine function.
- Use derivatives to find rates of change.

The Constant Rule

In Section 2.1 you used the limit definition to find derivatives. In this and the next two sections you will be introduced to several "differentiation rules" that allow you to find derivatives without the *direct* use of the limit definition.



he Constant Rule
igure 2.14

THEOREM 2.2 The Constant Rule

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0.$$

Proof Let $f(x) = c$. Then, by the limit definition of the derivative,

$$\begin{aligned}\frac{d}{dx}[c] &= f'(x)\\ &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{c-c}{\Delta x}\\ &= 0.\end{aligned}$$

OTE In Figure 2.14, note that the onstant Rule is equivalent to saying at the slope of a horizontal line is 0. his demonstrates the relationship etween slope and derivative.

Example 1 Using the Constant Rule

Function	*Derivative*
a. $y = 7$	$\dfrac{dy}{dx} = 0$
b. $f(x) = 0$	$f'(x) = 0$
c. $s(t) = -3$	$s'(t) = 0$
d. $y = k\pi^2$, k is constant	$y' = 0$

EXPLORATION

Writing a Conjecture Use the definition of the derivative given in Section 2.1 to find the derivative of each of the following. What patterns do you see? Use your results to write a conjecture about the derivative of $f(x) = x^n$.

a. $f(x) = x^1$ **b.** $f(x) = x^2$ **c.** $f(x) = x^3$
d. $f(x) = x^4$ **e.** $f(x) = x^{1/2}$ **f.** $f(x) = x^{-1}$

The Power Rule

Before proving the next rule, we review the procedure for expanding a binomial.

$$(x+\Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$
$$(x+\Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

The general binomial expansion for a positive integer n is

$$(x+\Delta x)^n = x^n + nx^{n-1}(\Delta x) + \underbrace{\frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdot\cdot\cdot + (\Delta x)^n}_{(\Delta x)^2 \text{ is a factor of these terms.}}.$$

This binomial expansion is used in proving a special case of the Power Rule.

THEOREM 2.3 The Power Rule

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

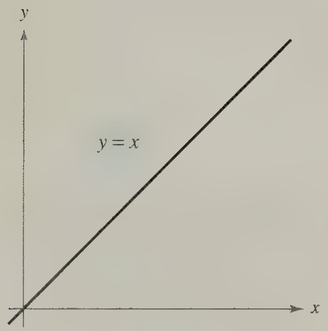
$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0.

Proof If n is a positive integer greater than 1, then the binomial expansion produ the following.

$$\begin{aligned}\frac{d}{dx}[x^n] &= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^n - x^n}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{x^n + nx^{n-1}(\Delta x) + \frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdot\cdot\cdot + (\Delta x)^n - x^n}{\Delta x}\\ &= \lim_{\Delta x\to 0}\left[nx^{n-1} + \frac{n(n-1)x^{n-2}}{2}(\Delta x) + \cdot\cdot\cdot + (\Delta x)^{n-1}\right]\\ &= nx^{n-1} + 0 + \cdot\cdot\cdot + 0\\ &= nx^{n-1}\end{aligned}$$

This proves the case for which n is a positive integer greater than 1. We leave it to y to prove the case for $n = 1$. Example 7 in Section 2.3 proves the case for which a negative integer. In Exercise 63 in Section 2.5 you are asked to prove the case which n is rational. (In Section 5.5, the Power Rule will be extended to cover irratio values of n.)



The slope of the line $y = x$ is 1.
Figure 2.15

When using the Power Rule, the case for which $n = 1$ is best thought of a separate differentiation rule. That is,

$$\frac{d}{dx}[x] = 1. \qquad \text{Power Rule when } n = 1$$

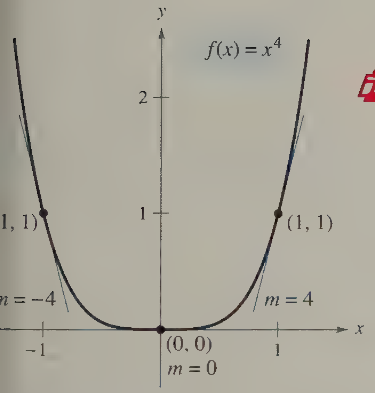
This rule is consistent with the fact that the slope of the line $y = x$ is 1, as show Figure 2.15.

Example 2 Using the Power Rule

Function	*Derivative*
a. $f(x) = x^3$	$f'(x) = 3x^2$
b. $g(x) = \sqrt[3]{x}$	$g'(x) = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$
c. $y = \frac{1}{x^2}$	$\frac{dy}{dx} = \frac{d}{dx}[x^{-2}] = (-2)x^{-3} = -\frac{2}{x^3}$

In Example 2c, note that *before* differentiating, $1/x^2$ was rewritten as x^{-2}. Rewriting is the first step in *many* differentiation problems.

Given: $y = \frac{1}{x^2}$ ⇨ Rewrite: $y = x^{-2}$ ⇨ Differentiate: $\frac{dy}{dx} = (-2)x^{-3}$ ⇨ Simplify: $\frac{dy}{dx} = -\frac{2}{x^3}$



e slope of a graph at a point is the value
the derivative at that point.
gure 2.16

Example 3 Finding the Slope of a Graph

Find the slope of the graph of $f(x) = x^4$ when

a. $x = -1$ **b.** $x = 0$ **c.** $x = 1$.

Solution The derivative of f is $f'(x) = 4x^3$.

a. When $x = -1$, the slope is $f'(-1) = 4(-1)^3 = -4$. Slope is negative.

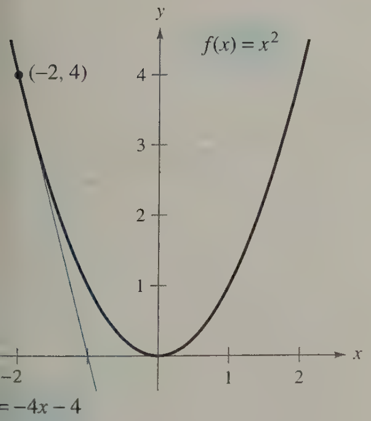
b. When $x = 0$, the slope is $f'(0) = 4(0)^3 = 0$. Slope is zero.

c. When $x = 1$, the slope is $f'(1) = 4(1)^3 = 4$. Slope is positive.

In Figure 2.16, note that the slope of the graph is negative at the point $(-1, 1)$, the slope is zero at the point $(0, 0)$, and the slope is positive at the point $(1, 1)$.

Example 4 Finding an Equation of a Tangent Line

Find an equation of the tangent line to the graph of $f(x) = x^2$ when $x = -2$.



e line $y = -4x - 4$ is tangent to the
ph of $f(x) = x^2$ at the point $(-2, 4)$.
gure 2.17

Solution To find the *point* on the graph of f, evaluate the original function at $x = -2$.

$(-2, f(-2)) = (-2, 4)$ Point on graph

To find the *slope* of the graph when $x = -2$, evaluate the derivative, $f'(x) = 2x$, at $x = -2$.

$m = f'(-2) = -4$ Slope of graph at $(-2, 4)$

Now, using the point-slope form of the equation of a line, you can write

$$
\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Point-slope form}\\
y - 4 &= -4[x - (-2)] && \text{Substitute for } y_1, m, \text{ and } x_1.\\
y &= -4x - 4. && \text{Simplify.}
\end{aligned}
$$

(See Figure 2.17.)

The Constant Multiple Rule

THEOREM 2.4 The Constant Multiple Rule

If f is a differentiable function and c is a real number, then cf is also differentiable and

$$\frac{d}{dx}[cf(x)] = cf'(x).$$

Proof

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x} && \text{Definition of derivative}\\ &= \lim_{\Delta x \to 0} c\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]\\ &= c\left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right]\\ &= cf'(x)\end{aligned}$$

Informally, the Constant Multiple Rule states that constants can be factored of the differentiation process, even if the constants appear in the denominator.

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[\;f(x)] = cf'(x)$$

$$\begin{aligned}\frac{d}{dx}\left[\frac{f(x)}{c}\right] &= \frac{d}{dx}\left[\left(\frac{1}{c}\right)f(x)\right]\\ &= \left(\frac{1}{c}\right)\frac{d}{dx}[\;f(x)] = \left(\frac{1}{c}\right)f'(x)\end{aligned}$$

Example 5 Using the Constant Multiple Rule

Function	*Derivative*
a. $y = \frac{2}{x}$	$\frac{dy}{dx} = \frac{d}{dx}[2x^{-1}] = 2\frac{d}{dx}[x^{-1}] = 2(-1)x^{-2} = -\frac{2}{x^2}$
b. $f(t) = \frac{4t^2}{5}$	$f'(t) = \frac{d}{dt}\left[\frac{4}{5}t^2\right] = \frac{4}{5}\frac{d}{dt}[t^2] = \frac{4}{5}(2t) = \frac{8}{5}t$
c. $y = 2\sqrt{x}$	$\frac{dy}{dx} = \frac{d}{dx}[2x^{1/2}] = 2\left(\frac{1}{2}x^{-1/2}\right) = x^{-1/2} = \frac{1}{\sqrt{x}}$
d. $y = \frac{1}{2\sqrt[3]{x^2}}$	$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}x^{-2/3}\right] = \frac{1}{2}\left(-\frac{2}{3}\right)x^{-5/3} = -\frac{1}{3x^{5/3}}$
e. $y = -\frac{3x}{2}$	$y' = \frac{d}{dx}\left[-\frac{3}{2}x\right] = -\frac{3}{2}(1) = -\frac{3}{2}$

NOTE The Constant Multiple Rule and the Power Rule can be combined into one rule. combination rule is $D_x[cx^n] = cnx^{n-1}$.

Example 6 Using Parentheses When Differentiating

Original Function	*Rewrite*	*Differentiate*	*Simplify*
a. $y = \dfrac{5}{2x^3}$	$y = \dfrac{5}{2}(x^{-3})$	$y' = \dfrac{5}{2}(-3x^{-4})$	$y' = -\dfrac{15}{2x^4}$
b. $y = \dfrac{5}{(2x)^3}$	$y = \dfrac{5}{8}(x^{-3})$	$y' = \dfrac{5}{8}(-3x^{-4})$	$y' = -\dfrac{15}{8x^4}$
c. $y = \dfrac{7}{3x^{-2}}$	$y = \dfrac{7}{3}(x^2)$	$y' = \dfrac{7}{3}(2x)$	$y' = \dfrac{14x}{3}$
d. $y = \dfrac{7}{(3x)^{-2}}$	$y = 63(x^2)$	$y' = 63(2x)$	$y' = 126x$

The Sum and Difference Rules

THEOREM 2.5 The Sum and Difference Rules

The sum (or difference) of two differentiable functions is differentiable and is the sum (or difference) of their derivatives.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \qquad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \qquad \text{Difference Rule}$$

Proof A proof of the Sum Rule follows from Theorem 1.2. (The Difference Rule can be proved in a similar way.)

$$\begin{aligned}
\frac{d}{dx}[f(x) + g(x)] &= \lim_{\Delta x \to 0} \frac{[f(x + \Delta x) + g(x + \Delta x)] - [f(x) + g(x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x)
\end{aligned}$$

The Sum and Difference Rules can be extended to any finite number of functions. For instance, if $F(x) = f(x) + g(x) - h(x)$, then $F'(x) = f'(x) + g'(x) - h'(x)$.

Example 7 Using the Sum and Difference Rules

Function	*Derivative*
a. $f(x) = x^3 - 4x + 5$	$f'(x) = 3x^2 - 4$
b. $g(x) = -\dfrac{x^4}{2} + 3x^3 - 2x$	$g'(x) = -2x^3 + 9x^2 - 2$

Derivatives of Sine and Cosine Functions

In Section 1.3, you studied the following limits.

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0$$

These two limits can be used to prove differentiation rules for the sine and cos functions. (The derivatives of the other four trigonometric functions are discusse Section 2.3.)

FOR FURTHER INFORMATION For the outline of a geometric proof of the derivatives of the sine and cosine functions, see the article "The Spider's Spacewalk Derivation of sin′ and cos′" by Tim Hesterberg in *The College Mathematics Journal.* To view this article, go to the website *www.matharticles.com.*

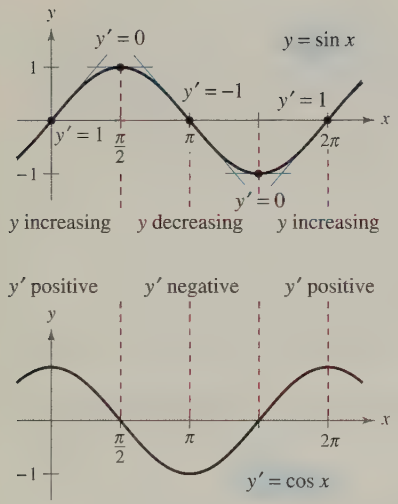
THEOREM 2.6 Derivatives of Sine and Cosine Functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

Proof

$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} && \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cos x \sin \Delta x - (\sin x)(1 - \cos \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[(\cos x)\left(\frac{\sin \Delta x}{\Delta x}\right) - (\sin x)\left(\frac{1 - \cos \Delta x}{\Delta x}\right) \right] \\
&= \cos x \left(\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \right) - \sin x \left(\lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} \right) \\
&= (\cos x)(1) - (\sin x)(0) \\
&= \cos x
\end{aligned}$$

This differentiation rule is shown graphically in Figure 2.18. Note that for each x, *slope* of the sine curve is equal to the value of the cosine. The proof of the second is left as an exercise (see Exercise 113).



The derivative of the sine function is the cosine function.

Figure 2.18

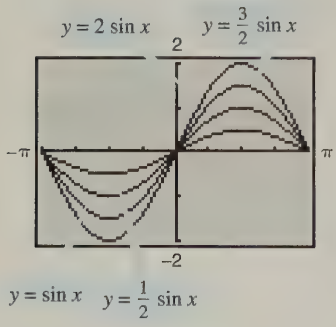
Example 8 Derivatives Involving Sines and Cosines

Function	*Derivative*
a. $y = 2 \sin x$	$y' = 2 \cos x$
b. $y = \dfrac{\sin x}{2} = \dfrac{1}{2} \sin x$	$y' = \dfrac{1}{2} \cos x = \dfrac{\cos x}{2}$
c. $y = x + \cos x$	$y' = 1 - \sin x$

TECHNOLOGY A graphing utility can provide insight into the interpretation derivative. For instance, Figure 2.19 shows the graphs of

$$y = a \sin x$$

for $a = \frac{1}{2}, 1, \frac{3}{2}$, and 2. Estimate the slope of each graph at the point $(0, 0)$. T verify your estimates analytically by evaluating the derivative of each func when $x = 0$.



$\frac{d}{dx}[a \sin x] = a \cos x$

Figure 2.19

Rates of Change

You have seen how the derivative is used to determine slope. The derivative can also be used to determine the rate of change of one variable with respect to another. Applications involving rates of change occur in a wide variety of fields. A few examples are population growth rates, production rates, water flow rates, velocity, and acceleration.

A common use of rate of change is to describe the motion of an object moving in a straight line. In such problems, it is customary to use either a horizontal or a vertical line with a designated origin to represent the line of motion. On such lines, movement to the right (or upward) is considered to be in the positive direction, and movement to the left (or downward) is considered to be in the negative direction.

The function s that gives the position (relative to the origin) of an object as a function of time t is called a **position function.** If, over a period of time Δt, the object changes its position by the amount $\Delta s = s(t + \Delta t) - s(t)$, then, by the familiar formula

$$\text{Rate} = \frac{\text{distance}}{\text{time}}$$

the **average velocity** is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t}.$$

Average velocity

Example 9 Finding Average Velocity of a Falling Object

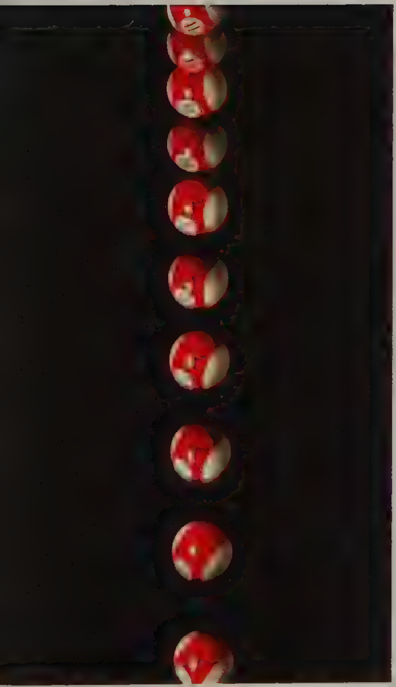
If a billiard ball is dropped from a height of 100 feet, its height s at time t is given by the position function

$$s = -16t^2 + 100$$

Position function

where s is measured in feet and t is measured in seconds. Find the average velocity over each of the following time intervals.

a. $[1, 2]$ **b.** $[1, 1.5]$ **c.** $[1, 1.1]$



ime-lapse photograph of a free-falling lliard ball

Solution

a. For the interval $[1, 2]$ the object falls from a height of $s(1) = -16(1)^2 + 100 = 84$ feet to a height of $s(2) = -16(2)^2 + 100 = 36$ feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{36 - 84}{2 - 1} = \frac{-48}{1} = -48 \text{ feet per second.}$$

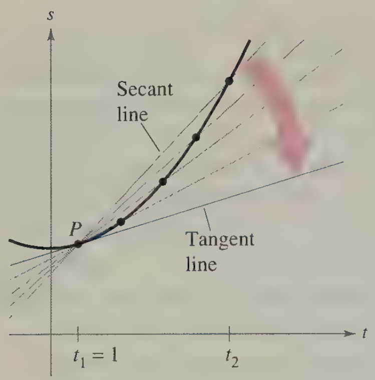
b. For the interval $[1, 1.5]$, the object falls from a height of 84 feet to a height of 64 feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{64 - 84}{1.5 - 1} = \frac{-20}{0.5} = -40 \text{ feet per second.}$$

c. For the interval $[1, 1.1]$, the object falls from a height of 84 feet to a height of 80.64 feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{80.64 - 84}{1.1 - 1} = \frac{-3.36}{0.1} = -33.6 \text{ feet per second.}$$

Note that the average velocities are *negative*, indicating that the object is moving downward.



The average velocity between t_1 and t_2 is the slope of the secant line, and the instantaneous velocity at t_1 is the slope of the tangent line.
Figure 2.20

Suppose that in Example 9 you wanted to find the *instantaneous* velocity (simply the velocity) of the object when $t = 1$. Just as you can approximate the slo of the tangent line by calculating the slope of the secant line, you can approximate t velocity at $t = 1$ by calculating the average velocity over a small interval $[1, 1 + \Delta$ (see Figure 2.20). By taking the limit as Δt approaches zero, you obtain the veloci when $t = 1$. Try doing this—you will find that the velocity when $t = 1$ is -32 fe per second.

In general, if $s = s(t)$ is the position function for an object moving along straight line, the **velocity** of the object at time t is

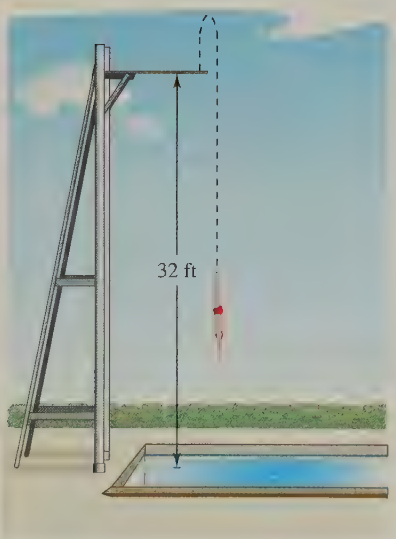
$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t).$$ Velocity function

In other words, the velocity function is the derivative of the position function. Veloci can be negative, zero, or positive. The **speed** of an object is the absolute value of velocity. Speed cannot be negative.

The position of a free-falling object (neglecting air resistance) under the influen of gravity can be represented by the equation

$$s(t) = \frac{1}{2}gt^2 + v_0t + s_0$$ Position function

where s_0 is the initial height of the object, v_0 is the initial velocity of the object, and is the acceleration due to gravity. On earth, the value of g is approximately -32 fe per second per second or -9.8 meters per second per second.



Velocity is positive when an object is rising, and is negative when an object is falling.
Figure 2.21

NOTE In Figure 2.21, note that the diver moves upward for the first half-second because the velocity is positive for $0 < t < \frac{1}{2}$. When the velocity is 0, the diver has reached the maximum height of the dive.

Example 10 Using the Derivative to Find Velocity

At time $t = 0$, a diver jumps from a platform diving board that is 32 feet above t water (see Figure 2.21). The position of the diver is given by

$$s(t) = -16t^2 + 16t + 32$$ Position function

where s is measured in feet and t is measured in seconds.

a. When does the diver hit the water?

b. What is the diver's velocity at impact?

Solution

a. To find the time t when the diver hits the water, let $s = 0$ and solve for t.

$$-16t^2 + 16t + 32 = 0$$ Set position function equal to 0.

$$-16(t + 1)(t - 2) = 0$$ Factor.

$$t = -1 \text{ or } 2$$ Solve for t.

Because $t \geq 0$, choose the positive value to conclude that the diver hits the wa at $t = 2$ seconds.

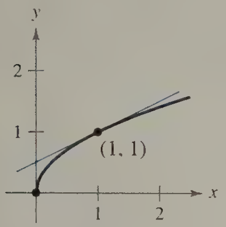
b. The velocity at time t is given by the derivative $s'(t) = -32t + 16$. Therefore, t velocity at time $t = 2$ is

$$s'(2) = -32(2) + 16 = -48 \text{ feet per second.}$$

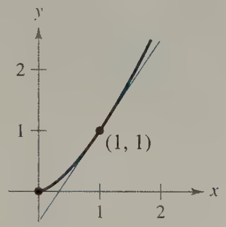
EXERCISES FOR SECTION 2.2

Exercises 1 and 2, use the graph to estimate the slope of the ngent line to $y = x^n$ at the point (1, 1). Verify your answer alytically. To print an enlarged copy of the graph, go to the ebsite *www.mathgraphs.com*.

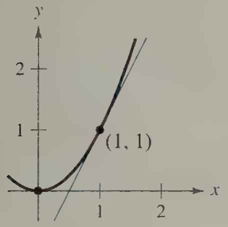
. (a) $y = x^{1/2}$



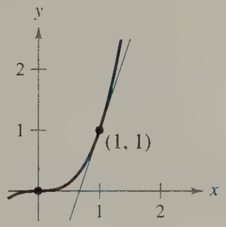
(b) $y = x^{3/2}$



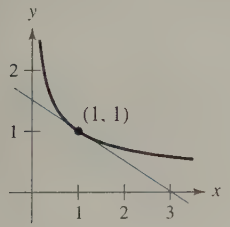
(c) $y = x^2$



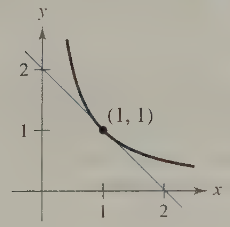
(d) $y = x^3$



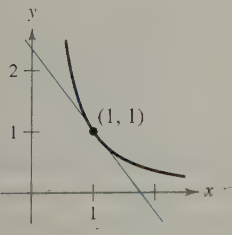
. (a) $y = x^{-1/2}$



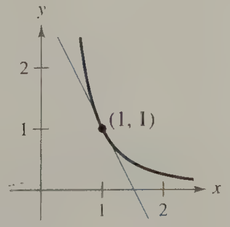
(b) $y = x^{-1}$



(c) $y = x^{-3/2}$



(d) $y = x^{-2}$



Exercises 3–24, find the derivative of the function.

. $y = 8$
4. $f(x) = -2$
. $y = x^6$
6. $y = x^8$
. $y = \dfrac{1}{x^7}$
8. $y = \dfrac{1}{x^8}$
. $f(x) = \sqrt[5]{x}$
10. $g(x) = \sqrt[4]{x}$
. $f(x) = x + 1$
12. $g(x) = 3x - 1$
. $f(t) = -2t^2 + 3t - 6$
14. $y = t^2 + 2t - 3$
. $g(x) = x^2 + 4x^3$
16. $y = 8 - x^3$
. $s(t) = t^3 - 2t + 4$
18. $f(x) = 2x^3 - x^2 + 3x$
19. $y = \dfrac{\pi}{2}\sin\theta - \cos\theta$
20. $g(t) = \pi\cos t$
21. $y = x^2 - \frac{1}{2}\cos x$
22. $y = 5 + \sin x$
23. $y = \dfrac{1}{x} - 3\sin x$
24. $y = \dfrac{5}{(2x)^3} + 2\cos x$

In Exercises 25–30, complete the table, using Example 6 as a model.

Original Function	*Rewrite*	*Differentiate*	*Simplify*
25. $y = \dfrac{5}{2x^2}$			
26. $y = \dfrac{2}{3x^2}$			
27. $y = \dfrac{3}{(2x)^3}$			
28. $y = \dfrac{\pi}{(3x)^2}$			
29. $y = \dfrac{\sqrt{x}}{x}$			
30. $y = \dfrac{4}{x^{-3}}$			

In Exercises 31–38, find the slope of the graph of the function at the indicated point. Use the *derivative* feature of a graphing utility to confirm your results.

Function	*Point*
31. $f(x) = \dfrac{3}{x^2}$	$(1, 3)$
32. $f(t) = 3 - \dfrac{3}{5t}$	$\left(\frac{3}{5}, 2\right)$
33. $f(x) = -\frac{1}{2} + \frac{7}{5}x^3$	$\left(0, -\frac{1}{2}\right)$
34. $y = 3x^3 - 6$	$(2, 18)$
35. $y = (2x + 1)^2$	$(0, 1)$
36. $f(x) = 3(5 - x)^2$	$(5, 0)$
37. $f(\theta) = 4\sin\theta - \theta$	$(0, 0)$
38. $g(t) = 2 + 3\cos t$	$(\pi, -1)$

In Exercises 39–52, find the derivative of the function.

39. $f(x) = x^2 + 5 - 3x^{-2}$
40. $f(x) = x^2 - 3x - 3x^{-2}$
41. $g(t) = t^2 - \dfrac{4}{t^3}$
42. $f(x) = x + \dfrac{1}{x^2}$
43. $f(x) = \dfrac{x^3 - 3x^2 + 4}{x^2}$
44. $h(x) = \dfrac{2x^2 - 3x + 1}{x}$
45. $y = x(x^2 + 1)$
46. $y = 3x(6x - 5x^2)$
47. $f(x) = \sqrt{x} - 6\sqrt[3]{x}$
48. $f(x) = \sqrt[3]{x} + \sqrt[5]{x}$
49. $h(s) = s^{4/5} - s^{2/3}$
50. $f(t) = t^{2/3} - t^{1/3} + 4$
51. $f(x) = 6\sqrt{x} + 5\cos x$
52. $f(x) = \dfrac{2}{\sqrt[3]{x}} + 3\cos x$

In Exercises 53–56, (a) find an equation of the tangent line to the graph of f at the indicated point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

	Function	*Point*
53.	$y = x^4 - 3x^2 + 2$	$(1, 0)$
54.	$y = x^3 + x$	$(-1, -2)$
55.	$f(x) = \dfrac{2}{\sqrt[4]{x^3}}$	$(1, 2)$
56.	$y = (x^2 + 2x)(x + 1)$	$(1, 6)$

In Exercises 57–62, determine the point(s) (if any) at which the graph of the function has a horizontal tangent line.

57. $y = x^4 - 8x^2 + 2$

58. $y = x^3 + x$

59. $y = \dfrac{1}{x^2}$

60. $y = x^2 + 1$

61. $y = x + \sin x, \quad 0 \le x < 2\pi$

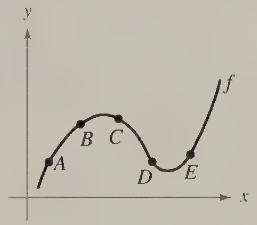
62. $y = \sqrt{3}x + 2\cos x, \quad 0 \le x < 2\pi$

In Exercises 63–66, find k such that the line is tangent to the graph of the function.

	Function	*Line*
63.	$f(x) = x^2 - kx$	$y = 4x - 9$
64.	$f(x) = k - x^2$	$y = -4x + 7$
65.	$f(x) = \dfrac{k}{x}$	$y = -\dfrac{3}{4}x + 3$
66.	$f(x) = k\sqrt{x}$	$y = x + 4$

Getting at the Concept

67. Use the graph of f to answer each question. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*



(a) Between which two consecutive points is the average rate of change of the function greatest?

(b) Is the average rate of change of the function between A and B greater than or less than the instantaneous rate of change at B?

(c) Sketch a tangent line to the graph between C and D such that the slope of the tangent line is the same as the average rate of change of the function between C and D.

68. Sketch the graph of a function f such that $f' > 0$ for all x and the rate of change of the function is decreasing.

Getting at the Concept *(continued)*

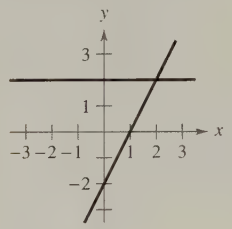
In Exercises 69 and 70, the relationship between f and g is given. Give the relationship between f' and g'.

69. $g(x) = f(x) + 6$

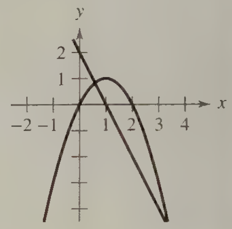
70. $g(x) = -5f(x)$

In Exercises 71 and 72, the graphs of a function f and its derivative f' are shown on the same set of coordinate axes. Label the graphs as f or f' and write a short paragraph stating the criteria used in making the selection. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*

71.



72.



73. Sketch the graphs of $y = x^2$ and $y = -x^2 + 6x - 5$, and sket the two lines that are tangent to both graphs. Find equations these lines.

74. Show that the graphs of the two equations $y = x$ and $y = 1$ have tangent lines that are perpendicular to each other at th point of intersection.

In Exercises 75 and 76, find an equation of the tangent line the graph of the function f through the point (x_0, y_0) not on t graph. To find the point of tangency (x, y) on the graph of solve the equation

$$f'(x) = \frac{y_0 - y}{x_0 - x}.$$

75. $f(x) = \sqrt{x}$

$(x_0, y_0) = (-4, 0)$

76. $f(x) = \dfrac{2}{x}$

$(x_0, y_0) = (5, 0)$

77. ***Linear Approximation*** Use a graphing utility (in squa mode) to zoom in on the graph of $f(x) = 4 - \frac{1}{2}x^2$ to approx mate $f'(1)$. Use the derivative to find $f'(1)$.

78. ***Linear Approximation*** Use a graphing utility (in squa mode) to zoom in on the graph of $f(x) = 4\sqrt{x} + 1$ to appro imate $f'(4)$. Use the derivative to find $f'(4)$.

79. ***Linear Approximation*** Consider the function $f(x) = x$ with the solution point $(4, 8)$.

(a) Use a graphing utility to obtain the graph of f. Use the *zo* feature to obtain successive magnifications of the graph the neighborhood of the point $(4, 8)$. After zooming in a f times, the graph should appear nearly linear. Use the *tra* feature to determine the coordinates of a point "near" (4, Find an equation of the secant line $S(x)$ through the t points.

(b) Find the equation of the line

$$T(x) = f'(4)(x - 4) + f(4)$$

tangent to the graph of f passing through the given point. Why are the linear functions S and T nearly the same?

(c) Use a graphing utility to graph f and T on the same set of coordinate axes. Note that T is a "good" approximation of f when x is "close to" 4. What happens to the accuracy of the approximation as you move farther away from the point of tangency?

(d) Demonstrate the conclusion in part (c) by completing the table.

Δx	-3	-2	-1	-0.5	-0.1	0
$f(4 + \Delta x)$						
$T(4 + \Delta x)$						

Δx	0.1	0.5	1	2	3
$f(4 + \Delta x)$					
$T(4 + \Delta x)$					

). ***Linear Approximation*** Repeat Exercise 79 for the function $f(x) = x^3$ where $T(x)$ is the line tangent to the graph at the point $(1, 1)$. Explain why the accuracy of the linear approximation decreases more rapidly than in Exercise 79.

rue or False? **In Exercises 81–86, determine whether the atement is true or false. If it is false, explain why or give an ample that shows it is false.**

1. If $f'(x) = g'(x)$, then $f(x) = g(x)$.
2. If $f(x) = g(x) + c$, then $f'(x) = g'(x)$.
3. If $y = \pi^2$, then $dy/dx = 2\pi$.
4. If $y = x/\pi$, then $dy/dx = 1/\pi$.
5. If $g(x) = 3f(x)$, then $g'(x) = 3f'(x)$.
6. If $f(x) = 1/x^n$, then $f'(x) = 1/(nx^{n-1})$.

Exercises 87–90, find the average rate of change of the func- on over the indicated interval. Compare this average rate of ange with the instantaneous rates of change at the endpoints the interval.

Function	*Interval*
. $f(t) = 2t + 7$	$[1, 2]$
. $f(t) = t^2 - 3$	$[2, 2.1]$
. $f(x) = \dfrac{-1}{x}$	$[1, 2]$
. $f(x) = \sin x$	$\left[0, \dfrac{\pi}{6}\right]$

Vertical Motion **In Exercises 91 and 92, use the position function $s(t) = -16t^2 + v_0 t + s_0$ for free-falling objects.**

91. A silver dollar is dropped from the top of a building that is 1362 feet tall.

(a) Determine the position and velocity functions for the coin.

(b) Determine the average velocity on the interval $[1, 2]$.

(c) Find the instantaneous velocities when $t = 1$ and $t = 2$.

(d) Find the time required for the coin to reach ground level.

(e) Find the velocity of the coin at impact.

92. A ball is thrown straight down from the top of a 220-foot building with an initial velocity of -22 feet per second. What is its velocity after 3 seconds? What is its velocity after falling 108 feet?

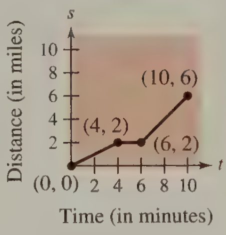
Vertical Motion **In Exercises 93 and 94, use the position function $s(t) = -4.9t^2 + v_0 t + s_0$ for free-falling objects.**

93. A projectile is shot upward from the surface of earth with an initial velocity of 120 meters per second. What is its velocity after 5 seconds? After 10 seconds?

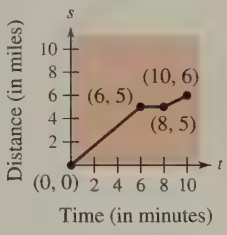
94. To estimate the height of a building, a stone is dropped from the top of the building into a pool of water at ground level. How high is the building if the splash is seen 6.8 seconds after the stone is dropped?

Think About It **In Exercises 95 and 96, the graph of a position function is shown. It represents the distance in miles that a person drives during a 10-minute trip to work. Make a sketch of the corresponding velocity function.**

95.

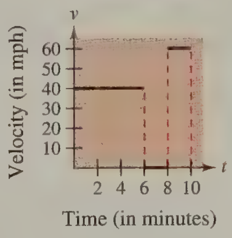


96.

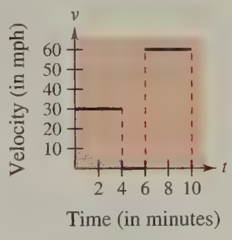


Think About It **In Exercises 97 and 98, the graph of a velocity function is shown. It represents the velocity in miles per hour during a 10-minute drive to work. Make a sketch of the corresponding position function.**

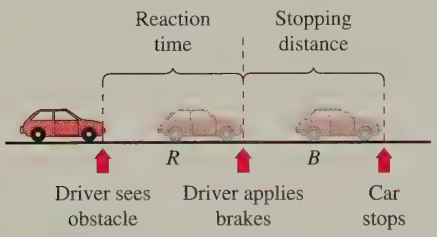
97.



98.



99. ***Modeling Data*** The stopping distance of an automobile traveling at a speed v (kilometers per hour) is the distance R (meters) the car travels during the reaction time of the driver plus the distance B (meters) the car travels after the brakes are applied (see figure). The table shows the results of an experiment.



v	20	40	60	80	100
R	3.3	6.7	10.0	13.3	16.7
B	2.3	8.9	20.2	35.9	56.7

(a) Use the regression capabilities of a graphing utility to find a linear model for reaction time.

(b) Use the regression capabilities of a graphing utility to find a quadratic model for braking time.

(c) Determine the polynomial giving the total stopping distance T.

(d) Use a graphing utility to graph the functions R, B, and T in the same viewing window.

(e) Find the derivative of T and the rate of change of the total stopping distance for $v = 40$, $v = 80$, and $v = 100$.

(f) Use the results of this exercise to draw conclusions about the total stopping distance as speed increases.

100. ***Velocity*** Verify that the average velocity over the time interval $[t_0 - \Delta t, t_0 + \Delta t]$ is the same as the instantaneous velocity at $t = t_0$ for the position function

$$s(t) = -\tfrac{1}{2}at^2 + c.$$

101. ***Area*** The area of a square with sides of length s is given by $A = s^2$. Find the rate of change of the area with respect to s when $s = 4$ meters.

102. ***Volume*** The volume of a cube with sides of length s is given by $V = s^3$. Find the rate of change of the volume with respect to s when $s = 4$ centimeters.

103. ***Inventory Management*** The annual inventory cost C for a certain manufacturer is

$$C = \frac{1{,}008{,}000}{Q} + 6.3Q$$

where Q is the order size when the inventory is replenished. Find the change in annual cost when Q is increased from 350 to 351, and compare this with the instantaneous rate of change when $Q = 350$.

104. ***Fuel Cost*** A car is driven 15,000 miles a year and gets miles per gallon. Assume that the average fuel cost is $1. per gallon. Find the annual cost of fuel C as a function o and use this function to complete the table.

x	10	15	20	25	30	35	40
C							
$\dfrac{dC}{dx}$							

Who would benefit more from a 1-mile-per-gallon increase fuel efficiency—the driver of a car that gets 15 miles per gall or the driver of a car that gets 35 miles per gallon? Explain.

105. ***Writing*** The number of gallons N of regular unleaded gas line sold by a gasoline station at a price of p dollars per gall is given by $N = f(p)$.

(a) Describe the meaning of $f'(1.479)$.

(b) Is $f'(1.479)$ usually positive or negative? Explain.

106. ***Newton's Law of Cooling*** This law states that the rate change of the temperature of an object is proportional to t difference between the object's temperature T and t temperature T_a of the surrounding medium. Write an equati for this law.

107. Find an equation of the parabola $y = ax^2 + bx + c$ that pass through $(0, 1)$ and is tangent to the line $y = x - 1$ at $(1, 0)$.

108. Let (a, b) be an arbitrary point on the graph of $y = 1/$ $x > 0$. Prove that the area of the triangle formed by the ta gent line through (a, b) and the coordinate axes is 2.

109. Find the tangent line(s) to the curve $y = x^3 - 9x$ through t point $(1, -9)$.

110. Find the equation(s) of the tangent line(s) to the parabo $y = x^2$ through the given point.

(a) $(0, a)$ (b) $(a, 0)$

Are there any restrictions on the constant a?

111. Find a and b such that

$$f(x) = \begin{cases} ax^3, & x \le 2 \\ x^2 + b, & x > 2 \end{cases}$$

is differentiable everywhere.

112. Where are the functions $f_1(x) = |\sin x|$ and $f_2(x) = \sin |$ differentiable?

113. Prove that $\dfrac{d}{dx}[\cos x] = -\sin x$.

FOR FURTHER INFORMATION For a geometric interpretation the derivatives of trigonometric functions, see the article "Sines and Cosines of the Times" by Victor J. Katz in *Math Horizons*. To view this article, go to the website *www.matharticles.com*.

Section 2.3 The Product and Quotient Rules and Higher-Order Derivatives

- Find the derivative of a function using the Product Rule.
- Find the derivative of a function using the Quotient Rule.
- Find the derivative of a trigonometric function.
- Find a higher-order derivative of a function.

The Product Rule

In Section 2.2 you learned that the derivative of the sum of two functions is simply the sum of their derivatives. The rules for the derivatives of the product and quotient of two functions are not as simple.

THEOREM 2.7 The Product Rule

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Proof Some mathematical proofs, such as the proof of the Sum Rule, are straightforward. Others involve clever steps that may appear unmotivated to a reader. This proof involves such a step—subtracting and adding the same quantity—which is shown in color.

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\left[f(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x} + g(x)\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]\\
&= \lim_{\Delta x\to 0}\left[f(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x}\right] + \lim_{\Delta x\to 0}\left[g(x)\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]\\
&= \lim_{\Delta x\to 0} f(x+\Delta x)\cdot\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x} + \lim_{\Delta x\to 0} g(x)\cdot\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

THE PRODUCT RULE

When Leibniz originally wrote a formula for the Product Rule, he was motivated by the expression

$$(x + dx)(y + dy) - xy$$

from which he subtracted $dx\,dy$ (as being negligible) and obtained the differential form $x\,dy + y\,dx$. This derivation resulted in the traditional form of the Product Rule. *(Source: The History of Mathematics by David M. Burton)*

A version of the Product Rule that ome people prefer is

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

The advantage of this form is that it generalizes easily to products involving three or more factors.

The Product Rule can be extended to cover products involving more than two factors. For example, if f, g, and h are differentiable functions of x, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

For instance, the derivative of $y = x^2 \sin x \cos x$ is

$$\begin{aligned}
\frac{dy}{dx} &= 2x\sin x\cos x + x^2\cos x\cos x + x^2\sin x(-\sin x)\\
&= 2x\sin x\cos x + x^2(\cos^2 x - \sin^2 x).
\end{aligned}$$

The derivative of a product of two functions is not (in general) given by the pro uct of the derivatives of the two functions. To see this, try comparing the product of th derivatives of $f(x) = 3x - 2x^2$ and $g(x) = 5 + 4x$ with the derivative in Example 1.

Example 1 Using the Product Rule

Find the derivative of $h(x) = (3x - 2x^2)(5 + 4x)$.

Solution

$$\begin{aligned} h'(x) &= \overbrace{(3x - 2x^2)}^{\text{First}}\overbrace{\frac{d}{dx}[5 + 4x]}^{\text{Derivative of second}} + \overbrace{(5 + 4x)}^{\text{Second}}\overbrace{\frac{d}{dx}[3x - 2x^2]}^{\text{Derivative of first}} && \text{Apply Product Rule.} \\ &= (3x - 2x^2)(4) + (5 + 4x)(3 - 4x) \\ &= (12x - 8x^2) + (15 - 8x - 16x^2) \\ &= -24x^2 + 4x + 15 \end{aligned}$$

In Example 1, you have the option of finding the derivative with or without th Product Rule. To find the derivative without the Product Rule, you can write

$$\begin{aligned} D_x[(3x - 2x^2)(5 + 4x)] &= D_x[-8x^3 + 2x^2 + 15x] \\ &= -24x^2 + 4x + 15. \end{aligned}$$

In the next example, you must use the Product Rule.

Example 2 Using the Product Rule

Find the derivative of $y = x \sin x$.

Solution

$$\begin{aligned} \frac{d}{dx}[x \sin x] &= x\frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[x] && \text{Apply Product Rule.} \\ &= x \cos x + (\sin x)(1) \\ &= x \cos x + \sin x \end{aligned}$$

Example 3 Using the Product Rule

Find the derivative of $y = 2x \cos x - 2 \sin x$.

Solution

$$\begin{aligned} \frac{dy}{dx} &= \overbrace{(2x)\left(\frac{d}{dx}[\cos x]\right) + (\cos x)\left(\frac{d}{dx}[2x]\right)}^{\text{Product Rule}} - \overbrace{2\frac{d}{dx}[\sin x]}^{\text{Constant Multiple Rule}} \\ &= (2x)(-\sin x) + (\cos x)(2) - 2(\cos x) \\ &= -2x \sin x \end{aligned}$$

NOTE In Example 3, notice that you use the Product Rule when both factors of the product are variable, and you use the Constant Multiple Rule when one of the factors is a constant.

The Quotient Rule

THEOREM 2.8 The Quotient Rule

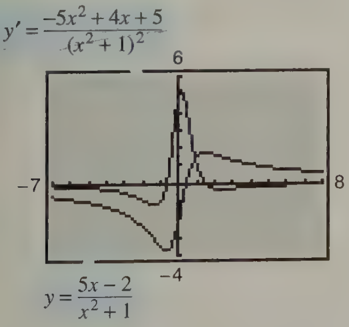
The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \qquad g(x) \neq 0$$

Proof As with the proof of Theorem 2.7, the key to this proof is subtracting and adding the same quantity.

$$\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{\Delta x \to 0} \frac{\dfrac{f(x+\Delta x)}{g(x+\Delta x)} - \dfrac{f(x)}{g(x)}}{\Delta x} && \text{Definition of derivative}\\
&= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x+\Delta x)}{\Delta x g(x)g(x+\Delta x)}\\
&= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x) + f(x)g(x) - f(x)g(x+\Delta x)}{\Delta x g(x)g(x+\Delta x)}\\
&= \frac{\displaystyle\lim_{\Delta x \to 0} \frac{g(x)[f(x+\Delta x) - f(x)]}{\Delta x} - \lim_{\Delta x \to 0} \frac{f(x)[g(x+\Delta x) - g(x)]}{\Delta x}}{\displaystyle\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]}\\
&= \frac{g(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right] - f(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}\right]}{\displaystyle\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]}\\
&= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
\end{aligned}$$

TECHNOLOGY Graphing utilities can be used to compare the graph of a unction with the graph of its derivaive. For instance, in Figure 2.22, the graph of the function in Example 4 appears to have two points that have horizontal tangent lines. What are the values of y' at these two points?



Graphical comparison of a function and its derivative
Figure 2.22

Example 4 Using the Quotient Rule

Find the derivative of $y = \dfrac{5x-2}{x^2+1}$.

Solution

$$\begin{aligned}
\frac{d}{dx}\left[\frac{5x-2}{x^2+1}\right] &= \frac{(x^2+1)\dfrac{d}{dx}[5x-2] - (5x-2)\dfrac{d}{dx}[x^2+1]}{(x^2+1)^2} && \text{Apply Quotient Rule.}\\
&= \frac{(x^2+1)(5) - (5x-2)(2x)}{(x^2+1)^2}\\
&= \frac{(5x^2+5) - (10x^2-4x)}{(x^2+1)^2}\\
&= \frac{-5x^2+4x+5}{(x^2+1)^2}
\end{aligned}$$

Note the use of parentheses in Example 4. A liberal use of parentheses is recc mended for *all* types of differentiation problems. For instance, with the Quotient Ru it is a good idea to enclose all factors and derivatives in parentheses, and to pay spec attention to the subtraction required in the numerator.

When we introduced differentiation rules in the preceding section, we emp sized the need for rewriting *before* differentiating. The next example illustrates t point with the Quotient Rule.

Example 5 Rewriting Before Differentiating

Find the derivative of $y = \dfrac{3 - (1/x)}{x + 5}$.

Solution Begin by rewriting the function.

$$y = \frac{3 - (1/x)}{x + 5} \qquad \text{Write original function.}$$

$$= \frac{x\left(3 - \dfrac{1}{x}\right)}{x(x + 5)} \qquad \text{Multiply numerator and denominator by}$$

$$= \frac{3x - 1}{x^2 + 5x} \qquad \text{Rewrite.}$$

$$\frac{dy}{dx} = \frac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2} \qquad \text{Quotient Rule}$$

$$= \frac{(3x^2 + 15x) - (6x^2 + 13x - 5)}{(x^2 + 5x)^2}$$

$$= \frac{-3x^2 + 2x + 5}{(x^2 + 5x)^2} \qquad \text{Simplify.}$$

Not every quotient needs to be differentiated by the Quotient Rule. For examp each quotient in the next example can be considered as the product of a constant tin a function of x. In such cases it is more convenient to use the Constant Multiple Ru

Example 6 Using the Constant Multiple Rule

	Original Function	*Rewrite*	*Differentiate*	*Simplify*
a.	$y = \dfrac{x^2 + 3x}{6}$	$y = \dfrac{1}{6}(x^2 + 3x)$	$y' = \dfrac{1}{6}(2x + 3)$	$y' = \dfrac{2x +}{6}$
b.	$y = \dfrac{5x^4}{8}$	$y = \dfrac{5}{8}x^4$	$y' = \dfrac{5}{8}(4x^3)$	$y' = \dfrac{5}{2}x^3$
c.	$y = \dfrac{-3(3x - 2x^2)}{7x}$	$y = -\dfrac{3}{7}(3 - 2x)$	$y' = -\dfrac{3}{7}(-2)$	$y' = \dfrac{6}{7}$
d.	$y = \dfrac{9}{5x^2}$	$y = \dfrac{9}{5}(x^{-2})$	$y' = \dfrac{9}{5}(-2x^{-3})$	$y' = -\dfrac{18}{5x^3}$

NOTE To see the benefit of using the Constant Multiple Rule for some quotients, try us the Quotient Rule to differentiate the functions in Example 6—you should obtain the sa results, but with more work.

In Section 2.2, we proved the Power Rule only for the case where the exponent n is a positive integer greater than 1. The next example extends the proof to include negative integer exponents.

Example 7 Proof of the Power Rule (Negative Integer Exponents)

If n is a negative integer, there exists a positive integer k such that $n = -k$. So, by the Quotient Rule, you can write

$$\begin{aligned}
\frac{d}{dx}[x^n] &= \frac{d}{dx}\left[\frac{1}{x^k}\right] \\
&= \frac{x^k(0) - (1)(kx^{k-1})}{(x^k)^2} && \text{Quotient Rule} \\
&= \frac{0 - kx^{k-1}}{x^{2k}} \\
&= -kx^{-k-1} \\
&= nx^{n-1}. && n = -k
\end{aligned}$$

So, the Power Rule

$$D_x[x^n] = nx^{n-1} \qquad \text{Power Rule}$$

is valid for any integer. In Exercise 63 in Section 2.5, you are asked to prove the case for which n is any rational number.

Derivatives of Trigonometric Functions

Knowing the derivatives of the sine and cosine functions, you can use the Quotient Rule to find the derivatives of the four remaining trigonometric functions.

THEOREM 2.9 Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

Proof Considering $\tan x = (\sin x)/(\cos x)$ and applying the Quotient Rule, you obtain

$$\begin{aligned}
\frac{d}{dx}[\tan x] &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} && \text{Apply Quotient Rule.} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x.
\end{aligned}$$

The proofs of the other three parts of the theorem are left as an exercise (see Exercise 81).

NOTE Because of trigonometric identities, the derivative of a trigonometric function can take many forms. This presents a challenge when you are trying to match your answers to those given in the back of the text.

Example 8 Differentiating Trigonometric Functions

Function	*Derivative*
a. $y = x - \tan x$	$\dfrac{dy}{dx} = 1 - \sec^2 x$
b. $y = x \sec x$	$y' = x(\sec x \tan x) + (\sec x)(1)$ $= (\sec x)(1 + x \tan x)$

Example 9 Different Forms of a Derivative

Differentiate both forms of $y = \dfrac{1 - \cos x}{\sin x} = \csc x - \cot x.$

Solution

First form: $y = \dfrac{1 - \cos x}{\sin x}$

$$\begin{aligned} y' &= \frac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x} \\ &= \frac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x} \\ &= \frac{1 - \cos x}{\sin^2 x} \end{aligned}$$

Second form: $y = \csc x - \cot x$

$$y' = -\csc x \cot x + \csc^2 x$$

To show that the two derivatives are equal, you can write

$$\frac{1 - \cos x}{\sin^2 x} = \frac{1}{\sin^2 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) = \csc^2 x - \csc x \cot x.$$

The following summary shows that much of the work in obtaining a simplifi form of a derivative occurs *after* differentiating. Note that two characteristics of simplified form are the absence of negative exponents and the combining of li terms.

	$f'(x)$ After Differentiating	*$f'(x)$ After Simplifying*
Example 1	$(3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$	$-24x^2 + 4x + 15$
Example 3	$(2x)(-\sin x) + (\cos x)(2) - 2(\cos x)$	$-2x \sin x$
Example 4	$\dfrac{(x^2 + 1)(5) - (5x - 2)(2x)}{(x^2 + 1)^2}$	$\dfrac{-5x^2 + 4x + 5}{(x^2 + 1)^2}$
Example 5	$\dfrac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2}$	$\dfrac{-3x^2 + 2x + 5}{(x^2 + 5x)^2}$
Example 9	$\dfrac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}$	$\dfrac{1 - \cos x}{\sin^2 x}$

Higher-Order Derivatives

Just as you can obtain a velocity function by differentiating a position function, you can obtain an **acceleration** function by differentiating a velocity function. Another way of looking at this is that you can obtain an acceleration function by differentiating a position function *twice*.

$$\begin{aligned} &s(t) && \text{Position function} \\ v(t) = {} &s'(t) && \text{Velocity function} \\ a(t) = v'(t) = {} &s''(t) && \text{Acceleration function} \end{aligned}$$

The function given by $a(t)$ is the **second derivative** of $s(t)$ and is denoted by $s''(t)$.

The second derivative is an example of a **higher-order derivative.** You can define derivatives of any positive integer order. For instance, the **third derivative** is the derivative of the second derivative. Higher-order derivatives are denoted as follows.

First derivative:	y',	$f'(x)$,	$\dfrac{dy}{dx}$,	$\dfrac{d}{dx}[f(x)]$,	$D_x[y]$
Second derivative:	y'',	$f''(x)$,	$\dfrac{d^2y}{dx^2}$,	$\dfrac{d^2}{dx^2}[f(x)]$,	$D_x^2[y]$
Third derivative:	y''',	$f'''(x)$,	$\dfrac{d^3y}{dx^3}$,	$\dfrac{d^3}{dx^3}[f(x)]$,	$D_x^3[y]$
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\dfrac{d^4y}{dx^4}$,	$\dfrac{d^4}{dx^4}[f(x)]$,	$D_x^4[y]$
⋮					
nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\dfrac{d^ny}{dx^n}$,	$\dfrac{d^n}{dx^n}[f(x)]$,	$D_x^n[y]$

Example 10 Finding the Acceleration Due to Gravity

Because the moon has no atmosphere, a falling object on the moon encounters no air resistance. In 1971, astronaut David Scott demonstrated that a feather and a hammer fall at the same rate on the moon. The position function for each of these falling objects is given by

$$s(t) = -0.81t^2 + 2$$

where $s(t)$ is the height in meters and t is the time in seconds. What is the ratio of earth's gravitational force to the moon's?

Solution To find the acceleration, differentiate the position function twice.

$$\begin{aligned} s(t) &= -0.81t^2 + 2 && \text{Position function} \\ s'(t) &= -1.62t && \text{Velocity function} \\ s''(t) &= -1.62 && \text{Acceleration function} \end{aligned}$$

So, the acceleration due to gravity on the moon is -1.62 meters per second per second. Because the acceleration due to gravity on earth is -9.8 meters per second per second, the ratio of earth's gravitational force to the moon's is

$$\frac{\text{Earth's gravitational force}}{\text{Moon's gravitational force}} = \frac{-9.8}{-1.62} \approx 6.05.$$

NASA

THE MOON

The moon's mass is 7.354×10^{22} kilograms, nd earth's mass is 5.979×10^{24} kilograms. The moon's radius is 1738 kilometers, and arth's radius is 6371 kilometers. Because the ravitational force on the surface of a planet is irectly proportional to its mass and inversely roportional to the square of its radius, the atio of the gravitational force on earth to the ravitational force on the moon is

$$\frac{(5.979 \times 10^{24})/6371^2}{(7.354 \times 10^{22})/1738^2} \approx 6.05.$$

Lab Series LAB 3

EXERCISES FOR SECTION 2.3

In Exercises 1–6, use the Product Rule to differentiate the function.

1. $g(x) = (x^2 + 1)(x^2 - 2x)$
2. $f(x) = (6x + 5)(x^3 - 2)$
3. $h(t) = \sqrt[3]{t}(t^2 + 4)$
4. $g(s) = \sqrt{s}(4 - s^2)$
5. $f(x) = x^3 \cos x$
6. $g(x) = \sqrt{x} \sin x$

In Exercises 7–12, use the Quotient Rule to differentiate the function.

7. $f(x) = \dfrac{x}{x^2 + 1}$
8. $g(t) = \dfrac{t^2 + 2}{2t - 7}$
9. $h(x) = \dfrac{\sqrt[3]{x}}{x^3 + 1}$
10. $h(s) = \dfrac{s}{\sqrt{s} - 1}$
11. $g(x) = \dfrac{\sin x}{x^2}$
12. $f(t) = \dfrac{\cos t}{t^3}$

In Exercises 13–18, find $f'(x)$ and $f'(c)$.

	Function	*Value of c*
13.	$f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$	$c = 0$
14.	$f(x) = (x^2 - 2x + 1)(x^3 - 1)$	$c = 1$
15.	$f(x) = \dfrac{x^2 - 4}{x - 3}$	$c = 1$
16.	$f(x) = \dfrac{x + 1}{x - 1}$	$c = 2$
17.	$f(x) = x \cos x$	$c = \dfrac{\pi}{4}$
18.	$f(x) = \dfrac{\sin x}{x}$	$c = \dfrac{\pi}{6}$

In Exercises 19–24, complete the table without using the Quotient Rule (see Example 6).

	Function	*Rewrite*	*Differentiate*	*Simplify*
19.	$y = \dfrac{x^2 + 2x}{3}$			
20.	$y = \dfrac{5x^2 - 3}{4}$			
21.	$y = \dfrac{7}{3x^3}$			
22.	$y = \dfrac{4}{5x^2}$			
23.	$y = \dfrac{4x^{3/2}}{x}$			
24.	$y = \dfrac{3x^2 - 5}{7}$			

In Exercises 25–38, find the derivative of the algebraic function.

25. $f(x) = \dfrac{3 - 2x - x^2}{x^2 - 1}$
26. $f(x) = \dfrac{x^3 + 3x + 2}{x^2 - 1}$
27. $f(x) = x\left(1 - \dfrac{4}{x + 3}\right)$
28. $f(x) = x^4\left(1 - \dfrac{2}{x + 1}\right)$
29. $f(x) = \dfrac{2x + 5}{\sqrt{x}}$
30. $f(x) = \sqrt[3]{x}(\sqrt{x} + 3)$
31. $h(s) = (s^3 - 2)^2$
32. $h(x) = (x^2 - 1)^2$
33. $f(x) = \dfrac{2 - \dfrac{1}{x}}{x - 3}$
34. $g(x) = x^2\left(\dfrac{2}{x} - \dfrac{1}{x + 1}\right)$
35. $f(x) = (3x^3 + 4x)(x - 5)(x + 1)$
36. $f(x) = (x^2 - x)(x^2 + 1)(x^2 + x + 1)$
37. $f(x) = \dfrac{x^2 + c^2}{x^2 - c^2}$, c is a constant
38. $f(x) = \dfrac{c^2 - x^2}{c^2 + x^2}$, c is a constant

In Exercises 39–54, find the derivative of the trigonometr function.

39. $f(t) = t^2 \sin t$
40. $f(\theta) = (\theta + 1) \cos \theta$
41. $f(t) = \dfrac{\cos t}{t}$
42. $f(x) = \dfrac{\sin x}{x}$
43. $f(x) = -x + \tan x$
44. $y = x + \cot x$
45. $g(t) = \sqrt[4]{t} + 8 \sec t$
46. $h(s) = \dfrac{1}{s} - 10 \csc s$
47. $y = \dfrac{3(1 - \sin x)}{2 \cos x}$
48. $y = \dfrac{\sec x}{x}$
49. $y = -\csc x - \sin x$
50. $y = x \sin x + \cos x$
51. $f(x) = x^2 \tan x$
52. $f(x) = \sin x \cos x$
53. $y = 2x \sin x + x^2 \cos x$
54. $h(\theta) = 5\theta \sec \theta + \theta \tan$

In Exercises 55–58, use a computer algebra system to differe tiate the function.

55. $g(x) = \left(\dfrac{x + 1}{x + 2}\right)(2x - 5)$
56. $f(x) = \left(\dfrac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$
57. $g(\theta) = \dfrac{\theta}{1 - \sin \theta}$
58. $f(\theta) = \dfrac{\sin \theta}{1 - \cos \theta}$

In Exercises 59–62, evaluate the derivative of the function at t indicated point. Use a graphing utility to verify your result.

	Function	*Point*
59.	$y = \dfrac{1 + \csc x}{1 - \csc x}$	$\left(\dfrac{\pi}{6}, -3\right)$
60.	$f(x) = \tan x \cot x$	$(1, 1)$
61.	$h(t) = \dfrac{\sec t}{t}$	$\left(\pi, -\dfrac{1}{\pi}\right)$
62.	$f(x) = \sin x(\sin x + \cos x)$	$\left(\dfrac{\pi}{4}, 1\right)$

Exercises 63–68, (a) find an equation of the tangent line to e graph of f at the indicated point, (b) use a graphing utility graph the function and its tangent line at the point, and (c) se the *derivative* feature of a graphing utility to confirm your sults.

Function	*Point*
3. $f(x) = (x^3 - 3x + 1)(x + 2)$	$(1, -3)$
4. $f(x) = (x - 1)(x^2 - 2)$	$(0, 2)$
5. $f(x) = \dfrac{x}{x - 1}$	$(2, 2)$
6. $f(x) = \dfrac{(x - 1)}{(x + 1)}$	$\left(2, \dfrac{1}{3}\right)$
7. $f(x) = \tan x$	$\left(\dfrac{\pi}{4}, 1\right)$
3. $f(x) = \sec x$	$\left(\dfrac{\pi}{3}, 2\right)$

Exercises 69 and 70, determine the point(s) at which the aph of the function has a horizontal tangent.

. $f(x) = \dfrac{x^2}{x - 1}$ **70.** $f(x) = \dfrac{x^2}{x^2 + 1}$

Exercises 71 and 72, verify that $f'(x) = g'(x)$, and explain the lationship between f and g.

1. $f(x) = \dfrac{3x}{x + 2}, \quad g(x) = \dfrac{5x + 4}{x + 2}$

2. $f(x) = \dfrac{\sin x - 3x}{x}, \quad g(x) = \dfrac{\sin x + 2x}{x}$

Exercises 73 and 74, find the derivative of the function f for $= 1, 2, 3$, and 4. Use the result to write a general rule for $f'(x)$ terms of n.

3. $f(x) = x^n \sin x$ **74.** $f(x) = \dfrac{\cos x}{x^n}$

5. ***Area*** The length of a rectangle is given by $2t + 1$ and its height is $\sqrt{t}$, where t is time in seconds and the dimensions are in centimeters. Find the rate of change of the area with respect to time.

. ***Volume*** The radius of a right circular cylinder is given by $\sqrt{t + 2}$ and its height is $\frac{1}{2}\sqrt{t}$, where t is time in seconds and the dimensions are in inches. Find the rate of change of the volume with respect to time.

. ***Inventory Replenishment*** The ordering and transportation cost C for the components used in manufacturing a certain product is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \geq 1$$

where C is measured in thousands of dollars and x is the order size in hundreds. Find the rate of change of C with respect to x when (a) $x = 10$, (b) $x = 15$, and (c) $x = 20$. What do these rates of change imply about increasing order size?

78. ***Boyle's Law*** This law states that if the temperature of a gas remains constant, its pressure is inversely proportional to its volume. Use the derivative to show that the rate of change of the pressure is inversely proportional to the square of the volume.

79. ***Population Growth*** A population of 500 bacteria is introduced into a culture and grows in number according to the equation

$$P(t) = 500\left(1 + \frac{4t}{50 + t^2}\right)$$

where t is measured in hours. Find the rate at which the population is growing when $t = 2$.

80. ***Rate of Change*** Determine whether there exist any values of x in the interval $[0, 2\pi)$ such that the rate of change of $f(x) = \sec x$ and the rate of change of $g(x) = \csc x$ are equal.

81. Prove the following differentiation rules.

(a) $\dfrac{d}{dx}[\sec x] = \sec x \tan x$

(b) $\dfrac{d}{dx}[\csc x] = -\csc x \cot x$

(c) $\dfrac{d}{dx}[\cot x] = -\csc^2 x$

82. ***Modeling Data*** The table shows the number of motor homes n (in thousands) in the United States and the retail value v (in millions of dollars) of these motor homes for the years 1992 through 1997. The year is represented by t, with $t = 2$ corresponding to 1992. *(Source: Recreation Vehicle Industry Association)*

Year	1992	1993	1994	1995	1996	1997
n	226.3	243.8	306.7	281.0	274.6	239.3
v	\$6963	\$7544	\$9897	\$9768	\$9788	\$9139

(a) Use a graphing utility to find quadratic models for the number of motor homes $n(t)$ and the total retail value $v(t)$ of the motor homes.

(b) Find $A = v(t)/n(t)$. What does this function represent?

(c) Find $A'(t)$. Interpret the derivative in the context of these data.

In Exercises 83–88, find the second derivative of the function.

83. $f(x) = 4x^{3/2}$ **84.** $f(x) = x + 32x^{2}$

85. $f(x) = \dfrac{x}{x - 1}$ **86.** $f(x) = \dfrac{x^2 + 2x - 1}{x}$

87. $f(x) = 3 \sin x$ **88.** $f(x) = \sec x$

In Exercises 89–92, find the higher-order derivative.

Given	*Find*
89. $f'(x) = x^2$	$f''(x)$
90. $f''(x) = 2 - \dfrac{2}{x}$	$f'''(x)$
91. $f'''(x) = 2\sqrt{x}$	$f^{(4)}(x)$
92. $f^{(4)}(x) = 2x + 1$	$f^{(6)}(x)$

Getting at the Concept

93. Sketch the graph of a differentiable function f such that $f(2) = 0$, $f' < 0$ for $-\infty < x < 2$, and $f' > 0$ for $2 < x < \infty$.

94. Sketch the graph of a differentiable function f such that $f > 0$ and $f' < 0$ for all real numbers x.

In Exercises 95–98, find $f'(2)$ given the following.

$g(2) = 3$ **and** $g'(2) = -2$

$h(2) = -1$ **and** $h'(2) = 4$

95. $f(x) = 2g(x) + h(x)$

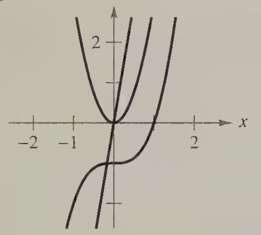
96. $f(x) = 4 - h(x)$

97. $f(x) = \dfrac{g(x)}{h(x)}$

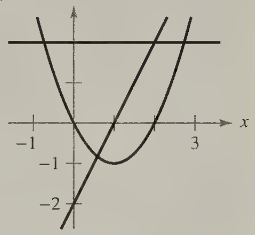
98. $f(x) = g(x)h(x)$

In Exercises 99 and 100, the graphs of f, f', and f'' are shown on the same set of coordinate axes. Which is which? To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

99.



100.



101. ***Acceleration*** The velocity of an object in meters per second is

$$v(t) = 36 - t^2, \quad 0 \le t \le 6.$$

Find the velocity and acceleration of the object when $t = 3$. What can be said about the speed of the object when the velocity and acceleration have opposite signs?

102. ***Stopping Distance*** A car is traveling at a rate of 66 feet per second (45 miles per hour) when the brakes are applied. The position function for the car is

$$s(t) = -8.25t^2 + 66t$$

where s is measured in feet and t is measured in seconds. Use this function to complete the table, and find the average velocity during each time interval.

t	0	1	2	3	4
$s(t)$					
$v(t)$					
$a(t)$					

103. ***Acceleration*** An automobile's velocity starting from rest

$$v(t) = \frac{100t}{2t + 15}$$

where v is measured in feet per second. Find the acceleratio at each of the following times.

(a) 5 seconds (b) 10 seconds (c) 20 seconds

104. ***Finding a Pattern*** Develop a general rule for $f^{(n)}(x)$ if

(a) $f(x) = x^n$ and (b) $f(x) = \dfrac{1}{x}$.

105. ***Finding a Pattern*** Consider the function $f(x) = g(x)h(x)$.

(a) Use the product rule to generate rules for finding $f''(x$ $f'''(x)$, and $f^{(4)}(x)$.

(b) Use the results in part (a) to write a general rule for $f^{(n)}(x$

106. ***Finding a Pattern*** Develop a general rule for $[xf(x)]$ where f is a differentiable function of x.

Linear and Quadratic Approximations **The linear and quadrat approximations of a function f at $x = a$ are**

$$P_1(x) = f'(a)(x - a) + f(a) \text{ and}$$

$$P_2(x) = \tfrac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a).$$

In Exercises 107 and 108, (a) find the specified linear an quadratic approximations of f, (b) use a graphing utility t graph f and the approximations, (c) determine whether P_1 or is the better approximation, and (d) state how the accurac changes as you move farther from $x = a$.

107. $f(x) = \cos x$

$a = \dfrac{\pi}{3}$

108. $f(x) = \sin x$

$a = \dfrac{\pi}{2}$

True or False? **In Exercises 109–114, determine whether th statement is true or false. If it is false, explain why or give a example that shows it is false.**

109. If $y = f(x)g(x)$, then $dy/dx = f'(x)g'(x)$.

110. If $y = (x + 1)(x + 2)(x + 3)(x + 4)$, then $d^5y/dx^5 = 0$.

111. If $f'(c)$ and $g'(c)$ are zero and $h(x) = f(x)g(x)$, then $h'(c) =$

112. If $f(x)$ is an nth-degree polynomial, then $f^{(n+1)}(x) = 0$.

113. The second derivative represents the rate of change of the fir derivative.

114. If the velocity of an object is constant, then its acceleration zero.

115. Find the derivative of $f(x) = x|x|$. Does $f''(0)$ exist?

116. ***Think About It*** Let f and g be functions whose first ar second derivatives exist on an interval I. Which of the following formulas is (are) true?

(a) $fg'' - f''g = (fg' - f'g)'$

(b) $fg'' + f''g = (fg)''$

Section 2.4 The Chain Rule

- Find the derivative of a composite function using the Chain Rule.
- Find the derivative of a function using the General Power Rule.
- Simplify the derivative of a function using algebra.
- Find the derivative of a trigonometric function using the Chain Rule.

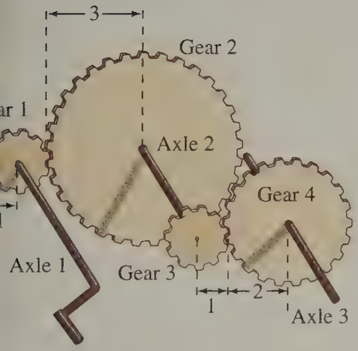
The Chain Rule

We have yet to discuss one of the most powerful differentiation rules—the **Chain Rule.** This rule deals with composite functions and adds a surprising versatility to the rules discussed in the two previous sections. For example, compare the following functions. Those on the left can be differentiated without the Chain Rule, and those on the right are best done with the Chain Rule.

Without the Chain Rule	*With the Chain Rule*
$y = x^2 + 1$	$y = \sqrt{x^2 + 1}$
$y = \sin x$	$y = \sin 6x$
$y = 3x + 2$	$y = (3x + 2)^5$
$y = x + \tan x$	$y = x + \tan x^2$

Basically, the Chain Rule states that if y changes dy/du times as fast as u, and u changes du/dx times as fast as x, then y changes $(dy/du)(du/dx)$ times as fast as x.

Example 1 The Derivative of a Composite Function



e 1: y revolutions per minute
e 2: u revolutions per minute
e 3: x revolutions per minute
ure 2.23

A set of gears is constructed, as shown in Figure 2.23, such that the second and third gears are on the same axle. As the first axle revolves, it drives the second axle, which in turn drives the third axle. Let y, u, and x represent the numbers of revolutions per minute of the first, second, and third axles. Find dy/du, du/dx, and dy/dx, and show that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Solution Because the circumference of the second gear is three times that of the first, the first axle must make three revolutions to turn the second axle once. Similarly, the second axle must make two revolutions to turn the third axle once, and you can write

$$\frac{dy}{du} = 3 \quad \text{and} \quad \frac{du}{dx} = 2.$$

Combining these two results, you know that the first axle must make six revolutions to turn the third axle once. So, you can write

$$\begin{aligned}\frac{dy}{dx} &= \boxed{\begin{matrix}\text{Rate of change of first axle}\\ \text{with respect to second axle}\end{matrix}} \cdot \boxed{\begin{matrix}\text{Rate of change of second axle}\\ \text{with respect to third axle}\end{matrix}}\\ &= \frac{dy}{du} \cdot \frac{du}{dx} = 3 \cdot 2 = 6\\ &= \boxed{\begin{matrix}\text{Rate of change of first axle}\\ \text{with respect to third axle}\end{matrix}}.\end{aligned}$$

In other words, the rate of change of y with respect to x is the product of the rate of change of y with respect to u and the rate of change of u with respect to x.

EXPLORATION

Using the Chain Rule Each of the following functions can be differentiated using rules that you studied in Sections 2.2 and 2.3. For each function, find the derivative using those rules. Then find the derivative using the Chain Rule. Compare your results. Which method is simpler?

a. $\dfrac{2}{3x+1}$

b. $(x+2)^3$

c. $\sin 2x$

Example 1 illustrates a simple case of the Chain Rule. The general rule is sta below.

THEOREM 2.10 The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

Proof Let $h(x) = f(g(x))$. Then, using the alternative form of the derivative, y need to show that, for $x = c$,

$$h'(c) = f'(g(c))g'(c).$$

An important consideration in this proof is the behavior of g as x approaches c. problem occurs if there are values of x, other than c, such that $g(x) = g(c)$. Appendix B we show how to use the differentiability of f and g to overcome t problem. For now, assume that $g(x) \neq g(c)$ for values of x other than c. In the pro of the Product Rule and the Quotient Rule, we added and subtracted the same quant to obtain the desired form. This proof uses a similar technique—multiplying a dividing by the same (nonzero) quantity. Note that because g is differentiable, it is a continuous, and it follows that $g(x) \to g(c)$ as $x \to c$.

$$\begin{aligned}
h'(c) &= \lim_{x\to c} \frac{f(g(x)) - f(g(c))}{x - c} \\
&= \lim_{x\to c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c}\right], \quad g(x) \neq g(c) \\
&= \left[\lim_{x\to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)}\right]\left[\lim_{x\to c} \frac{g(x) - g(c)}{x - c}\right] \\
&= f'(g(c))g'(c)
\end{aligned}$$

When applying the Chain Rule, it is helpful to think of the composite functi $f \circ g$ as having two parts—an inner part and an outer part.

Outer function

$$y = f(g(x)) = f(u)$$

Inner function

The derivative of $y = f(u)$ is the derivative of the outer function (at the inner functi u) *times* the derivative of the inner function.

$$y' = f'(u) \cdot u'$$

Example 2 Decomposition of a Composite Function

	$y = f(g(x))$	$u = g(x)$	$y = f(u)$
a.	$y = \dfrac{1}{x+1}$	$u = x + 1$	$y = \dfrac{1}{u}$
b.	$y = \sin 2x$	$u = 2x$	$y = \sin u$
c.	$y = \sqrt{3x^2 - x + 1}$	$u = 3x^2 - x + 1$	$y = \sqrt{u}$
d.	$y = \tan^2 x$	$u = \tan x$	$y = u^2$

Example 3 Using the Chain Rule

Find dy/dx for $y = (x^2 + 1)^3$.

Solution For this function, you can consider the inside function to be $u = x^2 + 1$. By the Chain Rule, you obtain

$$\frac{dy}{dx} = \underbrace{3(x^2+1)^2}_{\frac{dy}{du}}\underbrace{(2x)}_{\frac{du}{dx}} = 6x(x^2+1)^2.$$

TUDY TIP You could also solve the oblem in Example 3 without using the ain Rule by observing that

$$y = x^6 + 3x^4 + 3x^2 + 1$$

d

$$y' = 6x^5 + 12x^3 + 6x.$$

rify that this is the same as the deriva- e in Example 3. Which method would u use to find

$$\frac{d}{dx}(x^2 + 1)^{50}?$$

The General Power Rule

The function in Example 3 is an example of one of the most common types of composite functions, $y = [u(x)]^n$. The rule for differentiating such functions is called the **General Power Rule,** and it is a special case of the Chain Rule.

THEOREM 2.11 The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = nu^{n-1}\,u'.$$

Proof Because $y = u^n$, you apply the Chain Rule to obtain

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$
$$= \frac{d}{du}[u^n]\frac{du}{dx}.$$

By the (Simple) Power Rule in Section 2.2, you have $D_u[u^n] = nu^{n-1}$, and it follows that $dy/dx = n[u(x)]^{n-1}(du/dx)$.

Example 4 Applying the General Power Rule

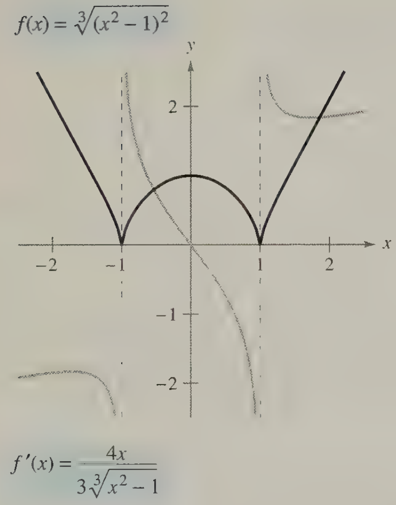
Find the derivative of $f(x) = (3x - 2x^2)^3$.

Solution Let $u = 3x - 2x^2$. Then

$$f(x) = (3x - 2x^2)^3 = u^3$$

and, by the General Power Rule, the derivative is

$$\begin{aligned} f'(x) &= \overset{n}{3}\overbrace{(3x - 2x^2)^2}^{u^{n-1}}\overbrace{\frac{d}{dx}[3x - 2x^2]}^{u'} && \text{Apply General Power Rule.} \\ &= 3(3x - 2x^2)^2(3 - 4x). && \text{Differentiate } 3x - 2x^2. \end{aligned}$$



The derivative of f is 0 at $x = 0$ and is undefined at $x = \pm 1$.
Figure 2.24

Example 5 Differentiating Functions Involving Radicals

Find all points on the graph of $f(x) = \sqrt[3]{(x^2 - 1)^2}$ for which $f'(x) = 0$ and those which $f'(x)$ does not exist.

Solution Begin by rewriting the function as

$$f(x) = (x^2 - 1)^{2/3}.$$

Then, applying the General Power Rule (with $u = x^2 - 1$) produces

$$\begin{aligned} f'(x) &= \overset{n}{\frac{2}{3}}\overbrace{(x^2 - 1)^{-1/3}}^{u^{n-1}}\overbrace{(2x)}^{u'} && \text{Apply General Power Rule.} \\ &= \frac{4x}{3\sqrt[3]{x^2 - 1}}. && \text{Write in radical form.} \end{aligned}$$

So, $f'(x) = 0$ when $x = 0$ and $f'(x)$ does not exist when $x = \pm 1$, as indicated Figure 2.24.

Example 6 Differentiating Quotients with Constant Numerators

Differentiate $g(t) = \dfrac{-7}{(2t - 3)^2}$.

Solution Begin by rewriting the function as

$$g(t) = -7(2t - 3)^{-2}.$$

Then, applying the General Power Rule produces

$$\begin{aligned} g'(t) &= \underbrace{(-7)\overbrace{(-2)}^{n}}_{\substack{\text{Constant} \\ \text{Multiple Rule}}}\overbrace{(2t - 3)^{-3}}^{u^{n-1}}\overbrace{(2)}^{u'} && \text{Apply General Power Rule.} \\ &= 28(2t - 3)^{-3} && \text{Simplify.} \\ &= \frac{28}{(2t - 3)^3}. && \text{Write with positive exponent.} \end{aligned}$$

NOTE Try differentiating the function in Example 6 using the Quotient Rule. You should obtain the same result, but using the Quotient Rule is less efficient than using the General Power Rule.

Simplifying Derivatives

The next three examples illustrate some techniques for simplifying the "raw derivatives" of functions involving products, quotients, and composites.

Example 7 Simplifying by Factoring Out the Least Powers

$$\begin{aligned}
f(x) &= x^2\sqrt{1-x^2} && \text{Original function}\\
&= x^2(1-x^2)^{1/2} && \text{Rewrite.}\\
f'(x) &= x^2\frac{d}{dx}\left[(1-x^2)^{1/2}\right] + (1-x^2)^{1/2}\frac{d}{dx}\left[x^2\right] && \text{Product Rule}\\
&= x^2\left[\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right] + (1-x^2)^{1/2}(2x) && \text{General Power Rule}\\
&= -x^3(1-x^2)^{-1/2} + 2x(1-x^2)^{1/2} && \text{Simplify.}\\
&= x(1-x^2)^{-1/2}\left[-x^2(1) + 2(1-x^2)\right] && \text{Factor.}\\
&= \frac{x(2-3x^2)}{\sqrt{1-x^2}} && \text{Simplify.}
\end{aligned}$$

Example 8 Simplifying the Derivative of a Quotient

ECHNOLOGY Symbolic differentiation utilities are capable of ifferentiating very complicated unctions. Often, however, the result s given in unsimplified form. If you ave access to such a utility, use it to ind the derivatives of the functions iven in Examples 7, 8, and 9. Then ompare the results with those given n this page.

$$\begin{aligned}
f(x) &= \frac{x}{\sqrt[3]{x^2+4}} && \text{Original function}\\
&= \frac{x}{(x^2+4)^{1/3}} && \text{Rewrite.}\\
f'(x) &= \frac{(x^2+4)^{1/3}(1) - x(1/3)(x^2+4)^{-2/3}(2x)}{(x^2+4)^{2/3}} && \text{Quotient Rule}\\
&= \frac{1}{3}(x^2+4)^{-2/3}\left[\frac{3(x^2+4) - (2x^2)(1)}{(x^2+4)^{2/3}}\right] && \text{Factor.}\\
&= \frac{x^2+12}{3(x^2+4)^{4/3}} && \text{Simplify.}
\end{aligned}$$

Example 9 Simplifying the Derivative of a Power

$$\begin{aligned}
y &= \left(\frac{3x-1}{x^2+3}\right)^2 && \text{Original function}\\
y' &= \overset{n}{2}\overbrace{\left(\frac{3x-1}{x^2+3}\right)}^{u^{n-1}}\overbrace{\frac{d}{dx}\left[\frac{3x-1}{x^2+3}\right]}^{u'} && \text{General Power Rule}\\
&= \left[\frac{2(3x-1)}{x^2+3}\right]\left[\frac{(x^2+3)(3) - (3x-1)(2x)}{(x^2+3)^2}\right] && \text{Quotient Rule}\\
&= \frac{2(3x-1)(3x^2+9-6x^2+2x)}{(x^2+3)^3} && \text{Multiply.}\\
&= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3} && \text{Simplify.}
\end{aligned}$$

Trigonometric Functions and the Chain Rule

The "Chain Rule versions" of the derivatives of the six trigonometric functions are follows.

$$\frac{d}{dx}[\sin u] = (\cos u)\,u' \qquad \frac{d}{dx}[\cos u] = -(\sin u)\,u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)\,u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)\,u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)\,u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)\,u'$$

Example 10 Applying the Chain Rule to Trigonometric Functions

a. $y = \sin \overbrace{2x}^{u}$ $\qquad y' = \overbrace{\cos 2x}^{\cos u}\,\overbrace{\frac{d}{dx}[2x]}^{u'} = (\cos 2x)(2) = 2 \cos 2x$

b. $y = \cos(x - 1)$ $\qquad y' = -\sin(x - 1)$

c. $y = \tan 3x$ $\qquad y' = 3 \sec^2 3x$

Be sure that you understand the mathematical conventions regarding parenthes and trigonometric functions. For instance, in Example 10a, $\sin 2x$ is written to me $\sin(2x)$.

Example 11 Parentheses and Trigonometric Functions

a. $y = \cos 3x^2 = \cos(3x^2)$ $\qquad y' = (-\sin 3x^2)(6x) = -6x \sin 3x^2$

b. $y = (\cos 3)x^2$ $\qquad y' = (\cos 3)(2x) = 2x \cos 3$

c. $y = \cos (3x)^2 = \cos(9x^2)$ $\qquad y' = (-\sin 9x^2)(18x) = -18x \sin 9x^2$

d. $y = \cos^2 x = (\cos x)^2$ $\qquad y' = 2(\cos x)(-\sin x) = -2 \cos x \sin x$

e. $y = \sqrt{\cos x} = (\cos x)^{1/2}$ $\qquad y' = \frac{1}{2}(\cos x)^{-1/2}(-\sin x) = -\frac{\sin x}{2\sqrt{\cos x}}$

To find the derivative of a function of the form $k(x) = f(g(h(x)))$, you need apply the Chain Rule twice, as shown in Example 12.

Example 12 Repeated Application of the Chain Rule

$$\begin{aligned}
f(t) &= \sin^3 4t && \text{Original function}\\
&= (\sin 4t)^3 && \text{Rewrite.}\\
f'(t) &= 3(\sin 4t)^2 \frac{d}{dt}[\sin 4t] && \text{Apply Chain Rule once.}\\
&= 3(\sin 4t)^2(\cos 4t)\frac{d}{dt}[4t] && \text{Apply Chain Rule a second time.}\\
&= 3(\sin 4t)^2(\cos 4t)(4) &&\\
&= 12 \sin^2 4t \cos 4t && \text{Simplify.}
\end{aligned}$$

We conclude this section with a summary of the differentiation rules studied so far. To become skilled at differentiation, you should memorize each rule.

Summary of Differentiation Rules

General Differentiation Rules Let f, g, and u be differentiable functions of x.

Constant Multiple Rule:

$$\frac{d}{dx}[cf] = cf'$$

Sum or Difference Rule:

$$\frac{d}{dx}[f \pm g] = f' \pm g'$$

Product Rule:

$$\frac{d}{dx}[fg] = fg' + gf'$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$$

Derivatives of Algebraic Functions

Constant Rule:

$$\frac{d}{dx}[c] = 0$$

(Simple) Power Rule:

$$\frac{d}{dx}[x^n] = nx^{n-1}, \quad \frac{d}{dx}[x] = 1$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

Chain Rule

Chain Rule:

$$\frac{d}{dx}[f(u)] = f'(u)\, u'$$

General Power Rule:

$$\frac{d}{dx}[u^n] = nu^{n-1}\, u'$$

STUDY TIP As an aid to memorization, note that the cofunctions (cosine, cotangent, and cosecant) require a negative sign as part of their derivatives.

EXERCISES FOR SECTION 2.4

Exercises 1–6, complete the table using Example 2 as a model.

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
$y = (6x - 5)^4$		
$y = \dfrac{1}{\sqrt{x+1}}$		
$y = \sqrt{x^2 - 1}$		
$y = 3\tan(\pi x^2)$		
$y = \csc^3 x$		
$y = \cos\dfrac{3x}{2}$		

Exercises 7–34, find the derivative of the function.

$y = (2x - 7)^3$

8. $y = (2x^3 + 1)^2$

$g(x) = 3(4 - 9x)^4$

10. $y = 3(4 - x^2)^5$

$f(x) = (9 - x^2)^{2/3}$

12. $f(t) = (9t + 2)^{2/3}$

$f(t) = \sqrt{1 - t}$

14. $g(x) = \sqrt{5 - 3x}$

15. $y = \sqrt[3]{9x^2 + 4}$

16. $g(x) = \sqrt{x^2 - 2x + 1}$

17. $y = 2\sqrt[4]{4 - x^2}$

18. $f(x) = -3\sqrt[4]{2 - 9x}$

19. $y = \dfrac{1}{x - 2}$

20. $s(t) = \dfrac{1}{t^2 + 3t - 1}$

21. $f(t) = \left(\dfrac{1}{t - 3}\right)^2$

22. $y = -\dfrac{5}{(t + 3)^3}$

23. $y = \dfrac{1}{\sqrt{x + 2}}$

24. $g(t) = \sqrt{\dfrac{1}{t^2 - 2}}$

25. $f(x) = x^2(x - 2)^4$

26. $f(x) = x(3x - 9)^3$

27. $y = x\sqrt{1 - x^2}$

28. $y = \frac{1}{2}x^2\sqrt{16 - x^2}$

29. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

30. $y = \dfrac{x}{\sqrt{x^4 + 4}}$

31. $g(x) = \left(\dfrac{x + 5}{x^2 + 2}\right)^2$

32. $h(t) = \left(\dfrac{t^2}{t^3 + 2}\right)^2$

33. $f(v) = \left(\dfrac{1 - 2v}{1 + v}\right)^3$

34. $g(x) = \left(\dfrac{3x^2 - 2}{2x + 3}\right)^3$

In Exercises 35–44, use a computer algebra system to find the derivative of the function. Then use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.

35. $y = \dfrac{\sqrt{x}+1}{x^2+1}$

36. $y = \sqrt{\dfrac{2x}{x+1}}$

37. $g(t) = \dfrac{3t^2}{\sqrt{t^2+2t-1}}$

38. $f(x) = \sqrt{x}(2-x)^2$

39. $y = \sqrt{\dfrac{x+1}{x}}$

40. $y = (t^2-9)\sqrt{t+2}$

41. $s(t) = \dfrac{-2(2-t)\sqrt{1+t}}{3}$

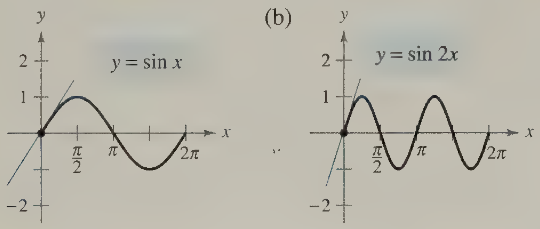
42. $g(x) = \sqrt{x-1}+\sqrt{x+1}$

43. $y = \dfrac{\cos \pi x + 1}{x}$

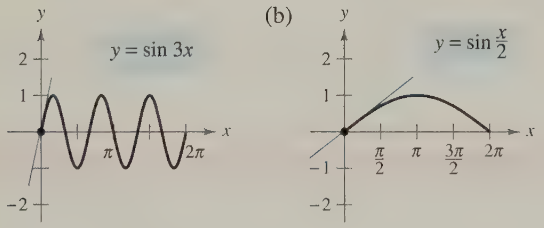
44. $y = x^2 \tan \dfrac{1}{x}$

In Exercises 45 and 46, find the slope of the tangent line to the sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$. What can you conclude about the slope of the sine function sin ax at the origin?

45. (a) (b)



46. (a) (b)



In Exercises 47–66, find the derivative of the function.

47. $y = \cos 3x$

48. $y = \sin \pi x$

49. $g(x) = 3 \tan 4x$

50. $h(x) = \sec x^2$

51. $y = \sin(\pi x)^2$

52. $y = \cos(1-2x)^2$

53. $h(x) = \sin 2x \cos 2x$

54. $g(\theta) = \sec\left(\frac{1}{2}\theta\right)\tan\left(\frac{1}{2}\theta\right)$

55. $f(x) = \dfrac{\cot x}{\sin x}$

56. $g(v) = \dfrac{\cos v}{\csc v}$

57. $y = 4\sec^2 x$

58. $y = 2\tan^3 x$

59. $f(\theta) = \frac{1}{4}\sin^2 2\theta$

60. $g(t) = 5\cos^2 \pi t$

61. $f(t) = 3\sec^2(\pi t - 1)$

62. $h(t) = 2\cot^2(\pi t + 2)$

63. $y = \sqrt{x} + \frac{1}{4}\sin(2x)^2$

64. $y = 3x - 5\cos(\pi x)^2$

65. $y = \sin(\cos x)$

66. $y = \sin \sqrt[3]{x} + \sqrt[3]{\sin x}$

In Exercises 67–74, evaluate the derivative of the function at indicated point. Use a graphing utility to verify your result.

Function	*Point*
67. $s(t) = \sqrt{t^2+2t+8}$	$(2, 4)$
68. $y = \sqrt[5]{3x^3+4x}$	$(2, 2)$
69. $f(x) = \dfrac{3}{x^3-4}$	$\left(-1, -\frac{3}{5}\right)$
70. $f(x) = \dfrac{1}{(x^2-3x)^2}$	$\left(4, \frac{1}{16}\right)$
71. $f(t) = \dfrac{3t+2}{t-1}$	$(0, -2)$
72. $f(x) = \dfrac{x+1}{2x-3}$	$(2, 3)$
73. $y = 37 - \sec^3(2x)$	$(0, 36)$
74. $y = \dfrac{1}{x} + \sqrt{\cos x}$	$\left(\frac{\pi}{2}, \frac{2}{\pi}\right)$

In Exercises 75–78, (a) find an equation of the tangent lin the graph of f at the indicated point, (b) use a graphing uti to graph the function and its tangent line at the point, and use the *derivative* feature of a graphing utility to confirm y results.

Function	*Point*
75. $f(x) = \sqrt{3x^2-2}$	$(3, 5)$
76. $f(x) = \frac{1}{3}x\sqrt{x^2+5}$	$(2, 2)$
77. $f(x) = \sin 2x$	$(\pi, 0)$
78. $f(x) = \tan^2 x$	$\left(\frac{\pi}{4}, 1\right)$

In Exercises 79–82, find the second derivative of the functio

79. $f(x) = 2(x^2-1)^3$

80. $f(x) = \dfrac{1}{x-2}$

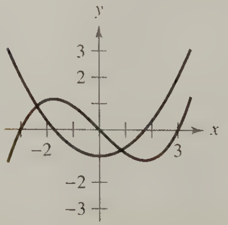
81. $f(x) = \sin x^2$

82. $f(x) = \sec^2 \pi x$

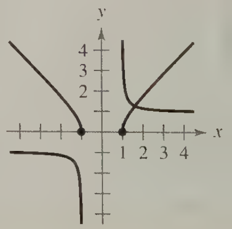
Getting at the Concept

In Exercises 83–86, the graphs of a function f and its derivative f' are shown. Label the graphs as f or f' and write a short paragraph stating the criteria used in making th selection. To print an enlarged copy of the graph, go to th website *www.mathgraphs.com*.

83.

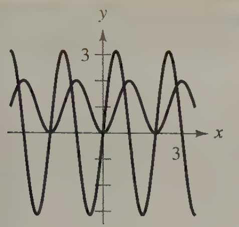


84.

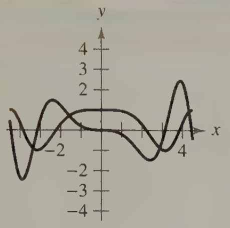


Getting at the Concept *(continued)*

85.



86.



In Exercises 87 and 88, the relationship between f and g is given. State the relationship between f' and g'.

87. $g(x) = f(3x)$

88. $g(x) = f(x^2)$

89. Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ (if possible) for each of the following. If it is not possible, state what additional information is required.

(a) $f(x) = g(x)h(x)$

(b) $f(x) = g(h(x))$

(c) $f(x) = \dfrac{g(x)}{h(x)}$

(d) $f(x) = [g(x)]^3$

). (a) Find the derivative of the function $g(x) = \sin^2 x + \cos^2 x$ in two ways.

(b) For $f(x) = \sec^2 x$ and $g(x) = \tan^2 x$, show that $f'(x) = g'(x)$.

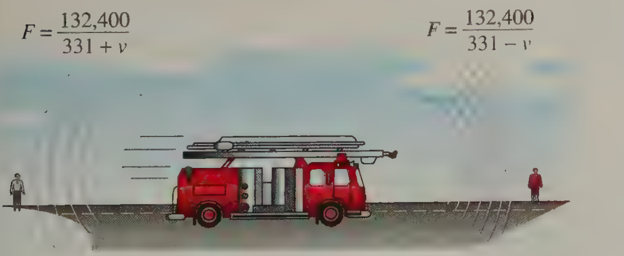
1. ***Doppler Effect*** The frequency F of a fire truck siren heard by a stationary observer is

$$F = \frac{132{,}400}{331 \pm v}$$

where $\pm v$ represents the velocity of the accelerating fire truck in meters per second (see figure). Find the rate of change of F with respect to v when

(a) the fire truck is approaching at a velocity of 30 meters per second (use $-v$).

(b) the fire truck is moving away at a velocity of 30 meters per second (use $+v$).



$F = \dfrac{132{,}400}{331 + v}$
$F = \dfrac{132{,}400}{331 - v}$

. ***Harmonic Motion*** The displacement from equilibrium of an object in harmonic motion on the end of a spring is

$$y = \tfrac{1}{3}\cos 12t - \tfrac{1}{4}\sin 12t$$

where y is measured in feet and t is the time in seconds. Determine the position and velocity of the object when $t = \pi/8$.

93. ***Pendulum*** A 15-centimeter pendulum moves according to the equation

$$\theta = 0.2 \cos 8t$$

where θ is the angular displacement from the vertical in radians and t is the time in seconds. Determine the maximum angular displacement and the rate of change of θ when $t = 3$ seconds.

94. ***Wave Motion*** A buoy oscillates in simple harmonic motion

$$y = A \cos \omega t$$

as waves move past it. The buoy moves a total of 3.5 feet (vertically) from its low point to its high point. It returns to its high point every 10 seconds.

(a) Write an equation describing the motion of the buoy if it is at its high point at $t = 0$.

(b) Determine the velocity of the buoy as a function of t.

95. ***Circulatory System*** The speed S of blood that is r centimeters from the center of an artery is

$$S = C(R^2 - r^2)$$

where C is a constant, R is the radius of the artery, and S is measured in centimeters per second. Suppose a drug is administered and the artery begins to dilate at a rate of dR/dt. At a constant distance r, find the rate at which S changes with respect to t for $C = 1.76 \times 10^5$, $R = 1.2 \times 10^{-2}$, and $dR/dt = 10^{-5}$.

96. ***Modeling Data*** The normal daily maximum temperature T (in degrees Fahrenheit) for Denver, Colorado, is shown in the table. *(Source: National Oceanic and Atmospheric Administration)*

Month	Jan	Feb	Mar	Apr	May	Jun
Temperature	43.2	46.6	52.2	61.8	70.8	81.4

Month	Jul	Aug	Sep	Oct	Nov	Dec
Temperature	88.2	85.8	76.9	66.3	52.5	44.5

(a) Use a graphing utility to plot the data and find a model for the data of the form

$$T(t) = a + b \sin(\pi t/6 - c)$$

where T is the temperature and t is the time in months, with $t = 1$ corresponding to January.

(b) Use a graphing utility to graph the model. How well does the model fit the data?

(c) Find T' and use a graphing utility to graph the derivative.

(d) Based on the graph of the derivative, during what times does the temperature change most rapidly? Most slowly? Do your answers agree with your observations of the temperature changes? Explain.

97. ***Modeling Data*** The cost of producing x units of a product is $C = 60x + 1350$. For one week management determined the number of units produced at the end of t hours during an 8-hour shift. The average values of x for the week are shown in the table.

t	0	1	2	3	4	5	6	7	8
x	0	16	60	130	205	271	336	384	392

(a) Use a graphing utility to fit a cubic model to the data.

(b) Use the Chain Rule to find dC/dt.

(c) Explain why the cost function is not increasing at a constant rate during the 8-hour shift.

98. ***Think About It*** The table shows some values of the derivative of an unknown function f. Complete the table by finding (if possible) the derivative of each transformation of f.

(a) $g(x) = f(x) - 2$ (b) $h(x) = 2f(x)$

(c) $r(x) = f(-3x)$ (d) $s(x) = f(x + 2)$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						

99. ***Finding a Pattern*** Consider the function $f(x) = \sin \beta x$, where β is a constant.

(a) Find the first-, second-, third-, and fourth-order derivatives of the function.

(b) Verify that the function and its second derivative satisfy the equation $f''(x) + \beta^2 f(x) = 0$.

(c) Use the results in part (a) to write general rules for the even- and odd-order derivatives

$f^{(2k)}(x)$ and $f^{(2k-1)}(x)$.

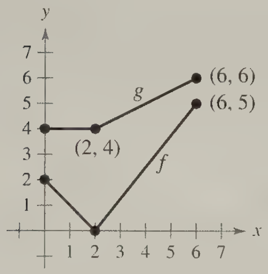
[*Hint:* $(-1)^k$ is positive if k is even and negative if k is odd.]

100. ***Conjecture*** Let f be a differentiable function of period p.

(a) Is the function f' periodic? Verify your answer.

(b) Consider the function $g(x) = f(2x)$. Is the function $g'(x)$ periodic? Verify your answer.

101. ***Think About It*** Let $r(x) = f(g(x))$ and $s(x) = g(f(x))$ where f and g are shown in the figure. Find (a) $r'(1)$ and (b) $s'(4)$.



102. Show that the derivative of an odd function is even. That is, $f(-x) = -f(x)$, then $f'(-x) = f'(x)$.

103. The geometric mean of x and $x + n$ is $g = \sqrt{x(x + n)}$, a the arithmetic mean is $a = [x + (x + n)]/2$. Show that

$$\frac{dg}{dx} = \frac{a}{g}.$$

104. Let u be a differentiable function of x. Use the fact th $|u| = \sqrt{u^2}$ to prove that

$$\frac{d}{dx}[|u|] = u'\frac{u}{|u|}, \quad u \neq 0.$$

In Exercises 105–108, use the result of Exercise 104 to find t derivative of the function.

105. $g(x) = |2x - 3|$

106. $f(x) = |x^2 - 4|$

107. $h(x) = |x| \cos x$

108. $f(x) = |\sin x|$

Linear and Quadratic Approximations **The linear and qua ratic approximations of a function f at $x = a$ are**

$$P_1(x) = f'(a)(x - a) + f(a) \text{ and}$$
$$P_2(x) = \tfrac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a).$$

In Exercises 109 and 110, (a) find the specified linear a quadratic approximations of f, (b) use a graphing utility graph f and the approximations, (c) determine whether P_1 P_2 is the better approximation, and (d) state how the accura changes as you move farther from $x = a$.

109. $f(x) = \tan \dfrac{\pi x}{4}$

$a = 1$

110. $f(x) = \sec 2x$

$a = \dfrac{\pi}{6}$

True or False? **In Exercises 111–114, determine whether t statement is true or false. If it is false, explain why or give example that shows it is false.**

111. If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}$.

112. If $f(x) = \sin^2(2x)$, then $f'(x) = 2(\sin 2x)(\cos 2x)$.

113. If y is a differentiable function of u, u is a differentiable fur tion of v, and v is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

114. You would first apply the General Power Rule to find t derivative of $y = x \sin^3 x$.

Section 2.5 Implicit Differentiation

- Distinguish between functions written in implicit form and explicit form.
- Use implicit differentiation to find the derivative of a function.

EXPLORATION

raphing an Implicit Equation How ould you use a graphing utility to ketch the graph of the equation

$$x^2 - 2y^3 + 4y = 2?$$

Iere are two possible approaches.

. Solve the equation for x. Switch the roles of x and y and graph the two resulting equations. The combined graphs will show a 90° rotation of the graph of the original equation.

. Set the graphing utility to *parametric mode* and graph the equations

$$x = -\sqrt{2t^3 - 4t + 2}$$
$$y = t$$

and

$$x = \sqrt{2t^3 - 4t + 2}$$
$$y = t.$$

rom either of these two approaches, an you decide whether the graph as a tangent line at the point $(0, 1)$? xplain your reasoning.

Implicit and Explicit Functions

Up to this point in the text, most functions have been expressed in **explicit form.** For example, in the equation

$$y = 3x^2 - 5 \qquad \text{Explicit form}$$

the variable y is explicitly written as a function of x. Some functions, however, are only implied by an equation. For instance, the function $y = 1/x$ is defined **implicitly** by the equation $xy = 1$. Suppose you were asked to find dy/dx for this equation. For this equation, you could begin by writing y explicitly as a function of x and then differentiating.

Implicit Form	*Explicit Form*	*Derivative*
$xy = 1$	$y = \frac{1}{x} = x^{-1}$	$\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$

This strategy works well whenever you can solve for the function explicitly. You cannot, however, use this procedure when you are unable to solve for y as a function of x. For instance, how would you find dy/dx for the equation

$$x^2 - 2y^3 + 4y = 2$$

where it is very difficult to express y as a function of x explicitly? To do this, you can use **implicit differentiation.**

To understand how to find dy/dx implicitly, you must realize that the differentiation is taking place *with respect to x*. This means that when you differentiate terms involving x alone, you can differentiate as usual. However, when you differentiate terms involving y, you must apply the Chain Rule, because you are assuming that y is defined implicitly as a differentiable function of x.

Example 1 Differentiating with Respect to x

a. $\frac{d}{dx}[x^3] = 3x^2$ Variables agree: use Simple Power Rule.

Variables agree

b. $\frac{d}{dx}\overbrace{[y^3]}^{u^n} = \overbrace{3y^2}^{nu^{n-1}}\overbrace{\frac{dy}{dx}}^{u'}$ Variables disagree: use Chain Rule.

Variables disagree

c. $\frac{d}{dx}[x + 3y] = 1 + 3\frac{dy}{dx}$ Chain Rule: $\frac{d}{dx}[3y] = 3y'$

d.
$$\begin{aligned} \frac{d}{dx}[xy^2] &= x\frac{d}{dx}[y^2] + y^2\frac{d}{dx}[x] && \text{Product Rule} \\ &= x\left(2y\frac{dy}{dx}\right) + y^2(1) && \text{Chain Rule} \\ &= 2xy\frac{dy}{dx} + y^2 && \text{Simplify.} \end{aligned}$$

Implicit Differentiation

Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation *with respect to x*.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx by dividing both sides of the equation by the left-hand factor that does not contain dy/dx.

Example 2 Implicit Differentiation

Find dy/dx given that $y^3 + y^2 - 5y - x^2 = -4$.

NOTE In Example 2, note that implicit differentiation can produce an expression for dy/dx that contains both x and y.

Solution

1. Differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}[y^3 + y^2 - 5y - x^2] = \frac{d}{dx}[-4]$$

$$\frac{d}{dx}[y^3] + \frac{d}{dx}[y^2] - \frac{d}{dx}[5y] - \frac{d}{dx}[x^2] = \frac{d}{dx}[-4]$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

2. Collect the dy/dx terms on the left side of the equation.

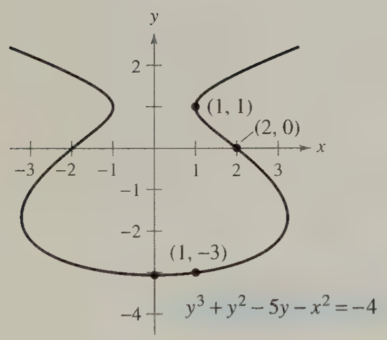
$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} = 2x$$

3. Factor dy/dx out of the left side of the equation.

$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x$$

4. Solve for dy/dx by dividing by $(3y^2 + 2y - 5)$.

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$



Point on Graph	*Slope of Graph*
$(2, 0)$	$-\frac{4}{5}$
$(1, -3)$	$\frac{1}{8}$
$x = 0$	0
$(1, 1)$	Undefined

The implicit equation

$$y^3 + y^2 - 5y - x^2 = -4$$

has the derivative

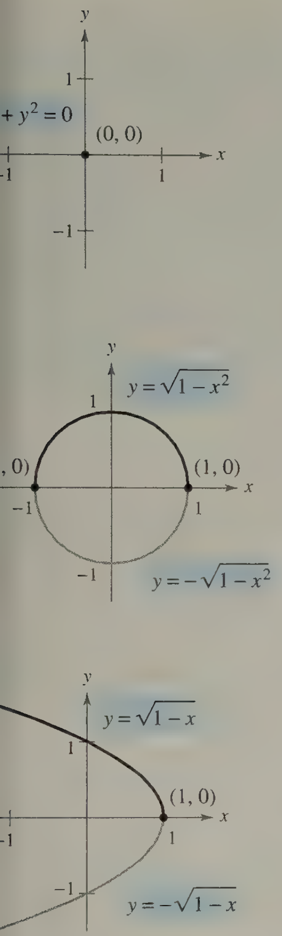
$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}.$$

Figure 2.25

To see how you can use an *implicit derivative*, consider the graph shown in Figu 2.25. From the graph, you can see that y is not a function of x. Even so, the derivati found in Example 2 gives a formula for the slope of the tangent line at a point on t graph. The slopes at several points on the graph are shown below the graph.

TECHNOLOGY With most graphing utilities, it is easy to sketch the graph an equation that explicitly represents y as a function of x. Sketching graphs other equations, however, can require some ingenuity. For instance, to ske the graph of the equation given in Example 2, try using a graphing utility, set parametric mode, to sketch the graphs given by $x = \sqrt{t^3 + t^2 - 5t + 4}$, $y = t$, a $x = -\sqrt{t^3 + t^2 - 5t + 4}$, $y = t$, for $-5 \le t \le 5$. How does the result comp with the graph shown in Figure 2.25?

It is meaningless to solve for dy/dx in an equation that has no solution points. (For example, $x^2 + y^2 = -4$ has no solution points.) If, however, a segment of a graph can be represented by a differentiable function, dy/dx will have meaning as the slope at each point on the segment. Recall that a function is not differentiable at (1) points with vertical tangents and (2) points at which the function is not continuous.



ne graph segments can be repre-
ted by differentiable functions.

ure 2.26

Example 3 Representing a Graph by Differentiable Functions

If possible, represent y as a differentiable function of x (see Figure 2.26).

a. $x^2 + y^2 = 0$ **b.** $x^2 + y^2 = 1$ **c.** $x + y^2 = 1$

Solution

a. The graph of this equation is a single point. Therefore, it does not define y as a differentiable function of x.

b. The graph of this equation is the unit circle, centered at (0, 0). The upper semicircle is given by the differentiable function

$$y = \sqrt{1 - x^2}, \quad -1 < x < 1$$

and the lower semicircle is given by the differentiable function

$$y = -\sqrt{1 - x^2}, \quad -1 < x < 1.$$

At the points $(-1, 0)$ and $(1, 0)$, the slope of the graph is undefined.

c. The upper half of this parabola is given by the differentiable function

$$y = \sqrt{1 - x}, \quad x < 1$$

and the lower half of this parabola is given by the differentiable function

$$y = -\sqrt{1 - x}, \quad x < 1.$$

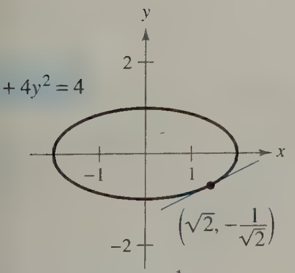
At the point $(1, 0)$, the slope of the graph is undefined.

Example 4 Finding the Slope of a Graph Implicitly

Determine the slope of the tangent line to the graph of

$$x^2 + 4y^2 = 4$$

at the point $(\sqrt{2}, -1/\sqrt{2})$. (See Figure 2.27.)



pe of tangent line is $\frac{1}{2}$.

ure 2.27

Solution

$$x^2 + 4y^2 = 4 \qquad \text{Write original equation.}$$

$$2x + 8y\frac{dy}{dx} = 0 \qquad \text{Differentiate with respect to } x.$$

$$\frac{dy}{dx} = \frac{-2x}{8y} = \frac{-x}{4y}. \qquad \text{Solve for } \frac{dy}{dx}.$$

So, at $(\sqrt{2}, -1/\sqrt{2})$, the slope is

$$\frac{dy}{dx} = \frac{-\sqrt{2}}{-4/\sqrt{2}} = \frac{1}{2}. \qquad \text{Evaluate } \frac{dy}{dx} \text{ when } x = \sqrt{2} \text{ and } y = -\frac{1}{\sqrt{2}}.$$

NOTE To see the benefit of implicit differentiation, try doing Example 4 using the explicit function $y = -\frac{1}{2}\sqrt{4 - x^2}$.

Example 5 Finding the Slope of a Graph Implicitly

Determine the slope of the graph of $3(x^2 + y^2)^2 = 100xy$ at the point $(3, 1)$.

Solution

$$\frac{d}{dx}[3(x^2 + y^2)^2] = \frac{d}{dx}[100xy]$$

$$3(2)(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 100\left[x\frac{dy}{dx} + y(1)\right]$$

$$12y(x^2 + y^2)\frac{dy}{dx} - 100x\frac{dy}{dx} = 100y - 12x(x^2 + y^2)$$

$$[12y(x^2 + y^2) - 100x]\frac{dy}{dx} = 100y - 12x(x^2 + y^2)$$

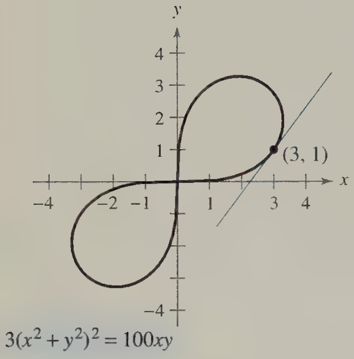
$$\frac{dy}{dx} = \frac{100y - 12x(x^2 + y^2)}{-100x + 12y(x^2 + y^2)}$$

$$= \frac{25y - 3x(x^2 + y^2)}{-25x + 3y(x^2 + y^2)}$$

At the point $(3, 1)$, the slope of the graph is

$$\frac{dy}{dx} = \frac{25(1) - 3(3)(3^2 + 1^2)}{-25(3) + 3(1)(3^2 + 1^2)} = \frac{25 - 90}{-75 + 30} = \frac{-65}{-45} = \frac{13}{9}$$

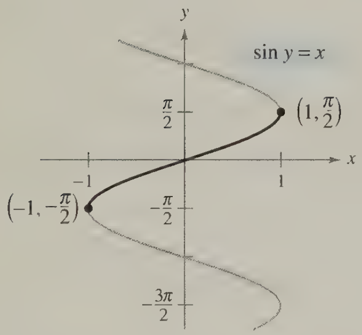
as shown in Figure 2.28. This graph is called a **lemniscate.**



Lemniscate
Figure 2.28

Example 6 Determining a Differentiable Function

Find dy/dx implicitly for the equation $\sin y = x$. Then find the largest interval of form $-a < y < a$ such that y is a differentiable function of x (see Figure 2.29).



The derivative is $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$.

Figure 2.29

Solution

$$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$$

$$\cos y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

The largest interval about the origin for which y is a differentiable function of $-\pi/2 < y < \pi/2$. To see this, note that $\cos y$ is positive for all y in this interval is 0 at the endpoints. If you restrict y to the interval $-\pi/2 < y < \pi/2$, you should able to write dy/dx explicitly as a function of x. To do this, you can use

$$\cos y = \sqrt{1 - \sin^2 y}$$

$$= \sqrt{1 - x^2}, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

and conclude that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

The Granger Collection

ISAAC BARROW (1630–1677)

The graph in Example 8 is called the **kappa curve** because it resembles the Greek letter kappa, κ. The general solution for the tangent line to this curve was discovered by the English mathematician Isaac Barrow. Newton was Barrow's student and they corresponded frequently regarding their work in the early development of calculus.

With implicit differentiation, the form of the derivative often can be simplified (as in Example 6) by an appropriate use of the *original* equation. A similar technique can be used to find and simplify higher-order derivatives obtained implicitly.

Example 7 Finding the Second Derivative Implicitly

Given $x^2 + y^2 = 25$, find $\dfrac{d^2y}{dx^2}$.

Solution Differentiating each term with respect to x produces

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} = \frac{-2x}{2y} &= -\frac{x}{y}. \end{aligned}$$

Differentiating a second time with respect to x yields

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{(y)(1) - (x)(dy/dx)}{y^2} && \text{Quotient Rule} \\ &= -\frac{y - (x)(-x/y)}{y^2} && \text{Substitute } -x/y \text{ for } \frac{dy}{dx}. \\ &= -\frac{y^2 + x^2}{y^3} && \text{Simplify.} \\ &= -\frac{25}{y^3}. && \text{Substitute 25 for } x^2 + y^2. \end{aligned}$$

Example 8 Finding a Tangent Line to a Graph

Find the tangent line to the graph given by $x^2(x^2 + y^2) = y^2$ at the point $(\sqrt{2}/2, \sqrt{2}/2)$, as shown in Figure 2.30.

Solution By rewriting and differentiating implicitly, you obtain

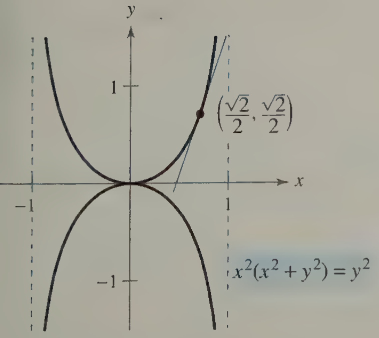
$$\begin{aligned} x^4 + x^2y^2 - y^2 &= 0 \\ 4x^3 + x^2\left(2y\frac{dy}{dx}\right) + 2xy^2 - 2y\frac{dy}{dx} &= 0 \\ 2y(x^2 - 1)\frac{dy}{dx} &= -2x(2x^2 + y^2) \\ \frac{dy}{dx} &= \frac{x(2x^2 + y^2)}{y(1 - x^2)}. \end{aligned}$$

At the point $(\sqrt{2}/2, \sqrt{2}/2)$, the slope is

$$\frac{dy}{dx} = \frac{(\sqrt{2}/2)[2(1/2) + (1/2)]}{(\sqrt{2}/2)[1 - (1/2)]} = \frac{3/2}{1/2} = 3$$

and the equation of the tangent line at this point is

$$\begin{aligned} y - \frac{\sqrt{2}}{2} &= 3\left(x - \frac{\sqrt{2}}{2}\right) \\ y &= 3x - \sqrt{2}. \end{aligned}$$



he kappa curve
gure 2.30

EXERCISES FOR SECTION 2.5

In Exercises 1–16, find dy/dx by implicit differentiation.

1. $x^2 + y^2 = 36$

2. $x^2 - y^2 = 16$

3. $x^{1/2} + y^{1/2} = 9$

4. $x^3 + y^3 = 8$

5. $x^3 - xy + y^2 = 4$

6. $x^2y + y^2x = -2$

7. $x^3y^3 - y = x$

8. $\sqrt{xy} = x - 2y$

9. $x^3 - 3x^2y + 2xy^2 = 12$

10. $2 \sin x \cos y = 1$

11. $\sin x + 2 \cos 2y = 1$

12. $(\sin \pi x + \cos \pi y)^2 = 2$

13. $\sin x = x(1 + \tan y)$

14. $\cot y = x - y$

15. $y = \sin(xy)$

16. $x = \sec \frac{1}{y}$

In Exercises 17–20, (a) find two explicit functions by solving the equation for y in terms of x, (b) sketch the graph of the equation and label the parts given by the corresponding explicit functions, (c) differentiate the explicit functions, and (d) find dy/dx implicitly and show that the result is equivalent to that of part (c).

17. $x^2 + y^2 = 16$

18. $x^2 + y^2 - 4x + 6y + 9 = 0$

19. $9x^2 + 16y^2 = 144$

20. $9y^2 - x^2 = 9$

In Exercises 21–28, find dy/dx by implicit differentiation and evaluate the derivative at the indicated point.

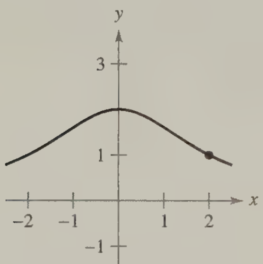
	Equation	*Point*
21.	$xy = 4$	$(-4, -1)$
22.	$x^2 - y^3 = 0$	$(1, 1)$
23.	$y^2 = \frac{x^2 - 4}{x^2 + 4}$	$(2, 0)$
24.	$(x + y)^3 = x^3 + y^3$	$(-1, 1)$
25.	$x^{2/3} + y^{2/3} = 5$	$(8, 1)$
26.	$x^3 + y^3 = 4xy + 1$	$(2, 1)$
27.	$\tan(x + y) = x$	$(0, 0)$
28.	$x \cos y = 1$	$\left(2, \frac{\pi}{3}\right)$

In Exercises 29–32, find the slope of the tangent line to the graph at the indicated point.

29. Witch of Agnesi:

$(x^2 + 4)y = 8$

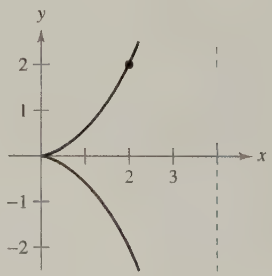
Point: $(2, 1)$



30. Cissoid:

$(4 - x)y^2 = x^3$

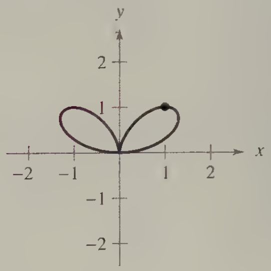
Point: $(2, 2)$



31. Bifolium:

$(x^2 + y^2)^2 = 4x^2y$

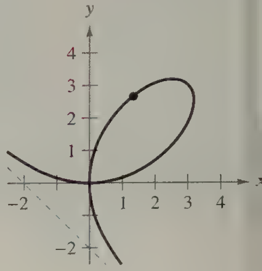
Point: $(1, 1)$



32. Folium of Descartes:

$x^3 + y^3 - 6xy = 0$

Point: $\left(\frac{4}{3}, \frac{8}{3}\right)$



In Exercises 33 and 34, find dy/dx implicitly and find the large interval of the form $-a < y < a$ such that y is a differentiab function of x. Then express dy/dx as a function of x.

33. $\tan y = x$

34. $\cos y = x$

In Exercises 35–40, find d^2y/dx^2 in terms of x and y.

35. $x^2 + y^2 = 36$

36. $x^2y^2 - 2x = 3$

37. $x^2 - y^2 = 16$

38. $1 - xy = x - y$

39. $y^2 = x^3$

40. $y^2 = 4x$

In Exercises 41 and 42, use a graphing utility to graph the equ tion. Find an equation of the tangent line to the graph at t indicated point and sketch its graph.

41. $\sqrt{x} + \sqrt{y} = 4, \quad (9, 1)$

42. $y^2 = \frac{x - 1}{x^2 + 1}, \quad \left(2, \frac{\sqrt{5}}{5}\right)$

In Exercises 43 and 44, find equations for the tangent line ar normal line to the circle at the indicated points. (The *norm line* at a point is perpendicular to the tangent line at the poin Use a graphing utility to graph the equation, tangent line, ar normal line.

43. $x^2 + y^2 = 25$

$(4, 3), (-3, 4)$

44. $x^2 + y^2 = 9$

$(0, 3), (2, \sqrt{5})$

45. Show that the normal line at any point on the circ $x^2 + y^2 = r^2$ passes through the origin.

46. Two circles of radius 4 are tangent to the graph of $y^2 = 4x$ the point $(1, 2)$. Find equations of these two circles.

In Exercises 47 and 48, find the points at which the graph of t equation has a vertical or horizontal tangent line.

47. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

48. $4x^2 + y^2 - 8x + 4y + 4 = 0$

hogonal Trajectories **In Exercises 49–52, use a graphing ty to sketch the intersecting graphs of the equations and w that they are orthogonal. [Two graphs are *orthogonal* if at r point(s) of intersection their tangent lines are perpendicu- to each other.]**

$2x^2 + y^2 = 6$ **50.** $y^2 = x^3$

$y^2 = 4x$ $2x^2 + 3y^2 = 5$

$x + y = 0$ **52.** $x^3 = 3(y - 1)$

$x = \sin y$ $x(3y - 29) = 3$

hogonal Trajectories **In Exercises 53 and 54, verify that the families of curves are orthogonal where C and K are real mbers. Use a graphing utility to graph the two families for values of C and two values of K.**

$xy = C$ **54.** $x^2 + y^2 = C^2$

$x^2 - y^2 = K$ $y = Kx$

Exercises 55–58, differentiate (a) with respect to x (y is a func- of x) and (b) with respect to t (x and y are functions of t).

$2y^2 - 3x^4 = 0$ **56.** $x^2 - 3xy^2 + y^3 = 10$

$\cos \pi y - 3 \sin \pi x = 1$ **58.** $4 \sin x \cos y = 1$

etting at the Concept

9. Describe the difference between the explicit form of a function and an implicit equation. Give an example of each.

0. In your own words, state the guidelines for implicit differentiation.

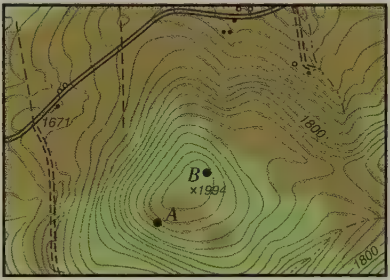
61. Consider the equation $x^4 = 4(4x^2 - y^2)$.

(a) Use a graphing utility to graph the equation.

(b) Find and graph the four tangent lines to the curve for $y = 3$.

(c) Find the exact coordinates of the point of intersection of the two tangent lines in the first quadrant.

62. ***Orthogonal Trajectories*** The figure below gives the topographic map carried by a group of hikers. The hikers are in a wooded area on top of the hill shown on the map and they decide to follow a path of steepest descent (orthogonal trajectories to the contours on the map). Draw their routes if they start from point A and if they start from point B. If their goal is to reach the road along the top of the map, which starting point should they use? To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



63. Prove (Theorem 2.3) that

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

for the case in which n is a rational number. (*Hint:* Write $y = x^{p/q}$ in the form $y^q = x^p$ and differentiate implicitly. Assume that p and q are integers, where $q > 0$.)

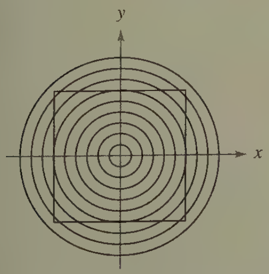
64. Let L be any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$. Show that the sum of the x- and y-intercepts of L is c.

ECTION PROJECT OPTICAL ILLUSIONS

each graph below, an optical illusion is created by having es intersect a family of curves. In each case, the lines appear be curved. Find the value of dy/dx for the indicated values x and y.

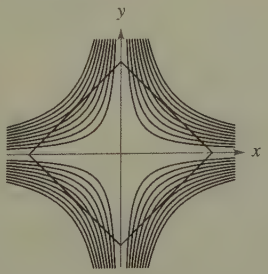
) Circles: $x^2 + y^2 = C^2$

$x = 3, y = 4, C = 5$



(b) Hyperbolas: $xy = C$

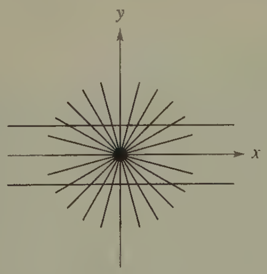
$x = 1, y = 4, C = 4$



(c) Lines: $ax = by$

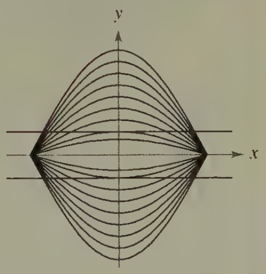
$x = \sqrt{3}, y = 3,$

$a = \sqrt{3}, b = 1$



(d) Cosine curves: $y = C \cos x$

$x = \frac{\pi}{3}, y = \frac{1}{3}, C = \frac{2}{3}$



FOR FURTHER INFORMATION For more information on the mathematics of optical illusions, see the article "Descriptive Models for Perception of Optical Illusions" by David A. Smith in *The UMAP Journal*. To view this article, go to the website *www.matharticles.com*.

Section 2.6 Related Rates

- Find a related rate.
- Use related rates to solve real-life problems.

Finding Related Rates

You have seen how the Chain Rule can be used to find dy/dx implicitly. Ano important use of the Chain Rule is to find the rates of change of two or more rel variables that are changing with respect to *time*.

For example, when water is drained out of a conical tank (see Figure 2.31), volume V, the radius r, and the height h of the water level are all functions of tim Knowing that these variables are related by the equation

$$V = \frac{\pi}{3}r^2h \qquad \text{Original equation}$$

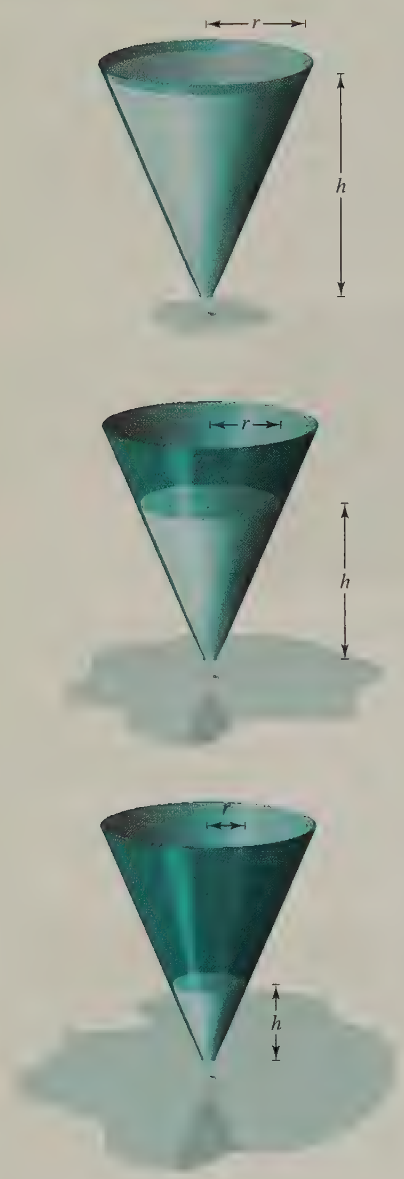
you can differentiate implicitly with respect to t to obtain the **related-rate** equati

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi}{3}r^2h\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2\frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right] \qquad \text{Differentiate with respect to } t.$$

$$= \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right).$$

From this equation you can see that the rate of change of V is related to the rate change of both h and r.



Volume is related to radius and height.
Figure 2.31

FOR FURTHER INFORMATION To learn more about the history of related-rate problems, see the article "The Lengthening Shadow: The Story of Related Rates" by Bill Austin, Don Barry, and David Berman in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com.*

EXPLORATION

Finding a Related Rate In the conical tank shown in Figure 2.31, suppose tha the height is changing at a rate of -0.2 foot per minute and the radius is changing at a rate of -0.1 foot per minute. What is the rate of change in the volume when the radius is $r = 1$ foot and the height is $h = 2$ feet? Does the rate of change in the volume depend on the values of r and h? Explain.

Example 1 Two Rates That Are Related

Suppose x and y are both differentiable functions of t and are related by the equa $y = x^2 + 3$. Find dy/dt when $x = 1$, given that $dx/dt = 2$ when $x = 1$.

Solution Using the Chain Rule, you can differentiate both sides of the equation *respect to t.*

$$y = x^2 + 3 \qquad \text{Write original equation.}$$

$$\frac{d}{dt}[y] = \frac{d}{dt}[x^2 + 3] \qquad \text{Differentiate with respect to } t.$$

$$\frac{dy}{dt} = 2x\frac{dx}{dt} \qquad \text{Chain Rule}$$

When $x = 1$ and $dx/dt = 2$, you have

$$\frac{dy}{dt} = 2(1)(2) = 4.$$

Problem Solving with Related Rates

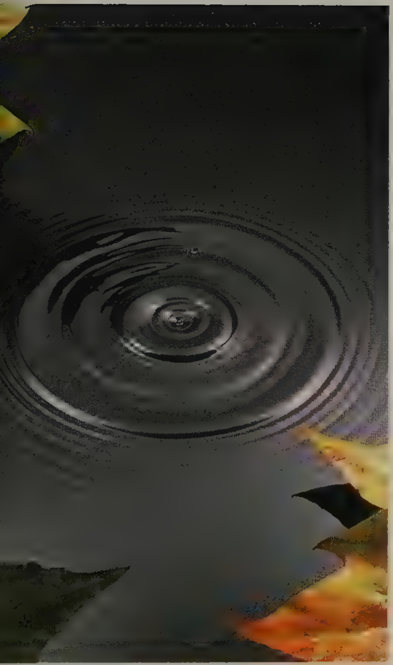
In Example 1, you were *given* an equation that related the variables x and y and were asked to find the rate of change of y when $x = 1$.

Equation: $y = x^2 + 3$

Given rate: $\frac{dx}{dt} = 2$ when $x = 1$

Find: $\frac{dy}{dt}$ when $x = 1$

In each of the remaining examples in this section, you must *create* a mathematical model from a verbal description.



tal area increases as the outer radius
creases.

gure 2.32

Example 2 Ripples in a Pond

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles, as shown in Figure 2.32. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

Solution The variables r and A are related by $A = \pi r^2$. The rate of change of the radius r is $dr/dt = 1$.

Equation: $A = \pi r^2$

Given rate: $\frac{dr}{dt} = 1$

Find: $\frac{dA}{dt}$ when $r = 4$

With this information, you can proceed as in Example 1.

$$\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2] \qquad \text{Differentiate with respect to } t.$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \qquad \text{Chain Rule}$$

$$\frac{dA}{dt} = 2\pi(4)(1) = 8\pi \qquad \text{Substitute 4 for } r \text{ and 1 for } dr/dt.$$

When the radius is 4 feet, the area is changing at a rate of 8π square feet per second.

Guidelines For Solving Related-Rate Problems

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t*.
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

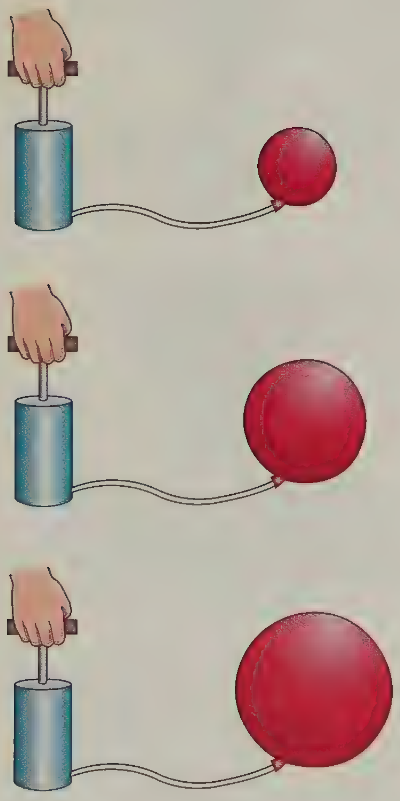
OTE When using these guidelines, be
re you perform Step 3 *before* Step 4.
bstituting the known values of the
riables before differentiating will
oduce an inappropriate derivative.

The following table lists examples of mathematical models involving rates change. For instance, the rate of change in the first example is the velocity of a car

Verbal Statement	Mathematical Model
The velocity of a car after traveling for 1 hour is 50 miles per hour.	$x =$ distance traveled $\frac{dx}{dt} = 50$ when $t = 1$
Water is being pumped into a swimming pool at a rate of 10 cubic meters per hour.	$V =$ volume of water in pool $\frac{dV}{dt} = 10\text{m}^3/\text{hr}$
A gear is revolving at a rate of 25 revolutions per minute (1 revolution $= 2\pi$ rad).	$\theta =$ angle of revolution $\frac{d\theta}{dt} = 25(2\pi)$ rad/min

Example 3 An Inflating Balloon

Air is being pumped into a spherical balloon (see Figure 2.33) at a rate of 4.5 cub feet per minute. Find the rate of change of the radius when the radius is 2 feet.



Inflating a balloon
Figure 2.33

Solution Let V be the volume of the balloon and let r be its radius. Because t volume is increasing at a rate of 4.5 cubic feet per minute, you know that at time t t rate of change of the volume is $dV/dt = \frac{9}{2}$. So, the problem can be stated as follows

Given rate: $\frac{dV}{dt} = \frac{9}{2}$ (constant rate)

Find: $\frac{dr}{dt}$ when $r = 2$

To find the rate of change of the radius, you must find an equation that relates t radius r to the volume V.

Equation: $V = \frac{4}{3}\pi r^3$ Volume of a sphere

Implicit differentiation with respect to t produces

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad \text{Differentiate with respect to } t.$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right). \qquad \text{Solve for } dr/dt.$$

Finally, when $r = 2$, the rate of change of the radius is

$$\frac{dr}{dt} = \frac{1}{16\pi}\left(\frac{9}{2}\right) \approx 0.09 \text{ foot per minute.}$$

In Example 3, note that the volume is increasing at a *constant* rate but the radi is increasing at a *variable* rate. Just because two rates are related does not mean t they are proportional. In this particular case, the radius is growing more and mo slowly as t increases. Do you see why?

Example 4 The Speed of an Airplane Tracked by Radar

An airplane is flying on a flight path that will take it directly over a radar tracking station, as shown in Figure 2.34. If s is decreasing at a rate of 400 miles per hour when $s = 10$ miles, what is the speed of the plane?

Solution Let x be the horizontal distance from the station, as shown in Figure 2.34. Notice that when $s = 10$, $x = \sqrt{10^2 - 36} = 8$.

Given rate: $ds/dt = -400$ when $s = 10$

Find: dx/dt when $s = 10$ and $x = 8$

You can find the velocity of the plane as follows.

Equation: $x^2 + 6^2 = s^2$ Pythagorean Theorem

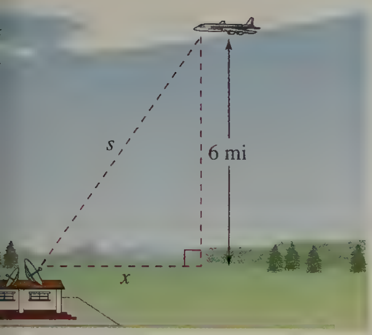
$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$ Differentiate with respect to t.

$$\frac{dx}{dt} = \frac{s}{x}\left(\frac{ds}{dt}\right)$$ Solve for dx/dt.

$$\frac{dx}{dt} = \frac{10}{8}(-400)$$ Substitute for s, x, and ds/dt.

$$= -500 \text{ miles per hour}$$ Simplify.

Because the velocity is -500 miles per hour, the *speed* is 500 miles per hour.



n airplane is flying at an altitude of 6 iles, s miles from the station.
igure 2.34

Example 5 A Changing Angle of Elevation

Find the rate of change in the angle of elevation of the camera shown in Figure 2.35 at 10 seconds after lift-off.

Solution Let θ be the angle of elevation, as shown in Figure 2.35. When $t = 10$, the height s of the rocket is $s = 50t^2 = 50(10)^2 = 5000$ feet.

Given rate: $ds/dt = 100t =$ velocity of rocket

Find: $d\theta/dt$ when $t = 10$ and $s = 5000$

Using Figure 2.35, you can relate s and θ by the equation $\tan\theta = s/2000$.

Equation: $$\tan\theta = \frac{s}{2000}$$ See Figure 2.35.

$$(\sec^2\theta)\frac{d\theta}{dt} = \frac{1}{2000}\left(\frac{ds}{dt}\right)$$ Differentiate with respect to t.

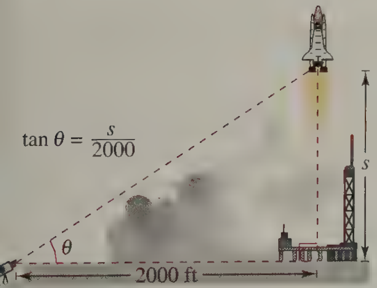
$$\frac{d\theta}{dt} = \cos^2\theta\,\frac{100t}{2000}$$ Substitute $100t$ for ds/dt.

$$= \left(\frac{2000}{\sqrt{s^2 + 2000^2}}\right)^2\frac{100t}{2000}$$ $\cos\theta = 2000/\sqrt{s^2 + 2000^2}$

When $t = 10$ and $s = 5000$, you have

$$\frac{d\theta}{dt} = \frac{2000(100)(10)}{5000^2 + 2000^2} = \frac{2}{29} \text{ radian per second.}$$

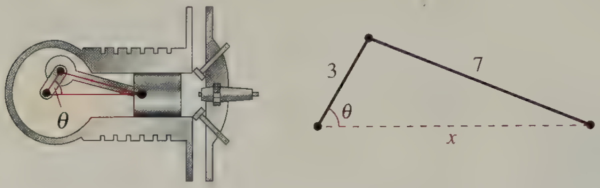
So, when $t = 10$, θ is changing at a rate of $\frac{2}{29}$ radian per second.



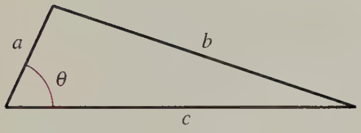
television camera at ground level is filming e lift-off of a space shuttle that is rising rtically according to the position equation $= 50t^2$, where s is measured in feet and t is easured in seconds. The camera is 2000 feet om the launch pad.
gure 2.35

Example 6 The Velocity of a Piston

In the engine shown in Figure 2.36, a 7-inch connecting rod is fastened to a cran of radius 3 inches. The crankshaft rotates counterclockwise at a constant rate of 2 revolutions per minute. Find the velocity of the piston when $\theta = \pi/3$.



The velocity of a piston is related to the angle of the crankshaft.
Figure 2.36



Law of Cosines:
$b^2 = a^2 + c^2 - 2ac \cos \theta$
Figure 2.37

Solution Label the distances as shown in Figure 2.36. Because a complete revolutio corresponds to 2π radians, it follows that $d\theta/dt = 200(2\pi) = 400\pi$ radians per minut

Given rate: $\dfrac{d\theta}{dt} = 400\pi$ (constant rate)

Find: $\dfrac{dx}{dt}$ when $\theta = \dfrac{\pi}{3}$

You can use the Law of Cosines (Figure 2.37) to find an equation that relates x and

Equation:
$$7^2 = 3^2 + x^2 - 2(3)(x) \cos \theta$$
$$0 = 2x\frac{dx}{dt} - 6\left(-x \sin \theta \frac{d\theta}{dt} + \cos \theta \frac{dx}{dt}\right)$$
$$(6 \cos \theta - 2x)\frac{dx}{dt} = 6x \sin \theta \frac{d\theta}{dt}$$
$$\frac{dx}{dt} = \frac{6x \sin \theta}{6 \cos \theta - 2x}\left(\frac{d\theta}{dt}\right)$$

When $\theta = \pi/3$, you can solve for x as follows.

$$7^2 = 3^2 + x^2 - 2(3)(x) \cos \frac{\pi}{3}$$
$$49 = 9 + x^2 - 6x\left(\frac{1}{2}\right)$$
$$0 = x^2 - 3x - 40$$
$$0 = (x - 8)(x + 5)$$
$$x = 8 \qquad \text{Choose positive solution.}$$

So, when $x = 8$ and $\theta = \pi/3$, the velocity of the piston is

$$\frac{dx}{dt} = \frac{6(8)\left(\sqrt{3}/2\right)}{6(1/2) - 16}(400\pi)$$
$$= \frac{9600\pi\sqrt{3}}{-13}$$
$$\approx -4018 \text{ inches per minute.}$$

NOTE Note that the velocity in Example 6 is negative because x represents a distance that decreasing.

XERCISES FOR SECTION 2.6

Exercises 1–4, assume that x and y are both differentiable ctions of t and find the required values of dy/dt and dx/dt.

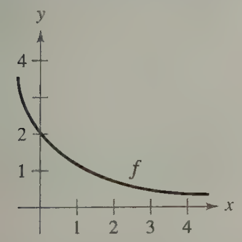
Equation	*Find*	*Given*
$= \sqrt{x}$	(a) $\frac{dy}{dt}$ when $x = 4$	$\frac{dx}{dt} = 3$
	(b) $\frac{dx}{dt}$ when $x = 25$	$\frac{dy}{dt} = 2$
$= 2(x^2 - 3x)$	(a) $\frac{dy}{dt}$ when $x = 3$	$\frac{dx}{dt} = 2$
	(b) $\frac{dx}{dt}$ when $x = 1$	$\frac{dy}{dt} = 5$
$y = 4$	(a) $\frac{dy}{dt}$ when $x = 8$	$\frac{dx}{dt} = 10$
	(b) $\frac{dx}{dt}$ when $x = 1$	$\frac{dy}{dt} = -6$
$^2 + y^2 = 25$	(a) $\frac{dy}{dt}$ when $x = 3, y = 4$	$\frac{dx}{dt} = 8$
	(b) $\frac{dx}{dt}$ when $x = 4, y = 3$	$\frac{dy}{dt} = -2$

Exercises 5–8, a point is moving along the graph of the func- such that dx/dt is 2 centimeters per second. Find dy/dt for specified values of x.

Function	*Values of x*		
$= x^2 + 1$	(a) $x = -1$	(b) $x = 0$	(c) $x = 1$
$= \frac{1}{1 + x^2}$	(a) $x = -2$	(b) $x = 0$	(c) $x = 2$
$= \tan x$	(a) $x = -\frac{\pi}{3}$	(b) $x = -\frac{\pi}{4}$	(c) $x = 0$
$= \sin x$	(a) $x = \frac{\pi}{6}$	(b) $x = \frac{\pi}{4}$	(c) $x = \frac{\pi}{3}$

etting at the Concept

Exercises 9 and 10, using the graph of f, (a) determine hether dy/dt is positive or negative given that dx/dt is egative, and (b) determine whether dx/dt is positive or egative given that dy/dt is positive.



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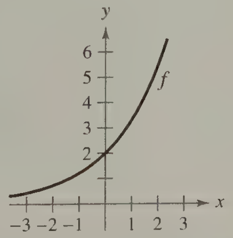


1. Consider the linear function $y = ax + b$. If x changes at a constant rate, does y change at a constant rate? If so, does it change at the same rate as x? Explain.

12. In your own words, state the guidelines for solving related rate problems.

13. Find the rate of change of the distance between the origin and a moving point on the graph of $y = x^2 + 1$ if $dx/dt = 2$ centimeters per second.

14. Find the rate of change of the distance between the origin and a moving point on the graph of $y = \sin x$ if $dx/dt = 2$ centimeters per second.

15. ***Area*** The radius r of a circle is increasing at a rate of 3 centimeters per minute. Find the rate of change of the area when (a) $r = 6$ centimeters and (b) $r = 24$ centimeters.

16. ***Area*** Let A be the area of a circle of radius r that is changing with respect to time. If dr/dt is constant, is dA/dt constant? Explain.

17. ***Area*** The included angle of the two sides of constant equal length s of an isosceles triangle is θ.

(a) Show that the area of the triangle is given by $A = \frac{1}{2}s^2 \sin \theta$.

(b) If θ is increasing at the rate of $\frac{1}{2}$ radian per minute, find the rate of change of the area when $\theta = \pi/6$ and $\theta = \pi/3$.

(c) Explain why the rate of change of the area of the triangle is not constant even though $d\theta/dt$ is constant.

18. ***Volume*** The radius r of a sphere is increasing at a rate of 2 inches per minute.

(a) Find the rate of change of the volume when $r = 6$ inches and $r = 24$ inches.

(b) Explain why the rate of change of the volume of the sphere is not constant even though dr/dt is constant.

19. ***Volume*** A spherical balloon is inflated with gas at the rate of 800 cubic centimeters per minute. How fast is the radius of the balloon increasing at the instant the radius is (a) 30 centimeters and (b) 60 centimeters?

20. ***Volume*** All edges of a cube are expanding at a rate of 3 centimeters per second. How fast is the volume changing when each edge is (a) 1 centimeter and (b) 10 centimeters?

21. ***Surface Area*** The conditions are the same as in Exercise 20. Determine how fast the *surface area* is changing when each edge is (a) 1 centimeter and (b) 10 centimeters.

22. ***Volume*** The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Find the rate of change of the volume if dr/dt is 2 inches per minute and $h = 3r$ when (a) $r = 6$ inches and (b) $r = 24$ inches.

23. ***Volume*** At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when the pile is 15 feet high?

24. ***Depth*** A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

25. ***Depth*** A swimming pool is 12 meters long, 6 meters wide, 1 meter deep at the shallow end, and 3 meters deep at the deep end (see figure). Water is being pumped into the pool at $\frac{1}{4}$ cubic meter per minute, and there is 1 meter of water at the deep end.

(a) What percent of the pool is filled?

(b) At what rate is the water level rising?

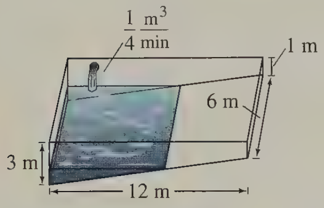


Figure for 25

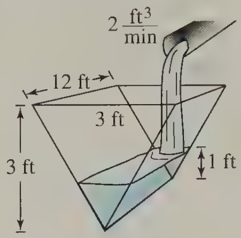


Figure for 26

26. ***Depth*** A trough is 12 feet long and 3 feet across the top (see figure). Its ends are isosceles triangles with altitudes of 3 feet.

(a) If water is being pumped into the trough at 2 cubic feet per minute, how fast is the water level rising when it is 1 foot deep?

(b) If the water is rising at a rate of $\frac{3}{8}$ inch per minute when $h = 2$, determine the rate at which water is being pumped into the trough.

27. ***Moving Ladder*** A ladder 25 feet long is leaning against the wall of a house (see figure). The base of the ladder is pulled away from the wall at a rate of 2 feet per second.

(a) How fast is the top moving down the wall when the base of the ladder is 7 feet, 15 feet, and 24 feet from the wall?

(b) Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.

(c) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 feet from the wall.

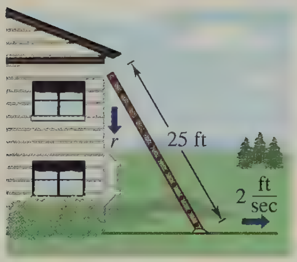


Figure for 27

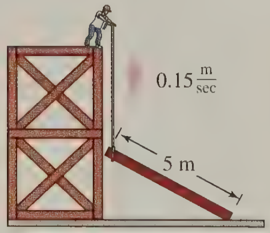


Figure for 28

FOR FURTHER INFORMATION For more information on the mathematics of moving ladders, see the article "The Falling Ladder Paradox" by Paul Scholten and Andrew Simoson in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

28. ***Construction*** A construction worker pulls a 5-meter plank up the side of a building under construction by means of a rope tied to one end of the plank (see figure). Assume the opposite end of the plank follows a path perpendicular to the wall of the building and the worker pulls the rope at a rate of 0.15 m per second. How fast is the end of the plank sliding along ground when it is 2.5 meters from the wall of the building

29. ***Construction*** A winch at the top of a 12-meter building p a pipe of the same length to a vertical position, as shown in figure. The winch pulls in rope at a rate of −0.2 meter second. Find the rate of vertical change and the rate of horiz tal change at the end of the pipe when $y = 6$.

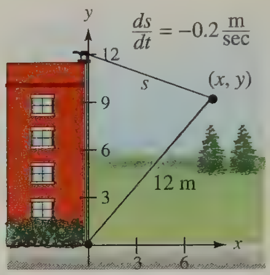
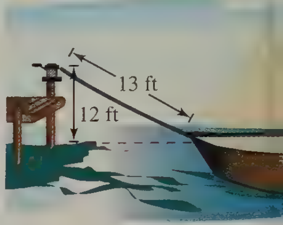


Figure for 29



Not drawn to s

Figure for 30

30. ***Boating*** A boat is pulled into a dock by means of a winc feet above the deck of the boat (see figure).

(a) The winch pulls in rope at a rate of 4 feet per sec Determine the speed of the boat when there is 13 fee rope out. What happens to the speed of the boat as it closer to the dock?

(b) Suppose the boat is moving at a constant rate of 4 feet second. Determine the speed at which the winch pull rope when there is a total of 13 feet of rope out. W happens to the speed at which the winch pulls in rope as boat gets closer to the dock?

31. ***Air Traffic Control*** An air traffic controller spots two pla at the same altitude converging on a point as they fly at r angles to each other (see figure). One plane is 150 miles f the point moving at 450 miles per hour. The other plane is miles from the point moving at 600 miles per hour.

(a) At what rate is the distance between the planes decreasi

(b) How much time does the air traffic controller have to one of the planes on a different flight path?

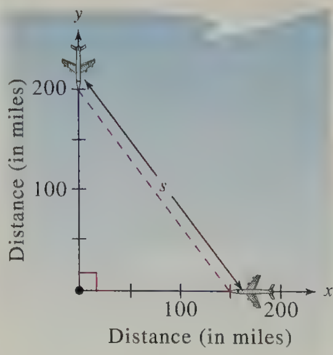


Figure for 31

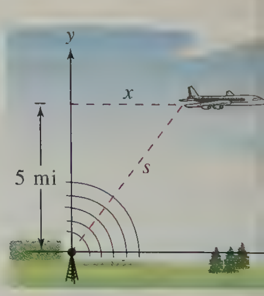
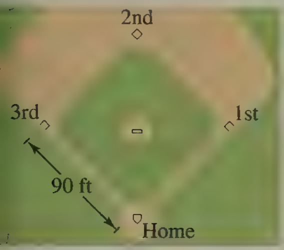


Figure for 32

32. ***Air Traffic Control*** An airplane is flying at an altitude miles and passes directly over a radar antenna (see fig When the plane is 10 miles away ($s = 10$), the radar de that the distance s is changing at a rate of 240 miles per h What is the speed of the plane?

3. ***Baseball*** A baseball diamond has the shape of a square with sides 90 feet long (see figure). A player running from second base to third base at a speed of 28 feet per second is 30 feet from third base. At what rate is the player's distance s from home plate changing?



igure for 33 and 34

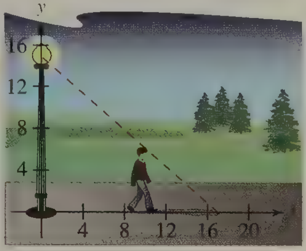


Figure for 35

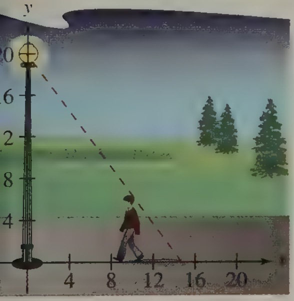
4. ***Baseball*** For the baseball diamond in Exercise 33, suppose the player is running from first to second at a speed of 28 feet per second. Find the rate at which the distance from home plate is changing when the player is 30 feet from second base.

5. ***Shadow Length*** A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 15 feet above the ground (see figure). When he is 10 feet from the base of the light,

(a) at what rate is the tip of his shadow moving?

(b) at what rate is the length of his shadow changing?

6. ***Shadow Length*** Repeat Exercise 35 for a man 6 feet tall walking at a rate of 5 feet per second *toward* a light that is 20 feet above the ground (see figure).



igure for 36

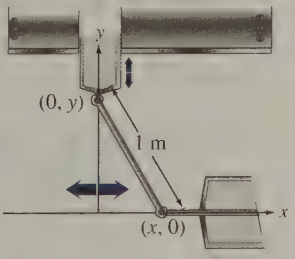


Figure for 37

7. ***Machine Design*** The endpoints of a movable rod of length 1 meter have coordinates $(x, 0)$ and $(0, y)$ (see figure). The position of the end on the x-axis is

$$x(t) = \frac{1}{2} \sin \frac{\pi t}{6}$$

where t is the time in seconds.

(a) Find the time of one complete cycle of the rod.

(b) What is the lowest point reached by the end of the rod on the y-axis?

(c) Find the speed of the y-axis endpoint when the x-axis endpoint is $\left(\frac{1}{4}, 0\right)$.

8. ***Machine Design*** Repeat Exercise 37 for a position function of $x(t) = \frac{3}{5} \sin \pi t$. Use the point $\left(\frac{3}{10}, 0\right)$ for part (c).

39. ***Evaporation*** As a spherical raindrop falls, it reaches a layer of dry air and begins to evaporate at a rate that is proportional to its surface area ($S = 4\pi r^2$). Show that the radius of the raindrop decreases at a constant rate.

40. ***Electricity*** The combined electrical resistance R of R_1 and R_2, connected in parallel, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

where R, R_1, and R_2 are measured in ohms. R_1 and R_2 are increasing at rates of 1 and 1.5 ohms per second, respectively. At what rate is R changing when $R_1 = 50$ ohms and $R_2 = 75$ ohms?

41. ***Adiabatic Expansion*** When a certain polyatomic gas undergoes adiabatic expansion, its pressure p and volume V satisfy the equation

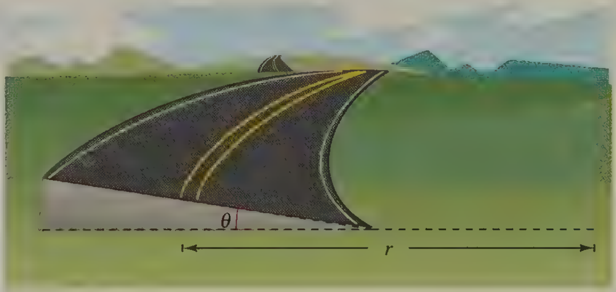
$$pV^{1.3} = k$$

where k is a constant. Find the relationship between the related rates dp/dt and dV/dt.

42. ***Roadway Design*** Cars on a certain roadway travel on a circular arc of radius r. In order not to rely on friction alone to overcome the centrifugal force, the road is banked at an angle of magnitude θ from the horizontal (see figure). The banking angle must satisfy the equation

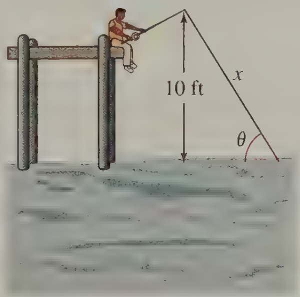
$$rg \tan \theta = v^2$$

where v is the velocity of the cars and $g = 32$ feet per second per second is the acceleration due to gravity. Find the relationship between the related rates dv/dt and $d\theta/dt$.

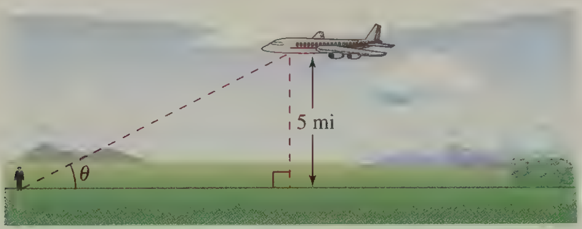


43. ***Angle of Elevation*** A balloon rises at a rate of 3 meters per second from a point on the ground 30 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 30 meters above the ground.

44. ***Angle of Elevation*** A fish is reeled in at a rate of 1 foot per second from a point 10 feet above the water (see figure). At what rate is the angle between the line and the water changing when there is a total of 25 feet of line out?



45. ***Angle of Elevation*** An airplane flies at an altitude of 5 miles toward a point directly over an observer (see figure). The speed of the plane is 600 miles per hour. Find the rate at which the angle of elevation θ is changing when the angle is (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 75°$.



46. ***Linear vs. Angular Speed*** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of 30 revolutions per minute. How fast is the light beam moving along the wall when the beam makes angles of (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 70°$ with the line perpendicular from the light to the wall?

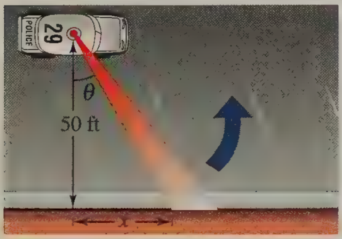


Figure for 46

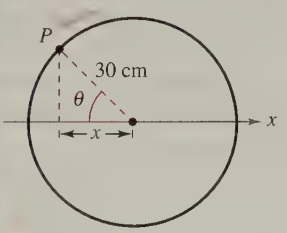


Figure for 47

47. ***Linear vs. Angular Speed*** A wheel of radius 30 centimeters revolves at a rate of 10 revolutions per second. A dot is painted at a point P on the rim of the wheel (see figure).

(a) Find dx/dt as a function of θ.

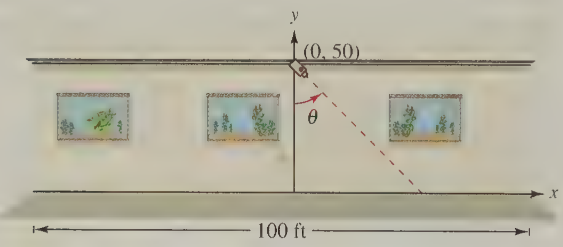
(b) Use a graphing utility to graph the function in part (a).

(c) When is the absolute value of the rate of change of x greatest? When is it least?

(d) Find dx/dt when $\theta = 30°$ and $\theta = 60°$.

48. ***Flight Control*** An airplane is flying in still air with an airspeed of 240 miles per hour. If it is climbing at an angle of 22°, find the rate at which it is gaining altitude.

49. ***Security Camera*** A security camera is centered 50 feet above a 100-foot hallway (see figure). It is easiest to design the camera with a constant angular rate of rotation, but this results in a variable rate at which the images of the surveillance area are recorded. Therefore, it is desirable to design a system with a variable rate of rotation and a constant rate of movement of the scanning beam along the hallway. Find a model for the variable rate of rotation if $|dx/dt| = 2$ feet per second.



50. ***Think About It*** Describe the relationship between the rate c change of y and the rate of change of x in each of the following. Assume all variables and derivatives are positive.

(a) $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$

(b) $\dfrac{dy}{dt} = x(L - x)\dfrac{dx}{dt}, \quad 0 \le x \le L$

Acceleration **In Exercises 51 and 52, find the acceleration** **the specified object. (*Hint:* Recall that if a variable is changin at a constant rate, its acceleration is zero.)**

51. Find the acceleration of the top of the ladder described i Exercise 27 when the base of the ladder is 7 feet from the wal

52. Find the acceleration of the boat in Exercise 30(a) when the is a total of 13 feet of rope out.

53. ***Modeling Data*** The table shows the numbers (in millions) c single women s and married women m in the civilian wor force in the United States for the years 1990 through 199 *(Source: U.S. Bureau of Labor Statistics)*

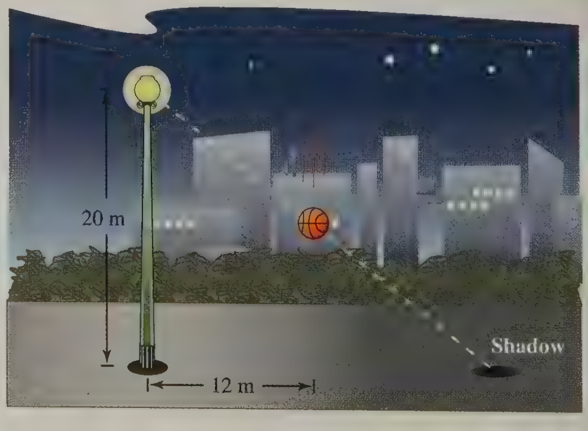
Year	1990	1991	1992	1993	1994	1995	1996	1997	1998
s	14.6	14.7	14.9	15.0	15.3	15.5	15.8	16.5	17.1
m	30.9	31.1	31.7	32.0	32.9	33.4	33.6	33.8	33.9

(a) Use the regression capabilities of a graphing utility to fin a model of the form $m(s) = as^2 + bs + c$ for the dat where t is the time in years, with $t = 0$ correspondin to 1990.

(b) Find $\dfrac{dm}{dt}$.

(c) Use the model to estimate dm/dt for $t = 5$ if it is predicte that the number of single women in the work force wi increase at the rate of 1.2 million per year.

54. A ball is dropped from a height of 20 meters, 12 meters awa from the top of a 20-meter lamppost (see figure). The ball shadow, caused by the light at the top of the lamppost, moving along the level ground. How fast is the shadow movin 1 second after the ball is released? *(Submitted by Denn Gittinger, St. Philips College, San Antonio, TX)*



REVIEW EXERCISES FOR CHAPTER 2

.1 **In Exercises 1–4, find the derivative of the function by ing the definition of the derivative.**

. $f(x) = x^2 - 2x + 3$

2. $f(x) = \dfrac{x+1}{x-1}$

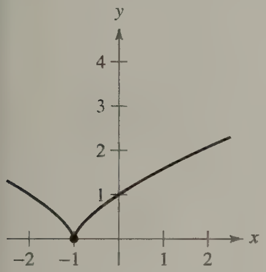
. $f(x) = \sqrt{x} + 1$

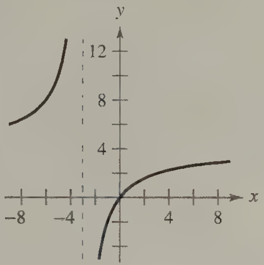
4. $f(x) = \dfrac{2}{x}$

Exercises 5 and 6, describe the x-values at which f is ferentiable.

. $f(x) = (x+1)^{2/3}$

6. $f(x) = \dfrac{4x}{x+3}$





. Sketch the graph of $f(x) = 4 - |x - 2|$.

(a) Is f continuous at $x = 2$?

(b) Is f differentiable at $x = 2$? Explain.

. Sketch the graph of $f(x) = \begin{cases} x^2 + 4x + 2, & x < -2 \\ 1 - 4x - x^2, & x \geq -2. \end{cases}$

(a) Is f continuous at $x = -2$?

(b) Is f differentiable at $x = -2$? Explain.

Exercises 9 and 10, find the slope of the tangent line to the aph of the function at the specified point.

$g(x) = \dfrac{2}{3}x^2 - \dfrac{x}{6}, \quad \left(-1, \dfrac{5}{6}\right)$

$h(x) = \dfrac{3x}{8} - 2x^2, \quad \left(-2, -\dfrac{35}{4}\right)$

Exercises 11 and 12, (a) find an equation of the tangent line the graph of f at the indicated point, (b) use a graphing utility graph the function and its tangent line at the point, and (c) the *derivative* feature of the graphing utility to confirm your ults.

$f(x) = x^3 - 1, \quad (-1, -2)$

$f(x) = \dfrac{2}{x+1}, \quad (0, 2)$

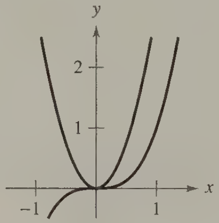
In Exercises 13 and 14, use the alternative form of the derivative to find the derivative at $x = c$ (if it exists).

13. $g(x) = x^2(x - 1), \quad c = 2$

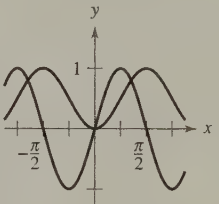
14. $f(x) = \dfrac{1}{x+1}, \quad c = 2$

Writing **In Exercises 15 and 16, the figure shows the graphs of a function and its derivative. Label the graphs as f or f' and write a short paragraph stating the criteria used in making the selection. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

15.



16.



2.2 **In Exercises 17–32, find the derivative of the function.**

17. $y = 25$

18. $y = -12$

19. $f(x) = x^8$

20. $g(x) = x^{12}$

21. $h(t) = 3t^4$

22. $f(t) = -8t^5$

23. $f(x) = x^3 - 3x^2$

24. $g(s) = 4s^4 - 5s^2$

25. $h(x) = 6\sqrt{x} + 3\sqrt[3]{x}$

26. $f(x) = x^{1/2} - x^{-1/2}$

27. $g(t) = \dfrac{2}{3t^2}$

28. $h(x) = \dfrac{2}{(3x)^2}$

29. $f(\theta) = 2\theta - 3 \sin \theta$

30. $g(\alpha) = 4 \cos \alpha + 6$

31. $f(\theta) = 3 \cos \theta - \dfrac{\sin \theta}{4}$

32. $g(\alpha) = \dfrac{5 \sin \alpha}{3} - 2\alpha$

33. ***Vibrating String*** When a guitar string is plucked, it vibrates with a frequency of $F = 200\sqrt{T}$, where F is measured in vibrations per second and the tension T is measured in pounds. Find the rate of change of F when (a) $T = 4$ and (b) $T = 9$.

34. ***Vertical Motion*** A ball is dropped from a height of 100 feet. One second later, another ball is dropped from a height of 75 feet. Which ball hits the ground first?

35. ***Vertical Motion*** To estimate the height of a building, a weight is dropped from the top of the building into a pool at ground level. How high is the building if the splash is seen 9.2 seconds after the weight is dropped?

36. ***Vertical Motion*** A bomb is dropped from an airplane at an altitude of 14,400 feet. How long will it take for the bomb to reach the ground? (Because of the motion of the plane, the fall will not be vertical, but the time will be the same as that for a vertical fall.) The plane is moving at 600 miles per hour. How far will the bomb move horizontally after it is released from the plane?

37. ***Projectile Motion*** A ball thrown follows a path described by $y = x - 0.02x^2$.

(a) Sketch a graph of the path.

(b) Find the total horizontal distance the ball was thrown.

(c) At what x-value does the ball reach its maximum height? (Use the symmetry of the path.)

(d) Find an equation that gives the instantaneous rate of change of the height of the ball with respect to the horizontal change. Evaluate the equation at $x = 0, 10, 25, 30$, and 50.

(e) What is the instantaneous rate of change of the height when the ball reaches its maximum height?

38. ***Projectile Motion*** The path of a projectile thrown at an angle of 45° with level ground is

$$y = x - \frac{32}{v_0^2}(x^2)$$

where the initial velocity is v_0 feet per second.

(a) Find the x-coordinate of the point where the projectile strikes the ground. Use the symmetry of the path of the projectile to locate the x-coordinate of the point where the projectile reaches its maximum height.

(b) What is the instantaneous rate of change of the height when the projectile is at its maximum height?

(c) Show that doubling the initial velocity of the projectile multiplies both the maximum height and the range by a factor of 4.

(d) Find the maximum height and range of a projectile thrown with an initial velocity of 70 feet per second. Use a graphing utility to sketch the path of the projectile.

39. ***Horizontal Motion*** The position function of a particle moving along the x-axis is

$$x(t) = t^2 - 3t + 2$$

for $-\infty < t < \infty$.

(a) Find the velocity of the particle.

(b) Find the open t-interval(s) in which the particle is moving to the left.

(c) Find the position of the particle when the velocity is 0.

(d) Find the speed of the particle when the position is 0.

40. ***Modeling Data*** The speed of a car in miles per hour and the stopping distance in feet are recorded in the table.

Speed (x)	20	30	40	50	60
Stopping Distance (y)	25	55	105	188	300

(a) Use the regression capabilities of a graphing utility to find a quadratic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use a graphing utility to graph dy/dx.

(d) Use the model to approximate the stopping distance at a speed of 65 miles per hour.

(e) Use the graphs in parts (b) and (c) to explain the change in stopping distance as the speed increases.

2.3 **In Exercises 41–57, find the derivative of the functi**

41. $f(x) = (3x^2 + 7)(x^2 - 2x + 3)$

42. $g(x) = (x^3 - 3x)(x + 2)$

43. $h(x) = \sqrt{x}\sin x$

44. $f(t) = t^3 \cos t$

45. $f(x) = \dfrac{2x^3 - 1}{x^2}$

46. $f(x) = \dfrac{x + 1}{x - 1}$

47. $f(x) = \dfrac{x^2 + x - 1}{x^2 - 1}$

48. $f(x) = \dfrac{6x - 5}{x^2 + 1}$

49. $f(x) = \dfrac{1}{4 - 3x^2}$

50. $f(x) = \dfrac{9}{3x^2 - 2x}$

51. $y = \dfrac{x^2}{\cos x}$

52. $y = \dfrac{\sin x}{x^2}$

53. $y = 3x^2 \sec x$

54. $y = 2x - x^2 \tan x$

55. $y = -x \tan x$

56. $y = \dfrac{1 + \sin x}{1 - \sin x}$

57. $y = x\cos x - \sin x$

58. ***Acceleration*** The velocity of an object in meters per secc is $v(t) = 36 - t^2$, $0 \le t \le 6$. Find the velocity and accele tion of the object when $t = 4$.

In Exercises 59–62, find the second derivative of the functio

59. $g(t) = t^3 - 3t + 2$

60. $f(x) = 12\sqrt[4]{x}$

61. $f(\theta) = 3\tan\theta$

62. $h(t) = 4\sin t - 5\cos t$

In Exercises 63 and 64, show that the function satisfies t equation.

Function	*Equation*
63. $y = 2\sin x + 3\cos x$	$y'' + y = 0$
64. $y = \dfrac{10 - \cos x}{x}$	$xy' + y = \sin x$

2.4 **In Exercises 65–80, find the derivative of the functi**

65. $f(x) = \sqrt{1 - x^3}$

66. $f(x) = \sqrt[3]{x^2 - 1}$

67. $h(x) = \left(\dfrac{x - 3}{x^2 + 1}\right)^2$

68. $f(x) = \left(x^2 + \dfrac{1}{x}\right)^5$

69. $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

70. $h(\theta) = \dfrac{\theta}{(1 - \theta)^3}$

71. $y = 3\cos(3x + 1)$

72. $y = 1 - \cos 2x + 2\cos$

73. $y = \frac{1}{2}\csc 2x$

74. $y = \csc 3x + \cot 3x$

75. $y = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$

76. $y = \dfrac{\sec^7 x}{7} - \dfrac{\sec^5 x}{5}$

77. $y = \dfrac{2}{3}\sin^{3/2} x - \dfrac{2}{7}\sin^{7/2} x$

78. $f(x) = \dfrac{3x}{\sqrt{x^2 + 1}}$

79. $y = \dfrac{\sin \pi x}{x + 2}$

80. $y = \dfrac{\cos(x - 1)}{x - 1}$

n Exercises 81–88, use a computer algebra system to find the erivative of the function. Use the utility to graph the function nd its derivative on the same set of coordinate axes. Describe e behavior of the function that corresponds to any zeros of the raph of the derivative.

31. $f(t) = t^2(t-1)^5$

82. $f(x) = [(x-2)(x+4)]^2$

33. $g(x) = \dfrac{2x}{\sqrt{x+1}}$

84. $g(x) = x\sqrt{x^2+1}$

35. $f(t) = \sqrt{t+1}\,\sqrt[3]{t+1}$

86. $y = \sqrt{3x}(x+2)^3$

37. $y = \tan\sqrt{1-x}$

88. $y = 2\csc^3\left(\sqrt{x}\right)$

n Exercises 89–92, find the second derivative of the function.

39. $y = 2x^2 + \sin 2x$

90. $y = \dfrac{1}{x} + \tan x$

91. $f(x) = \cot x$

92. $y = \sin^2 x$

n Exercises 93–96, use a computer algebra system to find the econd derivative of the function.

93. $f(t) = \dfrac{t}{(1-t)^2}$

94. $g(x) = \dfrac{6x-5}{x^2+1}$

95. $g(\theta) = \tan 3\theta - \sin(\theta - 1)$

96. $h(x) = x\sqrt{x^2-1}$

97. ***Refrigeration*** The temperature T of food put in a freezer is

$$T = \frac{700}{t^2 + 4t + 10}$$

where t is the time in hours. Find the rate of change of T with respect to t at each of the following times.

(a) $t = 1$ (b) $t = 3$ (c) $t = 5$ (d) $t = 10$

98. ***Fluid Flow*** The emergent velocity v of a liquid flowing from a hole in the bottom of a tank is given by $v = \sqrt{2gh}$, where g is the acceleration due to gravity (32 feet per second per second) and h is the depth of the liquid in the tank. Find the rate of change of v with respect to h when (a) $h = 9$ and (b) $h = 4$. (Note that $g = +32$ feet per second per second. The sign of g depends on how a problem is modeled. In this case, letting g be negative would produce an imaginary value for v.)

2.5 **In Exercises 99–104, use implicit differentiation to find /dx.**

99. $x^2 + 3xy + y^3 = 10$

100. $x^2 + 9y^2 - 4x + 3y = 0$

01. $y\sqrt{x} - x\sqrt{y} = 16$

102. $y^2 = (x-y)(x^2+y)$

03. $x \sin y = y \cos x$

104. $\cos(x+y) = x$

n Exercises 105 and 106, find the equations of the tangent line nd the normal line to the graph of the equation at the indicated oint. Use a graphing utility to graph the equation, the tangent ne, and the normal line.

05. $x^2 + y^2 = 20$, $(2, 4)$

106. $x^2 - y^2 = 16$, $(5, 3)$

2.6

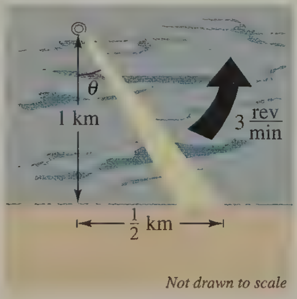
107. A point moves along the curve $y = \sqrt{x}$ in such a way that the y-value is increasing at a rate of 2 units per second. At what rate is x changing for each of the following values?

(a) $x = \frac{1}{2}$ (b) $x = 1$ (c) $x = 4$

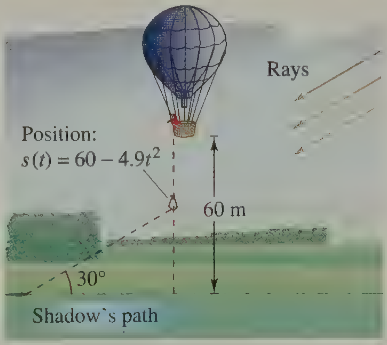
108. ***Surface Area*** The edges of a cube are expanding at a rate of 5 centimeters per second. How fast is the surface area changing when each edge is 4.5 centimeters?

109. ***Changing Depth*** The cross section of a 5-meter trough is an isosceles trapezoid with a 2-meter lower base, a 3-meter upper base, and an altitude of 2 meters. Water is running into the trough at a rate of 1 cubic meter per minute. How fast is the water level rising when the water is 1 meter deep?

110. ***Linear and Angular Velocity*** A rotating beacon is located 1 kilometer off a straight shoreline (see figure). If the beacon rotates at a rate of 3 revolutions per minute, how fast (in kilometers per hour) does the beam of light appear to be moving to a viewer who is $\frac{1}{2}$ kilometer down the shoreline?



111. ***Moving Shadow*** A sandbag is dropped from a balloon at a height of 60 meters when the angle of elevation to the sun is 30° (see figure). Find the rate at which the shadow of the sandbag is traveling along the ground when the sandbag is at a height of 35 meters. [*Hint:* The position of the sandbag is given by $s(t) = 60 - 4.9t^2$.]



P.S. Problem Solving

1. Consider the graph of the parabola $y = x^2$.
   (a) Find the radius r of the largest possible circle centered on the y-axis that is tangent to the parabola at the origin, as indicated in the figure. This circle is called the **circle of curvature** (see Section 11.5). Use a graphing utility to graph the circle and parabola in the same viewing window.
   (b) Find the center $(0, b)$ of the circle of radius 1 centered on the y-axis that is tangent to the parabola at two points, as indicated in the figure. Use a graphing utility to graph the circle and parabola in the same viewing window.

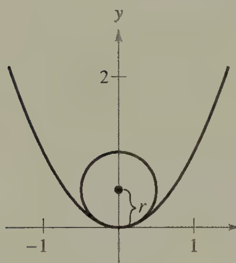


Figure for 1(a)

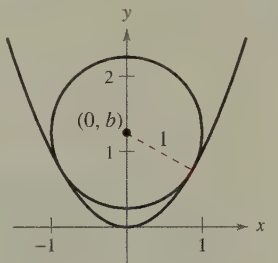


Figure for 1(b)

2. Graph the two parabolas $y = x^2$ and $y = -x^2 + 2x - 5$ in the same coordinate plane. Find equations of the two lines simultaneously tangent to both parabolas.

3. (a) Find the polynomial $P_1(x) = a_0 + a_1x$ whose value and slope agree with the value and slope of $f(x) = \cos x$ at the point $x = 0$.
   (b) Find the polynomial $P_2(x) = a_0 + a_1x + a_2x^2$ whose value and first two derivatives agree with the value and first two derivatives of $f(x) = \cos x$ at the point $x = 0$. This polynomial is called the second-degree **Taylor polynomial** of $f(x) = \cos x$ at $x = 0$.
   (c) Complete the table comparing the values of f and P_2. What do you observe?

x	-1.0	-0.1	-0.001	0	0.001	0.1	1.0
$\cos x$							
$P_2(x)$							

   (d) Find the third-degree Taylor polynomial of $f(x) = \sin x$ at $x = 0$.

4. (a) Find an equation of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$.
   (b) Find an equation of the normal line to $y = x^2$ at the point $(2, 4)$. (The normal line is perpendicular to the tangent line.) Where does this line intersect the parabola a second time?
   (c) Find equations of the tangent line and normal line to $y = x^2$ at the point $(0, 0)$.
   (d) Prove that for any point $(a, b) \neq (0, 0)$ on the parabola $y = x^2$, the normal line intersects the graph a second time.

5. Find a third-degree polynomial $p(x)$ that is tangent to the lin $y = 14x - 13$ at the point $(1, 1)$, and tangent to the lin $y = -2x - 5$ at the point $(-1, -3)$.

6. Find a function of the form $f(x) = a + b \cos cx$ that is tange to the line $y = 1$ at the point $(0, 1)$, and tangent to the line

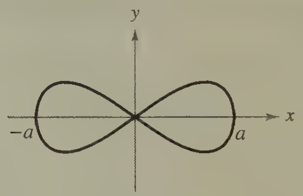
$$y = x + \frac{3}{2} - \frac{\pi}{4}$$

at the point $\left(\frac{\pi}{4}, \frac{3}{2}\right)$.

7. The graph of the **eight curve,**

$$x^4 = a^2(x^2 - y^2),\ a \neq 0,$$

is shown below.

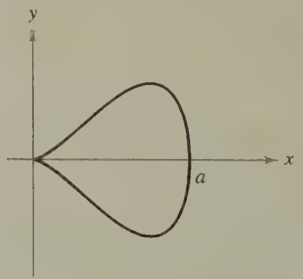


   (a) Explain how you could use a graphing utility to obtain th graph of this curve.
   (b) Use a graphing utility to graph the curve for various value of the constant a. Describe how a affects the shape of th curve.
   (c) Determine the points on the curve where the tangent line horizontal.

8. The graph of the **pear-shaped quartic,**

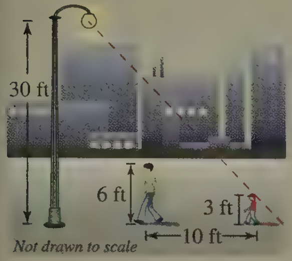
$$b^2y^2 = x^3(a - x),\ a, b > 0,$$

is shown below.



   (a) Explain how you could use a graphing utility to obtain th graph of this curve.
   (b) Use a graphing utility to graph the curve for various valu of the constants a and b. Describe how a and b affect th shape of the curve.
   (c) Determine the points on the curve where the tangent line horizontal.

A man 6 feet tall walks at a rate of 5 feet per second toward a street light that is 30 feet high. The man's 3-foot-tall child follows at the same speed, but 10 feet behind the man. At times, the shadow behind the child is caused by the man, and at other times, by the child.



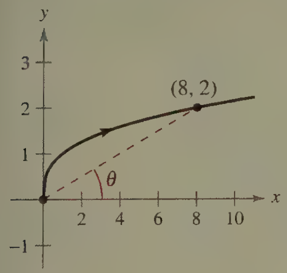
(a) Suppose the man is 90 feet from the street light. Show that the man's shadow extends beyond the child's shadow.

(b) Suppose the man is 60 feet from the street light. Show that the child's shadow extends beyond the man's shadow.

(c) Determine the distance d from the man to the street light at which the tips of the two shadows are exactly the same distance from the street light.

(d) Determine how fast the tip of the shadow is moving as a function of x, the distance between the man and the street light. Discuss the continuity of this shadow speed function.

A particle is moving along the graph of $y = \sqrt[3]{x}$. When $x = 8$, the y-component of its position is increasing at the rate of 1 centimeter per second.



(a) How fast is the x-component changing at this moment?

(b) How fast is the distance from the origin changing at this moment?

(c) How fast is the angle of inclination θ changing at this moment?

Let L be a differentiable function for all x. Prove that if $L(a + b) = L(a) + L(b)$ for all a and b, then $L'(x) = L'(0)$ for all x. What does the graph of L look like?

Let E be a function satisfying $E(0) = E'(0) = 1$. Prove that if $E(a + b) = E(a)E(b)$ for all a and b, then E is differentiable and $E'(x) = E(x)$ for all x. Find an example of a function satisfying $E(a + b) = E(a)E(b)$.

13. The fundamental limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

assumes that x is measured in radians. What happens if we assume that x is measured in degrees instead of radians?

(a) Set your calculator to degree mode and complete the table.

z (in degrees)	0.1	0.01	0.0001
$\dfrac{\sin z}{z}$			

(b) Use the table to estimate

$$\lim_{z \to 0} \frac{\sin z}{z}$$

for z in degrees. What is the exact value of this limit? (*Hint:* $180° = \pi$ radians)

(c) Use the limit definition of the derivative to find

$$\frac{d}{dz} \sin z$$

for z in degrees.

(d) Define the new functions $S(z) = \sin(cz)$ and $C(z) = \cos(cz)$, where $c = \pi/180$. Find $S(90)$ and $C(180)$. Use the Chain Rule to calculate

$$\frac{d}{dz} S(z).$$

(e) Explain why calculus is made easier by using radians instead of degrees.

14. An astronaut standing on the moon throws a rock into the air. The height of the rock is

$$s = -\frac{27}{10}t^2 + 27t + 6$$

where s is measured in feet and t is measured in seconds.

(a) Find expressions for the velocity and acceleration of the rock.

(b) Find the time when the rock is at its highest point by finding the time when the velocity is zero. What is its height at this time?

(c) How does the acceleration of the rock compare with the acceleration due to gravity on earth?

15. If a is the acceleration of an object, the *jerk* j is defined by $j = a'(t)$.

(a) Use this definition to give a physical interpretation of j.

(b) Find j for the slowing vehicle in Exercise 102 in Section 2.3 and interpret the result.

Packaging: The Optimal Form

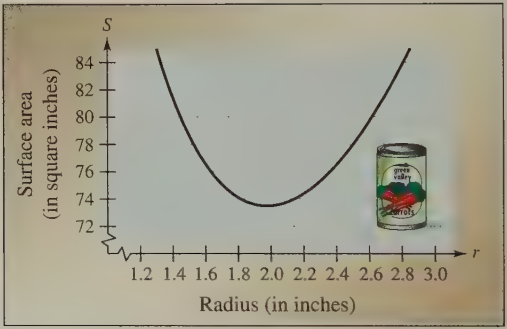
Many people are involved in deciding how to package the products you see in grocery stores. Packaging engineers select materials and package shapes to adequately protect the product through shipping at a reasonable cost.

A container's shape, as well as its material, is important in determining its strength. From an engineering perspective, the sphere is the strongest form, followed by the circular cylinder. The rectangular box comes in a poor third. From a cost perspective, it is preferable to use the smallest amount of material possible.

The table gives the approximate measurements in inches of several common items packed in cylindrical containers.

Product	*Radius (in.)*	*Height (in.)*	*Volume (in.^{3})*
Coffee creamer	1.50	6.85	48.42
Cleanser	1.45	7.50	49.54
Coffee	1.95	5.20	62.12
Pineapple juice	2.10	6.70	92.82
Frosting	1.63	3.60	30.05
Soup	1.30	3.80	20.18
Tomato puree	1.95	4.40	52.56
Baking powder	1.25	3.65	17.92

An infinite number of dimensions can be used to construct a right circular container of a given volume. The graph at the right shows the relationship between the radius and surface area for containers that have a volume of 48.4 cubic inches.



QUESTIONS

1. Create a table of values for the dimensions of a cylinder with a volume of 49.54 cubic inches. Does it appear that the cleanser container minimizes surface area?
2. Suppose you are designing a coffee creamer container that has a volume of 48.42 cubic inches. Use the equations for the surface area of a cylinder and the volume of a cylinder to develop an equation relating the radius r and surface area S.

   $S = 2\pi r^2 + 2\pi rh$ Surface area of a right circular cylinder

   $V = \pi r^2 h$ Volume of a right circular cylinder

3. Repeat Question 2 for each of the other containers in the table. Use a graphing utility to plot each equation. Determine whether the radius of each container is larger than, smaller than, or equal to the "optimal" radius.
4. Suppose, in order to fit more writing on the cylinder, you want to maximize the surface area of a cylinder that holds 49.5 cubic inches. Can you do this? Explain.

The concepts presented here will be explored further in this chapter. For an extension of this application, see Lab 5 in the lab series that accompanies this text at college.hmco.com.

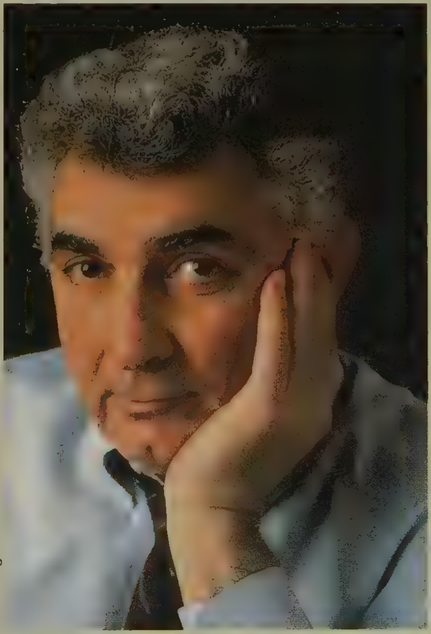
Applications of Differentiation

3

In addition to strength, engineers must consider not only how a package will fit into a shipping container but also how it will be displayed on a store shelf. The sphere may be the strongest form, but it would surely be impractical to use for product packaging.

By the time packaging engineers begin work on a container, design specialists ave already done their work. Designers use color, shape, and words to create n image that they think will appeal to their targeted market. Many designers elieve that the package is at least as important as the product inside.

Successful designer Primo Angeli feels so strongly about the importance of packaging that he has designed entire lines of packaged product ideas in realistic packages so that consumer response to these ideas can be measured before massive investments are made in product development.



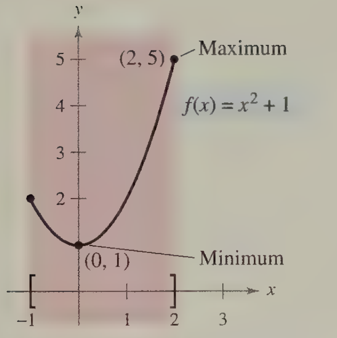
Primo Angeli

Section 3.1 Extrema on an Interval

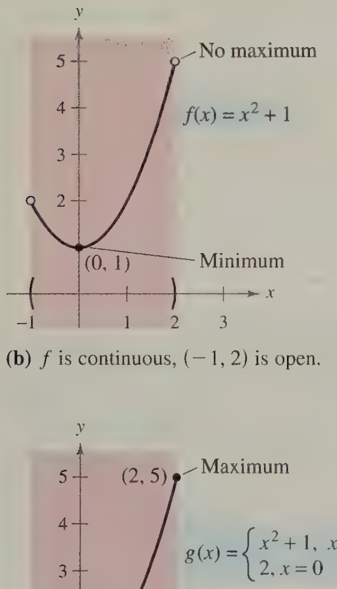
- Understand the definition of extrema of a function on an interval.
- Understand the definition of relative extrema of a function on an open interval.
- Find extrema on a closed interval.

Extrema of a Function

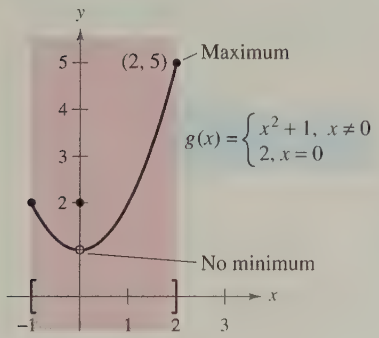
In calculus, much effort is devoted to determining the behavior of a function f on a interval I. Does f have a maximum value on I? Does it have a minimum value? Wher is the function increasing? Where is it decreasing? In this chapter you will learn ho derivatives can be used to answer these questions. You will also see why thes questions are important in real-life applications.



(a) f is continuous, $[-1, 2]$ is closed.



(b) f is continuous, $(-1, 2)$ is open.



(c) g is not continuous, $[-1, 2]$ is closed.

Extrema can occur at interior points or endpoints of an interval. Extrema that occur at the endpoints are called **endpoint extrema.**

Figure 3.1

Definition of Extrema

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum of f on I** if $f(c) \le f(x)$ for all x in I.
2. $f(c)$ is the **maximum of f on I** if $f(c) \ge f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the **extreme values,** or **extrema,** of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum** on the interval.

A function need not have a minimum or a maximum on an interval. For instance in Figure 3.1(a) and (b), you can see that the function $f(x) = x^2 + 1$ has both minimum and a maximum on the closed interval $[-1, 2]$, but does not have a maximum on the open interval $(-1, 2)$. Moreover, in Figure 3.1(c), you can see tha continuity (or the lack of it) can affect the existence of an extremum on the interval This suggests the following theorem. (Although the Extreme Value Theorem i intuitively plausible, a proof of this theorem is not within the scope of this text.)

THEOREM 3.1 The Extreme Value Theorem

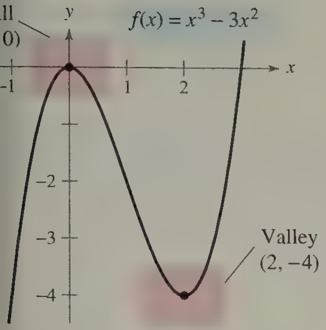
If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

EXPLORATION

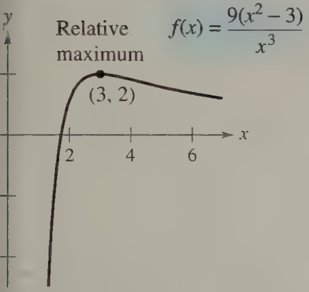
Finding Minimum and Maximum Values The Extreme Value Theorem (like the Intermediate Value Theorem) is an *existence theorem* because it tells of the existence of minimum and maximum values but does not show how to find these values. Use the extreme-value capability of a graphing utility to find the minimum and maximum values of each of the following. In each case, do you think the x-values are exact or approximate? Explain your reasoning.

a. $f(x) = x^2 - 4x + 5$ on the closed interval $[-1, 3]$

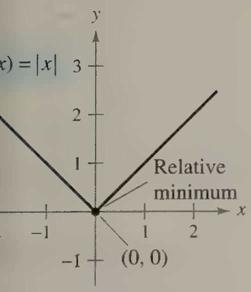
b. $f(x) = x^3 - 2x^2 - 3x - 2$ on the closed interval $[-1, 3]$



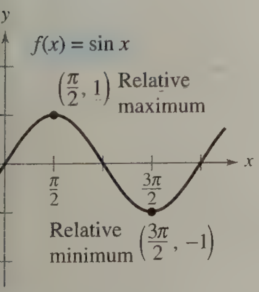
as a relative maximum at $(0, 0)$ and a
ative minimum at $(2, -4)$.
gure 3.2



$f'(3) = 0$



$f'(0)$ does not exist.



$f'\left(\frac{\pi}{2}\right) = 0; f'\left(\frac{3\pi}{2}\right) = 0$

ure 3.3

Relative Extrema and Critical Numbers

In Figure 3.2, the graph of $f(x) = x^3 - 3x^2$ has a **relative maximum** at the point $(0, 0)$ and a **relative minimum** at the point $(2, -4)$. Informally, you can think of a relative maximum as occurring on a "hill" on the graph, and a relative minimum as occurring in a "valley" on the graph. Such a hill and valley can occur in two ways. If the hill (or valley) is smooth and rounded, the graph has a horizontal tangent line at the high point (or low point). If the hill (or valley) is sharp and peaked, the graph represents a function that is not differentiable at the high point (or low point).

Definition of Relative Extrema

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f.
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f.

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima.

Example 1 examines the derivatives of functions at *given* relative extrema. (Much more is said about *finding* the relative extrema of a function in Section 3.3.)

Example 1 The Value of the Derivative at Relative Extrema

Find the value of the derivative at each of the relative extrema shown in Figure 3.3.

Solution

a. The derivative of $f(x) = \dfrac{9(x^2 - 3)}{x^3}$ is

$$f'(x) = \frac{x^3(18x) - (9)(x^2 - 3)(3x^2)}{(x^3)^2} \qquad \text{Differentiate using Quotient Rule.}$$

$$= \frac{9(9 - x^2)}{x^4}. \qquad \text{Simplify.}$$

At the point $(3, 2)$, the value of the derivative is $f'(3) = 0$ (see Figure 3.3a).

b. At $x = 0$, the derivative of $f(x) = |x|$ *does not exist* because the following one-sided limits differ (see Figure 3.3b).

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{|x|}{x} = -1 \qquad \text{Limit from the left}$$

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{|x|}{x} = 1 \qquad \text{Limit from the right}$$

c. The derivative of $f(x) = \sin x$ is

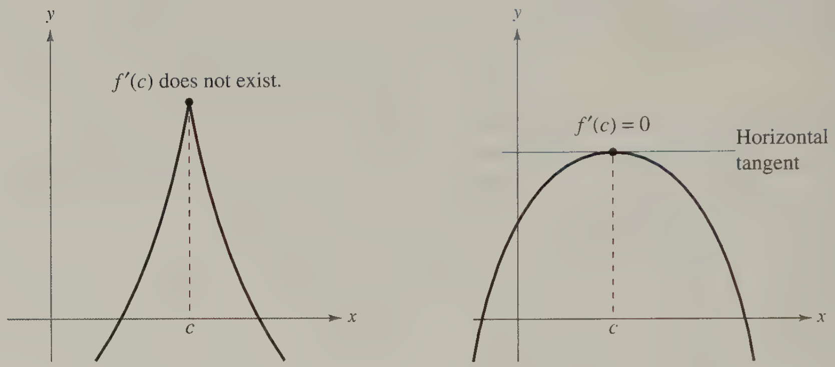
$$f'(x) = \cos x.$$

At the point $(\pi/2, 1)$, the value of the derivative is $f'(\pi/2) = \cos(\pi/2) = 0$. At the point $(3\pi/2, -1)$, the value of the derivative is $f'(3\pi/2) = \cos(3\pi/2) = 0$ (see Figure 3.3c).

Note in Example 1 that at the relative extrema, the derivative is either zero or d not exist. The x-values at these special points are called **critical numbers.** Figure illustrates the two types of critical numbers.

Definition of a Critical Number

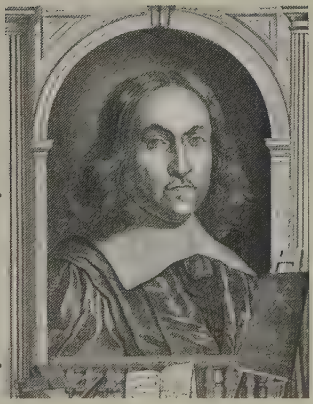
Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.



c is a critical number of f.
Figure 3.4

THEOREM 3.2 Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.



Mary Evans Picture Library

PIERRE DE FERMAT (1601–1665)

For Fermat, who was trained as a lawyer, mathematics was more of a hobby than a profession. Nevertheless, Fermat made many contributions to analytic geometry, number theory, calculus, and probability. In letters to friends, he wrote of many of the fundamental ideas of calculus, long before Newton or Leibniz. For instance, the theorem at the right is sometimes attributed to Fermat.

Proof

Case 1: If f is *not* differentiable at $x = c$, then, by definition, c is a critical number f and the theorem is valid.

Case 2: If f is differentiable at $x = c$, then $f'(c)$ must be positive, negative, or Suppose $f'(c)$ is positive. Then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0$$

which implies that there exists an interval (a, b) containing c such that

$$\frac{f(x) - f(c)}{x - c} > 0, \text{ for all } x \neq c \text{ in } (a, b). \qquad \text{(See Exercise 58, Section 1.2)}$$

Because this quotient is positive, the signs of the denominator and numerator m agree. This produces the following inequalities for x-values in the interval (a, b).

Left of c: $x < c$ and $f(x) < f(c)$ ⇨ $f(c)$ is not a relative minimum

Right of c: $x > c$ and $f(x) > f(c)$ ⇨ $f(c)$ is not a relative maximum

So, the assumption that $f'(c) > 0$ contradicts the hypothesis that $f(c)$ is a relat extremum. Assuming that $f'(c) < 0$ produces a similar contradiction, you are left w only one possibility—namely, $f'(c) = 0$. So, by definition, c is a critical number and the theorem is valid.

Finding Extrema on a Closed Interval

Theorem 3.2 states that the relative extrema of a function can occur *only* at the critical numbers of the function. Knowing this, you can use the following guidelines to find extrema on a closed interval.

Guidelines for Finding Extrema on a Closed Interval

To find the extrema of a continuous function f on a closed interval $[a, b]$, use the following steps.

1. Find the critical numbers of f in (a, b).
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint of $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

The next three examples show how to apply these guidelines. Be sure you see that finding the critical numbers of the function is only part of the procedure. Evaluating the function at the critical numbers *and* the endpoints is the other part.

Example 2 Finding Extrema on a Closed Interval

Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.

Solution Begin by differentiating the function.

$$f(x) = 3x^4 - 4x^3 \qquad \text{Write original function.}$$

$$f'(x) = 12x^3 - 12x^2 \qquad \text{Differentiate.}$$

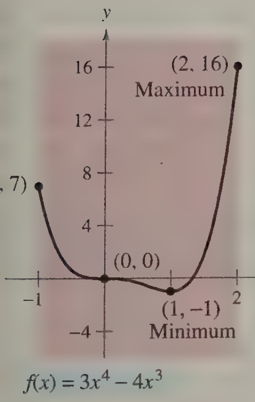
To find the critical numbers of f, you must find all x-values for which $f'(x) = 0$ and all x-values for which $f'(x)$ does not exist.

$$f'(x) = 12x^3 - 12x^2 = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

$$12x^2(x - 1) = 0 \qquad \text{Factor.}$$

$$x = 0, 1 \qquad \text{Critical numbers}$$

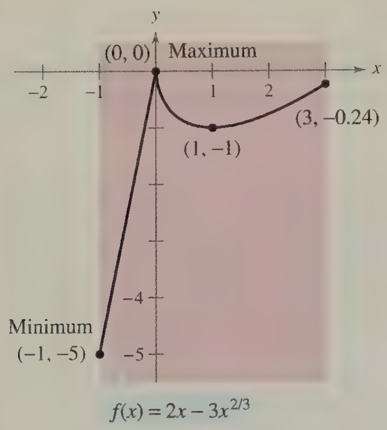
Because f' is defined for all x, you can conclude that these are the only critical numbers of f. By evaluating f at these two critical numbers and at the endpoints of $[-1, 2]$, you can determine that the maximum is $f(2) = 16$ and the minimum is $f(1) = -1$, as indicated in the table. The graph of f is shown in Figure 3.5.



the closed interval $[-1, 2]$, f has a
imum at $(1, -1)$ and a maximum
2, 16).
ure 3.5

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = 7$	$f(0) = 0$	$f(1) = -1$ Minimum	$f(2) = 16$ Maximum

In Figure 3.5, note that the critical number $x = 0$ does not yield a relative minimum or a relative maximum. This tells you that the converse of Theorem 3.2 is not true. In other words, *the critical numbers of a function need not produce relative extrema.*



On the closed interval $[-1, 3]$, f has a minimum at $(-1, -5)$ and a maximum at $(0, 0)$.
Figure 3.6

Example 3 Finding Extrema on a Closed Interval

Find the extrema of $f(x) = 2x - 3x^{2/3}$ on the interval $[-1, 3]$.

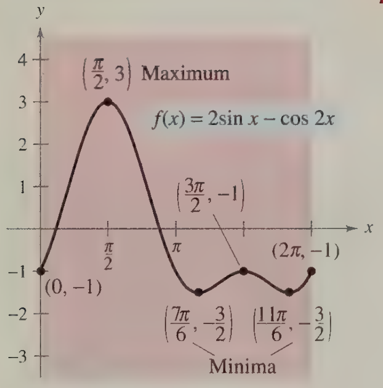
Solution Differentiating produces the following.

$$f(x) = 2x - 3x^{2/3} \qquad \text{Write original function.}$$

$$f'(x) = 2 - \frac{2}{x^{1/3}} = 2\left(\frac{x^{1/3} - 1}{x^{1/3}}\right) \qquad \text{Differentiate.}$$

From this derivative, you can see that the function has two critical numbers in t interval $[-1, 3]$. The number 1 is a critical number because $f'(1) = 0$, and t number 0 is a critical number because $f'(0)$ does not exist. By evaluating f at these t numbers and at the endpoints of the interval, you can conclude that the minimum $f(-1) = -5$ and the maximum is $f(0) = 0$, as indicated in the table. The graph o is shown in Figure 3.6.

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = -5$ Minimum	$f(0) = 0$ Maximum	$f(1) = -1$	$f(3) = 6 - 3\sqrt[3]{9} \approx -0.24$



On the closed interval $[0, 2\pi]$, f has two minima at $(7\pi/6, -3/2)$ and $(11\pi/6, -3/2)$ and a maximum at $(\pi/2, 3)$.
Figure 3.7

Example 4 Finding Extrema on a Closed Interval

Find the extrema of $f(x) = 2 \sin x - \cos 2x$ on the interval $[0, 2\pi]$.

Solution This function is differentiable for all real x, so you can find all critic numbers by differentiating the function and setting $f'(x)$ equal to zero, as follows.

$$f(x) = 2\sin x - \cos 2x \qquad \text{Write original function.}$$

$$f'(x) = 2\cos x + 2\sin 2x = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

$$2\cos x + 4\cos x \sin x = 0 \qquad \sin 2x = 2\cos x \sin x$$

$$2(\cos x)(1 + 2\sin x) = 0 \qquad \text{Factor.}$$

In the interval $[0, 2\pi]$, the factor $\cos x$ is zero when $x = \pi/2$ and when $x = 3\pi$ The factor $(1 + 2\sin x)$ is zero when $x = 7\pi/6$ and when $x = 11\pi/6$. By evaluati f at these four critical numbers and at the endpoints of the interval, you can conclu that the maximum is $f(\pi/2) = 3$ and the minimum occurs at *two* poin $f(7\pi/6) = -3/2$ and $f(11\pi/6) = -3/2$, as indicated in the table. The graph shown in Figure 3.7.

Left Endpoint	Critical Number	Critical Number	Critical Number	Critical Number	Right Endpoint
$f(0) = -1$	$f\left(\frac{\pi}{2}\right) = 3$ Maximum	$f\left(\frac{7\pi}{6}\right) = -\frac{3}{2}$ Minimum	$f\left(\frac{3\pi}{2}\right) = -1$	$f\left(\frac{11\pi}{6}\right) = -\frac{3}{2}$ Minimum	$f(2\pi) = -1$

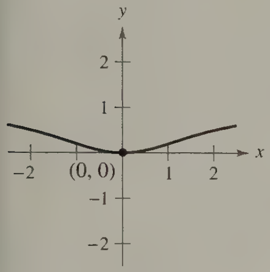
---

indicates that in the Interactive 3.0 *CD-ROM and* Internet 3.0 *versions of this (available at* college.hmco.com*) you will find an Open Exploration, which further explores example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

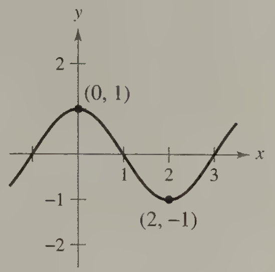
EXERCISES FOR SECTION 3.1

n Exercises 1–6, find the value of the derivative (if it exists) at ach indicated extremum.

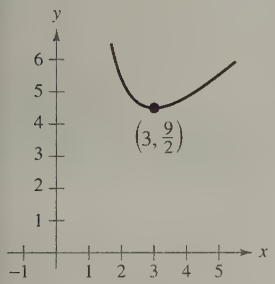
. $f(x) = \dfrac{x^2}{x^2+4}$



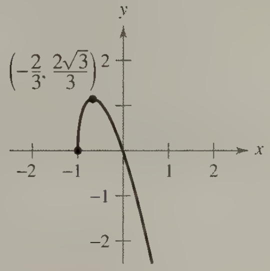
2. $f(x) = \cos\dfrac{\pi x}{2}$



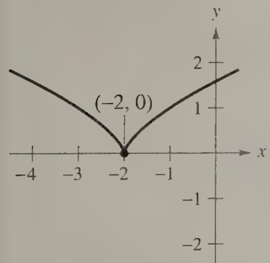
. $f(x) = x + \dfrac{27}{2x^2}$



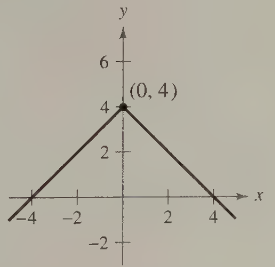
4. $f(x) = -3x\sqrt{x+1}$



. $f(x) = (x+2)^{2/3}$



6. $f(x) = 4 - |x|$



n Exercises 7–10, approximate the critical numbers of the unction shown in the graph. Determine whether the function as a relative maximum, relative minimum, absolute maximum, bsolute minimum, or none of these at each critical number on he interval shown.

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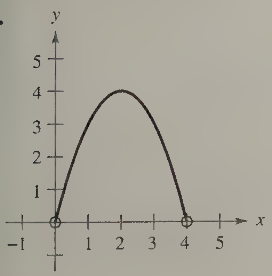
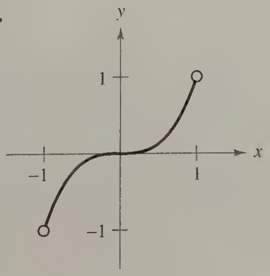


8.



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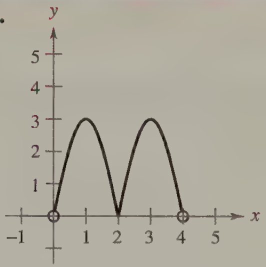
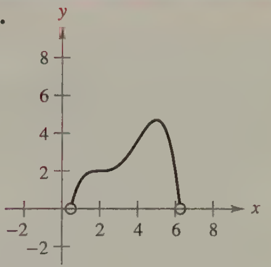


10.



In Exercises 11–16, find any critical numbers of the function.

11. $f(x) = x^2(x-3)$

12. $g(x) = x^2(x^2-4)$

13. $g(t) = t\sqrt{4-t},\ t < 3$

14. $f(x) = \dfrac{4x}{x^2+1}$

15. $h(x) = \sin^2 x + \cos x$
$0 < x < 2\pi$

16. $f(\theta) = 2\sec\theta + \tan\theta$
$0 < \theta < 2\pi$

In Exercises 17–32, locate the absolute extrema of the function on the closed interval.

17. $f(x) = 2(3-x),\ [-1, 2]$

18. $f(x) = \dfrac{2x+5}{3},\ [0, 5]$

19. $f(x) = -x^2 + 3x,\ [0, 3]$

20. $f(x) = x^2 + 2x - 4,\ [-1, 1]$

21. $f(x) = x^3 - \dfrac{3}{2}x^2,\ [-1, 2]$

22. $f(x) = x^3 - 12x,\ [0, 4]$

23. $y = 3x^{2/3} - 2x,\ [-1, 1]$

24. $g(x) = \sqrt[3]{x},\ [-1, 1]$

25. $g(t) = \dfrac{t^2}{t^2+3},\ [-1, 1]$

26. $y = 3 - |t-3|,\ [-1, 5]$

27. $h(s) = \dfrac{1}{s-2},\ [0, 1]$

28. $h(t) = \dfrac{t}{t-2},\ [3, 5]$

29. $f(x) = \cos\pi x,\ \left[0, \dfrac{1}{6}\right]$

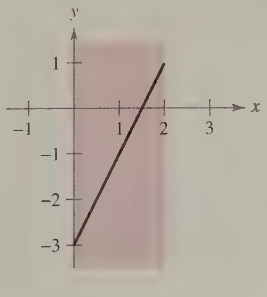
30. $g(x) = \sec x,\ \left[-\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

31. $y = \dfrac{4}{x} + \tan\left(\dfrac{\pi x}{8}\right),\ [1, 2]$

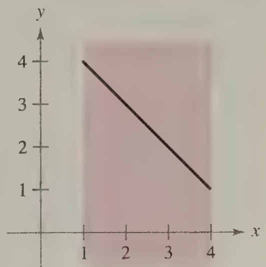
32. $y = x^2 - 2 - \cos x,\ [-1, 3]$

In Exercises 33–36, locate the absolute extrema of the function (if any exist) over the indicated intervals.

33. $f(x) = 2x - 3$
(a) $[0, 2]$ (b) $[0, 2)$
(c) $(0, 2]$ (d) $(0, 2)$



34. $f(x) = 5 - x$
(a) $[1, 4]$ (b) $[1, 4)$
(c) $(1, 4]$ (d) $(1, 4)$



35. $f(x) = x^2 - 2x$

(a) $[-1, 2]$ (b) $(1, 3)$

(c) $(0, 2)$ (d) $[1, 4)$

36. $f(x) = \sqrt{4 - x^2}$

(a) $[-2, 2]$ (b) $[-2, 0)$

(c) $(-2, 2)$ (d) $[1, 2)$

In Exercises 37–40, use a graphing utility to graph the function. Locate the absolute extrema of the function on the closed interval.

	Function	*Interval*
37.	$f(x) = \begin{cases} 2x + 2, & 0 \le x \le 1 \\ 4x^2, & 1 < x \le 3 \end{cases}$	$[0, 3]$
38.	$f(x) = \begin{cases} 2 - x^2, & 1 \le x < 3 \\ 2 - 3x, & 3 \le x \le 5 \end{cases}$	$[1, 5]$
39.	$f(x) = \dfrac{3}{x - 1}$	$(1, 4]$
40.	$f(x) = \dfrac{2}{2 - x}$	$[0, 2)$

In Exercises 41 and 42, (a) use a computer algebra system to graph the function and approximate any absolute extrema on the indicated interval. (b) Use the utility to find any critical numbers, and use them to find any absolute extrema not located at the endpoints. Compare the results with those in part (a).

	Function	*Interval*
41.	$f(x) = 3.2x^5 + 5x^3 - 3.5x$	$[0, 1]$
42.	$f(x) = \frac{4}{3}x\sqrt{3 - x}$	$[0, 3]$

In Exercises 43 and 44, use a computer algebra system to find the maximum value of $|f''(x)|$ on the closed interval. (This value is used in the error estimate for the Trapezoidal Rule, as discussed in Section 4.6.)

	Function	*Interval*
43.	$f(x) = \sqrt{1 + x^3}$	$[0, 2]$
44.	$f(x) = \dfrac{1}{x^2 + 1}$	$\left[\frac{1}{2}, 3\right]$

In Exercises 45 and 46, use a computer algebra system to find the maximum value of $|f^{4}(x)|$ on the closed interval. (This value is used in the error estimate for Simpson's Rule, as discussed in Section 4.6.)

	Function	*Interval*
45.	$f(x) = (x + 1)^{2/3}$	$[0, 2]$
46.	$f(x) = \dfrac{1}{x^2 + 1}$	$[-1, 1]$

47. Explain why the function $f(x) = \tan x$ has a maximum on $[0, \pi/4]$ but not on $[0, \pi]$.

48. ***Writing*** Write a short paragraph explaining why a continuous function on an open interval may not have a maximum or minimum. Illustrate your explanation with a sketch of the graph of a function.

Getting at the Concept

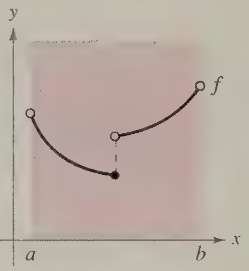
In Exercises 49 and 50, graph a function on the interval $[-2, 5]$ having the given characteristics.

49. Absolute maximum at $x = -2$
Absolute minimum at $x = 1$
Relative maximum at $x = 3$

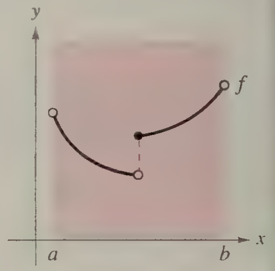
50. Relative minimum at $x = -1$
Critical number at $x = 0$, but no extrema
Absolute maximum at $x = 2$
Absolute minimum at $x = 5$

In Exercises 51–54, determine from the graph whether f has a minimum in the open interval (a, b).

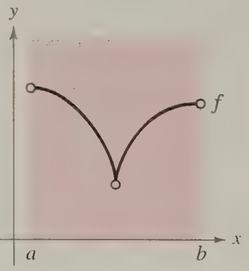
51. (a)



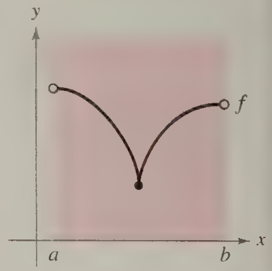
(b)



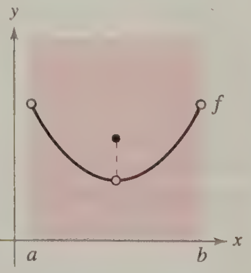
52. (a)



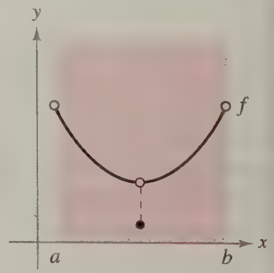
(b)



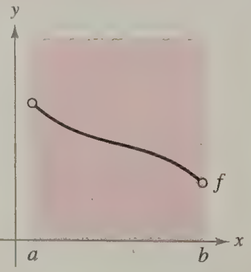
53. (a)



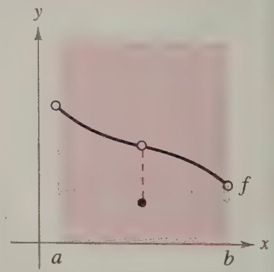
(b)



54. (a)



(b)

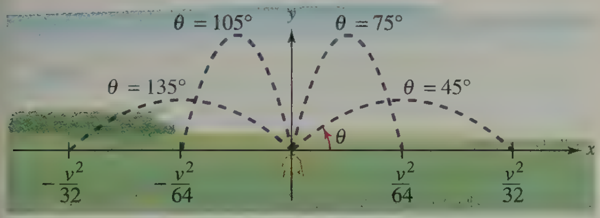


Power The formula for the power output P of a battery is $P = VI - RI^2$ where V is the electromotive force in volts, R is the resistance, and I is the current. Find the current (measured in amperes) that corresponds to a maximum value of P in a battery for which $V = 12$ volts and $R = 0.5$ ohm. Assume that a 15-ampere fuse bounds the output in the interval $0 \le I \le 15$. Could the power output be increased by replacing the 15-ampere fuse with a 20-ampere fuse? Explain.

Lawn Sprinkler A lawn sprinkler is constructed in such a way that $d\theta/dt$ is constant, where θ ranges between 45° and 135° (see figure). The distance the water travels horizontally is

$$x = \frac{v^2 \sin 2\theta}{32}, \qquad 45° \le \theta \le 135°$$

where v is the speed of the water. Find dx/dt and explain why this lawn sprinkler does not water evenly. What part of the lawn receives the most water?



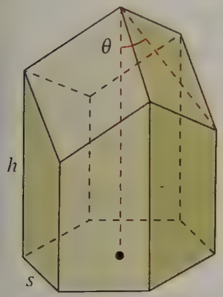
Water sprinkler: $45° \le \theta \le 135°$

FOR FURTHER INFORMATION For more information on the "calculus of lawn sprinklers," see the article "Design of an Oscillating Sprinkler" by Bart Braden in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

Honeycomb The surface area of a cell in a honeycomb is

$$S = 6hs + \frac{3s^2}{2}\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right)$$

where h and s are positive constants and θ is the angle at which the upper faces meet the altitude of the cell. Find the angle θ $(\pi/6 \le \theta \le \pi/2)$ that minimizes the surface area S.



FOR FURTHER INFORMATION For more information on the geometric structure of a honeycomb cell, see the article "The Design of Honeycombs" by Anthony L. Peressini in UMAP Module 502, published by COMAP, Inc., Suite 210, 57 Bedford Street, Lexington, MA. To view this article, go to the website *www.matharticles.com*.

58. ***Inventory Cost*** A retailer has determined that the cost C of ordering and storing x units of a certain product is

$$C = 2x + \frac{300{,}000}{x}, \qquad 1 \le x \le 300.$$

The delivery truck can bring at most 300 units per order. Find the order size that will minimize cost. Could the cost be decreased if the truck were replaced with one that could bring at most 400 units? Explain.

59. ***Highway Design*** In order to build a highway it is necessary to fill a section of a valley where the grades (slopes) of the sides are 9% and 6% (see figure). The top of the filled region will have the shape of a parabolic arc that is tangent to the two slopes at the points A and B. The horizontal distance between the points A and B is 1000 feet.

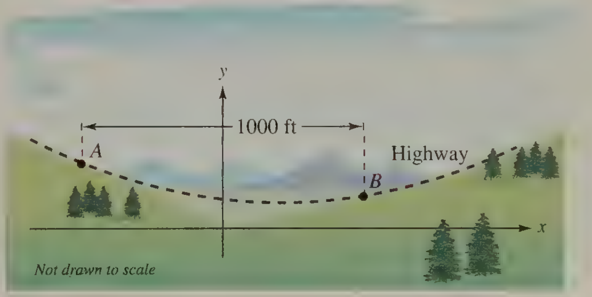
(a) Find a quadratic function $y = ax^2 + bx + c$, $-500 \le x \le 500$, that describes the top of the filled region.

(b) Complete the table giving the depths d of the fill at the specified values of x.

x	-500	-400	-300	-200	-100
d					

x	0	100	200	300	400	500
d						

(c) What will be the lowest point on the completed highway? Will it be directly over the point where the two hillsides come together?



60. Find all critical numbers of the greatest integer function $f(x) = [\![x]\!]$.

True or False? **In Exercises 61–64, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

61. The maximum of a function that is continuous on a closed interval can occur at two different values in the interval.

62. If a function is continuous on a closed interval, then it must have a minimum on the interval.

63. If $x = c$ is a critical number of the function f, then it is also a critical number of the function $g(x) = f(x) + k$, where k is a constant.

64. If $x = c$ is a critical number of the function f, then it is also a critical number of the function $g(x) = f(x - k)$, where k is a constant.

Section 3.2 Rolle's Theorem and the Mean Value Theorem

- Understand and use Rolle's Theorem.
- Understand and use the Mean Value Theorem.

Rolle's Theorem

ROLLE'S THEOREM

French mathematician Michel Rolle first published the theorem that bears his name in 1691. Before this time, however, Rolle was one of the most vocal critics of calculus, stating that the subject gave erroneous results and was based on unsound reasoning. Later in life, Rolle came to see the usefulness of calculus.

The Extreme Value Theorem (Section 3.1) states that a continuous function o closed interval $[a, b]$ must have both a minimum and a maximum on the interval. B of these values, however, can occur at the endpoints. **Rolle's Theorem,** named af the French mathematician Michel Rolle (1652–1719), gives conditions that guaran the existence of an extreme value in the *interior* of a closed interval.

EXPLORATION

Extreme Values in a Closed Interval Sketch a rectangular coordinate plane on piece of paper. Label the points (1, 3) and (5, 3). Using a pencil or pen, draw th graph of a differentiable function f that starts at (1, 3) and ends at (5, 3). Is there least one point on the graph for which the derivative is zero? Would it be possibl to draw the graph so that there *isn't* a point for which the derivative is zero? Explai your reasoning.

THEOREM 3.3 Rolle's Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If

$$f(a) = f(b)$$

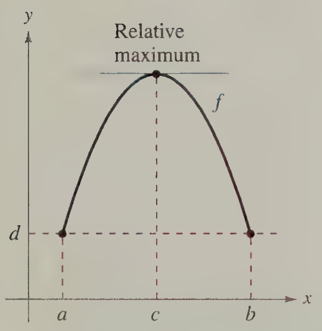
then there is at least one number c in (a, b) such that $f'(c) = 0$.

Proof Let $f(a) = d = f(b)$.

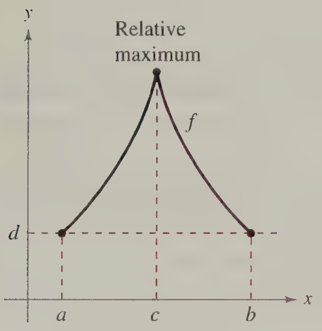
Case 1: If $f(x) = d$ for all x in $[a, b]$, f is constant on the interval and, by Theor 2.2, $f'(x) = 0$ for all x in (a, b).

Case 2: Suppose $f(x) > d$ for some x in (a, b). By the Extreme Value Theorem, y know that f has a maximum at some c in the interval. Moreover, because $f(c) >$ this maximum does not occur at either endpoint. So, f has a maximum in the op interval (a, b). This implies that $f(c)$ is a *relative* maximum and, by Theorem 3.2, c a critical number of f. Finally, because f is differentiable at c, you can conclude t $f'(c) = 0$.

Case 3: If $f(x) < d$ for some x in (a, b), you can use an argument similar to that Case 2, but involving the minimum instead of the maximum.



(a) f is continuous on $[a, b]$ and differentiable on (a, b).



(b) f is continuous on $[a, b]$.

Figure 3.8

From Rolle's Theorem, you can see that if a function f is continuous on $[a, b]$ a differentiable on (a, b), and if $f(a) = f(b)$, there must be at least one x-value betw a and b at which the graph of f has a horizontal tangent, as shown in Figure 3.8 If the differentiability requirement is dropped from Rolle's Theorem, f will still h a critical number in (a, b), but it may not yield a horizontal tangent. Such a case shown in Figure 3.8(b).

Example 1 Illustrating Rolle's Theorem

Find the two x-intercepts of

$$f(x) = x^2 - 3x + 2$$

and show that $f'(x) = 0$ at some point between the two intercepts.

Solution Note that f is differentiable on the entire real line. Setting $f(x)$ equal to 0 produces

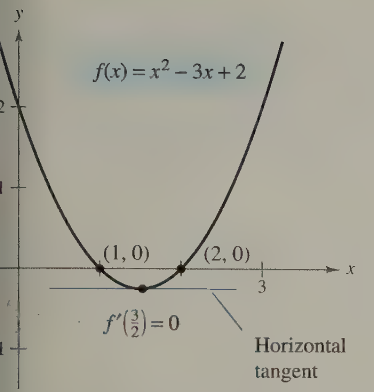
$$x^2 - 3x + 2 = 0 \qquad \text{Set } f(x) \text{ equal to 0.}$$

$$(x-1)(x-2) = 0. \qquad \text{Factor.}$$

So, $f(1) = f(2) = 0$, and from Rolle's Theorem you know that there *exists* at least one c in the interval $(1, 2)$ such that $f'(c) = 0$. To *find* such a c, you can solve the equation

$$f'(x) = 2x - 3 = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

and determine that $f'(x) = 0$ when $x = \frac{3}{2}$. Note that the x-value lies in the open interval $(1, 2)$, as shown in Figure 3.9.



he x-value for which $f'(x) = 0$ is between
e two x-intercepts.
igure 3.9

Rolle's Theorem states that if f satisfies the conditions of the theorem, there must be *at least* one point between a and b at which the derivative is 0. There may of course be more than one such point, as illustrated in the next example.

Example 2 Illustrating Rolle's Theorem

Let $f(x) = x^4 - 2x^2$. Find all values of c in the interval $(-2, 2)$ such that $f'(c) = 0$.

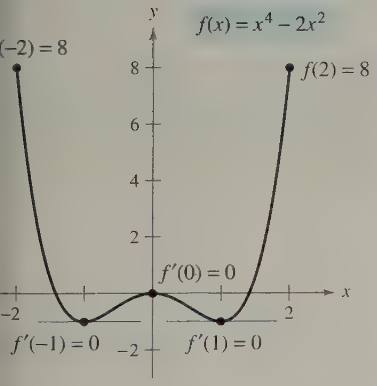
Solution To begin, note that the function satisfies the conditions of Rolle's Theorem. That is, f is continuous on the interval $[-2, 2]$ and differentiable on the interval $(-2, 2)$. Moreover, because $f(-2) = 8 = f(2)$, you can conclude that there exists at least one c in $(-2, 2)$ such that $f'(c) = 0$. Setting the derivative equal to 0 produces

$$f'(x) = 4x^3 - 4x = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

$$4x(x^2 - 1) = 0 \qquad \text{Factor.}$$

$$x = 0, 1, -1. \qquad x\text{-values for which } f'(x) = 0$$

So, in the interval $(-2, 2)$, the derivative is zero at three different values of x, as shown in Figure 3.10.

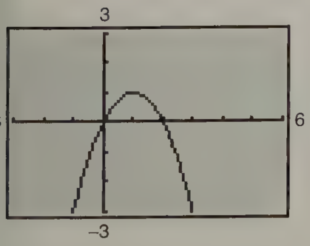


$x) = 0$ for more than one x-value in the
terval $(-2, 2)$.
igure 3.10

TECHNOLOGY PITFALL A graphing utility can be used to indicate whether the points on the graphs in Examples 1 and 2 are relative minima or relative maxima of the functions. When using a graphing utility, however, you should keep in mind that it can give misleading pictures of graphs. For example, try using a graphing utility to graph

$$f(x) = 1 - (x-1)^2 - \frac{1}{1000(x-1)^{1/7} + 1}.$$

With most viewing windows, it appears that the function has a maximum of 1 when $x = 1$ (see Figure 3.11). By evaluating the function at $x = 1$, however, you can see that $f(1) = 0$. To determine the behavior of this function near $x = 1$, you need to examine the graph analytically to get the complete picture.



gure 3.11

The Mean Value Theorem

Rolle's Theorem can be used to prove another theorem—the **Mean Value Theorem**

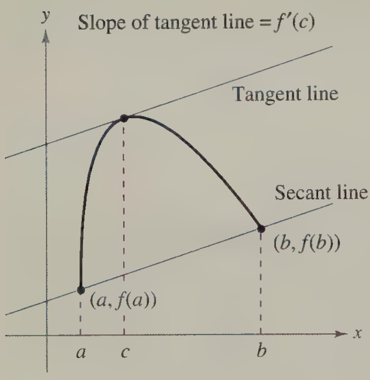


Figure 3.12

THEOREM 3.4 The Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof Refer to Figure 3.12. The equation of the secant line containing the poin $(a, f(a))$ and $(b, f(b))$ is

$$y = \left[\frac{f(b) - f(a)}{b - a}\right](x - a) + f(a).$$

Let $g(x)$ be the difference between $f(x)$ and y. Then

$$\begin{aligned} g(x) &= f(x) - y \\ &= f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a) - f(a). \end{aligned}$$

By evaluating g at a and b, you can see that $g(a) = 0 = g(b)$. Furthermore, becaus f is differentiable, g is also differentiable, and you can apply Rolle's Theorem to th function g. So, there exists a number c in (a, b) such that $g'(c) = 0$, which implies th

$$\begin{aligned} 0 &= g'(c) \\ &= f'(c) - \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

Therefore, there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

NOTE The "mean" in the Mean Value Theorem refers to the mean (or average) rate of chang of f in the interval $[a, b]$.

Although the Mean Value Theorem can be used directly in problem solving, it used more often to prove other theorems. In fact, some people consider this to be th most important theorem in calculus—it is closely related to the Fundamental Theore of Calculus discussed in Chapter 4. For now, you can get an idea of the versatility this theorem by looking at the results stated in Exercises 57–62 in this section.

The Mean Value Theorem has implications for both basic interpretations of th derivative. Geometrically, the theorem guarantees the existence of a tangent line th is parallel to the secant line through the points $(a, f(a))$ and $(b, f(b))$, as shown Figure 3.12. Example 3 illustrates this geometric interpretation of the Mean Valu Theorem. In terms of rates of change, the Mean Value Theorem implies that the must be a point in the open interval (a, b) at which the instantaneous rate of change equal to the average rate of change over the interval $[a, b]$. This is illustrated Example 4.

Mary Evans Picture Library

Joseph-Louis Lagrange (1736–1813)

The Mean Value Theorem was first proved by the famous mathematician Joseph-Louis Lagrange. Born in Italy, Lagrange held a position in the court of Frederick the Great in Berlin for 20 years. Afterward, he moved to France, where he met emperor Napoleon Bonaparte, who is quoted as saying, "Lagrange is the lofty pyramid of the mathematical sciences."

Example 3 Finding a Tangent Line

Given $f(x) = 5 - (4/x)$, find all values of c in the open interval $(1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

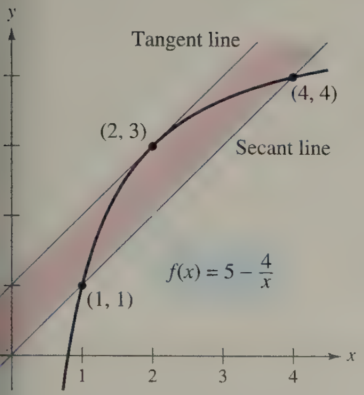
Solution The slope of the secant line through $(1, f(1))$ and $(4, f(4))$ is

$$\frac{f(4) - f(1)}{4 - 1} = \frac{4 - 1}{4 - 1} = 1.$$

Because f satisfies the conditions of the Mean Value Theorem, there exists at least one number c in $(1, 4)$ such that $f'(c) = 1$. Solving the equation $f'(x) = 1$ yields

$$f'(x) = \frac{4}{x^2} = 1$$

which implies that $x = \pm 2$. So, in the interval $(1, 4)$, you can conclude that $c = 2$, as shown in Figure 3.13.



e tangent line at (2, 3) is parallel to the
ant line through (1, 1) and (4, 4).
gure 3.13

Example 4 Finding an Instantaneous Rate of Change

Two stationary patrol cars equipped with radar are 5 miles apart on a highway, as shown in Figure 3.14. As a truck passes the first patrol car, its speed is clocked at 55 miles per hour. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 miles per hour. Prove that the truck must have exceeded the speed limit (of 55 miles per hour) at some time during the four minutes.

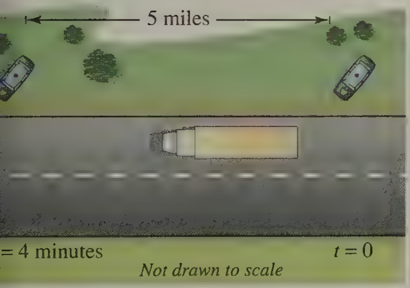
Solution Let $t = 0$ be the time (in hours) when the truck passes the first patrol car. The time when the truck passes the second patrol car is

$$t = \frac{4}{60} = \frac{1}{15} \text{ hour.}$$

By letting $s(t)$ represent the distance (in miles) traveled by the truck, you have $s(0) = 0$ and $s\left(\frac{1}{15}\right) = 5$. So, the average velocity of the truck over the 5-mile stretch of highway is

$$\begin{aligned}\text{Average velocity} &= \frac{s(1/15) - s(0)}{(1/15) - 0}\\ &= \frac{5}{1/15} = 75 \text{ mph.}\end{aligned}$$

Assuming that the position function is differentiable, you can apply the Mean Value Theorem to conclude that the truck must have been traveling at a rate of 75 miles per hour sometime during the four minutes.



some time t, the instantaneous velocity is
ual to the average velocity over four minutes.
gure 3.14

A useful alternative form of the Mean Value Theorem is as follows: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number c in (a, b) such that

$$f(b) = f(a) + (b - a)f'(c).$$

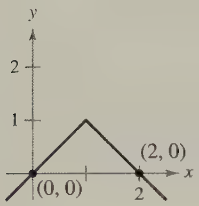
Alternative form of Mean Value Theorem

NOTE When working the exercises for this section, keep in mind that polynomial functions, rational functions, and trigonometric functions are differentiable at all points in their domains.

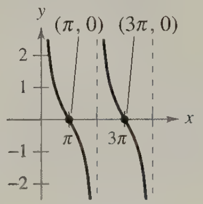
EXERCISES FOR SECTION 3.2

In Exercises 1 and 2, explain why Rolle's Theorem does not apply to the function even though there exist a and b such that $f(a) = f(b)$.

1. $f(x) = 1 - |x - 1|$



2. $f(x) = \cot \dfrac{x}{2}$



In Exercises 3–6, find the two x-intercepts of the function f and show that $f'(x) = 0$ at some point between the two intercepts.

3. $f(x) = x^2 - x - 2$

4. $f(x) = x(x - 3)$

5. $f(x) = x\sqrt{x + 4}$

6. $f(x) = -3x\sqrt{x + 1}$

In Exercises 7–20, determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$.

7. $f(x) = x^2 - 2x,\ [0, 2]$

8. $f(x) = x^2 - 5x + 4,\ [1, 4]$

9. $f(x) = (x - 1)(x - 2)(x - 3),\ [1, 3]$

10. $f(x) = (x - 3)(x + 1)^2,\ [-1, 3]$

11. $f(x) = x^{2/3} - 1,\ [-8, 8]$

12. $f(x) = 3 - |x - 3|,\ [0, 6]$

13. $f(x) = \dfrac{x^2 - 2x - 3}{x + 2},\ [-1, 3]$

14. $f(x) = \dfrac{x^2 - 1}{x},\ [-1, 1]$

15. $f(x) = \sin x,\ [0, 2\pi]$

16. $f(x) = \cos x,\ [0, 2\pi]$

17. $f(x) = \dfrac{6x}{\pi} - 4\sin^2 x,\ \left[0, \dfrac{\pi}{6}\right]$

18. $f(x) = \cos 2x,\ \left[-\dfrac{\pi}{12}, \dfrac{\pi}{6}\right]$

19. $f(x) = \tan x,\ [0, \pi]$

20. $f(x) = \sec x,\ \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

In Exercises 21–24, use a graphing utility to graph the function on the closed interval $[a, b]$. Determine whether Rolle's Theorem can be applied to f on the interval and, if so, find all values of c in the open interval (a, b) such that $f'(c) = 0$.

21. $f(x) = |x| - 1,\ [-1, 1]$

22. $f(x) = x - x^{1/3},\ [0, 1]$

23. $f(x) = 4x - \tan \pi x,\ \left[-\frac{1}{4}, \frac{1}{4}\right]$

24. $f(x) = \dfrac{x}{2} - \sin \dfrac{\pi x}{6},\ [-1, 0]$

25. ***Vertical Motion*** The height of a ball t seconds after it is thrown upward from a height of 32 feet and with an initial velocity of 48 feet per second is $f(t) = -16t^2 + 48t + 32$.

(a) Verify that $f(1) = f(2)$.

(b) According to Rolle's Theorem, what must be the velocity at some time in the interval $(1, 2)$? Find that time.

26. ***Reorder Costs*** The ordering and transportation cost C of co
ponents used in a manufacturing process is approximated by

$$C(x) = 10\left(\frac{1}{x} + \frac{x}{x + 3}\right)$$

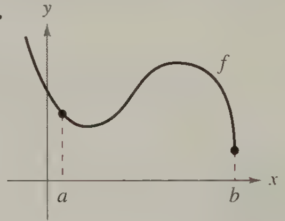
where C is measured in thousands of dollars and x is the or
size in hundreds.

(a) Verify that $C(3) = C(6)$.

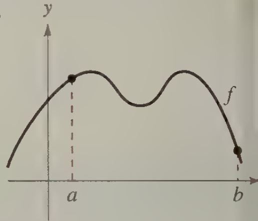
(b) According to Rolle's Theorem, the rate of change of c
must be 0 for some order size in the interval $(3, 6)$. Fi
that order size.

**In Exercises 27 and 28, copy the graph and sketch the seca
line to the graph through the points $(a, f(a))$ and $(b, f(b))$. Th
sketch any tangent lines to the graph for each value of c gua
anteed by the Mean Value Theorem. To print an enlarged co
of the graph, go to the website *www.mathgraphs.com*.**

27.



28.



**In Exercises 29 and 30, explain why the Mean Value Theore
does not apply to the function on the interval $[0, 6]$.**

29. $f(x) = \dfrac{1}{x - 3}$

30. $f(x) = |x - 3|$

**In Exercises 31–38, determine whether the Mean Val
Theorem can be applied to f on the closed interval $[a, b]$. If t
Mean Value Theorem can be applied, find all values of c in t
open interval (a, b) such that**

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

31. $f(x) = x^2,\ [-2, 1]$

32. $f(x) = x(x^2 - x - 2),\ [-1, 1]$

33. $f(x) = x^{2/3},\ [0, 1]$

34. $f(x) = \dfrac{x + 1}{x},\ \left[\frac{1}{2}, 2\right]$

35. $f(x) = \sqrt{2 - x},\ [-7, 2]$

36. $f(x) = x^3,\ [0, 1]$

37. $f(x) = \sin x,\ [0, \pi]$

38. $f(x) = 2\sin x + \sin 2x,\ [0, \pi]$

**In Exercises 39–42, use a graphing utility to (a) graph the fu
tion f on the indicated interval, (b) find and graph the seca
line through points on the graph of f at the endpoints of t
indicated interval, and (c) find and graph any tangent lines
the graph of f that are parallel to the secant line.**

39. $f(x) = \dfrac{x}{x + 1},\ \left[-\frac{1}{2}, 2\right]$

40. $f(x) = x - 2\sin x,\ [-\pi,$

41. $f(x) = \sqrt{x},\ [1, 9]$

42. $f(x) = -x^4 + 4x^3 + 8x^2 + 5,\ [0, 5]$

. ***Vertical Motion*** The height of an object t seconds after it is dropped from a height of 500 meters is $s(t) = -4.9t^2 + 500$.

(a) Find the average velocity of the object during the first 3 seconds.

(b) Use the Mean Value Theorem to verify that at some time during the first 3 seconds of fall the instantaneous velocity equals the average velocity. Find that time.

. ***Sales*** A company introduces a new product for which the number of units sold S is

$$S(t) = 200\left(5 - \frac{9}{2+t}\right)$$

where t is the time in months.

(a) Find the average value of $S(t)$ during the first year.

(b) During what month does $S'(t)$ equal the average value during the first year?

Getting at the Concept

45. Let f be continuous on $[a, b]$ and differentiable on (a, b). If there exists c in (a, b) such that $f'(c) = 0$, does it follow that $f(a) = f(b)$? Explain.

46. Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Also, suppose that $f(a) = f(b)$ and that c is a real number in the interval such that $f'(c) = 0$. Find an interval for the function g over which Rolle's Theorem can be applied, and find the corresponding critical number of g (k is a constant).

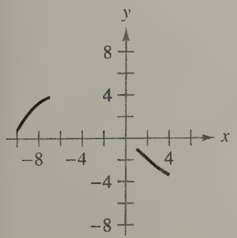
(a) $g(x) = f(x) + k$ (b) $g(x) = f(x - k)$

(c) $g(x) = f(kx)$

47. A plane begins its takeoff at 2:00 P.M. on a 2500-mile flight. The plane arrives at its destination at 7:30 P.M. Explain why there were at least two times during the flight when the speed of the plane was 400 miles per hour.

48. When an object is removed from a furnace and placed in an environment with a constant temperature of 90°F, its core temperature is 1500°F. Five hours later the core temperature is 390°F. Explain why there must exist a time in the interval when the temperature is decreasing at a rate of 222°F per hour.

. ***Graphical Reasoning*** The figure gives two parts of the graph of a continuous differentiable function f on $[-10, 4]$. The derivative f' is also continuous. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) Explain why f must have at least one zero in $[-10, 4]$.

(b) Explain why f' must also have at least one zero in the interval $[-10, 4]$. What are these zeros called?

(c) Make a possible sketch of the function with one zero of f' on the interval $[-10, 4]$.

(d) Make a possible sketch of the function with two zeros of f' on the interval $[-10, 4]$.

(e) Were the conditions of continuity of f and f' necessary to do parts (a) through (d)? Explain.

50. Consider the function $f(x) = 3\cos^2\left(\frac{\pi x}{2}\right)$.

(a) Use a graphing utility to graph f and f'.

(b) Is f a continuous function? Is f' a continuous function?

(c) Does Rolle's Theorem apply on the interval $[-1, 1]$? Does it apply on the interval $[1, 2]$? Explain.

(d) Evaluate, if possible, $\lim_{x\to 3^-} f'(x)$ and $\lim_{x\to 3^+} f'(x)$.

Think About It **In Exercises 51 and 52, sketch the graph of an arbitrary function f that satisfies the given condition but does not satisfy the conditions of the Mean Value Theorem on the interval $[-5, 5]$.**

51. f is continuous on $[-5, 5]$.

52. f is not continuous on $[-5, 5]$.

True or False? **In Exercises 53–56, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

53. The Mean Value Theorem can be applied to $f(x) = 1/x$ on the interval $[-1, 1]$.

54. If the graph of a function has three x-intercepts, then it must have at least two points at which its tangent line is horizontal.

55. If the graph of a polynomial function has three x-intercepts, then it must have at least two points at which its tangent line is horizontal.

56. If $f'(x) = 0$ for all x in the domain of f, then f is a constant function.

57. Prove that if $a > 0$ and n is any positive integer, then the polynomial function $p(x) = x^{2n+1} + ax + b$ cannot have two real roots.

58. Prove that if $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

59. Let $p(x) = Ax^2 + Bx + C$. Prove that for any interval $[a, b]$, the value c guaranteed by the Mean Value Theorem is the midpoint of the interval.

60. Prove that if f is differentiable on $(-\infty, \infty)$ and $f'(x) < 1$ for all real numbers, then f has at most one fixed point. A fixed point of a function f is a real number c such that $f(c) = c$.

61. Use the result of Exercise 60 to show that $f(x) = \frac{1}{2}\cos x$ has at most one fixed point.

62. Prove that $|\cos x - \cos y| \le |x - y|$ for all x and y.

Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

- Determine intervals on which a function is increasing or decreasing.
- Apply the First Derivative Test to find relative extrema of a function.

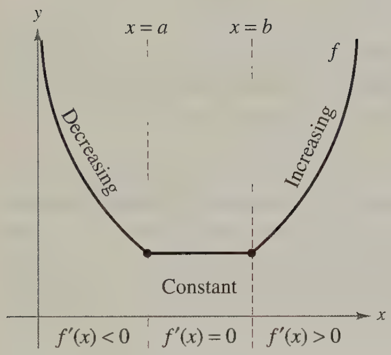
Increasing and Decreasing Functions

In this section you will learn how derivatives can be used to *classify* relative extrer as either relative minima or relative maxima. We begin by defining increasing a decreasing functions.

Definitions of Increasing and Decreasing Functions

A function f is **increasing** on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function f is **decreasing** on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.



The derivative is related to the slope of a function.
Figure 3.15

A function is increasing if, *as x moves to the right*, its graph moves up, and decreasing if its graph moves down. For example, the function in Figure 3.15 decreasing on the interval $(-\infty, a)$, is constant on the interval (a, b), and is increa ing on the interval (b, ∞). As shown in Theorem 3.5 below, a positive derivati implies that the function is increasing; a negative derivative implies that the functi is decreasing; and a zero derivative on an entire interval implies that the function constant on that interval.

THEOREM 3.5 Test for Increasing and Decreasing Functions

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

Proof To prove the first case, assume that $f'(x) > 0$ for all x in the interval (a, b) a let $x_1 < x_2$ be any two points in the interval. By the Mean Value Theorem, you kn that there exists a number c such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Because $f'(c) > 0$ and $x_2 - x_1 > 0$, you know that

$$f(x_2) - f(x_1) > 0$$

which implies that $f(x_1) < f(x_2)$. So, f is increasing on the interval. The second ca has a similar proof (see Exercise 77), and the third case was given as Exercise 58 Section 3.2.

NOTE The conclusions in the first two cases of Theorem 3.5 are valid even if $f'(x) = 0$ a finite number of x-values in (a, b).

Example 1 Intervals on Which f is Increasing or Decreasing

Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

Solution Note that f is continuous on the entire real line. To determine the critical numbers of f, set $f'(x)$ equal to zero.

$$f(x) = x^3 - \frac{3}{2}x^2 \qquad \text{Write original function.}$$

$$f'(x) = 3x^2 - 3x = 0 \qquad \text{Differentiate and set } f'(x) \text{ equal to 0.}$$

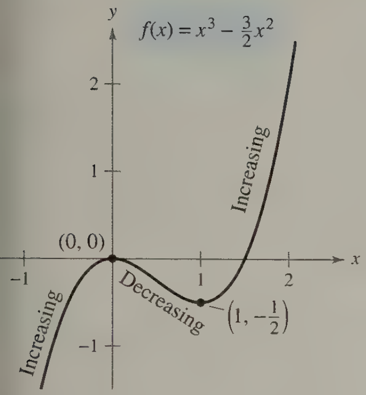
$$3(x)(x-1) = 0 \qquad \text{Factor.}$$

$$x = 0, 1 \qquad \text{Critical numbers}$$

Because there are no points for which f' does not exist, you can conclude that $x = 0$ and $x = 1$ are the only critical numbers. The table summarizes the testing of the three intervals determined by these two critical numbers.

Interval	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Value	$x = -1$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	$f'(-1) = 6 > 0$	$f'(\frac{1}{2}) = -\frac{3}{4} < 0$	$f'(2) = 6 > 0$
Conclusion	Increasing	Decreasing	Increasing

So, f is increasing on the intervals $(-\infty, 0)$ and $(1, \infty)$ and decreasing on the interval $(0, 1)$, as shown in Figure 3.16.



gure 3.16

Example 1 gives you one example of how to find intervals on which a function is increasing or decreasing. The guidelines below summarize the steps followed in the example.

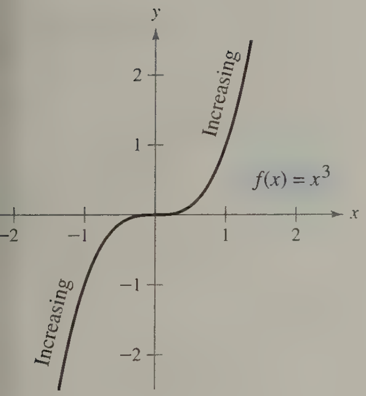
Guidelines for Finding Intervals on Which a Function Is Increasing or Decreasing

Let f be continuous on the interval (a, b). To find the open intervals on which f is increasing or decreasing, use the following steps.

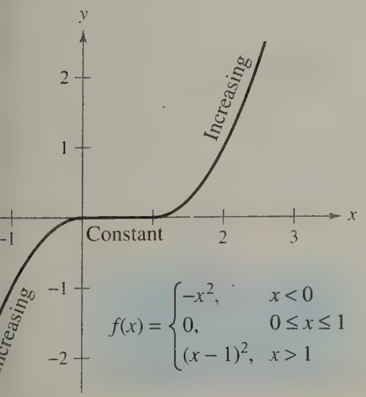
1. Locate the critical numbers of f in (a, b), and use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Use Theorem 3.5 to determine whether f is increasing or decreasing on each interval.

These guidelines are also valid if the interval (a, b) is replaced by an interval of the form $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$.

A function is **strictly monotonic** on an interval if it is either increasing on the entire interval or decreasing on the entire interval. For instance, the function $f(x) = x^3$ is strictly monotonic on the entire real line because it is increasing on the entire real line, as shown in Figure 3.17(a). The function shown in Figure 3.17(b) is not strictly monotonic on the entire real line because it is constant on the interval $[0, 1]$.

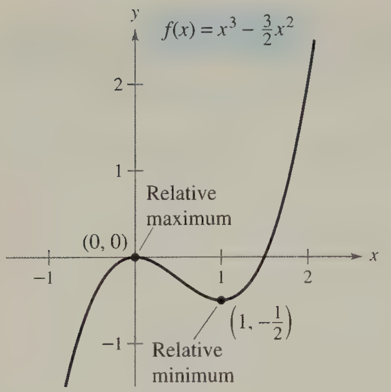


Strictly monotonic function



Not strictly monotonic

gure 3.17



Relative extrema of f
Figure 3.18

The First Derivative Test

After you have determined the intervals on which a function is increasing or decrea ing, it is not difficult to locate the relative extrema of the function. For instance, Figure 3.18 (from Example 1), the function

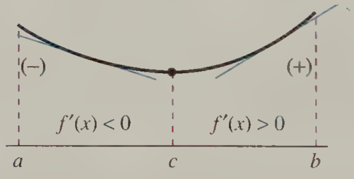
$$f(x) = x^3 - \frac{3}{2}x^2$$

has a relative maximum at the point (0, 0) because f is increasing immediately to t left of $x = 0$ and decreasing immediately to the right of $x = 0$. Similarly, f has relative minimum at the point $\left(1, -\frac{1}{2}\right)$ because f is decreasing immediately to the le of $x = 1$ and increasing immediately to the right of $x = 1$. The following theore called the First Derivative Test, makes this more explicit.

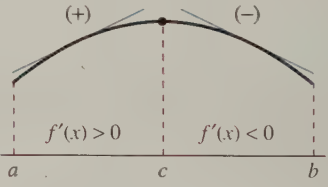
THEOREM 3.6 The First Derivative Test

Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.

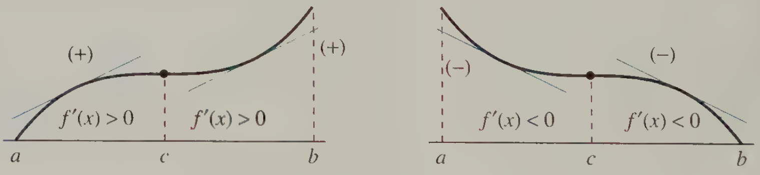
1. If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a *relative minimum* of f.
2. If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a *relative maximum* of f.
3. If $f'(x)$ does not change sign at c, then $f(c)$ is neither a relative minimum nor a relative maximum.



Relative minimum



Relative maximum



Neither relative minimum nor relative maximum

Proof Assume that $f'(x)$ changes from negative to positive at c. Then there exist and b in I such that

$$f'(x) < 0 \text{ for all } x \text{ in } (a, c)$$

and

$$f'(x) > 0 \text{ for all } x \text{ in } (c, b).$$

By Theorem 3.5, f is decreasing on (a, c) and increasing on (c, b). So, $f(c)$ is a mi mum of f on the open interval (a, b) and, consequently, a relative minimum of f. Th proves the first case of the theorem. The second case can be proved in a similar w (see Exercise 78).

Example 2 Applying the First Derivative Test

Find the relative extrema of the function $f(x) = \frac{1}{2}x - \sin x$ in the interval $(0, 2\pi)$.

Solution Note that f is continuous on the interval $(0, 2\pi)$. To determine the critical numbers of f in this interval, set $f'(x)$ equal to 0.

$$f'(x) = \frac{1}{2} - \cos x = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3} \qquad \text{Critical numbers}$$

Because there are no points for which f' does not exist, you can conclude that $x = \pi/3$ and $x = 5\pi/3$ are the only critical numbers. The table summarizes the testing of the three intervals determined by these two critical numbers.

Interval	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \frac{5\pi}{3}$	$\frac{5\pi}{3} < x < 2\pi$
Test Value	$x = \frac{\pi}{4}$	$x = \pi$	$x = \frac{7\pi}{4}$
Sign of $f'(x)$	$f'\left(\frac{\pi}{4}\right) < 0$	$f'(\pi) > 0$	$f'\left(\frac{7\pi}{4}\right) < 0$
Conclusion	Decreasing	Increasing	Decreasing

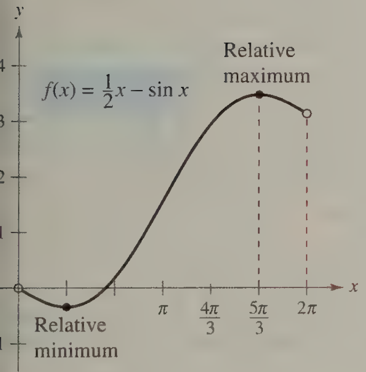
By applying the First Derivative Test, you can conclude that f has a relative minimum at

$$x = \frac{\pi}{3} \qquad x\text{-value of relative minimum}$$

and a relative maximum at

$$x = \frac{5\pi}{3} \qquad x\text{-value of relative maximum}$$

as shown in Figure 3.19.



relative minimum occurs where f changes om decreasing to increasing, and a relative aximum occurs where f changes from creasing to decreasing.
igure 3.19

EXPLORATION

Comparing Graphical and Analytical Approaches In Section 3.2, we pointed out that, *by itself*, a graphing utility can give misleading information about the relative extrema of a graph. *Used in conjunction with an analytical approach*, however, a graphing utility can provide a good way to reinforce your conclusions. Try using a graphing utility to graph the function in Example 2. Then use the *zoom* and *trace* features to estimate the relative extrema. How close are your graphical approximations?

Note that in Examples 1 and 2 the given functions are differentiable on the entire real line. For such functions, the only critical numbers are those for which $f'(x) = 0$. Example 3 concerns a function that has two types of critical numbers—those for which $f'(x) = 0$ and those for which f is not differentiable.

Example 3 Applying the First Derivative Test

Find the relative extrema of

$$f(x) = (x^2 - 4)^{2/3}.$$

Solution Begin by noting that f is continuous on the entire real line. The derivativ of f

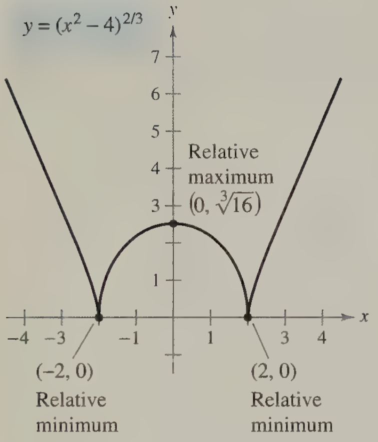
$$f'(x) = \frac{2}{3}(x^2 - 4)^{-1/3}(2x) \qquad \text{General Power Rule}$$

$$= \frac{4x}{3(x^2 - 4)^{1/3}} \qquad \text{Simplify.}$$

is 0 when $x = 0$ and does not exist when $x = \pm 2$. So, the critical numbers ar $x = -2$, $x = 0$, and $x = 2$. The table summarizes the testing of the four interva determined by these three critical numbers.

Interval	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value	$x = -3$	$x = -1$	$x = 1$	$x = 3$
Sign of $f'(x)$	$f'(-3) < 0$	$f'(-1) > 0$	$f'(1) < 0$	$f'(3) > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

By applying the First Derivative Test, you can conclude that f has a relative minimu at the point $(-2, 0)$, a relative maximum at the point $(0, \sqrt[3]{16})$, and another relativ minimum at the point $(2, 0)$, as shown in Figure 3.20.



You can apply the First Derivative Test to find relative extrema.
Figure 3.20

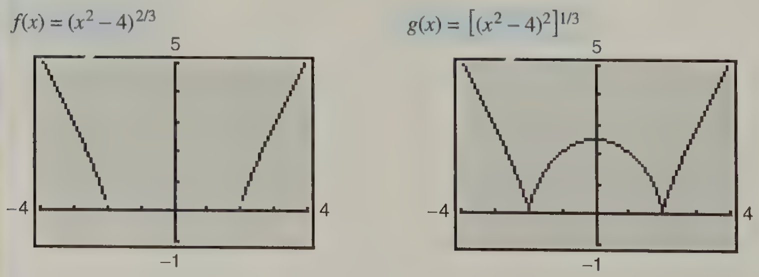
TECHNOLOGY PITFALL When using a graphing utility to graph a functic involving radicals or rational exponents, be sure you understand the way the utilit evaluates radical expressions. For instance, even though

$$f(x) = (x^2 - 4)^{2/3}$$

and

$$g(x) = [(x^2 - 4)^2]^{1/3}$$

are the same algebraically, many graphing utilities distinguish between these tw functions. Which of the graphs shown in Figure 3.21 is incorrect? Why did th graphing utility produce an incorrect graph?



Which graph is incorrect?
Figure 3.21

When using the First Derivative Test, be sure to consider the domain of the function. For instance, in the next example, the function

$$f(x) = \frac{x^4 + 1}{x^2}$$

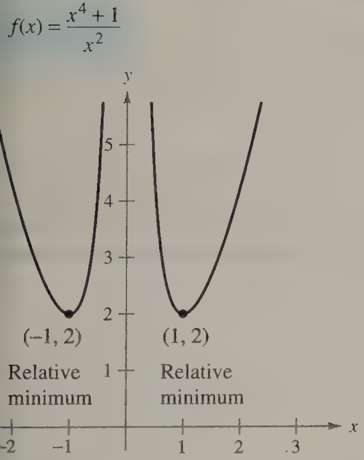
is not defined when $x = 0$. This x-value must be used with the critical numbers to determine the test intervals.

Example 4 Applying the First Derivative Test

Find the relative extrema of $f(x) = \dfrac{x^4 + 1}{x^2}$.

Solution

$$\begin{aligned} f(x) &= x^2 + x^{-2} && \text{Rewrite original function.} \\ f'(x) &= 2x - 2x^{-3} && \text{Differentiate.} \\ &= 2x - \frac{2}{x^3} \\ &= \frac{2(x^4 - 1)}{x^3} && \text{Simplify.} \\ &= \frac{2(x^2 + 1)(x - 1)(x + 1)}{x^3} && \text{Factor.} \end{aligned}$$



values that are not in the domain of *f*, well as critical numbers, determine test tervals for f'.
gure 3.22

So, $f'(x)$ is zero at $x = \pm 1$. Moreover, because $x = 0$ is not in the domain of f, you should use this x-value along with the critical numbers to determine the test intervals.

$x = \pm 1$ Critical numbers, $f'(\pm 1) = 0$

$x = 0$ 0 is not in the domain of f.

The table summarizes the testing of the four intervals determined by these three x-values.

Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Value	$x = -2$	$x = -\frac{1}{2}$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	$f'(-2) < 0$	$f'(-\frac{1}{2}) > 0$	$f'(\frac{1}{2}) < 0$	$f'(2) > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

By applying the First Derivative Test, you can conclude that f has one relative minimum at the point $(-1, 2)$ and another at the point $(1, 2)$, as shown in Figure 3.22.

TECHNOLOGY The most difficult step in applying the First Derivative Test is finding the values for which the derivative is equal to 0. For instance, the values of x for which the derivative of

$$f(x) = \frac{x^4 + 1}{x^2 + 1}$$

is equal to zero are 0 and $\pm\sqrt{\sqrt{2} - 1}$. If you have access to technology that can perform symbolic differentiation and solve equations, try using it to apply the First Derivative Test to this function.

Brian Bailey/Tony Stone Images

If a projectile is propelled from ground level and air resistance is neglected, the object will travel farthest with an initial angle of 45°. If, however, the projectile is propelled from a point above ground level, the angle that yields a maximum horizontal distance is not 45° (see Example 5).

Example 5 The Path of a Projectile

Neglecting air resistance, the path of a projectile that is propelled at an angle θ is

$$y = \frac{g \sec^2 \theta}{2v_0^2}x^2 + (\tan \theta)x + h, \qquad 0 \le \theta \le \frac{\pi}{2}$$

where y is the height, x is the horizontal distance, g is the acceleration due to gravit v_0 is the initial velocity, and h is the initial height. (This equation is derived in Sectio 11.3.) Let $g = -32$ feet per second per second, $v_0 = 24$ feet per second, and $h =$ feet. What value of θ will produce a maximum horizontal distance?

Solution To find the distance the projectile travels, let $y = 0$, and use the quadrati formula to solve for x.

$$\frac{g \sec^2 \theta}{2v_0^2}x^2 + (\tan \theta)x + h = 0$$

$$\frac{-32 \sec^2 \theta}{2(24^2)}x^2 + (\tan \theta)x + 9 = 0$$

$$-\frac{\sec^2 \theta}{36}x^2 + (\tan \theta)x + 9 = 0$$

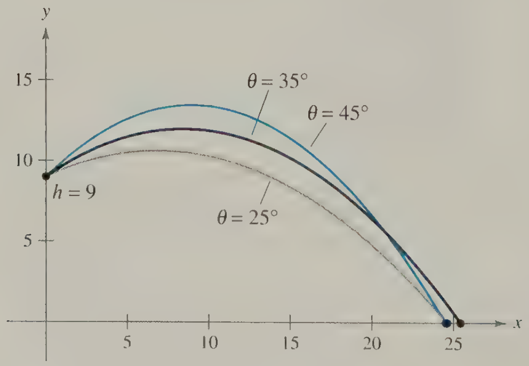
$$x = \frac{-\tan \theta \pm \sqrt{\tan^2 \theta + \sec^2 \theta}}{-\sec^2 \theta/18}$$

$$x = 18 \cos \theta\left(\sin \theta + \sqrt{\sin^2 \theta + 1}\right), \qquad x \ge 0$$

At this point, you need to find the value of θ that produces a maximum value of Applying the First Derivative Test by hand would be very tedious. Using technolog to solve the equation $dx/d\theta = 0$, however, eliminates most of the messy computa tions. The result is that the maximum value of x occurs when

$$\theta \approx 0.61548 \text{ radians, or } 35.3°.$$

This conclusion is reinforced by sketching the path of the projectile for differe values of θ, as shown in Figure 3.23. Of the three paths shown, note that the distanc traveled is greatest for $\theta = 35°$.



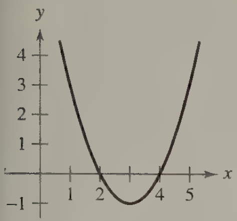
The path of a projectile with initial angle θ
Figure 3.23

NOTE A computer simulation of this example is given in the *Interactive* CD-ROM version this text (available at *college.hmco.com*). Using that simulation, you can experimental discover that the maximum value of x occurs when $\theta \approx 35.3°$.

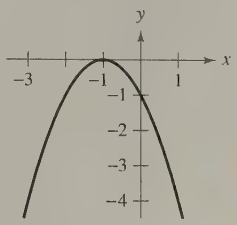
EXERCISES FOR SECTION 3.3

Exercises 1–10, identify the open intervals on which the func- on is increasing or decreasing.

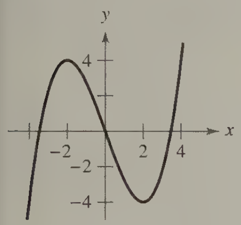
. $f(x) = x^2 - 6x + 8$



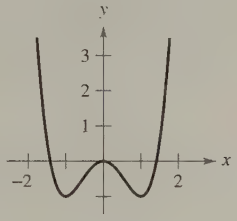
2. $y = -(x + 1)^2$



, $y = \dfrac{x^3}{4} - 3x$



4. $f(x) = x^4 - 2x^2$



. $f(x) = \dfrac{1}{x^2}$ **6.** $y = \dfrac{x^2}{x + 1}$

. $g(x) = x^2 - 2x - 8$ **8.** $h(x) = 27x - x^3$

. $y = x\sqrt{16 - x^2}$ **10.** $y = x + \dfrac{4}{x}$

Exercises 11–32, find the critical numbers of *f* (if any). Find e open intervals on which the function is increasing or ecreasing and locate all relative extrema. Use a graphing ility to confirm your results.

. $f(x) = x^2 - 6x$ **12.** $f(x) = x^2 + 8x + 10$

. $f(x) = -2x^2 + 4x + 3$ **14.** $f(x) = -(x^2 + 8x + 12)$

. $f(x) = 2x^3 + 3x^2 - 12x$ **16.** $f(x) = x^3 - 6x^2 + 15$

. $f(x) = x^2(3 - x)$ **18.** $f(x) = (x + 2)^2(x - 1)$

. $f(x) = \dfrac{x^5 - 5x}{5}$ **20.** $f(x) = x^4 - 32x + 4$

. $f(x) = x^{1/3} + 1$ **22.** $f(x) = x^{2/3} - 4$

. $f(x) = (x - 1)^{2/3}$ **24.** $f(x) = (x - 1)^{1/3}$

. $f(x) = 5 - |x - 5|$ **26.** $f(x) = |x + 3| - 1$

. $f(x) = x + \dfrac{1}{x}$ **28.** $f(x) = \dfrac{x}{x + 1}$

. $f(x) = \dfrac{x^2}{x^2 - 9}$ **30.** $f(x) = \dfrac{x + 3}{x^2}$

. $f(x) = \dfrac{x^2 - 2x + 1}{x + 1}$ **32.** $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$

In Exercises 33–36, consider the function on the interval $(0, 2\pi)$. Find the open intervals on which the function is increasing or decreasing and locate all relative extrema. Use a graphing utility to confirm your results.

33. $f(x) = \dfrac{x}{2} + \cos x$ **34.** $f(x) = \sin x \cos x$

35. $f(x) = \sin^2 x + \sin x$ **36.** $f(x) = \dfrac{\sin x}{1 + \cos^2 x}$

In Exercises 37–40, (a) use a computer algebra system to differentiate the function, (b) sketch the graphs of *f* and *f′* on the same set of coordinate axes over the indicated interval, (c) find the critical numbers of *f* in the open interval, and (d) find the interval(s) on which *f′* is positive and the interval(s) on which it is negative. Compare the behavior of *f* and the sign of *f′*.

37. $f(x) = 2x\sqrt{9 - x^2}$, $[-3, 3]$

38. $f(x) = 10\left(5 - \sqrt{x^2 - 3x + 16}\right)$, $[0, 5]$

39. $f(t) = t^2 \sin t$, $[0, 2\pi]$ **40.** $f(x) = \dfrac{x}{2} + \cos\dfrac{x}{2}$, $[0, 4\pi]$

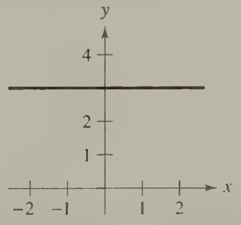
In Exercises 41 and 42, use symmetry, extrema, and zeros to sketch the graph of *f*. How do the functions *f* and *g* differ? Explain.

41. $f(x) = \dfrac{x^5 - 4x^3 + 3x}{x^2 - 1}$, $g(x) = x(x^2 - 3)$

42. $f(t) = \cos^2 t - \sin^2 t$, $g(t) = 1 - 2\sin^2 t$, $(-2, 2)$

Think About It **In Exercises 43–48, the graph of *f* is shown in the figure. Sketch a graph of the derivative of *f*. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

43.



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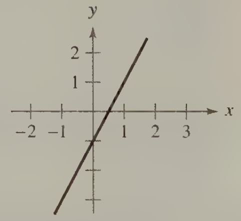

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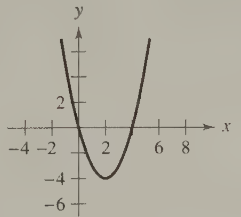
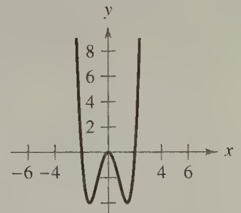


46.



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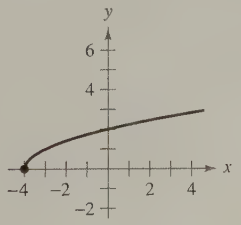

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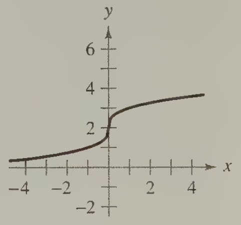


Getting at the Concept

In Exercises 49–54, assume that f is differentiable for all x. The sign of f' is as follows.

$f'(x) > 0$ on $(-\infty, -4)$

$f'(x) < 0$ on $(-4, 6)$

$f'(x) > 0$ on $(6, \infty)$

Supply the appropriate inequality for the indicated value of c.

	Function	*Sign of $g'(c)$*
49.	$g(x) = f(x) + 5$	$g'(0) \;\square\; 0$
50.	$g(x) = 3f(x) - 3$	$g'(-5) \;\square\; 0$
51.	$g(x) = -f(x)$	$g'(-6) \;\square\; 0$
52.	$g(x) = -f(x)$	$g'(0) \;\square\; 0$
53.	$g(x) = f(x - 10)$	$g'(0) \;\square\; 0$
54.	$g(x) = f(x - 10)$	$g'(8) \;\square\; 0$

55. Sketch the graph of an arbitrary function f such that

$$f'(x)\begin{cases} > 0, & x < 4 \\ \text{undefined}, & x = 4 \\ < 0, & x > 4. \end{cases}$$

56. A differentiable function f has one critical number at $x = 5$. Identify the relative extrema of f at the critical number if $f'(4) = -2.5$ and $f'(6) = 3$.

57. ***Think About It*** The function f is differentiable on the interval $[-1, 1]$. The table shows the values of f' for selected values of x. Sketch the graph of f, approximate the critical numbers, and identify the relative extrema.

x	-1	-0.75	-0.50	-0.25
$f'(x)$	-10	-3.2	-0.5	0.8

x	0	0.25	0.50	0.75	1
$f'(x)$	5.6	3.6	-0.2	-6.7	-20.1

58. ***Rolling a Ball Bearing*** A ball bearing is placed on an inclined plane and begins to roll. The angle of elevation of the plane is θ. The distance (in meters) the ball bearing rolls in t seconds is

$$s(t) = 4.9(\sin\theta)t^2.$$

(a) Determine the speed of the ball bearing after t seconds.

(b) What value of θ will produce the maximum speed at a particular time?

59. ***Numerical, Graphical, and Analytic Analysis*** Consider t functions $f(x) = x$ and $g(x) = \sin x$ on the interval $(0, \pi)$.

(a) Complete the table and make a conjecture about which the greater function on the interval $(0, \pi)$.

x	0.5	1	1.5	2	2.5	3
$f(x)$						
$g(x)$						

(b) Use a graphing utility to graph the functions and use t graphs to make a conjecture about which is the great function on the interval $(0, \pi)$.

(c) Prove that $f(x) > g(x)$ on the interval $(0, \pi)$. [*Hint:* Sho that $h'(x) > 0$ where $h = f - g$.]

60. ***Numerical, Graphical, and Analytic Analysis*** The conce tration C of a chemical in the bloodstream t hours after injecti into muscle tissue is

$$C(t) = \frac{3t}{27 + t^3}, \quad t \ge 0.$$

(a) Complete the table and use the table to approximate t time when the concentration is greatest.

t	0	0.5	1	1.5	2	2.5	3
$C(t)$							

(b) Use a graphing utility to graph the concentration functi and use the graph to approximate the time when the co centration is greatest.

(c) Use calculus to determine analytically the time when t concentration is greatest.

61. ***Trachea Contraction*** Coughing forces the trachea (win pipe) to contract, which affects the velocity v of the air passi through the trachea. Suppose the velocity of the air duri coughing is

$$v = k(R - r)r^2, \quad 0 \le r < R$$

where k is constant, R is the normal radius of the trachea, anc is the radius during coughing. What radius will produce t maximum air velocity?

62. ***Profit*** The profit P (in dollars) made by a fast-food restaura selling x hamburgers is

$$P = 2.44x - \frac{x^2}{20{,}000} - 5000, \quad 0 \le x \le 35{,}000.$$

Find the open intervals on which P is increasing or decreasir

63. ***Power*** The electric power P in watts in a direct-current circ with two resistors R_1 and R_2 connected in series is

$$P = \frac{vR_1R_2}{(R_1 + R_2)^2}$$

where v is the voltage. If v and R_1 are held constant, what res tance R_2 produces maximum power?

. ***Electrical Resistance*** The resistance R of a certain type of resistor is

$$R = \sqrt{0.001T^4 - 4T + 100}$$

where R is measured in ohms and the temperature T is measured in degrees Celsius.

(a) Use a computer algebra system to find dR/dT and the critical number of the function. Determine the minimum resistance for this type of resistor.

(b) Use a graphing utility to graph the function R and use the graph to approximate the minimum resistance for this type of resistor.

. ***Modeling Data*** The number of bankruptcies (in thousands) for the years 1981 through 1998 are as follows.

1981: 360.3; 1982: 367.9; 1983: 374.7; 1984: 344.3;

1985: 364.5; 1986: 477.9; 1987: 561.3; 1988: 594.6;

1989: 643.0; 1990: 725.5; 1991: 880.4; 1992: 972.5;

1993: 918.7; 1994: 845.3; 1995: 858.1; 1996: 1042.1;

1997: 1317.0; 1998: 1411.4

(Source: Administrative Office of the U.S. Courts)

(a) Use the regression capabilities of a graphing utility to find a model of the form

$$B = at^4 + bt^3 + ct^2 + dt + e$$

for the data. (Let $t = 1$ represent 1981.)

(b) Use a graphing utility to plot the data and graph the model.

(c) Analytically find the minimum of the model and compare the result with the actual data.

. Use a graphing utility to graph $f(x) = 2 \sin 3x + 4 \cos 3x$. Find the maximum value of f. How could you use calculus to estimate the maximum?

Creating Polynomial Functions **In Exercises 67–70, find a polynomial function**

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

that has only the specified extrema. (a) Determine the minimum degree of the function and give the criteria you used in determining the degree. (b) Using the fact that the coordinates of the extrema are solution points of the function, and that the x-coordinates are critical numbers, determine a system of linear equations whose solution yields the coefficients of the required function. (c) Use a graphing utility to solve the system of equations and determine the function. (d) Use a graphing utility to confirm your result graphically.

67. Relative minimum: $(0, 0)$; Relative maximum: $(2, 2)$

68. Relative minimum: $(0, 0)$; Relative maximum: $(4, 1000)$

69. Relative minima: $(0, 0)$, $(4, 0)$
Relative maximum: $(2, 4)$

70. Relative minimum: $(1, 2)$
Relative maxima: $(-1, 4)$, $(3, 4)$

True or False? **In Exercises 71–76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

71. The sum of two increasing functions is increasing.

72. The product of two increasing functions is increasing.

73. Every nth-degree polynomial has $(n - 1)$ critical numbers.

74. An nth-degree polynomial has at most $(n - 1)$ critical numbers.

75. There is a relative maximum or minimum at each critical number.

76. The relative maxima of the function f are $f(1) = 4$ and $f(3) = 10$. Therefore, f has at least one minimum for some x in the interval $(1, 3)$.

77. Prove the second case of Theorem 3.5.

78. Prove the second case of Theorem 3.6.

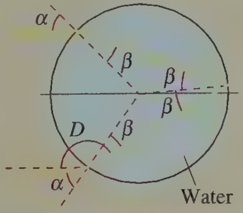
79. Let $x > 0$ and $n > 1$ be real numbers. Prove that $(1 + x)^n > 1 + nx$.

SECTION PROJECT RAINBOWS

Rainbows are formed when light strikes raindrops and is reflected and refracted, as shown in the figure. (This figure shows a cross section of a spherical raindrop.) The Law of Refraction tates that $(\sin \alpha)/(\sin \beta) = k$, where $k \approx 1.33$ (for water). The ngle of deflection is given by $D = \pi + 2\alpha - 4\beta$.

a) Sketch the graph of D for $0 \le \alpha \le \pi/2$. Use a graphing utility with

$$D = \pi + 2\alpha - 4 \sin^{-1}\left(\frac{1}{k} \sin \alpha\right).$$



(b) Prove that the minimum angle of deflection occurs when

$$\cos \alpha = \sqrt{\frac{k^2 - 1}{3}}.$$

For water, what is the minimum angle of deflection, D_{min}? (The angle $\pi - D_{min}$ is called the *rainbow angle*.) What value of α produces this minimum angle? (A ray of sunlight that strikes a raindrop at this angle, α, is called a *rainbow ray*.)

FOR FURTHER INFORMATION For more information about the mathematics of rainbows, see the article “Somewhere Within the Rainbow” by Steven Janke in *The UMAP Journal*. To view this article, go to the website *www.matharticles.com*.

Section 3.4 Concavity and the Second Derivative Test

- Determine intervals on which a function is concave upward or concave downward.
- Find any points of inflection of the graph of a function.
- Apply the Second Derivative Test to find relative extrema of a function.

Concavity

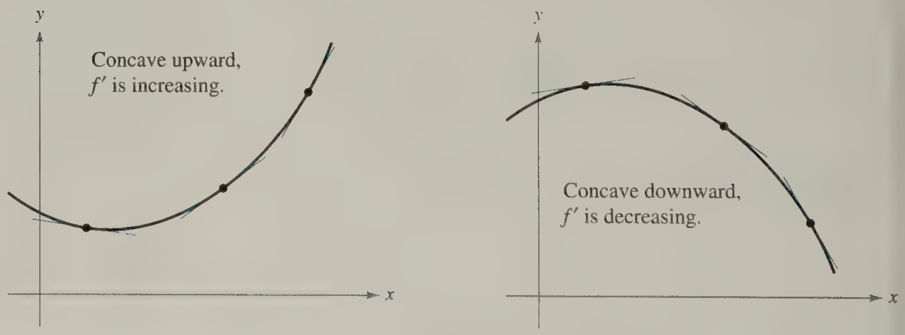
You have already seen that locating the intervals in which a function f increases decreases helps to describe its graph. In this section, you will see how locating th intervals in which f' increases or decreases can be used to determine where the gra of f is *curving upward* or *curving downward.*

Definition of Concavity

Let f be differentiable on an open interval I. The graph of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is decreasing on the interval.

The following graphical interpretation of concavity is useful. (See Appendix for a proof of these results.)

1. Let f be differentiable on an open interval I. If the graph of f is concave *upwa* on I, then the graph of f lies *above* all of its tangent lines on I. (See Figure 3.24a
2. Let f be differentiable on an open interval I. If the graph of f is concave *downwa* on I, then the graph of f lies *below* all of its tangent lines on I. (See Figure 3.24b



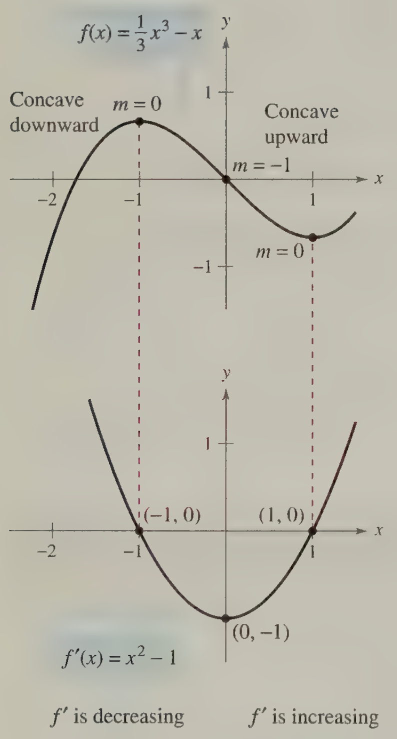
(a) The graph of f lies above its tangent lines. **(b)** The graph of f lies below its tangent lines

Figure 3.24

To find the open intervals on which the graph of a function f is concave upwa or downward, you need to find the intervals on which f' is increasing or decreasin For instance, the graph of

$$f(x) = \tfrac{1}{3}x^3 - x$$

is concave downward on the open interval $(-\infty, 0)$ because $f'(x) = x^2 - 1$ decreasing there. (See Figure 3.25.) Similarly, the graph of f is concave upward on t interval $(0, \infty)$ because f' is increasing on $(0, \infty)$.



The concavity of f is related to the slope of the derivative.

Figure 3.25

The following theorem shows how to use the *second* derivative of a function f to determine intervals on which the graph of f is concave upward or downward. A proof of this theorem follows directly from Theorem 3.5 and the definition of concavity.

THEOREM 3.7 Test for Concavity

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward in I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward in I.

NOTE A third case of Theorem 3.7 could be that if $f''(x) = 0$ for all x in I, then f is linear. Note, however, that concavity is not defined for a line. In other words, a straight line is neither concave upward nor concave downward.

To apply Theorem 3.7, locate the x-values at which $f''(x) = 0$ or f'' does not exist. Second, use these x-values to determine test intervals. Finally, test the sign of $f''(x)$ in each of the test intervals.

Example 1 Determining Concavity

Determine the open intervals on which the graph of

$$f(x) = \frac{6}{x^2 + 3}$$

is concave upward or downward.

Solution Begin by observing that f is continuous on the entire real line. Next, find the second derivative of f.

$$f(x) = 6(x^2 + 3)^{-1} \qquad \text{Rewrite original function.}$$

$$f'(x) = (-6)(x^2 + 3)^{-2}(2x) \qquad \text{Differentiate.}$$

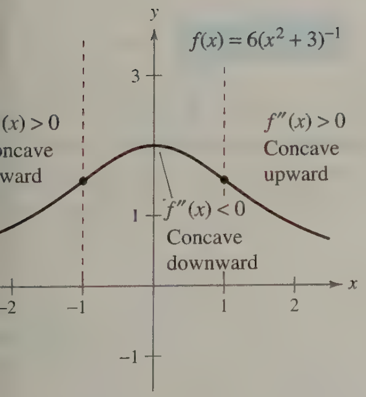
$$= \frac{-12x}{(x^2 + 3)^2} \qquad \text{First derivative}$$

$$f''(x) = \frac{(x^2 + 3)^2(-12) - (-12x)(2)(x^2 + 3)(2x)}{(x^2 + 3)^4} \qquad \text{Differentiate.}$$

$$= \frac{36(x^2 - 1)}{(x^2 + 3)^3} \qquad \text{Second derivative}$$

Because $f''(x) = 0$ when $x = \pm 1$ and f'' is defined on the entire real line, you should test f'' in the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. The results are shown in the table and in Figure 3.26.

Interval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Test value	$x = -2$	$x = 0$	$x = 2$
Sign of $f''(x)$	$f''(-2) > 0$	$f''(0) < 0$	$f''(2) > 0$
Conclusion	Concave upward	Concave downward	Concave upward



om the sign of f'' you can determine the
ncavity of the graph of f.
gure 3.26

The function given in Example 1 is continuous on the entire real line. If there a x-values at which the function is not continuous, these values should be used alo with the points at which $f''(x) = 0$ or points at which f is not differentiable to for the test intervals.

Example 2 Determining Concavity

Determine the open intervals in which the graph of $f(x) = \dfrac{x^2 + 1}{x^2 - 4}$ is concave upwa or downward.

Solution Differentiating twice produces the following.

$$f(x) = \frac{x^2 + 1}{x^2 - 4}$$ Write original equation.

$$f'(x) = \frac{(x^2 - 4)(2x) - (x^2 + 1)(2x)}{(x^2 - 4)^2}$$ Differentiate.

$$= \frac{-10x}{(x^2 - 4)^2}$$ First derivative

$$f''(x) = \frac{(x^2 - 4)^2(-10) - (-10x)(2)(x^2 - 4)(2x)}{(x^2 - 4)^4}$$ Differentiate.

$$= \frac{10(3x^2 + 4)}{(x^2 - 4)^3}$$ Second derivative

There are no points at which $f''(x) = 0$, but at $x = \pm 2$ the function f is not contin ous, so you test for concavity in the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$, shown in the table. The graph of f is shown in Figure 3.27.

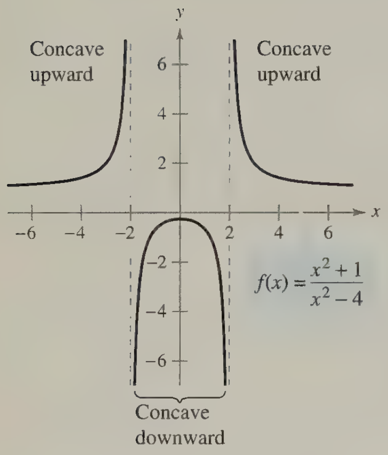
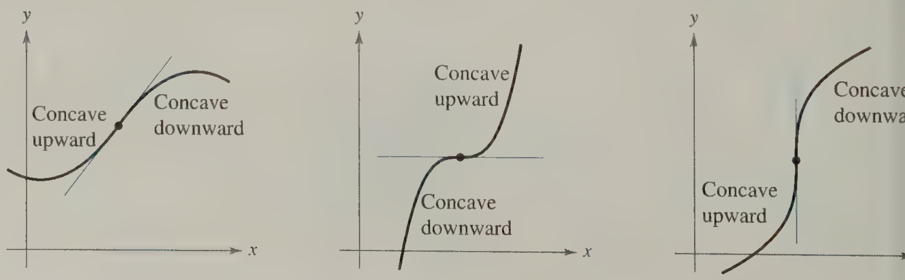


Figure 3.27

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Test value	$x = -3$	$x = 0$	$x = 3$
Sign of $f''(x)$	$f''(-3) > 0$	$f''(0) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of Inflection

The graph in Figure 3.26 has two points at which the concavity changes. If the tange line to the graph exists at such a point, that point is a **point of inflection.** Three typ of points of inflection are shown in Figure 3.28. Note that a graph crosses its tange line at a point of inflection.



The concavity of f changes at a point of inflection.
Figure 3.28

NOTE: The definition of *point of inflection* given in this book requires that the tangent line exists at the point of inflection. Some books do not require this. For instance, we do not consider the function

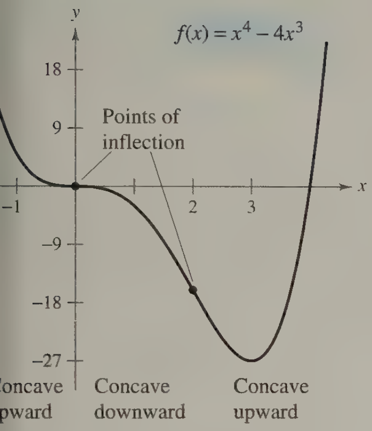
$$f(x) = \begin{cases} x^3, & x < 0 \\ x^2 + 2x, & x \ge 0 \end{cases}$$

to have a point of inflection at the origin, even though the concavity of the graph changes from concave downward to concave upward.

To locate *possible* points of inflection, you need only determine the values of x for which $f''(x) = 0$ or for which f is not differentiable. This is similar to the procedure for locating relative extrema of f.

THEOREM 3.8 Points of Inflection

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f is not differentiable at $x = c$.



oints of inflection can occur where $''(x) = 0$ or f'' does not exist.
igure 3.29

Example 3 Finding Points of Inflection

Determine the points of inflection and discuss the concavity of the graph of

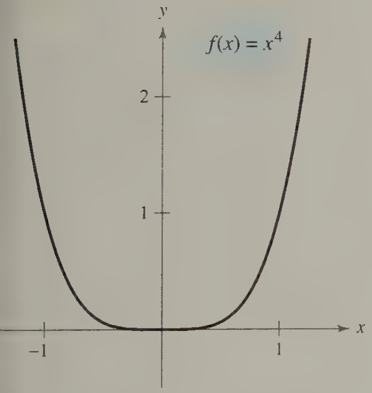
$$f(x) = x^4 - 4x^3.$$

Solution Differentiating twice produces the following.

$$f(x) = x^4 - 4x^3 \qquad \text{Write original function.}$$
$$f'(x) = 4x^3 - 12x^2 \qquad \text{Find first derivative.}$$
$$f''(x) = 12x^2 - 24x = 12x(x - 2) \qquad \text{Find second derivative.}$$

Setting $f''(x) = 0$, you can determine that the possible points of inflection occur at $x = 0$ and $x = 2$. By testing the intervals determined by these x-values, you can conclude that they both yield points of inflection. A summary of this testing is shown in the table, and the graph of f is shown in Figure 3.29.

Interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test value	$x = -1$	$x = 1$	$x = 3$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward



$''(0) = 0$, but $(0, 0)$ is not a point of flection.
igure 3.30

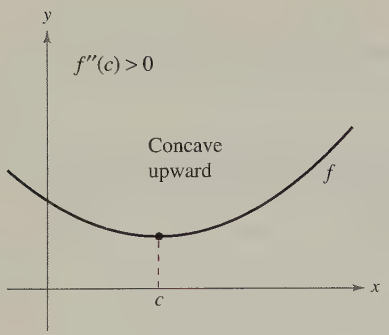
The converse of Theorem 3.8 is not generally true. That is, it is possible for the second derivative to be 0 at a point that is *not* a point of inflection. For instance, the graph of $f(x) = x^4$ is shown in Figure 3.30. The second derivative is 0 when $x = 0$, but the point $(0, 0)$ is not a point of inflection because the graph of f is concave upward in both intervals $-\infty < x < 0$ and $0 < x < \infty$.

EXPLORATION

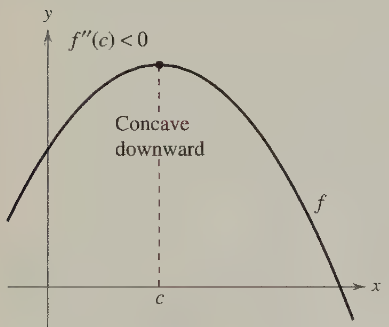
Consider a general cubic function of the form

$$f(x) = ax^3 + bx^2 + cx + d.$$

You know that the value of d has a bearing on the location of the graph but has no bearing on the value of the first derivative at given values of x. Graphically, this is true because changes in the value of d shift the graph up or down but do not change its basic shape. Use a graphing utility to graph several cubics with different values of c. Then give a graphical explanation of why changes in c do not affect the values of the second derivative.



If $f'(c) = 0$ and $f''(c) > 0$, $f(c)$ is a relative minimum.



If $f'(c) = 0$ and $f''(c) < 0$, $f(c)$ is a relative maximum.
Figure 3.31

The Second Derivative Test

In addition to testing for concavity, the second derivative can be used to perform simple test for relative maxima and minima. The test is based on the fact that if th graph of a function f is concave upward on an open interval containing c, an $f'(c) = 0$, $f(c)$ must be a relative minimum of f. Similarly, if the graph of a function is concave downward on an open interval containing c, and $f'(c) = 0$, $f(c)$ must be relative maximum of f (see Figure 3.31).

THEOREM 3.9 Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.

If $f''(c) = 0$, the test fails. In such cases, you can use the First Derivative Test.

Proof If $f'(c) = 0$ and $f''(c) > 0$, there exists an open interval I containing c fo which

$$\frac{f'(x) - f'(c)}{x - c} = \frac{f'(x)}{x - c} > 0$$

for all $x \neq c$ in I. If $x < c$, then $x - c < 0$ and $f'(x) < 0$. Also, if $x > c$, the $x - c > 0$ and $f'(x) > 0$. So, $f'(x)$ changes from negative to positive at c, and the Firs Derivative Test implies that $f(c)$ is a relative minimum. A proof of the second case i left to you.

Example 4 Using the Second Derivative Test

Find the relative extrema for $f(x) = -3x^5 + 5x^3$.

Solution Begin by finding the critical numbers of f.

$$f'(x) = -15x^4 + 15x^2 = 15x^2(1 - x^2) = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

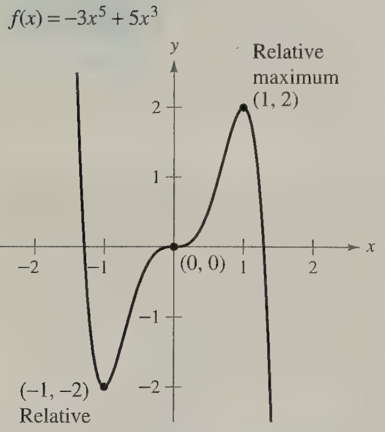
$$x = -1, 0, 1 \qquad \text{Critical numbers}$$

Using

$$f''(x) = -60x^3 + 30x = 30(-2x^3 + x),$$

you can apply the Second Derivative Test as follows.

Point	$(-1, -2)$	$(1, 2)$	$(0, 0)$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(0) = 0$
Conclusion	Relative minimum	Relative maximum	Test fails



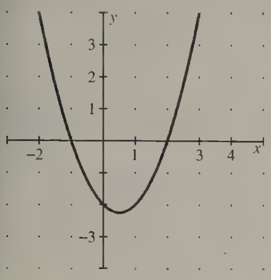
(0, 0) is neither a relative minimum nor a relative maximum.
Figure 3.32

Because the Second Derivative Test fails at (0, 0), you can use the First Derivative Te and observe that f increases to the left and right of $x = 0$. So, (0, 0) is neither a relativ minimum nor a relative maximum (even though the graph has a horizontal tangent lin at this point). The graph of f is shown in Figure 3.32.

EXERCISES FOR SECTION 3.4

Exercises 1–10, determine the open intervals on which the raph is concave upward or concave downward.

1. $y = x^2 - x - 2$



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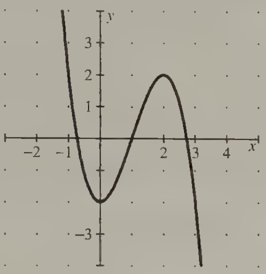
2. $y = -x^3 + 3x^2 - 2$



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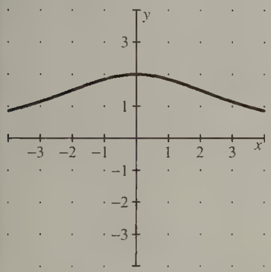
$f(x) = \dfrac{24}{x^2 + 12}$



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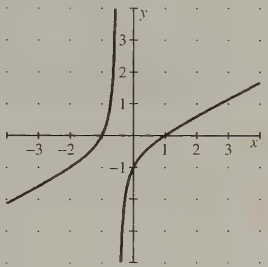
4. $f(x) = \dfrac{x^2 - 1}{2x + 1}$



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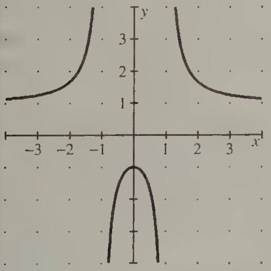
$f(x) = \dfrac{x^2 + 1}{x^2 - 1}$



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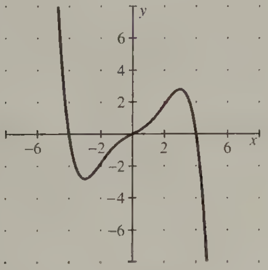
6. $y = \dfrac{-3x^5 + 40x^3 + 135x}{270}$



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$g(x) = 3x^2 - x^3$

8. $h(x) = x^5 - 5x + 2$

$y = 2x - \tan x, \quad \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

10. $y = x + \dfrac{2}{\sin x}, \quad (-\pi, \pi)$

Exercises 11–26, find the points of inflection and discuss the ncavity of the graph of the function.

$f(x) = x^3 - 6x^2 + 12x$

12. $f(x) = 2x^3 - 3x^2 - 12x + 5$

$f(x) = \frac{1}{4}x^4 - 2x^2$

14. $f(x) = 2x^4 - 8x + 3$

$f(x) = x(x - 4)^3$

16. $f(x) = x^3(x - 4)$

$f(x) = x\sqrt{x + 3}$

18. $f(x) = x\sqrt{x + 1}$

$f(x) = \dfrac{x}{x^2 + 1}$

20. $f(x) = \dfrac{x + 1}{\sqrt{x}}$

21. $f(x) = \sin \dfrac{x}{2}, \quad [0, 4\pi]$

22. $f(x) = 2 \csc \dfrac{3x}{2}, \quad (0, 2\pi)$

23. $f(x) = \sec\left(x - \dfrac{\pi}{2}\right), \quad (0, 4\pi)$

24. $f(x) = \sin x + \cos x, \quad [0, 2\pi]$

25. $f(x) = 2 \sin x + \sin 2x, \quad [0, 2\pi]$

26. $f(x) = x + 2 \cos x, \quad [0, 2\pi]$

In Exercises 27–40, find all relative extrema. Use the Second Derivative Test where applicable.

27. $f(x) = x^4 - 4x^3 + 2$

28. $f(x) = x^2 + 3x - 8$

29. $f(x) = (x - 5)^2$

30. $f(x) = -(x - 5)^2$

31. $f(x) = x^3 - 3x^2 + 3$

32. $f(x) = x^3 - 9x^2 + 27x$

33. $g(x) = x^2(6 - x)^3$

34. $g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2$

35. $f(x) = x^{2/3} - 3$

36. $f(x) = \sqrt{x^2 + 1}$

37. $f(x) = x + \dfrac{4}{x}$

38. $f(x) = \dfrac{x}{x - 1}$

39. $f(x) = \cos x - x, \quad [0, 4\pi]$

40. $f(x) = 2 \sin x + \cos 2x, \quad [0, 2\pi]$

In Exercises 41–44, use a computer algebra system to analyze the function over the indicated interval. (a) Find the first and second derivatives of the function. (b) Find any relative extrema and points of inflection. (c) Graph f, f', and f'' on the same set of coordinate axes and state the relationship between the behavior of f and the signs of f' and f''.

41. $f(x) = 0.2x^2(x - 3)^3, \quad [-1, 4]$

42. $f(x) = x^2\sqrt{6 - x^2}, \quad [-\sqrt{6}, \sqrt{6}]$

43. $f(x) = \sin x - \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x, \quad [0, \pi]$

44. $f(x) = \sqrt{2x} \sin x, \quad [0, 2\pi]$

Getting at the Concept

45. Consider a function f such that f' is increasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.

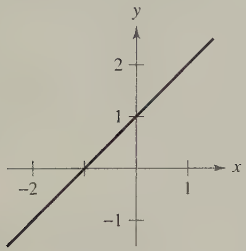
46. Consider a function f such that f' is decreasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.

47. Sketch the graph of a function f that does *not* have a point of inflection at $(c, f(c))$ even though $f''(c) = 0$.

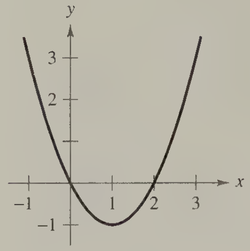
48. S represents weekly sales of a product. What can be said of S' and S'' for each of the following?
    - (a) The rate of change of sales is increasing.
    - (b) Sales are increasing at a slower rate.
    - (c) The rate of change of sales is constant.
    - (d) Sales are steady.
    - (e) Sales are declining, but at a slower rate.
    - (f) Sales have bottomed out and have started to rise.

In Exercises 49–52, graph f, f', and f'' on the same set of coordinate axes. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

49.



50.



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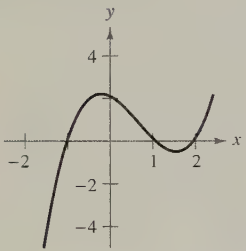
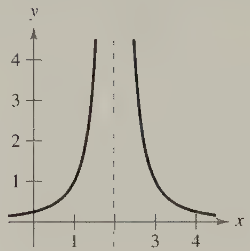


52.



Think About It **In Exercises 53–56, sketch the graph of a function f having the indicated characteristics.**

53. $f(2) = f(4) = 0$
$f(3)$ is defined.
$f'(x) < 0$ if $x < 3$
$f'(3)$ does not exist.
$f'(x) > 0$ if $x > 3$
$f''(x) < 0, x \neq 3$

54. $f(0) = f(2) = 0$
$f'(x) > 0$ if $x < 1$
$f'(1) = 0$
$f'(x) < 0$ if $x > 1$
$f''(x) < 0$

55. $f(2) = f(4) = 0$
$f'(x) > 0$ if $x < 3$
$f'(3)$ does not exist.
$f'(x) < 0$ if $x > 3$
$f''(x) > 0, x \neq 3$

56. $f(0) = f(2) = 0$
$f'(x) < 0$ if $x < 1$
$f'(1) = 0$
$f'(x) > 0$ if $x > 1$
$f''(x) > 0$

57. ***Think About It*** The figure shows the graph of f''. Sketch a graph of f. (The answer is not unique.)

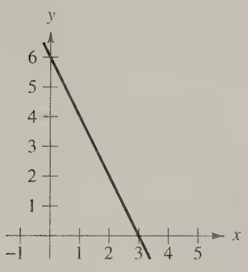


Figure for 57

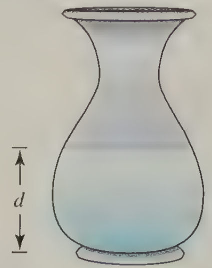


Figure for 58

58. ***Think About It*** Water is running into the vase shown in the figure at a constant rate.

(a) Graph the depth d of water in the vase as a function of time.

(b) Does the function have any extrema? Explain.

(c) Interpret the inflection points of the graph of d.

59. ***Conjecture*** Consider the function $f(x) = (x - 2)^n$.

(a) Use a graphing utility to graph f for $n = 1, 2, 3$, and 4. Us the graphs to make a conjecture about the relationshi between n and any inflection points of the graph of f.

(b) Verify your conjecture in part (a).

60. (a) Graph $f(x) = \sqrt[3]{x}$ and identify the inflection point.

(b) Does $f''(x)$ exist at the inflection point? Explain.

In Exercises 61 and 62, find a, b, c, and d such that the cubi $f(x) = ax^3 + bx^2 + cx + d$ satisfies the indicated conditions.

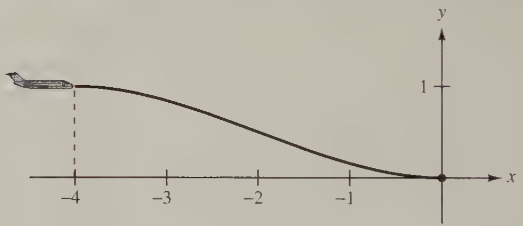
61. Relative maximum: (3, 3)
Relative minimum: (5, 1)
Inflection point: (4, 2)

62. Relative maximum: (2, 4)
Relative minimum: (4, 2)
Inflection point: (3, 3)

63. ***Aircraft Glide Path*** A small aircraft starts its descent from a altitude of 1 mile, 4 miles west of the runway (see figure).

(a) Find the cubic $f(x) = ax^3 + bx^2 + cx + d$ on the interva $[-4, 0]$ that describes a smooth glide path for the landing

(b) The function in part (a) models the glide path of the plane When would the plane be descending at the most rapid rate



FOR FURTHER INFORMATION For more information on this type of modeling, see the article "How Not to Land at Lake Tahoe!" by Richard Barshinger in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

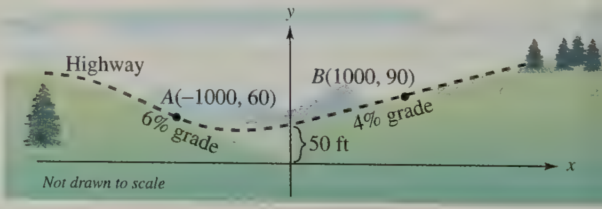
64. ***Highway Design*** A section of highway connecting tw hillsides with grades of 6% and 4% is to be built between tw points that are separated by a horizontal distance of 2000 fee (see figure). At the point where the two hillsides come togethe there is a 50-foot difference in elevation.

(a) Design a section of highway connecting the hillside modeled by the function $f(x) = ax^3 + bx^2 + cx +$ $(-1000 \le x \le 1000)$. At the points A and B, the slope o the model must match the grade of the hillside.

(b) Use a graphing utility to graph the model.

(c) Use a graphing utility to graph the derivative of the mode

(d) Determine the grade at the steepest part of the transitiona section of the highway.



5. ***Beam Deflection*** The deflection D of a particular beam of length L is

$$D = 2x^4 - 5Lx^3 + 3L^2x^2$$

where x is the distance from one end of the beam. Find the value of x that yields the maximum deflection.

6. ***Specific Gravity*** A model for the specific gravity of water S is

$$S = \frac{5.755}{10^8}T^3 - \frac{8.521}{10^6}T^2 + \frac{6.540}{10^5}T + 0.99987, \quad 0 < T < 25$$

where T is the water temperature in degrees Celsius.

(a) Use a computer algebra system to find the coordinates of the maximum value of the function.

(b) Sketch a graph of the function over the specified domain. (Use a setting in which $0.996 \le S \le 1.001$.)

(c) Estimate the specific gravity of water when $T = 20°$.

7. ***Average Cost*** A manufacturer has determined that the total cost C of operating a factory is

$$C = 0.5x^2 + 15x + 5000$$

where x is the number of units produced. At what level of production will the average cost per unit be minimized? (The average cost per unit is C/x.)

8. ***Inventory Cost*** The total cost C for ordering and storing x units is

$$C = 2x + \frac{300{,}000}{x}.$$

What order size will produce a minimum cost?

9. ***Sales Growth*** The annual sales S of a new product is given by

$$S = \frac{5000t^2}{8 + t^2}$$

where t is time in years. Find the time when sales are increasing at the greatest rate.

0. ***Modeling Data*** The average typing speed S of a typing student after t weeks of lessons is shown in the table.

t	5	10	15	20	25	30
S	38	56	79	90	93	94

A model for the data is $S = \dfrac{100t^2}{65 + t^2}$, $t > 0$.

(a) Use a graphing utility to plot the data and graph the model.

(b) Use the second derivative to determine the concavity of S. Compare the result with the graph in part (a).

(c) What is the sign of the first derivative for $t > 0$? Combining this information with the concavity of the model, what inferences can be made about the typing speed as t increases?

Linear and Quadratic Approximations **In Exercises 71–74, use a graphing utility to graph the function. Then graph the linear and quadratic approximations**

$$P_1(x) = f(a) + f'(a)(x - a)$$

and

$$P_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2$$

in the same viewing window. Compare the values of f, P_1, and P_2 and their first derivatives at $x = a$. How do the approximations change as you move farther away from $x = a$?

Function	*Value of a*
71. $f(x) = 2(\sin x + \cos x)$	$a = \dfrac{\pi}{4}$
72. $f(x) = 2(\sin x + \cos x)$	$a = 0$
73. $f(x) = \sqrt{1 - x}$	$a = 0$
74. $f(x) = \dfrac{\sqrt{x}}{x - 1}$	$a = 2$

75. Use a graphing utility to graph $y = x\sin(1/x)$. Show that the graph is concave downward to the right of $x = 1/\pi$.

76. Show that the point of inflection of $f(x) = x(x - 6)^2$ lies midway between the relative extrema of f.

77. Prove that every cubic function with three distinct real zeros has a point of inflection whose x-coordinate is the average of the three zeros.

78. Show that the cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ has exactly one point of inflection (x_0, y_0), where

$$x_0 = \frac{-b}{3a} \quad \text{and} \quad y_0 = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d.$$

Use this formula to find the point of inflection of $p(x) = x^3 - 3x^2 + 2$.

True or False? **In Exercises 79–84, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

79. The graph of every cubic polynomial has precisely one point of inflection.

80. The graph of $f(x) = 1/x$ is concave downward for $x < 0$ and concave upward for $x > 0$, and thus it has a point of inflection at $x = 0$.

81. The maximum value of $y = 3\sin x + 2\cos x$ is 5.

82. The maximum slope of the graph of $y = \sin(bx)$ is b.

83. If $f'(c) > 0$, then f is concave upward at $x = c$.

84. If $f''(2) = 0$, then the graph of f must have a point of inflection at $x = 2$.

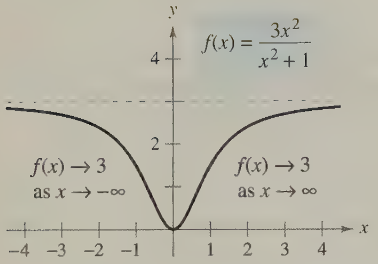
Section 3.5 Limits at Infinity

- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

Limits at Infinity

This section discusses the "end behavior" of a function on an *infinite* interva Consider the graph of

$$f(x) = \frac{3x^2}{x^2 + 1}$$



The limit of $f(x)$ as x approaches $-\infty$ or ∞ is 3.
Figure 3.33

as shown in Figure 3.33. Graphically, you can see that the values of $f(x)$ appear t approach 3 as x increases without bound or decreases without bound. You can com to the same conclusions numerically, as shown in the table.

x decreases without bound.
x increases without bound.

x	$-\infty \leftarrow$	-100	-10	-1	0	1	10	100	$\to \infty$
$f(x)$	$3 \leftarrow$	2.9997	2.97	1.5	0	1.5	2.97	2.9997	$\to 3$

$f(x)$ approaches 3.
$f(x)$ approaches 3.

The table suggests that the value of $f(x)$ approaches 3 as x increases without boun $(x \to \infty)$. Similarly, $f(x)$ approaches 3 as x decreases without bound $(x \to -\infty$ These **limits at infinity** are denoted by

$$\lim_{x\to-\infty} f(x) = 3 \qquad \text{Limit at negative infinity}$$

and

$$\lim_{x\to\infty} f(x) = 3. \qquad \text{Limit at positive infinity}$$

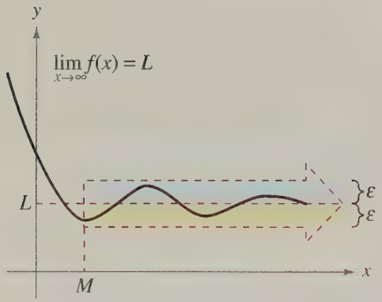
NOTE By writing $\lim_{x\to-\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = L$, we mean that the limit exists *and* the limit is equal to L.

To say that a statement is true as x increases *without bound* means that for som (large) real number M, the statement is true for *all* x in the interval $\{x\colon x > M\}$. Th following definition uses this concept.

Definition of Limits at Infinity

Let L be a real number.

1. The statement $\lim_{x\to\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$.
2. The statement $\lim_{x\to-\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.



$f(x)$ is within ε units of L as $x \to \infty$.
Figure 3.34

The definition of a limit at infinity is illustrated in Figure 3.34. In this figure, no that for a given positive number ε there exists a positive number M such that, f $x > M$, the graph of f will lie between the horizontal lines given by $y = L + \varepsilon$ ar $y = L - \varepsilon$.

EXPLORATION

Use a graphing utility to sketch the graph of

$$f(x) = \frac{2x^2 + 4x - 6}{3x^2 + 2x - 16}.$$

Describe all the important features of he graph. Can you find a single viewing window that shows all of hese features clearly? Explain your reasoning.

What are the horizontal asymptotes of the graph? How far to the right do you have to move on the graph so that the graph is within 0.001 unit of its horizontal asymptote? Explain your reasoning.

Horizontal Asymptotes

In Figure 3.34, the graph of f approaches the line $y = L$ as x increases without bound. We call the line $y = L$ a **horizontal asymptote** of the graph of f.

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of f if

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L.$$

Note that from this definition, it follows that the graph of a *function* of x can have at most two horizontal asymptotes—one to the right and one to the left.

Limits at infinity have many of the same properties of limits discussed in Section 1.3. For example, if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ both exist, then

$$\lim_{x\to\infty} [f(x) + g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$$

and

$$\lim_{x\to\infty} [f(x)g(x)] = \left[\lim_{x\to\infty} f(x)\right]\left[\lim_{x\to\infty} g(x)\right].$$

Similar properties hold for limits at $-\infty$.

When evaluating limits at infinity, the following theorem is helpful. (A proof of this theorem is given in Appendix B.)

THEOREM 3.10 Limits at Infinity

If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0.$$

Example 1 Evaluating a Limit at Infinity

Find the limit: $\lim_{x\to\infty} \left(5 - \frac{2}{x^2}\right)$.

Solution Using Theorem 3.10, you can write the following.

$$\begin{aligned}\lim_{x\to\infty} \left(5 - \frac{2}{x^2}\right) &= \lim_{x\to\infty} 5 - \lim_{x\to\infty} \frac{2}{x^2} && \text{Property of limits}\\ &= 5 - 0\\ &= 5\end{aligned}$$

Example 2 Evaluating a Limit at Infinity

Find the limit: $\lim_{x\to\infty} \frac{2x-1}{x+1}$.

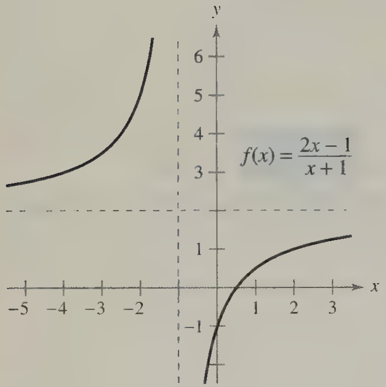
Solution Note that both the numerator and the denominator approach infinity as
approaches infinity.

$$\begin{tikzcd} & \lim_{x\to\infty} (2x-1) \to \infty \\ \lim_{x\to\infty} \frac{2x-1}{x+1} \arrow[ur] \arrow[dr] & \\ & \lim_{x\to\infty} (x+1) \to \infty \end{tikzcd}$$

This results in $\frac{\infty}{\infty}$, an **indeterminate form.** To resolve this problem, you can divi
both the numerator and the denominator by x. After dividing, the limit may be evalu
ated as follows.

$$\begin{aligned} \lim_{x\to\infty} \frac{2x-1}{x+1} &= \lim_{x\to\infty} \frac{\frac{2x-1}{x}}{\frac{x+1}{x}} && \text{Divide numerator and denominator by } x. \\ &= \lim_{x\to\infty} \frac{2-\frac{1}{x}}{1+\frac{1}{x}} \\ &= \frac{\lim_{x\to\infty} 2 - \lim_{x\to\infty} \frac{1}{x}}{\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{1}{x}} && \text{Take limits of numerator and denominator.} \\ &= \frac{2-0}{1+0} && \text{Apply Theorem 3.10.} \\ &= 2 \end{aligned}$$

So, the line $y = 2$ is a horizontal asymptote to the right. By taking the limit
$x \to -\infty$, you can see that $y = 2$ is also a horizontal asymptote to the left. The grap
of the function is shown in Figure 3.35.



$y = 2$ is a horizontal asymptote.
Figure 3.35

NOTE When you encounter an indeterminate form such as the one in Example 2, we suggest dividing the numerator and denominator by the highest power of x in the *denominator*.

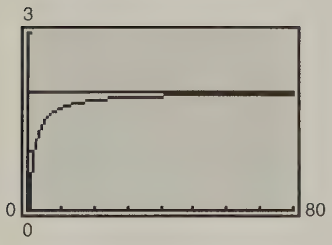
TECHNOLOGY You can test the reasonableness of the limit found in Example 2
evaluating $f(x)$ for a few large positive values of x. For instance,

$$f(100) \approx 1.9703, \quad f(1000) \approx 1.9970, \quad \text{and} \quad f(10{,}000) \approx 1.9997.$$

Another way to test the reasonableness of the limit is to use a graphing utility. F
instance, in Figure 3.36, the graph of

$$f(x) = \frac{2x-1}{x+1}$$

is shown with the horizontal line $y = 2$. Note that as x increases, the graph of
moves closer and closer to its horizontal asymptote.



As x increases, the graph of f moves closer and closer to the line $y = 2$.
Figure 3.36

Example 3 A Comparison of Three Rational Functions

Find each of the limits.

a. $\displaystyle\lim_{x\to\infty}\frac{2x+5}{3x^2+1}$ **b.** $\displaystyle\lim_{x\to\infty}\frac{2x^2+5}{3x^2+1}$ **c.** $\displaystyle\lim_{x\to\infty}\frac{2x^3+5}{3x^2+1}$

Solution In each case, attempting to evaluate the limit produces the indeterminate form ∞/∞.

a. Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty}\frac{2x+5}{3x^2+1}=\lim_{x\to\infty}\frac{(2/x)+(5/x^2)}{3+(1/x^2)}=\frac{0+0}{3+0}=\frac{0}{3}=0$$

b. Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty}\frac{2x^2+5}{3x^2+1}=\lim_{x\to\infty}\frac{2+(5/x^2)}{3+(1/x^2)}=\frac{2+0}{3+0}=\frac{2}{3}$$

c. Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty}\frac{2x^3+5}{3x^2+1}=\lim_{x\to\infty}\frac{2x+(5/x^2)}{3+(1/x^2)}=\frac{\infty}{3}$$

You can conclude that the limit *does not exist* because the numerator increases without bound while the denominator approaches 3.

The Granger Collection

Maria Agnesi (1718–1799)

Agnesi was one of a handful of women to receive credit for significant contributions to mathematics before the twentieth century. In ner early twenties, she wrote the first text that ncluded both differential and integral calculus. By age 30, she was an honorary member of the faculty at the University of Bologna.

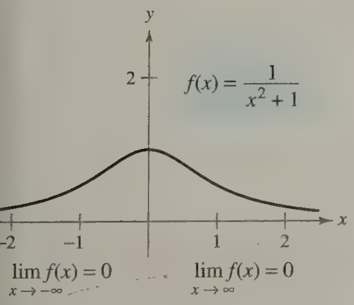
Guidelines for Finding Limits at Infinity of Rational Functions

1. If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exist.

Use these guidelines to check the results in Example 3. These limits seem reasonable when you consider that for large values of x, the highest-power term of the rational function is the most "influential" in determining the limit. For instance, the limit as x approaches infinity of the function

$$f(x)=\frac{1}{x^2+1}$$

is 0 because the denominator overpowers the numerator as x increases or decreases without bound, as shown in Figure 3.37.



as a horizontal asymptote at $y = 0$.
gure 3.37

The function shown in Figure 3.37 is a special case of a type of curve studied by the Italian mathematician Maria Gaetana Agnesi. The general form of this function is

$$f(x)=\frac{8a^3}{x^2+4a^2}\qquad\text{Witch of Agnesi}$$

and, through a mistranslation of the Italian word *vertéré,* the curve has come to be known as the Witch of Agnesi. Agnesi's work with this curve first appeared in a comprehensive text on calculus that was published in 1748.

R FURTHER INFORMATION For re information on the contributions of men to mathematics, see the article hy Women Succeed in Mathematics" Mona Fabricant, Sylvia Svitak, and ricia Clark Kenschaft in *Mathematics acher.* To view this article, go to the bsite *www.matharticles.com.*

In Figure 3.37, you can see that the function $f(x) = 1/(x^2 + 1)$ approaches th same horizontal asymptote to the right and to the left. This is always true of ration functions. Functions that are not rational, however, may approach different horizont asymptotes to the right and to the left. This is demonstrated in Example 4.

Example 4 A Function with Two Horizontal Asymptotes

Determine each of the limits.

a. $\displaystyle\lim_{x\to\infty} \frac{3x-2}{\sqrt{2x^2+1}}$ **b.** $\displaystyle\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}}$

Solution

a. For $x > 0$, you can write $x = \sqrt{x^2}$. So, dividing both the numerator and th denominator by x produces

$$\frac{3x-2}{\sqrt{2x^2+1}} = \frac{\dfrac{3x-2}{x}}{\dfrac{\sqrt{2x^2+1}}{\sqrt{x^2}}} = \frac{3-\dfrac{2}{x}}{\sqrt{\dfrac{2x^2+1}{x^2}}} = \frac{3-\dfrac{2}{x}}{\sqrt{2+\dfrac{1}{x^2}}}$$

and you can take the limit as follows.

$$\lim_{x\to\infty} \frac{3x-2}{\sqrt{2x^2+1}} = \lim_{x\to\infty} \frac{3-\dfrac{2}{x}}{\sqrt{2+\dfrac{1}{x^2}}} = \frac{3-0}{\sqrt{2+0}} = \frac{3}{\sqrt{2}}$$

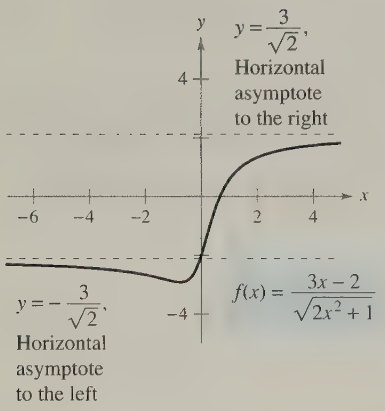
b. For $x < 0$, you can write $x = -\sqrt{x^2}$. So, dividing both the numerator and th denominator by x produces

$$\frac{3x-2}{\sqrt{2x^2+1}} = \frac{\dfrac{3x-2}{x}}{\dfrac{\sqrt{2x^2+1}}{-\sqrt{x^2}}} = \frac{3-\dfrac{2}{x}}{-\sqrt{\dfrac{2x^2+1}{x^2}}} = \frac{3-\dfrac{2}{x}}{-\sqrt{2+\dfrac{1}{x^2}}}$$

and you can take the limit as follows.

$$\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}} = \lim_{x\to-\infty} \frac{3-\dfrac{2}{x}}{-\sqrt{2+\dfrac{1}{x^2}}} = \frac{3-0}{-\sqrt{2+0}} = -\frac{3}{\sqrt{2}}$$

The graph of $f(x) = (3x-2)/\sqrt{2x^2+1}$ is shown in Figure 3.38.

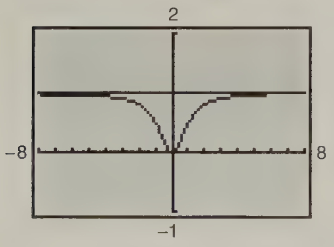


Functions that are not rational may have different right and left horizontal asymptotes.
Figure 3.38

TECHNOLOGY PITFALL If you use a graphing utility to help estimate a limit, sure that you also confirm the estimate analytically—the pictures shown by graphing utility can be misleading. For instance, Figure 3.39 shows one view of t graph of

$$y = \frac{2x^3 + 1000x^2 + x}{x^3 + 1000x^2 + x + 1000}.$$

From this view, one could be convinced that the graph has $y = 1$ as a horizon asymptote. An analytical approach shows that the horizontal asymptote is actua $y = 2$. Confirm this by enlarging the viewing window on the graphing utility.



The horizontal asymptote appears to be the line $y = 1$ but is actually the line $y = 2$.
Figure 3.39

In Section 1.3 (Example 9), you saw how the Squeeze Theorem can be used to evaluate limits involving trigonometric functions. This theorem is also valid for limits at infinity.

Example 5 Limits Involving Trigonometric Functions

Determine each of the limits.

a. $\lim_{x \to \infty} \sin x$ **b.** $\lim_{x \to \infty} \frac{\sin x}{x}$

Solution

a. As x approaches infinity, the sine function oscillates between 1 and -1. So, this limit does not exist.

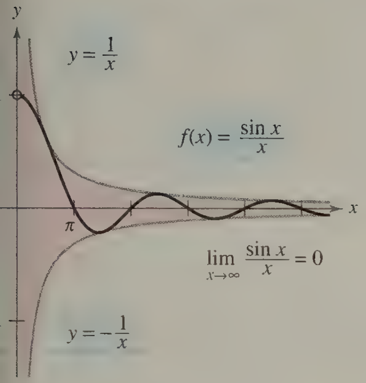
b. Because $-1 \leq \sin x \leq 1$, it follows that for $x > 0$,

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

where $\lim_{x \to \infty} (-1/x) = 0$ and $\lim_{x \to \infty} (1/x) = 0$. Therefore, by the Squeeze Theorem, you can obtain

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0$$

as indicated in Figure 3.40.



s x increases without bound, $f(x)$ pproaches 0.
igure 3.40

Example 6 Oxygen Level in a Pond

Suppose that $f(t)$ measures the level of oxygen in a pond, where $f(t) = 1$ is the normal (unpolluted) level and the time t is measured in weeks. When $t = 0$, organic waste is dumped into the pond, and as the waste material oxidizes, the level of oxygen in the pond is

$$f(t) = \frac{t^2 - t + 1}{t^2 + 1}.$$

What percent of the normal level of oxygen exists in the pond after 1 week? After 2 weeks? After 10 weeks? What is the limit as t approaches infinity?

Solution When $t = 1$, 2, and 10, the levels of oxygen are as follows.

$$f(1) = \frac{1^2 - 1 + 1}{1^2 + 1} = \frac{1}{2} = 50\% \quad \text{1 week}$$

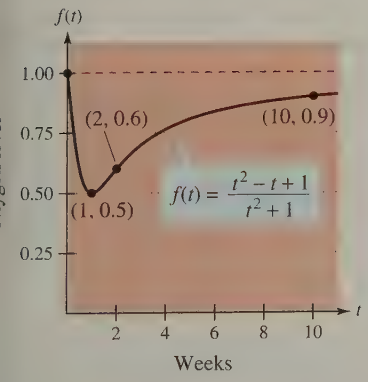
$$f(2) = \frac{2^2 - 2 + 1}{2^2 + 1} = \frac{3}{5} = 60\% \quad \text{2 weeks}$$

$$f(10) = \frac{10^2 - 10 + 1}{10^2 + 1} = \frac{91}{101} \approx 90.1\% \quad \text{10 weeks}$$

To take the limit as t approaches infinity, divide the numerator and the denominator by t^2 to obtain

$$\lim_{x \to \infty} \frac{t^2 - t + 1}{t^2 + 1} = \lim_{x \to \infty} \frac{1 - (1/t) + (1/t^2)}{1 + (1/t^2)} = \frac{1 - 0 + 0}{1 + 0} = 1 = 100\%.$$

(See Figure 3.41.)



e level of oxygen in a pond approaches the rmal level of 1 as t approaches ∞.
gure 3.41

Infinite Limits at Infinity

Many functions do not approach a finite limit as x increases (or decreases) withou bound. For instance, no polynomial function has a finite limit at infinity. To describe th behavior of polynomial and other functions at infinity, we use the following definition

NOTE Determining whether a function has an infinite limit at infinity is useful in analyzing the "end behavior" of its graph. You will see examples of this in Section 3.6 on curve sketching.

Definition of Infinite Limits at Infinity

Let f be a function defined on the interval (a, ∞).

1. The statement $\lim\limits_{x\to\infty} f(x) = \infty$ means that for each positive number M, there is a corresponding number $N > 0$ such that $f(x) > M$ whenever $x > N$.
2. The statement $\lim\limits_{x\to\infty} f(x) = -\infty$ means that for each negative number M, there is a corresponding number $N > 0$ such that $f(x) < M$ whenever $x > N$.

Similar statements can be made about the notations $\lim\limits_{x\to-\infty} f(x) = \infty$ an $\lim\limits_{x\to-\infty} f(x) = -\infty$.

Example 7 Finding Infinite Limits at Infinity

Find each limit.

a. $\lim\limits_{x\to\infty} x^3$ **b.** $\lim\limits_{x\to-\infty} x^3$

Solution

a. As x increases without bound, x^3 also increases without bound. So, you can writ $\lim\limits_{x\to\infty} x^3 = \infty$.

b. As x decreases without bound, x^3 also decreases without bound. So, you can writ $\lim\limits_{x\to -\infty} x^3 = -\infty$.

The graph of $f(x) = x^3$ in Figure 3.42 illustrates these two results. These results agre with the Leading Coefficient Test for polynomial functions as described in Section P.

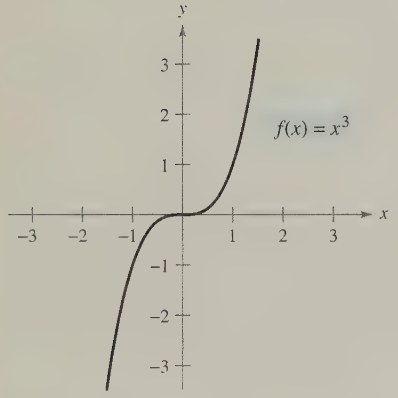


Figure 3.42

Example 8 Finding Infinite Limits at Infinity

Find each limit.

a. $\lim\limits_{x\to\infty} \dfrac{2x^2-4x}{x+1}$ **b.** $\lim\limits_{x\to-\infty} \dfrac{2x^2-4x}{x+1}$

Solution One way to evaluate these limits is to use long division to rewrite th improper rational function as the sum of a polynomial and a rational function.

$$\textbf{a.}\ \lim_{x\to\infty} \frac{2x^2-4x}{x+1} = \lim_{x\to\infty}\left(2x - 6 + \frac{6}{x+1}\right) = \infty$$

$$\textbf{b.}\ \lim_{x\to-\infty} \frac{2x^2-4x}{x+1} = \lim_{x\to-\infty}\left(2x - 6 + \frac{6}{x+1}\right) = -\infty$$

The above statements can be interpreted as saying that as x approaches ±∞, the fun tion $f(x) = (2x^2 - 4x)/(x + 1)$ behaves like the function $g(x) = 2x - 6$. In Sectio 3.6, you will see that this is graphically described by saying that the line $y = 2x -$ is a slant asymptote of the graph of f, as shown in Figure 3.43.

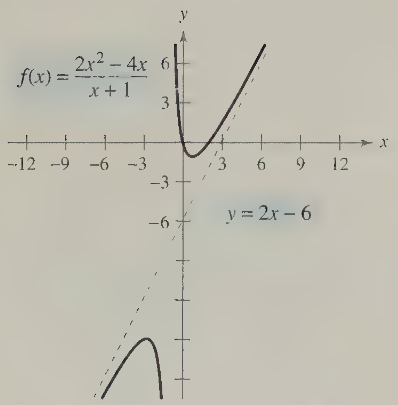
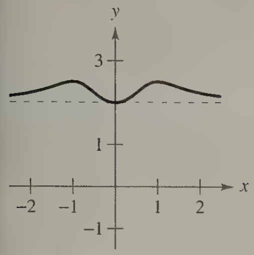


Figure 3.43

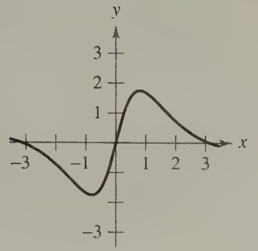
EXERCISES FOR SECTION 3.5

Exercises 1–6, match the function with one of the graphs [(a),), (c), (d), (e), or (f)] using horizontal asymptotes as an aid.

)



(b)



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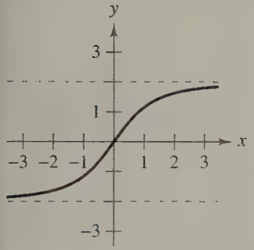

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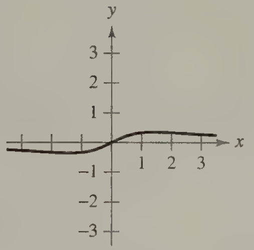

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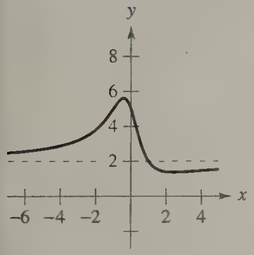
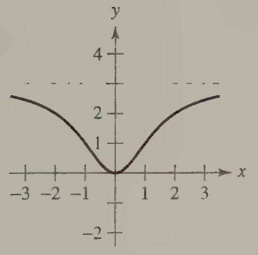


(f)



. $f(x) = \dfrac{3x^2}{x^2+2}$

2. $f(x) = \dfrac{2x}{\sqrt{x^2+2}}$

. $f(x) = \dfrac{x}{x^2+2}$

4. $f(x) = 2 + \dfrac{x^2}{x^4+1}$

. $f(x) = \dfrac{4\sin x}{x^2+1}$

6. $f(x) = \dfrac{2x^2-3x+5}{x^2+1}$

umerical and Graphical Analysis **In Exercises 7–12, use a raphing utility to complete the table and estimate the limit as approaches infinity. Then use a graphing utility to graph the nction and estimate the limit graphically.**

	10^0	10^1	10^2	10^3	10^4	10^5	10^6
(x)							

. $f(x) = \dfrac{4x+3}{2x-1}$

8. $f(x) = \dfrac{2x^2}{x+1}$

. $f(x) = \dfrac{-6x}{\sqrt{4x^2+5}}$

10. $f(x) = \dfrac{8x}{\sqrt{x^2-3}}$

. $f(x) = 5 - \dfrac{1}{x^2+1}$

12. $f(x) = 4 + \dfrac{3}{x^2+2}$

In Exercises 13 and 14, find $\lim\limits_{x\to\infty} h(x)$, if possible.

13. $f(x) = 5x^3 - 3x^2 + 10$

(a) $h(x) = \dfrac{f(x)}{x^2}$ (b) $h(x) = \dfrac{f(x)}{x^3}$

(c) $h(x) = \dfrac{f(x)}{x^4}$

14. $f(x) = 5x^2 - 3x + 7$

(a) $h(x) = \dfrac{f(x)}{x}$ (b) $h(x) = \dfrac{f(x)}{x^2}$

(c) $h(x) = \dfrac{f(x)}{x^3}$

In Exercises 15–18, find each of the limits, if possible.

15. (a) $\lim\limits_{x\to\infty} \dfrac{x^2+2}{x^3-1}$

(b) $\lim\limits_{x\to\infty} \dfrac{x^2+2}{x^2-1}$

(c) $\lim\limits_{x\to\infty} \dfrac{x^2+2}{x-1}$

16. (a) $\lim\limits_{x\to\infty} \dfrac{3-2x}{3x^3-1}$

(b) $\lim\limits_{x\to\infty} \dfrac{3-2x}{3x-1}$

(c) $\lim\limits_{x\to\infty} \dfrac{3-2x^2}{3x-1}$

17. (a) $\lim\limits_{x\to\infty} \dfrac{5-2x^{3/2}}{3x^2-4}$

(b) $\lim\limits_{x\to\infty} \dfrac{5-2x^{3/2}}{3x^{3/2}-4}$

(c) $\lim\limits_{x\to\infty} \dfrac{5-2x^{3/2}}{3x-4}$

18. (a) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4x^2+1}$

(b) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4x^{3/2}+1}$

(c) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4\sqrt{x}+1}$

In Exercises 19–32, find the limit.

19. $\lim\limits_{x\to\infty} \dfrac{2x-1}{3x+2}$

20. $\lim\limits_{x\to\infty} \dfrac{3x^3+2}{9x^3-2x^2+7}$

21. $\lim\limits_{x\to\infty} \dfrac{x}{x^2-1}$

22. $\lim\limits_{x\to\infty} \left(4+\dfrac{3}{x}\right)$

23. $\lim\limits_{x\to-\infty} \dfrac{5x^2}{x+3}$

24. $\lim\limits_{x\to-\infty} \left(\dfrac{1}{2}x-\dfrac{4}{x^2}\right)$

25. $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2-x}}$

26. $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2+1}}$

27. $\lim\limits_{x\to-\infty} \dfrac{2x+1}{\sqrt{x^2-x}}$

28. $\lim\limits_{x\to-\infty} \dfrac{-3x+1}{\sqrt{x^2+x}}$

29. $\lim\limits_{x\to\infty} \dfrac{\sin 2x}{x}$

30. $\lim\limits_{x\to\infty} \dfrac{x-\cos x}{x}$

31. $\lim\limits_{x\to\infty} \dfrac{1}{2x+\sin x}$

32. $\lim\limits_{x\to\infty} \cos\dfrac{1}{x}$

In Exercises 33 and 34, use a graphing utility to graph the function and verify that it has two horizontal asymptotes.

33. $f(x) = \dfrac{|x|}{x+1}$

34. $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$

In Exercises 35 and 36, find the limit. (*Hint:* Let $x = 1/t$ and find the limit as $t \to 0^+$.)

35. $\lim_{x\to\infty} x \sin \frac{1}{x}$

36. $\lim_{x\to\infty} x \tan \frac{1}{x}$

In Exercises 37–40, find the limit. (*Hint:* Treat the expression as a fraction whose denominator is 1, and rationalize the numerator.) Use a graphing utility to verify your result.

37. $\lim_{x\to-\infty} \left(x + \sqrt{x^2+3}\right)$

38. $\lim_{x\to\infty} \left(2x - \sqrt{4x^2+1}\right)$

39. $\lim_{x\to\infty} \left(x - \sqrt{x^2+x}\right)$

40. $\lim_{x\to-\infty} \left(3x + \sqrt{9x^2-x}\right)$

Numerical, Graphical, and Analytic Analysis **In Exercises 41–44, use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the function and estimate the limit graphically. Finally, find the limit analytically and compare your results with the estimates.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

41. $f(x) = x - \sqrt{x(x-1)}$

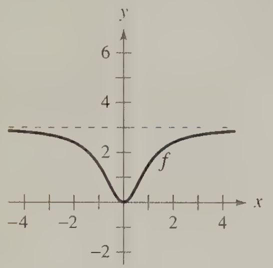
42. $f(x) = x^2 - x\sqrt{x(x-1)}$

43. $f(x) = x \sin \frac{1}{2x}$

44. $f(x) = \frac{x+1}{x\sqrt{x}}$

Getting at the Concept

45. The graph of a function f is shown below. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) Sketch f'.

(b) Use the graphs to estimate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$.

(c) Explain the answers you gave in part (b).

46. Sketch a graph of a differentiable function f that satisfies the following conditions and has $x = 2$ as its only critical number.

$f'(x) < 0$ for $x < 2$

$f'(x) > 0$ for $x > 2$

$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 6$

47. Is it possible to sketch a graph of a function that satisfies the conditions of Exercise 46 and has *no* points of inflection? Explain.

Getting at the Concept *(continued)*

48. If f is a continuous function such that $\lim_{x\to\infty} f(x) = 5$, find, if possible, $\lim_{x\to-\infty} f(x)$ for each specified condition.

(a) The graph of f is symmetric to the y-axis.

(b) The graph of f is symmetric to the origin.

In Exercises 49–66, sketch the graph of the equation. Look fo extrema, intercepts, symmetry, and asymptotes as necessar Use a graphing utility to verify your result.

49. $y = \frac{2+x}{1-x}$

50. $y = \frac{x-3}{x-2}$

51. $y = \frac{x}{x^2-4}$

52. $y = \frac{2x}{9-x^2}$

53. $y = \frac{x^2}{x^2+9}$

54. $y = \frac{x^2}{x^2-9}$

55. $y = \frac{2x^2}{x^2-4}$

56. $y = \frac{2x^2}{x^2+4}$

57. $xy^2 = 4$

58. $x^2y = 4$

59. $y = \frac{2x}{1-x}$

60. $y = \frac{2x}{1-x^2}$

61. $y = 2 - \frac{3}{x^2}$

62. $y = 1 + \frac{1}{x}$

63. $y = 3 + \frac{2}{x}$

64. $y = 4\left(1 - \frac{1}{x^2}\right)$

65. $y = \frac{x^3}{\sqrt{x^2-4}}$

66. $y = \frac{x}{\sqrt{x^2-4}}$

In Exercises 67–76, use a computer algebra system to analyz the graph of the function. Label any extrema and/or asymptote that exist.

67. $f(x) = 5 - \frac{1}{x^2}$

68. $f(x) = \frac{x^2}{x^2-1}$

69. $f(x) = \frac{x}{x^2-4}$

70. $f(x) = \frac{1}{x^2-x-2}$

71. $f(x) = \frac{x-2}{x^2-4x+3}$

72. $f(x) = \frac{x+1}{x^2+x+1}$

73. $f(x) = \frac{3x}{\sqrt{4x^2+1}}$

74. $g(x) = \frac{2x}{\sqrt{3x^2+1}}$

75. $g(x) = \sin \frac{x}{x-2}, \quad 3 < x < \infty$

76. $f(x) = \frac{2 \sin 2x}{x}$

n Exercises 77 and 78, (a) use a graphing utility to graph f and in the same viewing window, (b) verify algebraically that f nd g represent the same function, and (c) zoom out sufficiently r so that the graph appears as a line. What equation does this ne appear to have? (Note that the points at which the function not continuous are not readily seen when you zoom out.)

7. $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x-3)}$

$g(x) = x + \dfrac{2}{x(x-3)}$

78. $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$

$g(x) = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$

9. ***Average Cost*** A business has a cost of $C = 0.5x + 500$ for producing x units. The average cost per unit is

$$\overline{C} = \frac{C}{x}.$$

Find the limit of $\overline{C}$ as x approaches infinity.

0. ***Engine Efficiency*** The efficiency of an internal combustion engine is

$$\text{Efficiency (\%)} = 100\left[1 - \frac{1}{(v_1/v_2)^c}\right]$$

where v_1/v_2 is the ratio of the uncompressed gas to the compressed gas and c is a positive constant dependent on the engine design. Find the limit of the efficiency as the compression ratio approaches infinity.

1. A line with slope m passes through the point $(0, 4)$.

(a) Write the distance d between the line and the point $(3, 1)$ as a function of m.

(b) Use a graphing utility to graph the equation in part (a).

(c) Find $\lim\limits_{m\to\infty} d(m)$ and $\lim\limits_{m\to-\infty} d(m)$. Interpret the results geometrically.

2. ***Modeling Data*** The table shows the world record times for running one mile, where t represents the year with $t = 0$ corresponding to 1900, and y is the time in minutes and seconds.

t	23	33	45	54
y	4:10.4	4:07.6	4:01.3	3:59.4

t	58	66	79	85	99
y	3:54.5	3:51.3	3:48.9	3:46.3	3:43.1

A model for the data is

$$y = \frac{3.351t^2 + 42.461t - 543.730}{t^2}$$

where the seconds have been changed to a decimal part of a minute.

(a) Use a graphing utility to plot the data and graph the model.

(b) Does there appear to be a limiting time for running one mile? Explain.

83. ***Modeling Data*** A heat probe is attached to the heat exchanger of a heating system. The temperature T (degrees Celsius) is recorded t seconds after the furnace is started. The results for the first 2 minutes are recorded in the table.

t	0	15	30	45	60
T	25.2°	36.9°	45.5°	51.4°	56.0°

t	75	90	105	120
T	59.6°	62.0°	64.0°	65.2°

(a) Use the regression capabilities of a graphing utility to find a model of the form $T_1 = at^2 + bt + c$ for the data.

(b) Use a graphing utility to graph T_1.

(c) A rational model for the data is

$$T_2 = \frac{1451 + 86t}{58 + t}.$$

Use a graphing utility to graph the model.

(d) Find $T_1(0)$ and $T_2(0)$.

(e) Find $\lim\limits_{t\to\infty} T_2$.

(f) Interpret the result in part (e) in the context of the problem. Is it possible to do this type of analysis using T_1? Explain.

84. ***Modeling Data*** The average typing speed S of a typing student after t weeks of lessons is shown in the table.

t	5	10	15	20	25	30
S	28	56	79	90	93	94

A model for the data is $S = \dfrac{100t^2}{65 + t^2}$, $t > 0$.

(a) Use a graphing utility to plot the data and graph the model.

(b) Does there appear to be a limiting typing speed? Explain.

85. In your own words, state the guidelines for finding the limit of a rational function. Give examples.

86. Prove that if $p(x) = a_nx^n + \cdots + a_1x + a_0$ and $q(x) = b_mx^m + \cdots + b_1x + b_0$ $(a_n \neq 0, b_m \neq 0)$, then

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 0, & n < m \\ \dfrac{a_n}{b_m}, & n = m \\ \pm\infty, & n > m. \end{cases}$$

True or False? **In Exercises 87 and 88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

87. If $f'(x) > 0$ for all real numbers x, then f increases without bound.

88. If $f''(x) < 0$ for all real numbers x, then f decreases without bound.

Section 3.6 A Summary of Curve Sketching

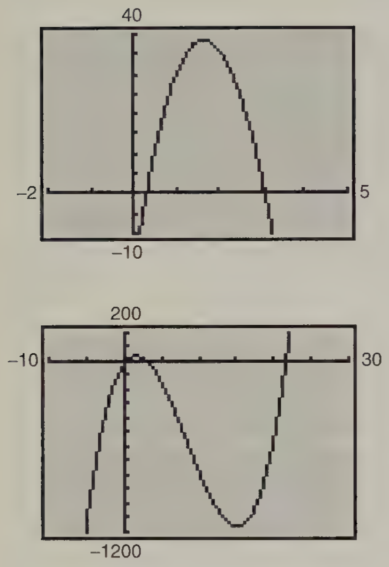
- Analyze and sketch the graph of a function.

Analyzing the Graph of a Function

It would be difficult to overstate the importance of using graphs in mathematic Descartes's introduction of analytic geometry contributed significantly to the rapi advances in calculus that began during the mid-seventeenth century. In the words c Lagrange, "As long as algebra and geometry traveled separate paths their advance wa slow and their applications limited. But when these two sciences joined company, the drew from each other fresh vitality and thenceforth marched on at a rapid pace towar perfection."

So far, you have studied several concepts that are useful in analyzing the graph c a function.

• x-intercepts and y-intercepts	(Section P.1)
• Symmetry	(Section P.1)
• Domain and range	(Section P.3)
• Continuity	(Section 1.4)
• Vertical asymptotes	(Section 1.5)
• Differentiability	(Section 2.1)
• Relative extrema	(Section 3.1)
• Concavity	(Section 3.4)
• Points of inflection	(Section 3.4)
• Horizontal asymptotes	(Section 3.5)
• Infinite limits at infinity	(Section 3.5)



Different viewing windows for the same graph
Figure 3.44

When you are sketching the graph of a function, either by hand or with a graphing utility, remember that normally you cannot show the *entire* graph. The decision a to which part of the graph you choose to show is often crucial. For instance, which c the viewing windows in Figure 3.44 better represents the graph of

$$f(x) = x^3 - 25x^2 + 74x - 20?$$

By seeing both views, it is clear that the second viewing window gives a more complete representation of the graph. But would a third viewing window reveal othe interesting portions of the graph? To answer this, you need to use calculus to interpre the first and second derivatives. Here are some guidelines for determining a goo viewing window for the graph of a function.

Guidelines for Analyzing the Graph of a Function

1. Determine the domain and range of the function.
2. Determine the intercepts, asymptotes, and symmetry of the graph.
3. Locate the x-values for which $f'(x)$ and $f''(x)$ are either zero or do not exist. Use the results to determine relative extrema and points of inflection.

NOTE In these guidelines, note the importance of *algebra* (as well as calculus) for solving th equations $f(x) = 0$, $f'(x) = 0$, and $f''(x) = 0$.

Example 1 Analyzing and Sketching the Graph of a Rational Function

Analyze and sketch the graph of $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$.

Solution

First derivative: $f'(x) = \dfrac{20x}{(x^2 - 4)^2}$

Second derivative: $f''(x) = \dfrac{-20(3x^2 + 4)}{(x^2 - 4)^3}$

x-intercepts: $(-3, 0), (3, 0)$

y-intercept: $\left(0, \frac{9}{2}\right)$

Vertical asymptotes: $x = -2, x = 2$

Horizontal asymptote: $y = 2$

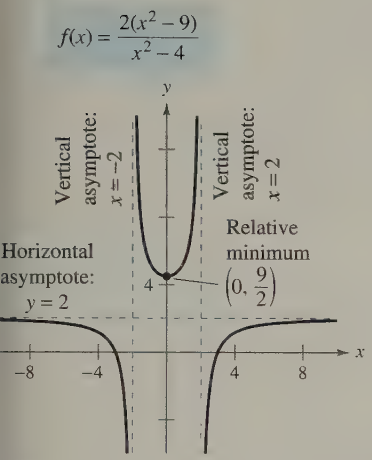
Critical number: $x = 0$

Possible points of inflection: None

Domain: All real numbers except $x = \pm 2$

Symmetry: With respect to y-axis

Test intervals: $(-\infty, -2), (-2, 0), (0, 2), (2, \infty)$



Jsing calculus, you can be certain that you nave determined all characteristics of the graph of *f*.
Figure 3.45

The table shows how the test intervals are used to determine several characteristics of the graph. The graph of f is shown in Figure 3.45.

	$f(x)$	$f'(x)$	$f''(x)$	**Characteristic of Graph**
$-\infty < x < -2$		$-$	$-$	Decreasing, concave downward
$x = -2$	Undef.	Undef.	Undef.	Vertical asymptote
$-2 < x < 0$		$-$	$+$	Decreasing, concave upward
$x = 0$	$\frac{9}{2}$	0	$+$	Relative minimum
$0 < x < 2$		$+$	$+$	Increasing, concave upward
$x = 2$	Undef.	Undef.	Undef.	Vertical asymptote
$2 < x < \infty$		$+$	$-$	Increasing, concave downward

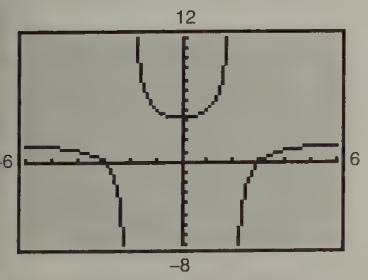
OR FURTHER INFORMATION For nore information on the use of technology to graph rational functions, see the rticle "Graphs of Rational Functions for Computer Assisted Calculus" by Stan 3yrd and Terry Walters in *The College Aathematics Journal*. To view this article, o to the website *www.matharticles.com*.

Be sure you understand all of the implications of creating a table such as that shown in Example 1. Because of the use of calculus, you can *be sure* that the graph has no relative extrema or points of inflection other than those indicated in Figure 3.45.

TECHNOLOGY PITFALL Without using the type of analysis outlined in Example 1, it is easy to obtain an incomplete view of a graph's basic characteristics. For instance, Figure 3.46 shows a view of the graph of

$$g(x) = \frac{2(x^2 - 9)(x - 20)}{(x^2 - 4)(x - 21)}.$$

From this view, it appears that the graph of g is about the same as the graph of f shown in Figure 3.45. The graphs of these two functions, however, differ significantly. Try enlarging the viewing window to see the differences.



3y not using calculus you may overlook mportant characteristics of the graph of *g*.
Figure 3.46

Example 2 Analyzing and Sketching the Graph of a Rational Function

Analyze and sketch the graph of $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$.

Solution

First derivative: $f'(x) = \dfrac{x(x-4)}{(x-2)^2}$

Second derivative: $f''(x) = \dfrac{8}{(x-2)^3}$

x-intercepts: None

y-intercept: $(0, -2)$

Vertical asymptote: $x = 2$

Horizontal asymptotes: None

End behavior: $\lim\limits_{x \to -\infty} f(x) = -\infty, \ \lim\limits_{x \to \infty} f(x) = \infty$

Critical numbers: $x = 0, x = 4$

Possible points of inflection: None

Domain: All real numbers except $x = 2$

Test intervals: $(-\infty, 0), (0, 2), (2, 4), (4, \infty)$

The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.47.

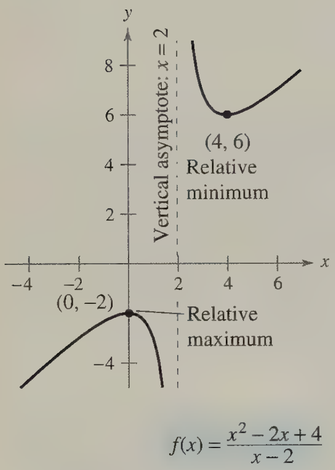


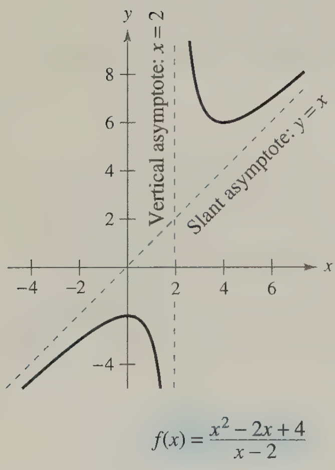
Figure 3.47

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		+	−	Increasing, concave downward
$x = 0$	−2	0	−	Relative maximum
$0 < x < 2$		−	−	Decreasing, concave downward
$x = 2$	Undef.	Undef.	Undef.	Vertical asymptote
$2 < x < 4$		−	+	Decreasing, concave upward
$x = 4$	6	0	+	Relative minimum
$4 < x < \infty$		+	+	Increasing, concave upward

Although the graph of the function in Example 2 has no horizontal asymptote, i does have a slant asymptote. The graph of a rational function (having no common factors) has a **slant asymptote** if the degree of the numerator exceeds the degree o the denominator by 1. To find the slant asymptote, use long division to rewrite the rational function as the sum of a first-degree polynomial and another rational function.

$$f(x) = \frac{x^2 - 2x + 4}{x - 2} \quad \text{Rewrite using long division.}$$

$$= x + \frac{4}{x - 2} \quad y = x \text{ is a slant asymptote.}$$



A slant asymptote
Figure 3.48

In Figure 3.48, note that the graph of f approaches the slant asymptote $y = x$ as approaches $-\infty$ or ∞.

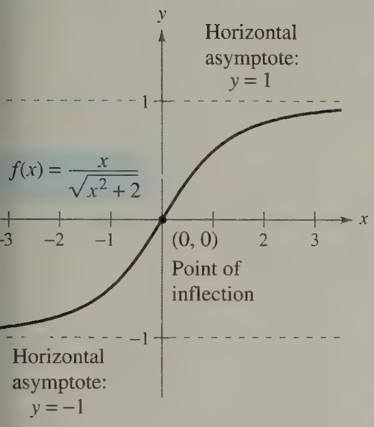
Example 3 Analyzing and Sketching the Graph of a Radical Function

Analyze and sketch the graph of $f(x) = \dfrac{x}{\sqrt{x^2+2}}$.

Solution

$$f'(x) = \frac{2}{(x^2+2)^{3/2}} \qquad f''(x) = -\frac{6x}{(x^2+2)^{5/2}}$$

The graph has only one intercept, (0, 0). It has no vertical asymptotes, but it has two horizontal asymptotes: $y = 1$ (to the right) and $y = -1$ (to the left). The function has no critical numbers and one possible point of inflection (at $x = 0$). The domain of the function is all real numbers, and the graph is symmetric with respect to the origin. The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.49.



igure 3.49

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		$+$	$+$	Increasing, concave upward
$x = 0$	0	$\frac{1}{\sqrt{2}}$	0	Point of inflection
$0 < x < \infty$		$+$	$-$	Increasing, concave downward

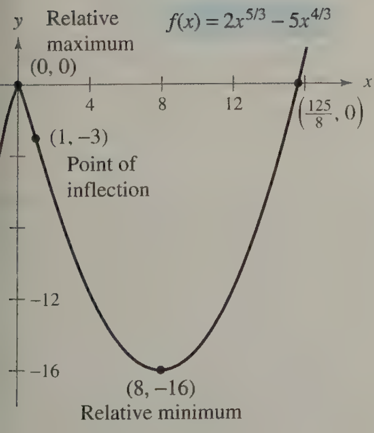
Example 4 Analyzing and Sketching the Graph of a Radical Function

Analyze and sketch the graph of $f(x) = 2x^{5/3} - 5x^{4/3}$.

Solution

$$f'(x) = \frac{10}{3}x^{1/3}(x^{1/3} - 2) \qquad f''(x) = \frac{20(x^{1/3} - 1)}{9x^{2/3}}$$

The function has two intercepts: (0, 0) and $\left(\frac{125}{8}, 0\right)$. There are no horizontal or vertical asymptotes. The function has two critical numbers ($x = 0$ and $x = 8$) and two possible points of inflection ($x = 0$ and $x = 1$). The domain is all real numbers. The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.50.



gure 3.50

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		$+$	$-$	Increasing, concave downward
$x = 0$	0	0	Undef.	Relative maximum
$0 < x < 1$		$-$	$-$	Decreasing, concave downward
$x = 1$	-3	$-$	0	Point of inflection
$1 < x < 8$		$-$	$+$	Decreasing, concave upward
$x = 8$	-16	0	$+$	Relative minimum
$8 < x < \infty$		$+$	$+$	Increasing, concave upward

Example 5 Analyzing and Sketching the Graph of a Polynomial Function

Analyze and sketch the graph of $f(x) = x^4 - 12x^3 + 48x^2 - 64x$.

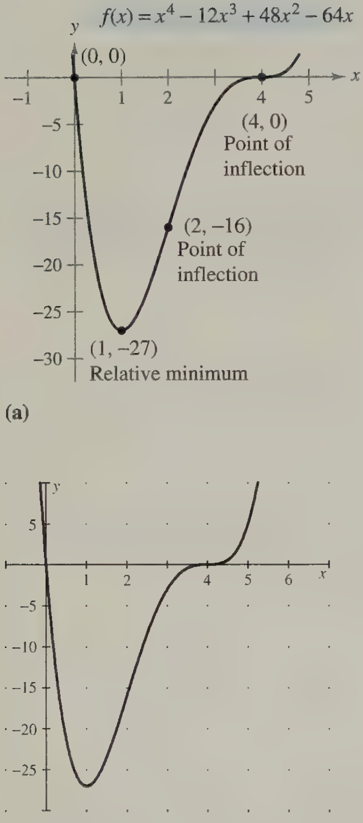
Solution Begin by factoring to obtain

$$\begin{aligned} f(x) &= x^4 - 12x^3 + 48x^2 - 64x \\ &= x(x-4)^3. \end{aligned}$$

Then, using the factored form of $f(x)$, you can perform the following analysis.

First derivative:	$f'(x) = 4(x-1)(x-4)^2$
Second derivative:	$f''(x) = 12(x-4)(x-2)$
x-intercepts:	$(0, 0), (4, 0)$
y-intercept:	$(0, 0)$
Vertical asymptotes:	None
Horizontal asymptotes:	None
End behavior:	$\lim\limits_{x\to-\infty} f(x) = \infty,\ \lim\limits_{x\to\infty} f(x) = \infty$
Critical numbers:	$x = 1, x = 4$
Possible points of inflection:	$x = 2, x = 4$
Domain:	All real numbers
Test intervals:	$(-\infty, 1), (1, 2), (2, 4), (4, \infty)$

The analysis of the graph of f is shown in the table, and the graph is shown in Figu 3.51(a). Using a computer algebra system such as Derive (see Figure 3.51b) can he you verify your analysis.



(a)

(b)
A polynomial function of even degree must have at least one relative extremum.
Figure 3.51

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 1$		$-$	$+$	Decreasing, concave upward
$x = 1$	-27	0	$+$	Relative minimum
$1 < x < 2$		$+$	$+$	Increasing, concave upward
$x = 2$	-16	$+$	0	Point of inflection
$2 < x < 4$		$+$	$-$	Increasing, concave downwarc
$x = 4$	0	0	0	Point of inflection
$4 < x < \infty$		$+$	$+$	Increasing, concave upward

The fourth-degree polynomial function in Example 5 has one relative minimu and no relative maxima. In general, a polynomial function of degree n can have *most* $n - 1$ relative extrema, and *at most* $n - 2$ points of inflection. Moreover, po nomial functions of even degree must have *at least* one relative extremum.

Remember from the leading coefficient test described in Section P.3 that the “e behavior” of the graph of a polynomial function is determined by its leading coef cient and its degree. For instance, because the polynomial in Example 5 has a positi leading coefficient, the graph rises to the right. Moreover, because the degree is ev the graph also rises to the left.

Example 6 Analyzing and Sketching the Graph of a Trigonometric Function

Analyze and sketch the graph of $f(x) = \dfrac{\cos x}{1 + \sin x}$.

Solution Because the function has a period of 2π, you can restrict the analysis of the graph to the interval $(-\pi/2, 3\pi/2)$.

First derivative: $f'(x) = -\dfrac{1}{1 + \sin x}$

Second derivative: $f''(x) = \dfrac{\cos x}{(1 + \sin x)^2}$

Period: 2π

x-intercept: $\left(\dfrac{\pi}{2}, 0\right)$

y-intercept: $(0, 1)$

Vertical asymptotes: $x = -\dfrac{\pi}{2},\ x = \dfrac{3\pi}{2}$

Horizontal asymptotes: None

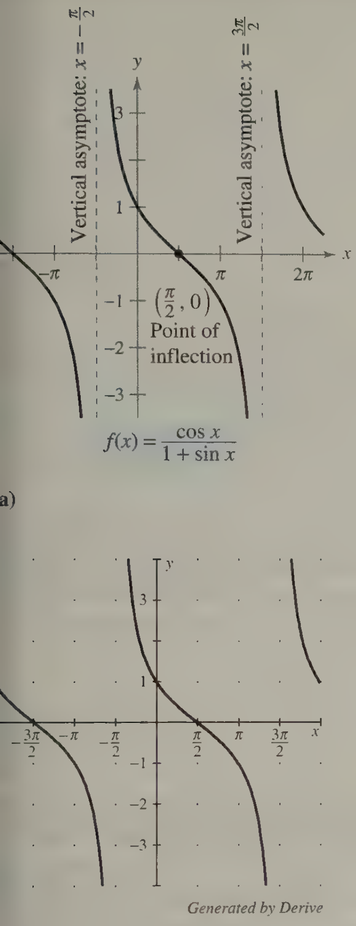
Critical numbers: None

Possible points of inflection: $x = \dfrac{\pi}{2}$

Domain: All real numbers except $x = \dfrac{3 + 4n}{2}\pi$

Test intervals: $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

The analysis of the graph of f on the interval $(-\pi/2, 3\pi/2)$ is shown in the table, and the graph is shown in Figure 3.52(a). Compare this with the graph generated by the computer algebra system Derive in Figure 3.52(b).



a)

Generated by Derive

b)

igure 3.52

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$x = -\frac{\pi}{2}$	Undef.	Undef.	Undef.	Vertical asymptote
$-\frac{\pi}{2} < x < \frac{\pi}{2}$		$-$	$+$	Decreasing, concave upward
$x = \frac{\pi}{2}$	0	$-\frac{1}{2}$	0	Point of inflection
$\frac{\pi}{2} < x < \frac{3\pi}{2}$		$-$	$-$	Decreasing, concave downward
$x = \frac{3\pi}{2}$	Undef.	Undef.	Undef.	Vertical asymptote

NOTE The work involved in sketching the graph of a trigonometric function can be lessened sometimes by using trigonometric identities. For instance, the function in Example 6 can be rewritten as

$$f(x) = \frac{\cos x}{1 + \sin x} = \cot\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

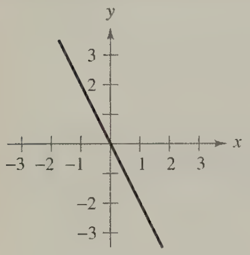
In this form, you can recognize the familiar cotangent graph shown in Figure 3.52.

EXERCISES FOR SECTION 3.6

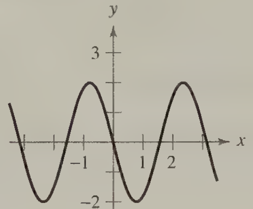
In Exercises 1–4, match the graph of *f* in the left column with that of its derivative in the right column.

Graph of f *Graph of f′*

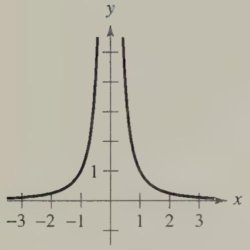
1.



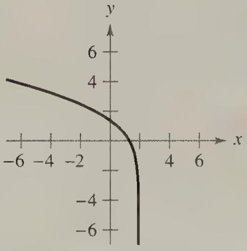
(a)



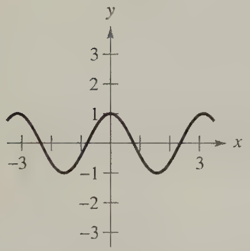
2.



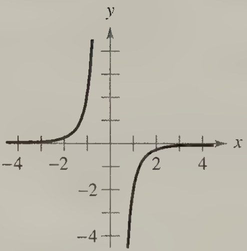
(b)



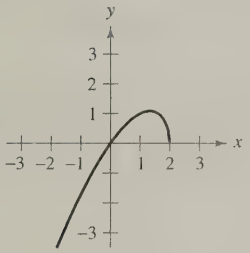
3.



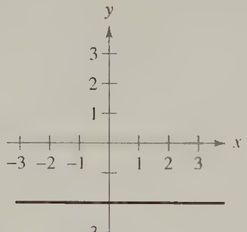
(c)



4.



(d)



5. ***Graphical Reasoning*** The graph of f is given in the figure.

(a) For which values of x is $f'(x)$ zero? Positive? Negative?

(b) For which values of x is $f''(x)$ zero? Positive? Negative?

(c) On what interval is f' an increasing function?

(d) For which value of x is $f'(x)$ minimum? For this value of x, how does the rate of change of f compare with the rate of change of f for other values of x? Explain.

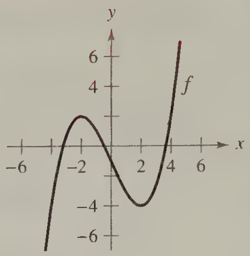


Figure for 5

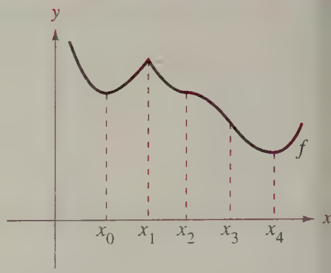


Figure for 6

6. ***Graphical Reasoning*** Identify the real numbers x_0, x_1, x_2, x and x_4 in the figure such that each of the following is true.

(a) $f'(x) = 0$ (b) $f''(x) = 0$

(c) $f'(x)$ does not exist. (d) f has a relative maximum.

(e) f has a point of inflection.

In Exercises 7–38, analyze and sketch a graph of the functio Label any intercepts, relative extrema, points of inflection, an asymptotes. Use a graphing utility to verify your results.

7. $y = \dfrac{x^2}{x^2+3}$ **8.** $y = \dfrac{x}{x^2+1}$

9. $y = \dfrac{1}{x-2} - 3$ **10.** $y = \dfrac{x^2+1}{x^2-9}$

11. $y = \dfrac{2x}{x^2-1}$ **12.** $f(x) = \dfrac{x+2}{x}$

13. $g(x) = x + \dfrac{4}{x^2+1}$ **14.** $f(x) = x + \dfrac{32}{x^2}$

15. $f(x) = \dfrac{x^2+1}{x}$ **16.** $f(x) = \dfrac{x^3}{x^2-4}$

17. $y = \dfrac{x^2-6x+12}{x-4}$ **18.** $y = \dfrac{2x^2-5x+5}{x-2}$

19. $y = x\sqrt{4-x}$ **20.** $g(x) = x\sqrt{9-x}$

21. $h(x) = x\sqrt{9-x^2}$ **22.** $y = x\sqrt{16-x^2}$

23. $y = 3x^{2/3} - 2x$ **24.** $y = 3(x-1)^{2/3} - (x - 1$

25. $y = x^3 - 3x^2 + 3$ **26.** $y = -\frac{1}{3}(x^3 - 3x + 2)$

27. $y = 2 - x - x^3$ **28.** $f(x) = \frac{1}{3}(x-1)^3 + 2$

29. $f(x) = 3x^3 - 9x + 1$

30. $f(x) = (x+1)(x-2)(x-5)$

31. $y = 3x^4 + 4x^3$ **32.** $y = 3x^4 - 6x^2 + \frac{5}{3}$

33. $f(x) = x^4 - 4x^3 + 16x$

34. $f(x) = x^4 - 8x^3 + 18x^2 - 16x + 5$

35. $y = x^5 - 5x$ **36.** $y = (x-1)^5$

37. $y = |2x - 3|$ **38.** $y = |x^2 - 6x + 5|$

**Exercises 39–46, sketch a graph of the function over the indi-
ted interval. Use a graphing utility to verify your graph.**

Function	*Interval*
$y = \sin x - \frac{1}{18}\sin 3x$	$0 \le x \le 2\pi$
$y = \cos x - \frac{1}{2}\cos 2x$	$0 \le x \le 2\pi$
$y = 2x - \tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$y = 2(x-2) + \cot x$	$0 < x < \pi$
$y = 2(\csc x + \sec x)$	$0 < x < \frac{\pi}{2}$
$y = \sec^2\left(\frac{\pi x}{8}\right) - 2\tan\left(\frac{\pi x}{8}\right) - 1$	$-3 < x < 3$
$g(x) = x\tan x$	$-\frac{3\pi}{2} < x < \frac{3\pi}{2}$
$g(x) = x\cot x$	$-2\pi < x < 2\pi$

**Exercises 47–50, use a computer algebra system to analyze
d graph the function. Identify any relative extrema, points of
flection, and asymptotes.**

$f(x) = \dfrac{20x}{x^2+1} - \dfrac{1}{x}$

48. $f(x) = 5\left(\dfrac{1}{x-4} - \dfrac{1}{x+2}\right)$

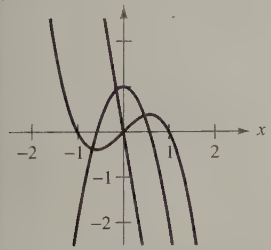
$f(x) = \dfrac{x}{\sqrt{x^2+7}}$

50. $f(x) = \dfrac{4x}{\sqrt{x^2+15}}$

etting at the Concept

In Exercises 51 and 52, the graphs of f, f', and f'' are shown on the same set of coordinate axes. Which is which? To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

51.



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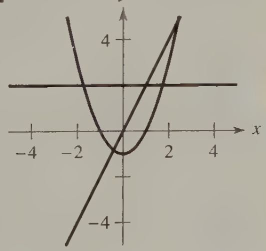
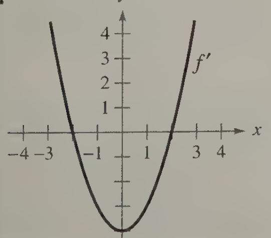


In Exercises 53–56, use the graph of f' to sketch a graph of f and the graph of f''. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

53.



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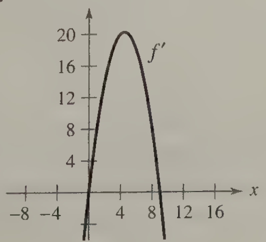
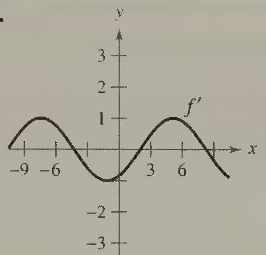


Getting at the Concept *(continued)*

55.



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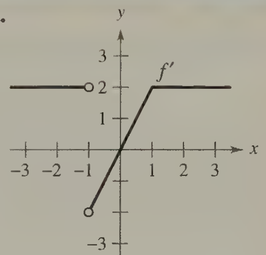


(Submitted by Bill Fox, Moberly Area Community College, Moberly, MO)

57. Suppose $f'(t) < 0$ for all t in the interval $(2, 8)$. Explain why $f(3) > f(5)$.

58. Suppose $f(0) = 3$ and $2 \le f'(x) \le 4$ for all x in the interval $[-5, 5]$. Determine the greatest and least possible values of $f(2)$.

In Exercises 59 and 60, use a graphing utility to graph the function. Use the graph to determine whether it is possible for the graph of a function to cross its horizontal asymptote. Do you think it is possible for the graph of a function to cross its vertical asymptote? Why or why not?

59. $f(x) = \dfrac{4(x-1)^2}{x^2-4x+5}$

60. $g(x) = \dfrac{3x^4-5x+3}{x^4+1}$

Writing **In Exercises 61 and 62, use a graphing utility to graph the function. Explain why there is no vertical asymptote when a superficial examination of the function may indicate that there should be one.**

61. $h(x) = \dfrac{6-2x}{3-x}$

62. $g(x) = \dfrac{x^2+x-2}{x-1}$

Writing **In Exercises 63 and 64, use a graphing utility to graph the function and determine the slant asymptote of the graph. Zoom out repeatedly and describe how the graph on the display appears to change. Why does this occur?**

63. $f(x) = -\dfrac{x^2-3x-1}{x-2}$

64. $g(x) = \dfrac{2x^2-8x-15}{x-5}$

65. ***Graphical Reasoning*** Consider the function

$$f(x) = \frac{\cos^2 \pi x}{\sqrt{x^2+1}}, \quad 0 < x < 4.$$

(a) Use a computer algebra system to graph the function and use the graph to visually approximate the critical numbers.

(b) Use a computer algebra system to find f' and approximate the critical numbers. Are the results the same as the visual approximation in part (a)? Explain.

66. ***Graphical Reasoning*** Consider the function

$$f(x) = \tan(\sin \pi x).$$

(a) Use a graphing utility to graph the function.

(b) Identify any symmetry of the graph.

(c) Is the function periodic? If so, what is the period?

(d) Identify any extrema on $(-1, 1)$.

(e) Use a graphing utility to determine the concavity of the graph on $(0, 1)$.

Think About It **In Exercises 67–70, create a function whose graph has the indicated characteristics. (There is more than one correct answer.)**

67. Vertical asymptote: $x = 5$
Horizontal asymptote: $y = 0$

68. Vertical asymptote: $x = -3$
Horizontal asymptote: None

69. Vertical asymptote: $x = 5$
Slant asymptote: $y = 3x + 2$

70. Vertical asymptote: $x = 0$
Slant asymptote: $y = -x$

71. ***Graphical Reasoning*** Consider the function

$$f(x) = \frac{ax}{(x - b)^2}.$$

(a) Determine the effect on the graph of f if $b \neq 0$ and a is varied. Consider cases where a is positive and a is negative.

(b) Determine the effect on the graph of f if $a \neq 0$ and b is varied.

72. Consider the function

$$f(x) = \tfrac{1}{2}(ax)^2 - (ax), \quad a \neq 0.$$

(a) Determine the changes (if any) in the intercepts, extrema, and concavity of the graph of f when a is varied.

(b) In the same viewing window, use a graphing utility to graph the function for four different values of a.

73. ***Investigation*** Consider the function

$$f(x) = \frac{3x^n}{x^4 + 1}$$

for nonnegative integer values of n.

(a) Discuss the relationship between the value of n and the symmetry of the graph.

(b) For which values of n will the x-axis be the horizontal asymptote?

(c) For which value of n will $y = 3$ be the horizontal asymptote?

(d) What is the asymptote of the graph when $n = 5$?

(e) Use a graphing utility to graph f for the indicated values of n in the table. Use the graph to determine the number of extrema M and the number of inflection points N of the graph.

n	0	1	2	3	4	5
M						
N						

Table for 73(e)

74. ***Investigation*** Let $P(x_0, y_0)$ be an arbitrary point on the gra of f such that $f'(x_0) \neq 0$, as indicated in the figure. Verify ea of the following.

(a) The x-intercept of the tangent line is $\left(x_0 - \dfrac{f(x_0)}{f'(x_0)}, 0\right)$.

(b) The y-intercept of the tangent line is $(0, f(x_0) - x_0 f'(x$

(c) The x-intercept of the normal line is $(x_0 + f(x_0)f'(x_0),$

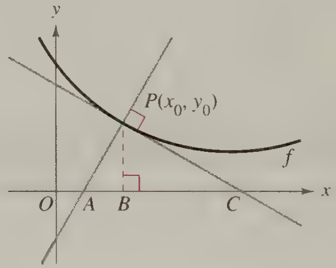
(d) The y-intercept of the normal line is $\left(0, y_0 + \dfrac{x_0}{f'(x_0)}\right)$.

(e) $|BC| = \left|\dfrac{f(x_0)}{f'(x_0)}\right|$

(f) $|PC| = \left|\dfrac{f(x_0)\sqrt{1 + [f'(x_0)]^2}}{f'(x_0)}\right|$

(g) $|AB| = |f(x_0)f'(x_0)|$

(h) $|AP| = |f(x_0)|\sqrt{1 + [f'(x_0)]^2}$



75. ***Modeling Data*** The data in the table show the number N bacteria in a culture at time t, where t is measured in days.

t	1	2	3	4	5	6	7	8
N	25	200	804	1756	2296	2434	2467	2473

A model for this data is given by

$$N = \frac{24{,}670 - 35{,}153t + 13{,}250t^2}{100 - 39t + 7t^2}, \quad 1 \leq t \leq 8.$$

(a) Use a graphing utility to plot the data and graph the moc

(b) Use the model to estimate the number of bacteria wh $t = 10$.

(c) Approximate the day when the number of bacteria v greatest.

(d) Use a computer algebra system to determine the time wh the rate of increase in the number of bacteria was greate

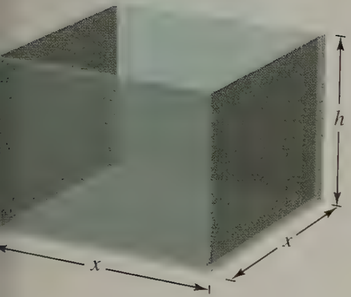
(e) Find $\lim\limits_{t\to\infty} N(t)$.

Section 3.7 Optimization Problems

- Solve applied minimum and maximum problems.

Applied Minimum and Maximum Problems

One of the most common applications of calculus involves the determination of minimum and maximum values. Consider how frequently you hear or read terms such as greatest profit, least cost, least time, greatest voltage, optimum size, least size, greatest strength, and greatest distance. Before outlining a general problem-solving strategy for such problems, let's look at an example.



pen box with square base: $= x^2 + 4xh = 108$
igure 3.53

Example 1 Finding Maximum Volume

A manufacturer wants to design an open box having a square base and a surface area of 108 square inches, as shown in Figure 3.53. What dimensions will produce a box with maximum volume?

Solution Because the box has a square base, its volume is

$$V = x^2h. \qquad \text{Primary equation}$$

(This equation is called the **primary equation** because it gives a formula for the quantity to be optimized.) The surface area of the box is

$$S = (\text{area of base}) + (\text{area of four sides})$$
$$S = x^2 + 4xh = 108. \qquad \text{Secondary equation}$$

Because V is to be maximized, you want to express V as a function of just one variable. To do this, you can solve the equation $x^2 + 4xh = 108$ for h in terms of x to obtain $h = (108 - x^2)/(4x)$. Substituting into the primary equation produces

$$\begin{aligned} V &= x^2h && \text{Function of two variables} \\ &= x^2\left(\frac{108 - x^2}{4x}\right) && \text{Substitute for } h. \\ &= 27x - \frac{x^3}{4}. && \text{Function of one variable} \end{aligned}$$

Before finding which x-value will yield a maximum value of V, you should determine the *feasible domain*. That is, what values of x make sense in this problem? You know that $V \geq 0$. You also know that x must be nonnegative and that the area of the base ($A = x^2$) is at most 108. So, the feasible domain is

$$0 \leq x \leq \sqrt{108}. \qquad \text{Feasible domain}$$

To maximize V, you can find the critical numbers of the volume function.

$$\begin{aligned} \frac{dV}{dx} = 27 - \frac{3x^2}{4} &= 0 && \text{Set derivative equal to 0.} \\ 3x^2 &= 108 \\ x &= \pm 6 && \text{Critical numbers} \end{aligned}$$

So, the critical numbers are $x = \pm 6$. You do not need to consider -6 because it is outside the domain. Evaluating V at the critical number $x = 6$ and at the endpoints of the domain produces $V(0) = 0$, $V(6) = 108$, and $V(\sqrt{108}) = 0$. Thus, V is maximum when $x = 6$ and the dimensions of the box are 6 × 6 × 3 inches.

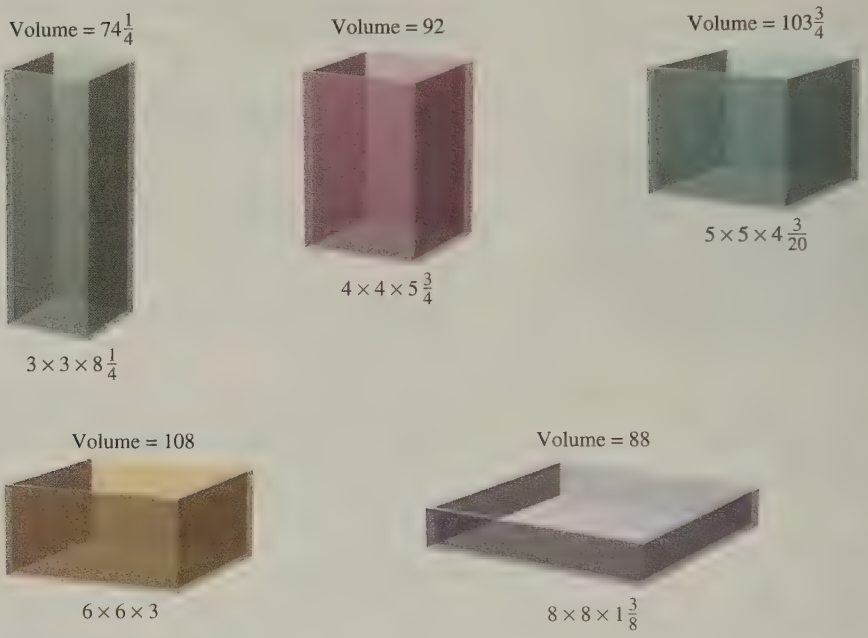
TECHNOLOGY You can verify your answer by using a graphing utility to graph the volume

$$V = 27x - \frac{x^3}{4}.$$

Use a viewing window in which $0 \leq x \leq \sqrt{108} \approx 10.4$ and $0 \leq y \leq 120$, and the *trace* feature to determine the maximum value of V.

In Example 1, you should realize that there are infinitely many open boxes ha ing 108 square inches of surface area. To begin solving the problem, you might a yourself which basic shape would seem to yield a maximum volume. Should the b be tall, squat, or nearly cubical?

You might even try calculating a few volumes, as shown in Figure 3.54, to see you can get a better feeling for what the optimum dimensions should be. Rememb that you are not ready to begin solving a problem until you have clearly identifi what the problem is.


Volume = $74\frac{1}{4}$
$3 \times 3 \times 8\frac{1}{4}$
Volume = 92
$4 \times 4 \times 5\frac{3}{4}$
Volume = $103\frac{3}{4}$
$5 \times 5 \times 4\frac{3}{20}$
Volume = 108
$6 \times 6 \times 3$
Volume = 88
$8 \times 8 \times 1\frac{3}{8}$

Which box has the greatest volume?
Figure 3.54

Example 1 illustrates the following guidelines for solving applied minimum a maximum problems.

Guidelines for Solving Applied Minimum and Maximum Problems

1. Identify all *given* quantities and quantities *to be determined.* When feasible, make a sketch.
2. Write a **primary equation** for the quantity that is to be maximized (or minimized). (A review of several useful formulas from geometry is presented inside the front cover.)
3. Reduce the primary equation to one having a *single independent variable.* This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques discussed in Sections 3.1 through 3.4.

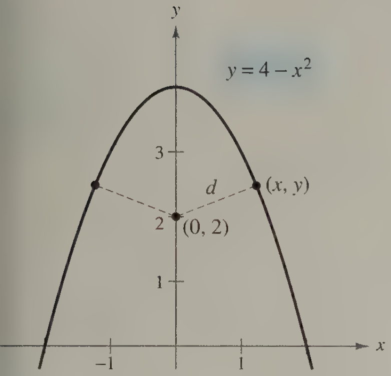
NOTE When performing Step 5, recall that to determine the maximum or minimum value of a continuous function f on a closed interval, you should compare the values of f at its critical numbers with the values of f at the endpoints of the interval.

Example 2 Finding Minimum Distance

Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?

Solution Figure 3.55 indicates that there are two points at a minimum distance from the point $(0, 2)$. The distance between the point $(0, 2)$ and a point (x, y) on the graph of $y = 4 - x^2$ is given by

$$d = \sqrt{(x-0)^2 + (y-2)^2}. \qquad \text{Primary equation}$$



The quantity to be minimized is distance:
$= \sqrt{(x-0)^2 + (y-2)^2}$.
Figure 3.55

Using the secondary equation $y = 4 - x^2$, you can rewrite the primary equation as

$$d = \sqrt{x^2 + (4 - x^2 - 2)^2} = \sqrt{x^4 - 3x^2 + 4}.$$

Because d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of $f(x) = x^4 - 3x^2 + 4$. Note that the domain of f is the entire real line. Moreover, setting $f'(x)$ equal to 0 yields

$$f'(x) = 4x^3 - 6x = 2x(2x^2 - 3) = 0$$
$$x = 0, \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}.$$

The First Derivative Test verifies that $x = 0$ yields a relative maximum, whereas both $x = \sqrt{3/2}$ and $x = -\sqrt{3/2}$ yield a minimum distance. Hence, the closest points are $\left(\sqrt{3/2}, 5/2\right)$ and $\left(-\sqrt{3/2}, 5/2\right)$.

Example 3 Finding Minimum Area

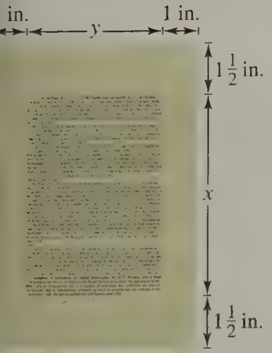
A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be $1\frac{1}{2}$ inches, and the margins on the left and right are to be 1 inch (see Figure 3.56). What should the dimensions of the page be so that the least amount of paper is used?

Solution Let A be the area to be minimized.

$$A = (x + 3)(y + 2) \qquad \text{Primary equation}$$

The printed area inside the margins is given by

$$24 = xy. \qquad \text{Secondary equation}$$



he quantity to be minimized is area:
$= (x + 3)(y + 2)$.
igure 3.56

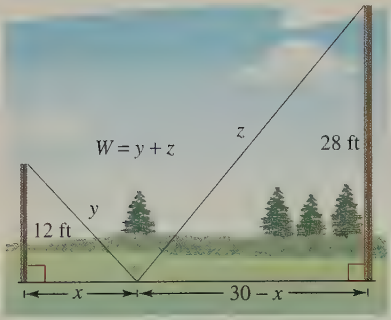
Solving this equation for y produces $y = 24/x$. Substitution into the primary equation produces

$$A = (x + 3)\left(\frac{24}{x} + 2\right) = 30 + 2x + \frac{72}{x}. \qquad \text{Function of one variable}$$

Because x must be positive, you are interested only in values of A for $x > 0$. To find the critical numbers, differentiate with respect to x.

$$\frac{dA}{dx} = 2 - \frac{72}{x^2} = 0 \quad \Longrightarrow \quad x^2 = 36$$

So, the critical numbers are $x = \pm 6$. You do not have to consider -6 because it is outside the domain. The First Derivative Test confirms that A is a minimum when $x = 6$. Therefore, $y = \frac{24}{6} = 4$ and the dimensions of the page should be $x + 3 = 9$ inches by $y + 2 = 6$ inches.



The quantity to be minimized is length. From the diagram, you can see that x varies between 0 and 30.
Figure 3.57

Example 4 Finding Minimum Length

Two posts, one 12 feet high and the other 28 feet high, stand 30 feet apart. They ar to be stayed by two wires, attached to a single stake, running from ground level to th top of each post. Where should the stake be placed to use the least wire?

Solution Let W be the wire length to be minimized. Using Figure 3.57, you can wri

$$W = y + z. \qquad \text{Primary equation}$$

In this problem, rather than solving for y in terms of z (or vice versa), you can solv for both y and z in terms of a third variable x, as shown in Figure 3.57. From th Pythagorean Theorem, you obtain

$$\begin{aligned} x^2 + 12^2 &= y^2 \\ (30 - x)^2 + 28^2 &= z^2 \end{aligned}$$

which implies that

$$\begin{aligned} y &= \sqrt{x^2 + 144} \\ z &= \sqrt{x^2 - 60x + 1684}. \end{aligned}$$

Thus, W is given by

$$\begin{aligned} W &= y + z \\ &= \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}, \qquad 0 \le x \le 30. \end{aligned}$$

Differentiating W with respect to x yields

$$\frac{dW}{dx} = \frac{x}{\sqrt{x^2 + 144}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}}.$$

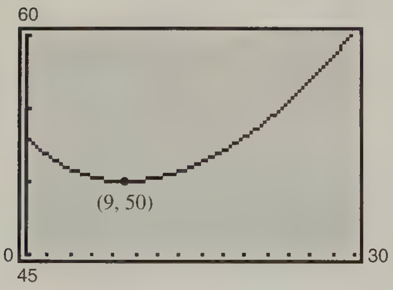
By letting $dW/dx = 0$, you obtain

$$\begin{aligned} \frac{x}{\sqrt{x^2 + 144}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}} &= 0 \\ x\sqrt{x^2 - 60x + 1684} &= (30 - x)\sqrt{x^2 + 144} \\ x^2(x^2 - 60x + 1684) &= (30 - x)^2(x^2 + 144) \\ x^4 - 60x^3 + 1684x^2 &= x^4 - 60x^3 + 1044x^2 - 8640x + 129{,}60 \\ 640x^2 + 8640x - 129{,}600 &= 0 \\ 320(x - 9)(2x + 45) &= 0 \\ x &= 9, -22.5. \end{aligned}$$

Because $x = -22.5$ is not in the domain and

$$W(0) \approx 53.04, \qquad W(9) = 50, \qquad \text{and} \qquad W(30) \approx 60.31$$

you can conclude that the wire should be staked at 9 feet from the 12-foot pole.



You can confirm the minimum value of W with a graphing utility.
Figure 3.58

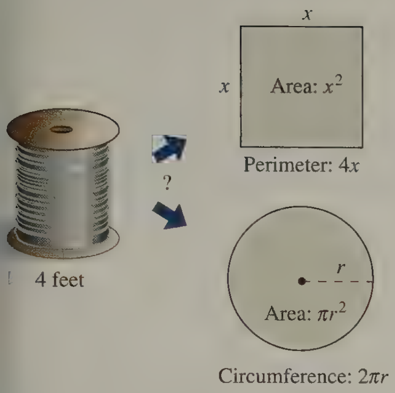
TECHNOLOGY From Example 4, you can see that applied optimization problem can involve a lot of algebra. If you have access to a graphing utility, you can cor firm that $x = 9$ yields a minimum value of W by sketching the graph of

$$W = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}$$

as shown in Figure 3.58.

In each of the first four examples, the extreme value occurs at a critical number. Although this happens often, remember that an extreme value can also occur at an endpoint of an interval, as shown in Example 5.

Example 5 An Endpoint Maximum



he quantity to be maximized is area:
$= x^2 + \pi r^2$.
igure 3.59

Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?

Solution The total area (see Figure 3.59) is given by

$$A = (\text{area of square}) + (\text{area of circle})$$
$$A = x^2 + \pi r^2. \qquad \text{Primary equation}$$

Because the total amount of wire is 4 feet, you obtain

$$4 = (\text{perimeter of square}) + (\text{circumference of circle})$$
$$4 = 4x + 2\pi r.$$

So, $r = 2(1 - x)/\pi$, and by substituting into the primary equation you have

$$\begin{aligned} A &= x^2 + \pi\left[\frac{2(1-x)}{\pi}\right]^2 \\ &= x^2 + \frac{4(1-x)^2}{\pi} \\ &= \frac{1}{\pi}[(\pi + 4)x^2 - 8x + 4]. \end{aligned}$$

The feasible domain is $0 \leq x \leq 1$ restricted by the square's perimeter. Because

$$\frac{dA}{dx} = \frac{2(\pi + 4)x - 8}{\pi}$$

the only critical number in $(0, 1)$ is $x = 4/(\pi + 4) \approx 0.56$. Therefore, using

$$A(0) \approx 1.273, \qquad A(0.56) \approx 0.56, \qquad \text{and} \qquad A(1) = 1$$

you can conclude that the maximum area occurs when $x = 0$. That is, *all* the wire is used for the circle.

EXPLORATION

What would the answer be if Example 5 asked for the dimensions needed to enclose the *minimum* total area?

Let's review the primary equations developed in the first five examples. As applications go, these five examples are fairly simple, and yet the resulting primary equations are quite complicated.

$$V = 27x - \frac{x^3}{4} \qquad W = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}$$

$$d = \sqrt{x^4 - 3x^2 + 4} \qquad A = \frac{1}{\pi}[(\pi + 4)x^2 - 8x + 4]$$

$$A = 30 + 2x + \frac{72}{x}$$

You must expect that real-life applications often involve equations that are *at least as complicated* as these five. Remember that one of the main goals of this course is to learn to use calculus to analyze equations that initially seem formidable.

Lab Series LAB 5

EXERCISES FOR SECTION 3.7

1. ***Numerical, Graphical, and Analytic Analysis*** Find two positive numbers whose sum is 110 and whose product is a maximum.

(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

First Number x	Second Number	Product P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$

(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the solution. (*Hint:* Use the *table* feature of the graphing utility.)

(c) Write the product P as a function of x.

(d) Use a graphing utility to graph the function in part (c) and estimate the solution from the graph.

(e) Use calculus to find the critical number of the function in part (c). Then find the two numbers.

In Exercises 2–6, find two positive numbers that satisfy the given requirements.

2. The sum is S and the product is a maximum.

3. The product is 192 and the sum is a minimum.

4. The product is 192 and the sum of the first plus three times the second is a minimum.

5. The second number is the reciprocal of the first and the sum is a minimum.

6. The sum of the first and twice the second is 100 and the product is a maximum.

In Exercises 7 and 8, find the length and width of a rectangle that has the given perimeter and a maximum area.

7. Perimeter: 100 meters

8. Perimeter: P units

In Exercises 9 and 10, find the length and width of a rectangle that has the given area and a minimum perimeter.

9. Area: 64 square feet

10. Area: A square centimeters

In Exercises 11–14, find the point on the graph of the function that is closest to the given point.

Function	*Point*	*Function*	*Point*
11. $f(x) = \sqrt{x}$	$(4, 0)$	**12.** $f(x) = \sqrt{x - 8}$	$(2, 0)$
13. $f(x) = x^2$	$\left(2, \frac{1}{2}\right)$	**14.** $f(x) = (x + 1)^2$	$(5, 3)$

15. ***Chemical Reaction*** In an autocatalytic chemical reaction, t product formed is a catalyst for the reaction. If Q_0 is the amou of the original substance and x is the amount of catalyst form the rate of chemical reaction is

$$\frac{dQ}{dx} = kx(Q_0 - x).$$

For what value of x will the rate of chemical reaction greatest?

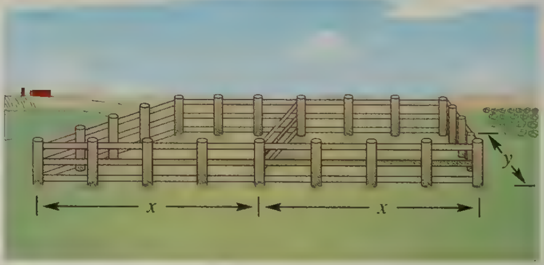
16. ***Traffic Control*** On a given day, the flow rate F (cars hour) on a congested roadway is

$$F = \frac{v}{22 + 0.02v^2}$$

where v is the speed of the traffic in miles per hour. What spe will maximize the flow rate on the road?

17. ***Area*** A farmer plans to fence a rectangular pasture adjac to a river. The pasture must contain 180,000 square met in order to provide enough grass for the herd. What dim sions would require the least amount of fencing if no fencin needed along the river?

18. ***Area*** A rancher has 200 feet of fencing with which to encl two adjacent rectangular corrals (see figure). What dimensic should be used so that the enclosed area will be a maximum

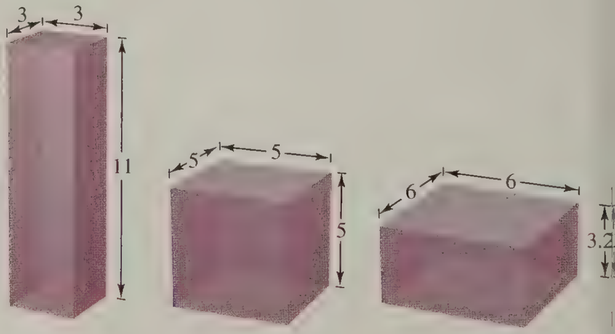


19. ***Volume***

(a) Verify that each of the rectangular solids shown in figure has a surface area of 150 square inches.

(b) Find the volume of each.

(c) Determine the dimensions of a rectangular solid (wit square base) of maximum volume if its surface area is square inches.



20. ***Numerical, Graphical, and Analytic Analysis*** An open box of maximum volume is to be made from a square piece of material, 24 inches on a side, by cutting equal squares from the corners and turning up the sides (see figure).

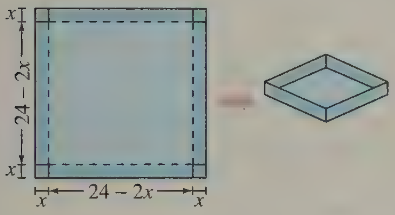
(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.) Use the table to guess the maximum volume.

Height	Length and Width	Volume
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$

(b) Write the volume V as a function of x.

(c) Use calculus to find the critical number of the function in part (b) and find the maximum value.

(d) Use a graphing utility to graph the function in part (b) and verify the maximum volume from the graph.



21. (a) Solve Exercise 20 given that the square piece of material is s meters on a side.

(b) If the dimensions of the square piece of material are doubled, how does the volume change?

22. ***Numerical, Graphical, and Analytic Analysis*** A physical fitness room consists of a rectangle with a semicircle on each end. A 200-meter running track runs around the outside of the room.

(a) Draw a figure to represent the problem. Let x and y represent the length and width of the rectangle.

(b) Analytically complete six rows of a table such as the one below. (The first two rows are shown.) Use the table to guess the maximum area of the rectangular region.

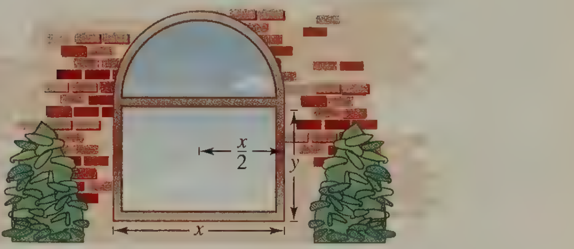
Length x	Width y	Area
10	$\frac{2}{\pi}(100 - 10)$	$(10)\frac{2}{\pi}(100 - 10) \approx 573$
20	$\frac{2}{\pi}(100 - 20)$	$(20)\frac{2}{\pi}(100 - 20) \approx 1019$

(c) Write the area A as a function of x.

(d) Use calculus to find the critical number of the function in part (c) and find the maximum value.

(e) Use a graphing utility to graph the function in part (c) and verify the maximum area from the graph.

23. ***Area*** A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window (see figure). Find the dimensions of a Norman window of maximum area if the total perimeter is 16 feet.



24. ***Area*** A rectangle is bounded by the x- and y-axes and the graph of $y = (6 - x)/2$ (see figure). What length and width should the rectangle have so that its area is a maximum?

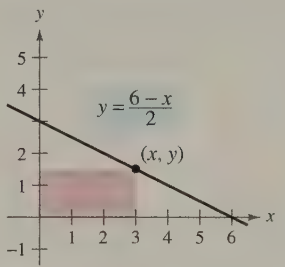


Figure for 24

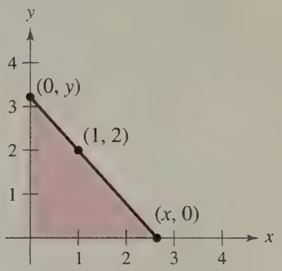


Figure for 25

25. ***Length*** A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point (1, 2) (see figure).

(a) Write the length L of the hypotenuse as a function of x.

(b) Use a graphing utility to graphically approximate x such that the length of the hypotenuse is a minimum.

(c) Find the vertices of the triangle such that its area is a minimum.

26. ***Area*** Find the area of the largest isosceles triangle that can be inscribed in a circle of radius 4 (see figure).

(a) Solve by writing the area as a function of h.

(b) Solve by writing the area as a function of α.

(c) Identify the type of triangle of maximum area.

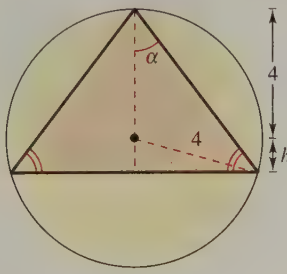


Figure for 26

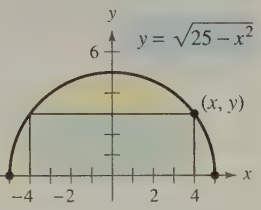


Figure for 27

27. ***Area*** A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$ (see figure). What length and width should the rectangle have so that its area is a maximum?

28. ***Area*** Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius r (see Exercise 27).

29. ***Area*** A rectangular page is to contain 30 square inches of print. The margins on each side are 1 inch. Find the dimensions of the page such that the least amount of paper is used.

30. ***Area*** A rectangular page is to contain 36 square inches of print. The margins on each side are to be $1\frac{1}{2}$ inches. Find the dimensions of the page such that the least amount of paper is used.

31. ***Numerical, Graphical, and Analytic Analysis*** A right circular cylinder is to be designed to hold 22 cubic inches of a soft drink (approximately 12 fluid ounces).

(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

Radius r	Height	Surface Area
0.2	$\dfrac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \dfrac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\dfrac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \dfrac{22}{\pi(0.4)^2}\right] \approx 111.1$

(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the minimum surface area. (*Hint:* Use the *table* feature of the graphing utility.)

(c) Write the surface area S as a function of r.

(d) Use a graphing utility to graph the function in part (c) and estimate the minimum surface area from the graph.

(e) Use calculus to find the critical number of the function in part (c) and find dimensions that will yield the minimum surface area.

32. ***Surface Area*** Use calculus to find the required dimensions for the cylinder in Exercise 31 if its volume is V_0 cubic units.

33. ***Volume*** A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches (see figure). Find the dimensions of the package of maximum volume that can be sent. (Assume the cross section is square.)

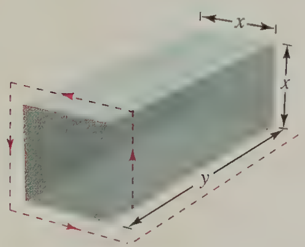


Figure for 33

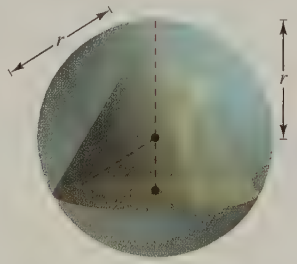


Figure for 35

34. ***Volume*** Rework Exercise 33 for a cylindrical package. (The cross section is circular.)

35. ***Volume*** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius r (see figure).

36. ***Volume*** Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius r.

Getting at the Concept

37. The perimeter of a rectangle is 20 feet. Of all possible dimensions, the maximum area is 25 square feet when its length and width are both 5 feet. Are there dimensions that yield a minimum area? Explain.

38. A plastic shampoo bottle is a right circular cylinder. Because the surface area of the bottle does not change when it is squeezed, is it true that the volume remains the same? Explain.

39. ***Surface Area*** A solid is formed by adjoining two hem spheres to the ends of a right circular cylinder. The total volum of the solid is 12 cubic centimeters. Find the radius of th cylinder that produces the minimum surface area.

40. ***Cost*** An industrial tank of the shape described in Exercise 3 must have a volume of 3000 cubic feet. The hemispherical enc cost twice as much per square foot of surface area as the side Find the dimensions that will minimize cost.

41. ***Area*** The sum of the perimeters of an equilateral triangle an a square is 10. Find the dimensions of the triangle and th square that produce a minimum total area.

42. ***Area*** Twenty feet of wire is to be used to form two figures. I each of the following cases, how much should be used for eac figure so that the total enclosed area is maximum?

(a) Equilateral triangle and square

(b) Square and regular pentagon

(c) Regular pentagon and regular hexagon

(d) Regular hexagon and circle

What can you conclude from this pattern? {*Hint:* The are of a regular polygon with n sides of length x is $A =$ $(n/4)[\cot(\pi/n)]x^2$.}

43. ***Beam Strength*** A wooden beam has a rectangular cro section of height h and width w (see figure). The strength S the beam is directly proportional to the width and the square the height. What are the dimensions of the strongest beam th can be cut from a round log of diameter 24 inches? (*Hin* $S = kh^2w$, where k is the proportionality constant.)

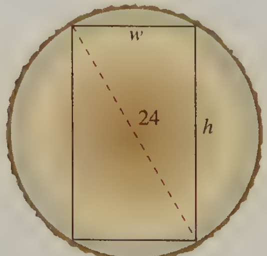


Figure for 43

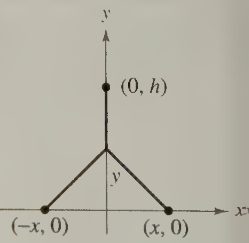


Figure for 44

44. ***Minimum Length*** Two factories are located at the coordinat $(-x, 0)$ and $(x, 0)$ with their power supply located at the poi $(0, h)$ (see figure). Find y such that the total amount of pow line from the power supply to the factories is a minimum.

Projectile Range The range R of a projectile fired with an initial velocity v_0 at an angle θ with the horizontal is

$$R = \frac{v_0^2 \sin 2\theta}{g}$$

where g is the acceleration due to gravity. Find the angle θ such that the range is a maximum.

Conjecture Consider the functions $f(x) = \frac{1}{2}x^2$ and $g(x) = \frac{1}{16}x^4 - \frac{1}{2}x^2$ on the domain $[0, 4]$.

(a) Use a graphing utility to graph the functions on the specified domain.

(b) Write the vertical distance d between the functions as a function of x and use calculus to find the value of x for which d is maximum.

(c) Find the equations of the tangent lines to the graphs of f and g at the critical number found in part (b). Graph the tangent lines. What is the relationship between the lines?

(d) Make a conjecture about the relationship between tangent lines to the graphs of two functions at the value of x at which the vertical distance between the functions is greatest, and prove your conjecture.

Illumination A light source is located over the center of a circular table of diameter 4 feet (see figure). Find the height h of the light source such that the illumination I at the perimeter of the table is maximum if $I = k(\sin \alpha)/s^2$, where s is the slant height, α is the angle at which the light strikes the table, and k is a constant.

Illumination The illumination from a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. Two light sources of intensities I_1 and I_2 are d units apart. What point on the line segment joining the two sources has the least illumination?

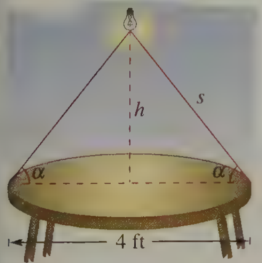


Figure for 47

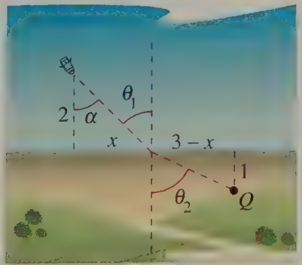


Figure for 49

Minimum Time A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point Q, located 3 miles down the coast and 1 mile inland (see figure). If he can row at 2 miles per hour and walk at 4 miles per hour, toward what point on the coast should he row in order to reach point Q in the least time?

Minimum Time Consider Exercise 49 if the point Q is on the shoreline rather than 1 mile inland.

(a) Write the travel time T as a function of α.

(b) Use the result of part (a) to find the minimum time to reach Q.

(c) Suppose the man can row at v_1 miles per hour and walk at v_2 miles per hour. Write the time T as a function of α. Show that the critical number of T depends only on v_1 and v_2 and not the distances. Explain how this result would be more beneficial to the man than the result of Exercise 49.

(d) Describe how to apply the result of part (c) to minimizing the cost of constructing a power transmission cable that costs c_1 dollars per mile under water and c_2 dollars per mile over land.

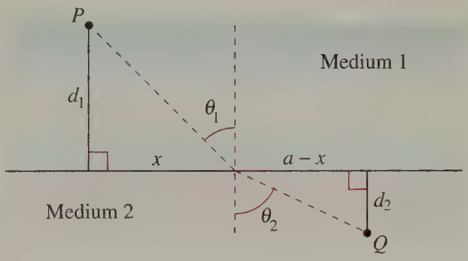
51. ***Minimum Time*** The conditions are the same as in Exercise 49 except that the man can row at v_1 miles per hour and walk at v_2 miles per hour. If θ_1 and θ_2 are the magnitudes of the angles, show that the man will reach point Q in the least time when

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$

52. ***Minimum Time*** When light waves, traveling in a transparent medium, strike the surface of a second transparent medium, they change directions. This change of direction is called *refraction* and is defined by **Snell's Law of Refraction,**

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

where θ_1 and θ_2 are the magnitudes of the angles shown in the figure and v_1 and v_2 are the velocities of light in the two media. Show that this problem is equivalent to that of Exercise 51, and that light waves traveling from P to Q follow the path of minimum time.



53. Sketch the graph of $f(x) = 2 - 2 \sin x$ on the interval $[0, \pi/2]$.

(a) Find the distance from the origin to the y-intercept and the distance from the origin to the x-intercept.

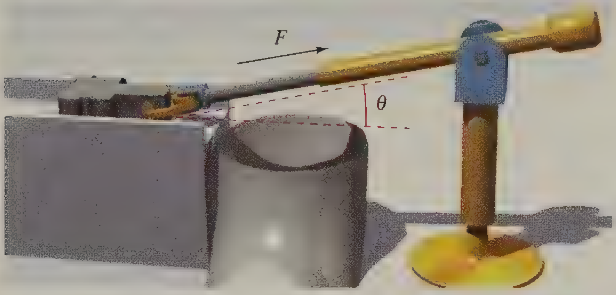
(b) Express the distance d from the origin to a point on the graph of f as a function of x. Use your graphing utility to graph d and find the minimum distance.

(c) Use calculus and the root finding capabilities of a graphing utility to find the value of x that minimizes the function d on the interval $[0, \pi/2]$. What is the minimum distance?

(Submitted by Tim Chapell, Penn Valley Community College, Kansas City, MO.)

54. ***Minimum Cost*** An offshore oil well is 2 kilometers off the coast. The refinery is 4 kilometers down the coast. If laying pipe in the ocean is twice as expensive as on land, what path should the pipe follow in order to minimize the cost?

55. ***Minimum Force*** A component is designed to slide a block of steel with weight W across a table and into a chute (see figure.) The motion of the block is resisted by a frictional force proportional to its apparent weight. (Let k be the constant of proportionality.) Find the minimum force F needed to slide the block, and find the corresponding value of θ. (*Hint:* $F \cos \theta$ is the force in the direction of motion, and $F \sin \theta$ is the amount of force tending to lift the block. Therefore, the apparent weight of the block is $W - F \sin \theta$.)



56. ***Volume*** A sector with central angle θ is cut from a circle of radius 12 inches (see figure), and the edges of the sector are brought together to form a cone. Find the magnitude of θ such that the volume of the cone is a maximum.

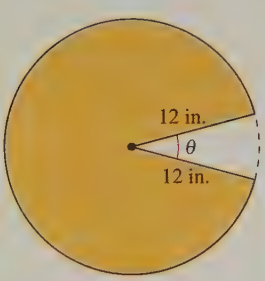


Figure for 56

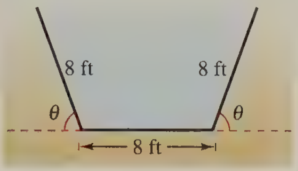


Figure for 57

57. ***Numerical, Graphical, and Analytic Analysis*** The cross sections of an irrigation canal are isosceles trapezoids of which three sides are 8 feet long (see figure). Determine the angle of elevation θ of the sides such that the area of the cross section is a maximum by completing the following.

(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

Base 1	Base 2	Altitude	Area
8	$8 + 16 \cos 10°$	$8 \sin 10°$	≈ 22.1
8	$8 + 16 \cos 20°$	$8 \sin 20°$	≈ 42.5

(b) Use a graphing utility to generate additional rows of the table and estimate the maximum cross-sectional area. (*Hint:* Use the *table* feature of the graphing utility.)

(c) Write the cross-sectional area A as a function of θ.

(d) Use calculus to find the critical number of the function in part (c) and find the angle that will yield the maximum cross-sectional area.

(e) Use a graphing utility to graph the function in part (c) and verify the maximum cross-sectional area.

58. ***Maximum Profit*** Assume that the amount of money depos in a bank is proportional to the square of the interest rate the b pays on this money. Furthermore, the bank can reinvest money at 12%. Find the interest rate the bank should pa maximize profit. (Use the simple interest formula.)

59. ***Minimum Cost*** The ordering and transportation cost C of components used in manufacturing a certain product is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \qquad x \geq 1$$

where C is measured in thousands of dollars and x is the o size in hundreds. Find the order size that minimizes the c (*Hint:* Use the *root* feature of a graphing utility.)

60. ***Diminishing Returns*** The profit P (in thousands of doll for a company spending an amount s (in thousands of doll on advertising is

$$P = -\tfrac{1}{10}s^3 + 6s^2 + 400.$$

(a) Find the amount of money the company should spenc advertising in order to yield a maximum profit.

(b) The *point of diminishing returns* is the point at which rate of growth of the profit function begins to decline. I the point of diminishing returns.

Minimum Distance **In Exercises 61–63, consider a fuel dis bution center located at the origin of the rectangular coordi system (units in miles; see figures). The center supplies th factories with coordinates (4, 1), (5, 6), and (10, 3). A trunk will run from the distribution center along the line $y = mx$, feeder lines will run to the three factories. The objective i find m such that the lengths of the feeder lines are minimiz**

61. Minimize the sum of the squares of the lengths of ver feeder lines given by

$$S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2.$$

Find the equation for the trunk line by this method and determine the sum of the lengths of the feeder lines.

62. Minimize the sum of the absolute values of the length vertical feeder lines given by

$$S_2 = |4m - 1| + |5m - 6| + |10m - 3|.$$

Find the equation for the trunk line by this method and determine the sum of the lengths of the feeder lines. (*Hint:* a graphing utility to graph the function S_2 and approximate required critical number.)

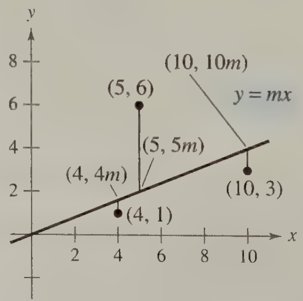
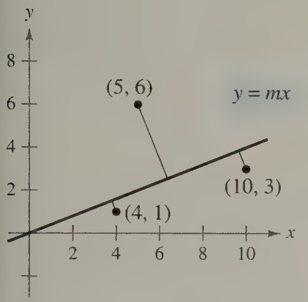


Figure for 61 and 62

. Minimize the sum of the perpendicular distances (see Exercises 83–88 in Section P.2) from the trunk line to the factories given by

$$S_3 = \frac{|4m - 1|}{\sqrt{m^2 + 1}} + \frac{|5m - 6|}{\sqrt{m^2 + 1}} + \frac{|10m - 3|}{\sqrt{m^2 + 1}}.$$

Find the equation for the trunk line by this method and then determine the sum of the lengths of the feeder lines. (*Hint:* Use a graphing utility to graph the function S_3 and approximate the required critical number.)

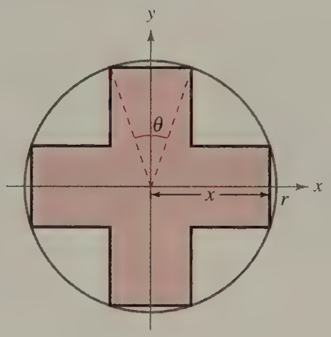


64. *Area* Consider a symmetric cross inscribed in a circle of radius r (see figure).

(a) Write the area A of the cross as a function of x and find the value of x that maximizes the area.

(b) Write the area A of the cross as a function of θ and find the value of θ that maximizes the area.

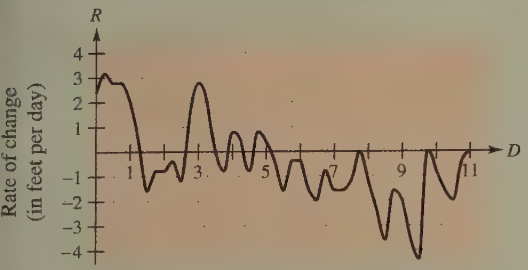
(c) Show that the critical numbers of parts (a) and (b) yield the same maximum area. What is that area?



SECTION PROJECT CONNECTICUT RIVER

Whenever the Connecticut River reaches a level of 105 feet above sea level, two Northampton, Massachusetts flood control station operators begin a round-the-clock river watch. Every two ours, they check the height of the river, using a scale marked off in tenths of a foot, and record the data in a log book. In the spring of 1996, the flood watch lasted from April 4, when the iver reached 105 feet and was rising at 0.2 foot per hour, until April 25, when the level subsided again to 105 feet. Between hose dates, their log shows that the river rose and fell several imes, at one point coming close to the 115-foot mark. If the iver had reached 115 feet, the city would have closed down Mount Tom Road (Route 5, south of Northampton).

The graph below shows the *rate of change* of the level of he river during one portion of the flood watch. Use the graph to nswer the following questions.



Day (0 ↔ 12:01 A.M. April 14)

(a) On what date was the river rising most rapidly? How do you know?

(b) On what date was the river falling most rapidly? How do you know?

(c) There were two dates in a row on which the river rose, then fell, then rose again during the course of the day. On which days did this occur, and how do you know?

(d) At one minute past midnight, April 14, the river level was 111.0 feet. Estimate its height 24 hours later and 48 hours later. Explain how you made your estimates.

(e) The river crested at 114.4 feet. On what date do you think this occurred?

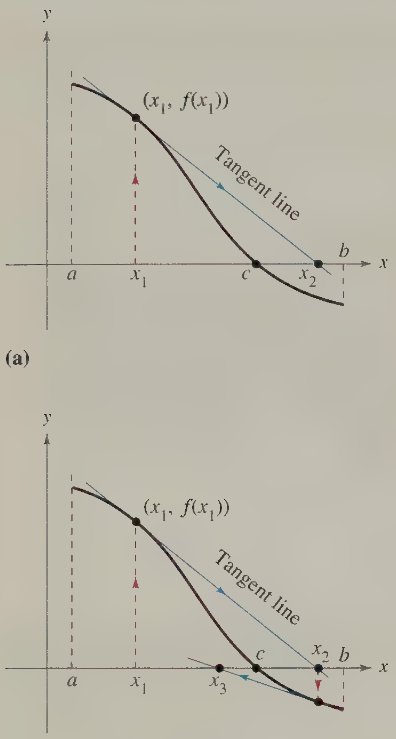
(Submitted by Mary Murphy, Smith College, Northampton, MA)

UPI/Corbis-Bettmann

Section 3.8

Newton's Method

- Approximate a zero of a function using Newton's Method.



(a)

(b)
The x-intercept of the tangent line approximates the zero of f.
Figure 3.60

Newton's Method

In this section you will study a technique for approximating the real zeros of a fun tion. The technique is called **Newton's Method,** and it uses tangent lines to approx mate the graph of the function near its x-intercepts.

To see how Newton's Method works, consider a function f that is continuous the interval $[a, b]$ and differentiable on the interval (a, b). If $f(a)$ and $f(b)$ differ in sig then, by the Intermediate Value Theorem, f must have at least one zero in the interv (a, b). Suppose you estimate this zero to occur at

$$x = x_1 \qquad \text{First estimate}$$

as shown in Figure 3.60(a). Newton's Method is based on the assumption that t graph of f and the tangent line at $(x_1, f(x_1))$ both cross the x-axis at *about* the sa point. Because you can easily calculate the x-intercept for this tangent line, you c use it as a second (and, usually, better) estimate for the zero of f. The tangent li passes through the point $(x_1, f(x_1))$ with a slope of $f'(x_1)$. In point-slope form, t equation of the tangent line is therefore

$$y - f(x_1) = f'(x_1)(x - x_1)$$
$$y = f'(x_1)(x - x_1) + f(x_1).$$

Letting $y = 0$ and solving for x produces

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

So, from the initial estimate x_1 you obtain a new estimate

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}. \qquad \text{Second estimate (see Figure 3.60b)}$$

You can improve on x_2 and calculate yet a third estimate

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}. \qquad \text{Third estimate}$$

Repeated application of this process is called Newton's Method.

Newton's Method for Approximating the Zeros of a Function

Let $f(c) = 0$, where f is differentiable on an open interval containing c. Then, to approximate c, use the following steps.

1. Make an initial estimate x_1 that is "close to" c. (A graph is helpful.)
2. Determine a new approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

3. If $|x_n - x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called an **iteration.**

NEWTON'S METHOD

Isaac Newton first described the method for approximating the real zeros of a function in his text *Method of Fluxions*. Although the book was written in 1671, it was not published until 1736. Meanwhile, in 1690, Joseph Raphson (1648–1715) published a paper describing a method for approximating the real zeros of a function that was very similar to Newton's. For this reason, the method is often referred to as the Newton-Raphson method.

TE For many functions, just a few
ations of Newton's Method will pro-
e approximations having very small
rs, as shown in Example 1.

Example 1 Using Newton's Method

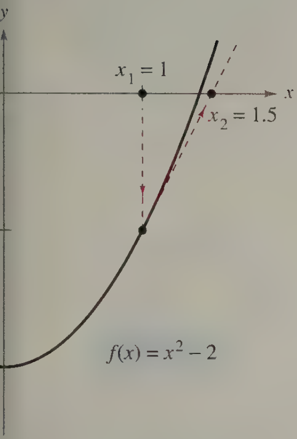
Calculate three iterations of Newton's Method to approximate a zero of $f(x) = x^2 - 2$. Use $x_1 = 1$ as the initial guess.

Solution Because $f(x) = x^2 - 2$, you have $f'(x) = 2x$, and the iterative process is given by the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{\,2} - 2}{2x_n}.$$

The calculations for three iterations are shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	1.000000	−1.000000	2.000000	−0.500000	1.500000
2	1.500000	0.250000	3.000000	0.083333	1.416667
3	1.416667	0.006945	2.833334	0.002451	1.414216
4	1.414216				



first iteration of Newton's Method
ure 3.61

Of course, in this case you know that the two zeros of the function are $\pm\sqrt{2}$. To six decimal places, $\sqrt{2} = 1.414214$. So, after only three iterations of Newton's Method, you have obtained an approximation that is within 0.000002 of an actual root. The first iteration of this process is shown in Figure 3.61.

Example 2 Using Newton's Method

Use Newton's Method to approximate the zeros of

$$f(x) = 2x^3 + x^2 - x + 1.$$

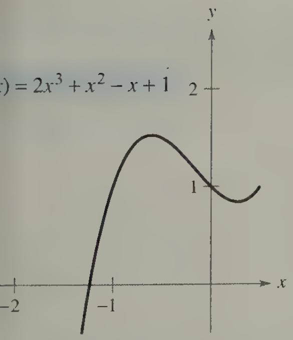
Continue the iterations until two successive approximations differ by less than 0.0001.

Solution Begin by sketching a graph of f, as shown in Figure 3.62. From the graph, you can observe that the function has only one zero, which occurs near $x = -1.2$. Next, differentiate f and form the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n^{\,3} + x_n^{\,2} - x_n + 1}{6x_n^{\,2} + 2x_n - 1}.$$

The calculations are shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	−1.20000	0.18400	5.24000	0.03511	−1.23511
2	−1.23511	−0.00771	5.68276	−0.00136	−1.23375
3	−1.23375	0.00001	5.66533	0.00000	−1.23375
4	−1.23375				

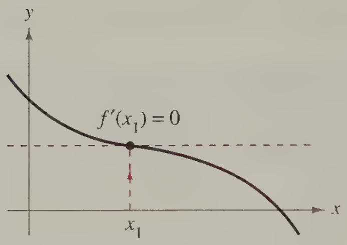


er three iterations of Newton's Method,
zero of f is approximated to the desired
uracy.
ure 3.62

Because two successive approximations differ by less than the required 0.0001, you can estimate the zero of f to be -1.23375.

When, as in Examples 1 and 2, the approximations approach a limit, the seque $x_1, x_2, x_3, \ldots, x_n, \ldots$ is said to **converge.** Moreover, if the limit is c, it can shown that c must be a zero of f.

Newton's Method does not always yield a convergent sequence. One way this happen is shown in Figure 3.63. Because Newton's Method involves division by $f'($ it is clear that the method will fail if the derivative is zero for any x_n in the sequen When you encounter this problem, you can usually overcome it by choosing different value for x_1. Another way Newton's Method can fail is illustrated in the n example.



Newton's Method fails to converge if $f'(x_n) = 0$.
Figure 3.63

Example 3 An Example in Which Newton's Method Fails

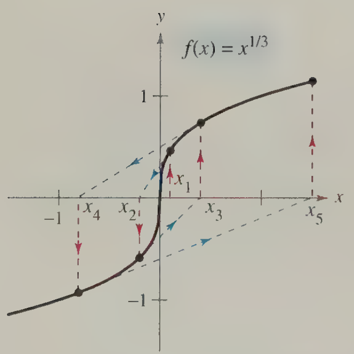
Using $x_1 = 0.1$, show that Newton's Method fails to converge for $f(x) = x^{1/3}$.

Solution Because $f'(x) = \frac{1}{3}x^{-2/3}$, the iterative formula is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} \\ &= x_n - 3x_n \\ &= -2x_n. \end{aligned}$$

The calculations are shown in the table. This table and Figure 3.64 indicate tha continues to increase in magnitude as $n \to \infty$, and thus the limit of the sequence d not exist.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.10000	0.46416	1.54720	0.30000	−0.20000
2	−0.20000	−0.58480	0.97467	−0.60000	0.40000
3	0.40000	0.73681	0.61401	1.20000	−0.80000
4	−0.80000	−0.92832	0.38680	−2.40000	1.60000



Newton's Method fails to converge for every x-value other than the actual zero of f.
Figure 3.64

NOTE In Example 3, the initial estimate $x_1 = 0.1$ fails to produce a convergent seque Try showing that Newton's Method also fails for every other choice of x_1 (other than actual zero).

The Granger Collection

NIELS HENRIK ABEL (1802–1829)

The Granger Collection

EVARISTE GALOIS (1811–1832)

lthough the lives of both Abel and Galois ere brief, their work in the fields of analysis nd abstract algebra was far-reaching.

It can be shown that a condition sufficient to produce convergence of Newton's Method to a zero of f is that

$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$$

Condition for convergence

on an open interval containing the zero. For instance, in Example 1 this test would yield $f(x) = x^2 - 2$, $f'(x) = 2x$, $f''(x) = 2$, and

$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| = \left|\frac{(x^2-2)(2)}{4x^2}\right| = \left|\frac{1}{2} - \frac{1}{x^2}\right|.$$

Example 1

On the interval $(1, 3)$, this quantity is less than 1 and therefore the convergence of Newton's Method is guaranteed. On the other hand, in Example 3, you have $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3}x^{-2/3}$, $f''(x) = -\frac{2}{9}x^{-5/3}$, and

$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| = \left|\frac{x^{1/3}(-2/9)(x^{-5/3})}{(1/9)(x^{-4/3})}\right| = 2$$

Example 3

which is not less than 1 for any value of x, so you cannot conclude that Newton's Method will converge.

Algebraic Solutions of Polynomial Equations

The zeros of some functions, such as

$$f(x) = x^3 - 2x^2 - x + 2$$

can be found by simple algebraic techniques, such as factoring. The zeros of other functions, such as

$$f(x) = x^3 - x + 1$$

cannot be found by *elementary* algebraic methods. This particular function has only one real zero, and by using more advanced algebraic techniques you can determine the zero to be

$$x = -\sqrt[3]{\frac{3 - \sqrt{23/3}}{6}} - \sqrt[3]{\frac{3 + \sqrt{23/3}}{6}}.$$

Because the *exact* solution is written in terms of square roots and cube roots, it is called a **solution by radicals.**

NOTE Try approximating the real zero of $f(x) = x^3 - x + 1$ and compare your result with the exact solution shown above.

The determination of radical solutions of a polynomial equation is one of the fundamental problems of algebra. The earliest such result is the Quadratic Formula, which dates back at least to Babylonian times. The general formula for the zeros of a cubic function was developed much later. In the sixteenth century an Italian mathematician, Jerome Cardan, published a method for finding radical solutions to cubic and quartic equations. Then, for 300 years, the problem of finding a general quintic formula remained open. Finally, in the nineteenth century, the problem was answered independently by two young mathematicians. Niels Henrik Abel, a Norwegian mathematician, and Evariste Galois, a French mathematician, proved that it is not possible to solve a *general* fifth- (or higher-) degree polynomial equation by radicals. Of course, you can solve particular fifth-degree equations such as $x^5 - 1 = 0$, but Abel and Galois were able to show that no general *radical* solution exists.

EXERCISES FOR SECTION 3.8

In Exercises 1–4, complete two iterations of Newton's Method for the function using the indicated initial guess.

1. $f(x) = x^2 - 3, \quad x_1 = 1.7$

2. $f(x) = 2x^2 - 3, \quad x_1 = 1$

3. $f(x) = \sin x, \quad x_1 = 3$

4. $f(x) = \tan x, \quad x_1 = 0.1$

In Exercises 5–14, approximate the zero(s) of the function. Use Newton's Method and continue the process until two successive approximations differ by less than 0.001. Then find the zero(s) using a graphing utility and compare the results.

5. $f(x) = x^3 + x - 1$

6. $f(x) = x^5 + x - 1$

7. $f(x) = 3\sqrt{x - 1} - x$

8. $f(x) = x - 2\sqrt{x + 1}$

9. $f(x) = x^3 + 3$

10. $f(x) = 1 - 2x^3$

11. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

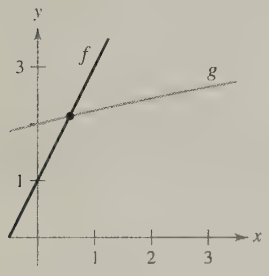
12. $f(x) = \frac{1}{2}x^4 - 3x - 3$

13. $f(x) = x + \sin(x + 1)$

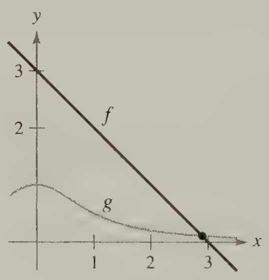
14. $f(x) = x^3 - \cos x$

In Exercises 15–18, apply Newton's Method to approximate the *x*-value of the indicated point(s) of intersection of the two graphs. Continue the process until two successive approximations differ by less than 0.001. [*Hint:* Let $h(x) = f(x) - g(x)$.]

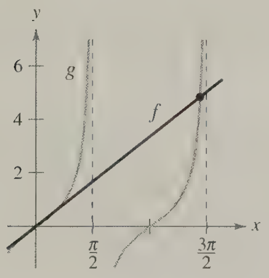
15. $f(x) = 2x + 1$
$g(x) = \sqrt{x + 4}$



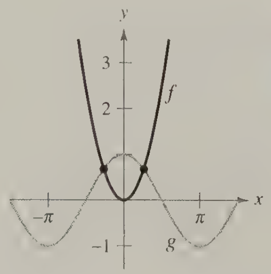
16. $f(x) = 3 - x$
$g(x) = 1/(x^2 + 1)$



17. $f(x) = x$
$g(x) = \tan x$



18. $f(x) = x^2$
$g(x) = \cos x$



In Exercises 19 and 20, use Newton's Method to obtain a general rule for approximating the required radical.

19. $x = \sqrt{a}$ [*Hint:* Consider $f(x) = x^2 - a$.]

20. $x = \sqrt[n]{a}$ [*Hint:* Consider $f(x) = x^n - a$.]

In Exercises 21–24, use the results of Exercises 19 and 20 approximate the indicated radical to three decimal places.

21. $\sqrt{7}$

22. $\sqrt{5}$

23. $\sqrt[4]{6}$

24. $\sqrt[3]{15}$

In Exercises 25 and 26, approximate π to three decimal plac using Newton's Method and the given function.

25. $f(x) = 1 + \cos x$

26. $f(x) = \tan x$

In Exercises 27–30, apply Newton's Method using the indicat initial guess, and explain why the method fails.

27. $y = 2x^3 - 6x^2 + 6x - 1, \quad x_1 = 1$

28. $y = 4x^3 - 12x^2 + 12x - 3, \quad x_1 = \frac{3}{2}$

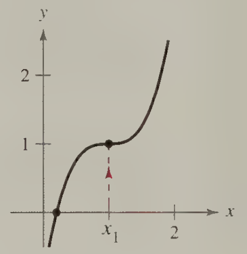


Figure for 27

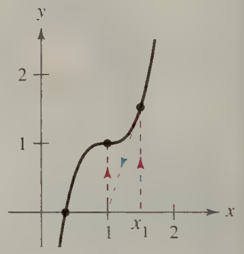


Figure for 28

29. $f(x) = -x^3 + 6x^2 - 10x + 6, \quad x_1 = 2$

30. $f(x) = 2\sin x + \cos 2x, \quad x_1 = \dfrac{3\pi}{2}$

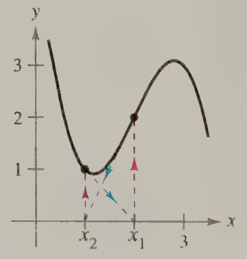


Figure for 29

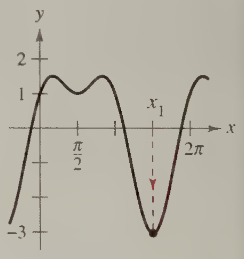


Figure for 30

Getting at the Concept

31. In your own words and using a sketch, describe Newton' Method for approximating the zeros of a function.

32. Under what conditions will Newton's Method fail?

Fixed Point **In Exercises 33 and 34, approximate the fix point of the function to two decimal places. [A *fixed point* x_0 a function *f* is a value of *x* such that $f(x_0) = x_0$.]**

33. $f(x) = \cos x$

34. $f(x) = \cot x, \quad 0 < x <$

. *Writing* Consider the function $f(x) = x^3 - 3x^2 + 3$.

(a) Use a graphing utility to obtain the graph of f.

(b) Use Newton's Method with $x_1 = 1$ as an initial guess.

(c) Repeat part (b) using $x_1 = \frac{1}{4}$ as an initial guess and observe that the result is different.

(d) To understand why the results in parts (b) and (c) are different, sketch the tangent lines to the graph of f at the points $(1, f(1))$ and $\left(\frac{1}{4}, f\left(\frac{1}{4}\right)\right)$. Find the x-intercept of each tangent line and compare the intercepts with the first iteration of Newton's Method using the respective initial guesses.

(e) Write a short paragraph summarizing how Newton's Method works. Use the results of this exercise to describe why it is important to select the initial guess carefully.

. *Writing* Repeat the steps in Exercise 35 for the function $f(x) = \sin x$ with initial guesses of $x_1 = 1.8$ and $x_1 = 3$.

. Use Newton's Method to show that the equation

$$x_{n+1} = x_n(2 - ax_n)$$

can be used to approximate $1/a$ if x_1 is an initial guess of the reciprocal of a. Note that this method of approximating reciprocals uses only the operations of multiplication and subtraction. [*Hint:* Consider $f(x) = (1/x) - a$.]

. Use the result of Exercise 37 to approximate the indicated reciprocal to three decimal places.

(a) $\frac{1}{3}$ (b) $\frac{1}{11}$

Exercises 39 and 40, approximate the critical number of f on e interval $(0, \pi)$. Sketch the graph of f, labeling any extrema.

. $f(x) = x \cos x$

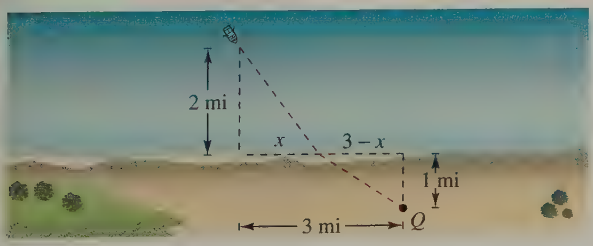
40. $f(x) = x \sin x$

Exercises 41–44, we review some typical problems from the evious sections of this chapter. In each case, use Newton's ethod to approximate the solution.

. *Minimum Distance* Find the point on the graph of $f(x) = 4 - x^2$ that is closest to the point $(1, 0)$.

. *Minimum Distance* Find the point on the graph of $f(x) = x^2$ that is closest to the point $(4, -3)$.

. *Minimum Time* You are in a boat 2 miles from the nearest point on the coast (see figure). You are to go to a point Q, which is 3 miles down the coast and 1 mile inland. You can row at 3 miles per hour and walk at 4 miles per hour. Toward what point on the coast should you row in order to reach Q in the least time?



44. *Medicine* The concentration C of a certain chemical in the bloodstream t hours after injection into muscle tissue is given by $C = (3t^2 + t)/(50 + t^3)$. When is the concentration greatest?

45. *Advertising Costs* A company that produces portable cassette players estimates that the profit for selling a particular model is

$$P = -76x^3 + 4830x^2 - 320{,}000, \quad 0 \le x \le 60$$

where P is the profit in dollars and x is the advertising expense in 10,000s of dollars (see figure). According to this model, find the smaller of two advertising amounts that yield a profit P of \$2,500,000.

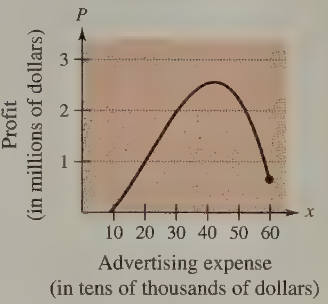


Figure for 45

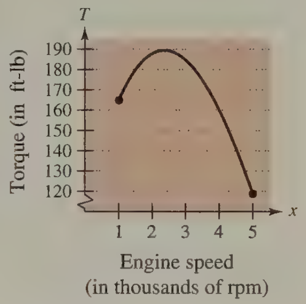


Figure for 46

46. *Engine Power* The torque produced by a compact automobile engine is approximated by the model

$$T = 0.808x^3 - 17.974x^2 + 71.248x + 110.843, \quad 1 \le x \le 5$$

where T is the torque in foot-pounds and x is the engine speed in thousands of revolutions per minute (see figure). Approximate the two engine speeds that yield a torque T of 170 foot-pounds.

True or False? **In Exercises 47–50, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

47. The zeros of $f(x) = p(x)/q(x)$ coincide with the zeros of $p(x)$.

48. If the coefficients of a polynomial function are all positive, then the polynomial has no positive zeros.

49. If $f(x)$ is a cubic polynomial such that $f'(x)$ is never zero, then any initial guess will force Newton's Method to converge to the zero of f.

50. The roots of $\sqrt{f(x)} = 0$ coincide with the roots of $f(x) = 0$.

In Exercises 51 and 52, *write a computer program or use a spreadsheet* to find the zeros of a function using Newton's Method. Approximate the zeros of the function accurate to three decimal places. The output should be a table with the following headings.

$$n, \quad x_n, \quad f(x_n), \quad f'(x_n), \quad \frac{f(x_n)}{f'(x_n)}, \quad x_n - \frac{f(x_n)}{f'(x_n)}$$

51. $f(x) = \frac{1}{4}x^3 - 3x^2 + \frac{3}{4}x - 2$

52. $f(x) = \sqrt{4 - x^2}\,\sin(x - 2)$

Section 3.9 Differentials

- Understand the concept of a tangent line approximation.
- Compare the value of the differential, dy, with the actual change in y, Δy.
- Estimate a propagated error using a differential.
- Find the differential of a function using differentiation formulas.

EXPLORATION

Tangent Line Approximation Use a graphing utility to sketch the graph of

$$f(x) = x^2.$$

In the same viewing window, sketch the graph of the tangent line to the graph of f at the point $(1, 1)$. Zoom in twice on the point of tangency. Does your graphing utility distinguish between the two graphs? Use the *trace* feature to compare the two graphs. As the x-values get closer to 1, what can you say about the y-values?

Linear Approximations

Newton's Method (Section 3.8) is an example of the use of a tangent line to a gra to approximate the graph. In this section, you will study other situations in which t graph of a function can be approximated by a straight line.

To begin, consider a function f that is differentiable at c. The equation for t tangent line at the point $(c, f(c))$ is given by

$$y - f(c) = f'(c)(x - c)$$
$$y = f(c) + f'(c)(x - c).$$

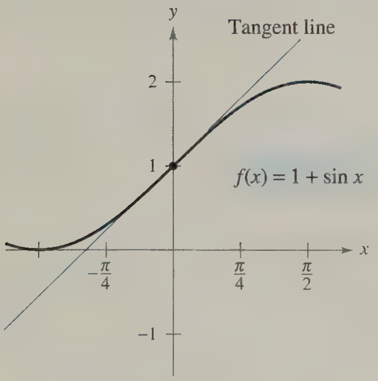
Because c is a constant, y is a linear function of x. Moreover, by restricting the valu of x to be sufficiently close to c, the values of y can be used as approximations (to a desired accuracy) of the values of the function f. In other words, as $x \to c$, the limit y is $f(c)$.

Example 1 Using a Tangent Line Approximation

Find the tangent line approximation of

$$f(x) = 1 + \sin x$$

at the point $(0, 1)$. Then use a table to compare the y-values of the linear function w those of $f(x)$ in an open interval containing $x = 0$.



The tangent line approximation of f at the point $(0, 1)$
Figure 3.65

Solution The derivative of f is

$$f'(x) = \cos x. \qquad \text{First derivative}$$

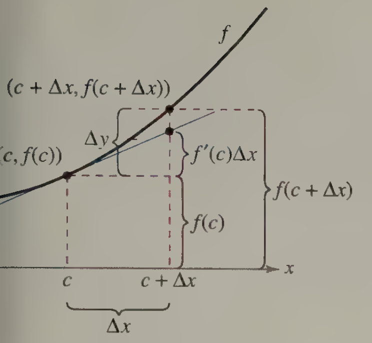
So, the equation of the tangent line to the graph of f at the point $(0, 1)$ is

$$y - f(0) = f'(0)(x - 0)$$
$$y - 1 = (1)(x - 0)$$
$$y = 1 + x. \qquad \text{Tangent line approximation}$$

The table compares the values of y given by this linear approximation with the valu of $f(x)$ near $x = 0$. Notice that the closer x is to 0, the better the approximation is. T conclusion is reinforced by the graph shown in Figure 3.65.

x	-0.5	-0.1	-0.01	0	0.01	0.1	0.5
$f(x) = 1 + \sin x$	0.521	0.9002	0.9900002	1	1.0099998	1.0998	1.479
$y = 1 + x$	0.5	0.9	0.99	1	1.01	1.1	1.5

NOTE Be sure you see that this linear approximation of $f(x) = 1 + \sin x$ depends on point of tangency. At a different point on the graph of f, you would obtain a different tang line approximation.



en Δx is small, $\Delta y = f(c + \Delta x) - f(c)$
pproximated by $f'(c)\Delta x$.
ure 3.66

Differentials

When the tangent line to the graph of f at the point $(c, f(c))$

$$y = f(c) + f'(c)(x - c)$$ Tangent line at $(c, f(c))$

is used as an approximation to the graph of f, the quantity $x - c$ is called the change in x, and is denoted by Δx, as shown in Figure 3.66. When Δx is small, the change in y (denoted by Δy) can be approximated as follows.

$$\Delta y = f(c + \Delta x) - f(c)$$ Actual change in y

$$\approx f'(c)\Delta x$$ Approximate change in y

For such an approximation, the quantity Δx is traditionally denoted by dx, and is called the **differential of x.** The expression $f'(x)\,dx$ is denoted by dy, and is called the **differential of y.**

Definition of Differentials

Let $y = f(x)$ represent a function that is differentiable in an open interval containing x. The **differential of x** (denoted by dx) is any nonzero real number. The **differential of y** (denoted by dy) is

$$dy = f'(x)\,dx.$$

In many types of applications, the differential of y can be used as an approximation of the change in y. That is,

$$\Delta y \approx dy \qquad \text{or} \qquad \Delta y \approx f'(x)dx.$$

Example 2 Comparing Δy and dy

Let $y = x^2$. Find dy when $x = 1$ and $dx = 0.01$. Compare this value with Δy for $x = 1$ and $\Delta x = 0.01$.

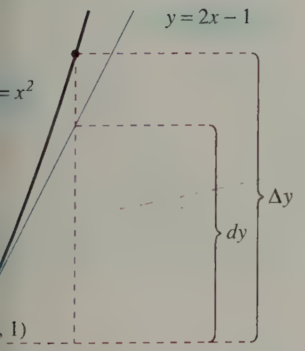
Solution Because $y = f(x) = x^2$, you have $f'(x) = 2x$, and the differential dy is given by

$$dy = f'(x)\,dx = f'(1)(0.01) = 2(0.01) = 0.02.$$ Differential of y

Now, using $\Delta x = 0.01$, the change in y is

$$\Delta y = f(x + \Delta x) - f(x) = f(1.01) - f(1) = (1.01)^2 - 1^2 = 0.0201.$$

Figure 3.67 shows the geometric comparison of dy and Δy. Try comparing other values of dy and Δy. You will see that the values become closer to each other as dx (or Δx) approaches 0.



change in y, Δy, is approximated by the
rential of y, dy.
ure 3.67

In Example 2, the tangent line to the graph of $f(x) = x^2$ at $x = 1$ is

$$y = 2x - 1 \qquad \text{or} \qquad g(x) = 2x - 1.$$ Tangent line to the graph of f at $x = 1$.

For x-values near 1, this line is close to the graph of f, as shown in Figure 3.67. For instance,

$$f(1.01) = 1.01^2 = 1.0201 \qquad \text{and} \qquad g(1.01) = 2(1.01) - 1 = 1.02.$$

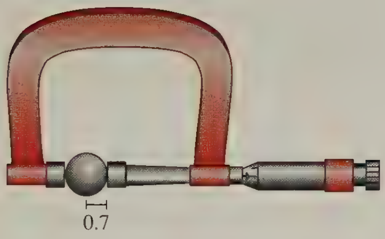
We say that the line $y = 2x - 1$ is the **linear approximation** or **tangent line approximation** to the graph of $f(x) = x^2$ at $x = 1$.

Error Propagation

Physicists and engineers tend to make liberal use of the approximation of Δy by One way this occurs in practice is in the estimation of errors propagated by physi measuring devices. For example, if you let x represent the measured value of a v able and let $x + \Delta x$ represent the exact value, then Δx is the *error in measurem* Finally, if the measured value x is used to compute another value $f(x)$, the differe between $f(x + \Delta x)$ and $f(x)$ is the **propagated error.**

$$\underbrace{f(x + \overbrace{\Delta x}^{\text{Measurement error}})}_{\text{Exact value}} - \underbrace{f(x)}_{\text{Measured value}} = \overbrace{\Delta y}^{\text{Propagated error}}$$

Example 3 Estimation of Error



Ball bearing with measured radius that is correct to within 0.01 inch
Figure 3.68

The radius of a ball bearing is measured to be 0.7 inch, as shown in Figure 3.68. If measurement is correct to within 0.01 inch, estimate the propagated error in volume V of the ball bearing.

Solution The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$, where r is the rad of the sphere. So, you can write

$$r = 0.7 \qquad \text{Measured radius}$$

and

$$-0.01 \le \Delta r \le 0.01. \qquad \text{Possible error}$$

To approximate the propagated error in the volume, differentiate V to ob $dV/dr = 4\pi r^2$ and write

$$\begin{aligned} \Delta V &\approx dV && \text{Approximate } \Delta V \text{ by } dV. \\ &= 4\pi r^2\,dr \\ &= 4\pi(0.7)^2(\pm 0.01) && \text{Substitute for } r \text{ and } dr. \\ &\approx \pm 0.06158 \text{ in}^3. \end{aligned}$$

So the volume has a propagated error of about 0.06 cubic inch.

Would you say that the propagated error in Example 3 is large or small? answer is best given in *relative* terms by comparing dV with V. The ratio

$$\begin{aligned} \frac{dV}{V} &= \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3} && \text{Ratio of } dV \text{ to } V \\ &= \frac{3\,dr}{r} && \text{Simplify.} \\ &\approx \frac{3}{0.7}(\pm 0.01) && \text{Substitute for } dr \text{ and } r. \\ &\approx \pm 0.0429 \end{aligned}$$

is called the **relative error.** The corresponding **percent error** is approxima 4.29%.

Calculating Differentials

Each of the differentiation rules that you studied in Chapter 2 can be written in **differential form.** For example, suppose u and v are differentiable functions of x. By the definition of differentials, you have

$$du = u'\,dx \quad \text{and} \quad dv = v'\,dx.$$

Therefore, you can write the differential form of the Product Rule as follows.

$$\begin{aligned} d[uv] &= \frac{d}{dx}[uv]\,dx && \text{Differential of } uv \\ &= [uv' + vu']\,dx && \text{Product Rule} \\ &= uv'\,dx + vu'\,dx \\ &= u\,dv + v\,du \end{aligned}$$

Differential Formulas

Let u and v be differentiable functions of x.

Constant multiple: $d[cu] = c\,du$

Sum or difference: $d[u \pm v] = du \pm dv$

Product: $d[uv] = u\,dv + v\,du$

Quotient: $d\left[\dfrac{u}{v}\right] = \dfrac{v\,du - u\,dv}{v^2}$

Example 4 Finding Differentials

	Function	*Derivative*	*Differential*
a.	$y = x^2$	$\dfrac{dy}{dx} = 2x$	$dy = 2x\,dx$
b.	$y = 2\sin x$	$\dfrac{dy}{dx} = 2\cos x$	$dy = 2\cos x\,dx$
c.	$y = x\cos x$	$\dfrac{dy}{dx} = -x\sin x + \cos x$	$dy = (-x\sin x + \cos x)\,dx$
d.	$y = \dfrac{1}{x}$	$\dfrac{dy}{dx} = -\dfrac{1}{x^2}$	$dy = -\dfrac{dx}{x^2}$

Mary Evans Picture Library

GOTTFRIED WILHELM LEIBNIZ (1646–1716)

oth Leibniz and Newton are credited with reating calculus. It was Leibniz, however, ho tried to broaden calculus by developing ules and formal notation. He often spent days hoosing an appropriate notation for a new oncept.

The notation in Example 4 is called the **Leibniz notation** for derivatives and differentials, named after the German mathematician Gottfried Wilhelm Leibniz. The beauty of this notation is that it provides an easy way to remember several important calculus formulas by making it seem as though the formulas were derived from algebraic manipulations of differentials. For instance, in Leibniz notation, the *Chain Rule*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

would appear to be true because the du's cancel. Even though this reasoning is *incorrect*, the notation does help one remember the Chain Rule.

Example 5 Finding the Differential of a Composite Function

$$y = f(x) = \sin 3x \qquad \text{Original function}$$

$$f'(x) = 3\cos 3x \qquad \text{Apply Chain Rule.}$$

$$dy = f'(x)\,dx = 3\cos 3x\,dx \qquad \text{Differential form}$$

Example 6 Finding the Differential of a Composite Function

$$y = f(x) = (x^2 + 1)^{1/2} \qquad \text{Original function}$$

$$f'(x) = \frac{1}{2}(x^2 + 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + 1}} \qquad \text{Apply Chain Rule.}$$

$$dy = f'(x)\,dx = \frac{x}{\sqrt{x^2 + 1}}\,dx \qquad \text{Differential form}$$

Differentials can be used to approximate function values. To do this for the fun tion given by $y = f(x)$, you use the formula

$$f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)\,dx$$

which is derived from the approximation $\Delta y = f(x + \Delta x) - f(x) \approx dy$. The key using this formula is to choose a value for x that makes the calculations easier, shown in Example 7.

Example 7 Approximating Function Values

Use differentials to approximate $\sqrt{16.5}$.

Solution Using $f(x) = \sqrt{x}$, you can write

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt{x} + \frac{1}{2\sqrt{x}}\,dx.$$

Now, choosing $x = 16$ and $dx = 0.5$, you obtain the following approximation.

$$f(x + \Delta x) = \sqrt{16.5} \approx \sqrt{16} + \frac{1}{2\sqrt{16}}(0.5) = 4 + \left(\frac{1}{8}\right)\left(\frac{1}{2}\right) = 4.0625$$

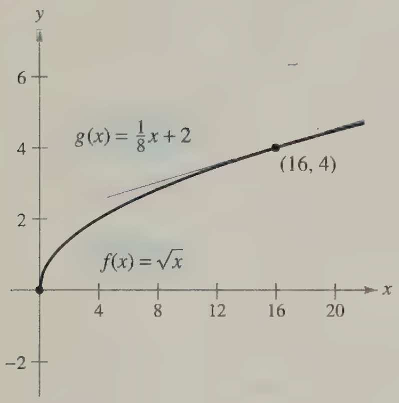


Figure 3.69

The tangent line approximation to $f(x) = \sqrt{x}$ at $x = 16$ is the li $g(x) = \frac{1}{8}x + 2$. For x-values near 16, the graphs of f and g are close together, shown in Figure 3.69. For instance,

$$f(16.5) = \sqrt{16.5} \approx 4.0620 \quad \text{and} \quad g(16.5) = \frac{1}{8}(16.5) + 2 = 4.0625.$$

In fact, if you use a graphing utility to zoom in near the point of tangency (16, 4), y will see that the two graphs appear to coincide. Notice also that as you move fart away from the point of tangency, the linear approximation is less accurate.

XERCISES FOR SECTION 3.9

xercises 1–6, find the equation of the tangent line T to the ph of f at the indicated point. Use this linear approximation omplete the table.

	1.9	1.99	2	2.01	2.1
x)					
x)					

unction	*Point*
$f(x) = x^2$	$(2, 4)$
$f(x) = \dfrac{6}{x^2}$	$\left(2, \dfrac{3}{2}\right)$
$f(x) = x^5$	$(2, 32)$
$f(x) = \sqrt{x}$	$(2, \sqrt{2})$
$f(x) = \sin x$	$(2, \sin 2)$
$f(x) = \csc x$	$(2, \csc 2)$

xercises 7–10, use the information to evaluate and compare and dy.

$y = \frac{1}{2}x^3$	$x = 2$	$\Delta x = dx = 0.1$
$y = 1 - 2x^2$	$x = 0$	$\Delta x = dx = -0.1$
$y = x^4 + 1$	$x = -1$	$\Delta x = dx = 0.01$
$y = 2x + 1$	$x = 2$	$\Delta x = dx = 0.01$

xercises 11–20, find the differential dy of the given function.

$y = 3x^2 - 4$ **12.** $y = 3x^{2/3}$

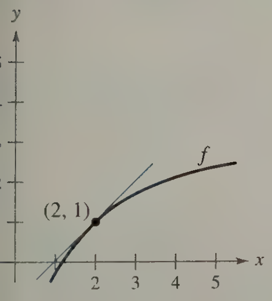
$y = \dfrac{x+1}{2x-1}$ **14.** $y = \sqrt{9 - x^2}$

$y = x\sqrt{1 - x^2}$ **16.** $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

$y = 2x - \cot^2 x$ **18.** $y = x \sin x$

$y = \dfrac{1}{3}\cos\left(\dfrac{6\pi x - 1}{2}\right)$ **20.** $y = \dfrac{\sec^2 x}{x^2 + 1}$

xercises 21–24, use differentials and the graph of f to roximate (a) $f(1.9)$ and (b) $f(2.04)$. To print an enlarged of the graph, go to the website *www.mathgraphs.com*.



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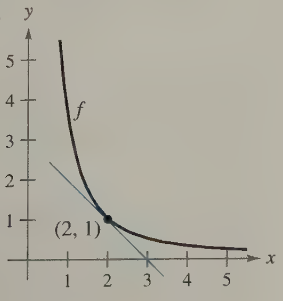

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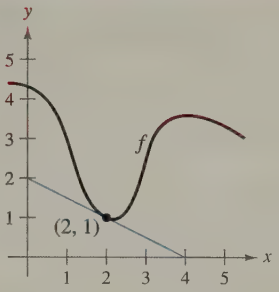

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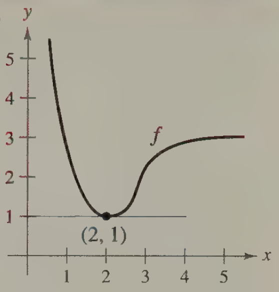
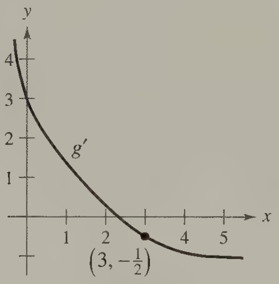


In Exercises 25–28, use differentials and the graph of g' to approximate (a) $g(2.93)$ and (b) $g(3.1)$ given that $g(3) = 8$.

25.



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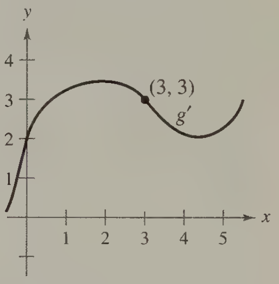

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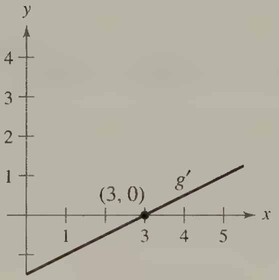

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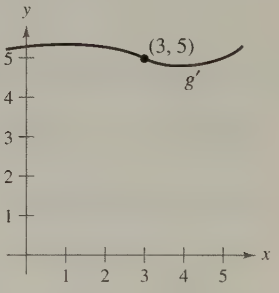


29. ***Area*** The measurement of the side of a square is found to be 12 inches, with a possible error of $\frac{1}{64}$ inch. Use differentials to approximate the possible propagated error in computing the area of the square.

30. ***Area*** The measurements of the base and altitude of a triangle are found to be 36 and 50 centimeters. The possible error in each measurement is 0.25 centimeter. Use differentials to approximate the possible propagated error in computing the area of the triangle.

31. ***Area*** The measurement of the radius of the end of a log is found to be 14 inches, with a possible error of $\frac{1}{4}$ inch. Use differentials to approximate the possible propagated error in computing the area of the end of the log.

32. ***Volume and Surface Area*** The measurement of the edge of a cube is found to be 12 inches, with a possible error of 0.03 inch. Use differentials to approximate the maximum possible propagated error in computing

(a) the volume of the cube.

(b) the surface area of the cube.

33. ***Area*** The measurement of a side of a square is found to be 15 centimeters. The possible error in measuring the side is 0.05 centimeter.

(a) Approximate the percent error in computing the area of the square.

(b) Estimate the maximum allowable percent error in measuring the side if the error in computing the area cannot exceed 2.5%.

34. ***Circumference*** The measurement of the circumference of a circle is found to be 56 centimeters. The possible error in measuring the circumference is 1.2 centimeters.

(a) Approximate the percent error in computing the area of the circle.

(b) Estimate the maximum allowable percent error in measuring the circumference if the error in computing the area cannot exceed 3%.

35. ***Volume and Surface Area*** The radius of a sphere is measured to be 6 inches, with a possible error of 0.02 inch. Use differentials to approximate the maximum possible error in calculating (a) the volume of the sphere, (b) the surface area of the sphere, and (c) the relative errors in parts (a) and (b).

36. ***Profit*** The profit P for a company is given by

$$P = (500x - x^2) - \left(\tfrac{1}{2}x^2 - 77x + 3000\right).$$

Approximate the change and percent change in profit as production changes from $x = 115$ to $x = 120$ units.

In Exercises 37 and 38, the thickness of the shell is 0.2 centimeter. Use differentials to approximate the volume of the shell.

37. A cylindrical shell with height 40 centimeters and radius 5 centimeters

38. A spherical shell of radius 100 centimeters

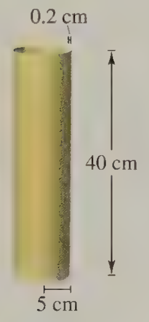


Figure for 37

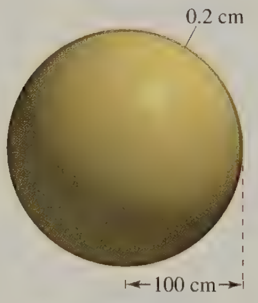


Figure for 38

39. ***Pendulum*** The period of a pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the pendulum in feet, g is the acceleration due to gravity, and T is the time in seconds. Suppose that the pendulum has been subjected to an increase in temperature such that the length has increased by $\frac{1}{2}\%$.

(a) Find the approximate percent change in the period.

(b) Using the result in part (a), find the approximate error in this pendulum clock in one day.

40. ***Ohm's Law*** A current of I amperes passes through a resi of R ohms. **Ohm's Law** states that the voltage E applied to resistor is

$$E = IR.$$

If the voltage is constant, show that the magnitude of the r tive error in R caused by a change in I is equal in magnitud the relative error in I.

41. ***Triangle Measurements*** The measurement of one side right triangle is found to be 9.5 inches, and the angle opp that side is 26°45′ with a possible error of 15′.

(a) Approximate the percent error in computing the lengt the hypotenuse.

(b) Estimate the maximum allowable percent error in mea ing the angle if the error in computing the length of hypotenuse cannot exceed 2%.

42. ***Area*** Approximate the percent error in computing the are the triangle in Exercise 41.

43. ***Projectile Motion*** The range R of a projectile is

$$R = \frac{v_0^2}{32}(\sin 2\theta)$$

where v_0 is the initial velocity in feet per second and θ is angle of elevation. If $v_0 = 2200$ feet per second and changed from 10° to 11°, use differentials to approximate change in the range.

44. ***Surveying*** A surveyor standing 50 feet from the base of a l tree measures the angle of elevation to the top of the tre 71.5°. How accurately must the angle be measured if the per error in estimating the height of the tree is to be less than 6

In Exercises 45–48, use differentials to approximate the valu the expression. Compare your answer with that of a calcul

45. $\sqrt{99.4}$ **46.** $\sqrt[3]{26}$ **47.** $\sqrt[4]{624}$ **48.** (2.

Writing **In Exercises 49 and 50, give a short explanatio why the approximation is valid.**

49. $\sqrt{4.02} \approx 2 + \frac{1}{4}(0.02)$ **50.** $\tan 0.05 \approx 0 + 1(0.05$

Getting at the Concept

51. Describe the change in accuracy of dy as an approximati for Δy when Δx is decreased.

52. When using differentials, what is meant by the terms pro agated error, relative error, and percent error?

True or False? **In Exercises 53–56, determine whether statement is true or false. If it is false, explain why or giv example that shows it is false.**

53. If $y = x + c$, then $dy = dx$.

54. If $y = ax + b$, then $\Delta y/\Delta x = dy/dx$.

55. If y is differentiable, then $\lim_{\Delta x \to 0} (\Delta y - dy) = 0$.

56. If $y = f(x)$, f is increasing and differentiable, and $\Delta x > 0$, $\Delta y \geq dy$.

EVIEW EXERCISES FOR CHAPTER 3

1

Give the definition of a critical number, and graph a function f showing the different types of critical numbers.

Consider the odd function f that is continuous, differentiable, and has the functional values shown in the table.

x	-5	-4	-1	0	2	3	6
$f(x)$	1	3	2	0	-1	-4	0

(a) Determine $f(4)$.

(b) Determine $f(-3)$.

(c) Plot the points and make a possible sketch of the graph of f on the interval $[-6, 6]$. What is the smallest number of critical points in the interval? Explain.

(d) Does there exist at least one real number c in the interval $(-6, 6)$ where $f'(c) = -1$? Explain.

(e) Is it possible that $\lim\limits_{x \to 0} f(x)$ does not exist? Explain.

(f) Is it necessary that $f'(x)$ exists at $x = 2$? Explain.

Exercises 3 and 4, find the absolute extrema of the function the closed interval. Use a graphing utility to graph the function over the indicated interval to confirm your results.

$g(x) = 2x + 5\cos x, \quad [0, 2\pi]$

4. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}, \quad [0, 2]$

2 **In Exercises 5 and 6, determine whether Rolle's eorem can be applied to f on the closed interval $[a, b]$. If le's Theorem can be applied, find all values of c in the open rval (a, b) such that $f'(c) = 0$.**

$f(x) = (x - 2)(x + 3)^2, \quad [-3, 2]$

$f(x) = |x - 2| - 2, \quad [0, 4]$

Consider the function $f(x) = 3 - |x - 4|$.

(a) Graph the function and verify that $f(1) = f(7)$.

(b) Note that $f'(x)$ is not equal to zero for any x in $[1, 7]$. Explain why this does not contradict Rolle's Theorem.

Can the Mean Value Theorem be applied to the function $f(x) = 1/x^2$ on the interval $[-2, 1]$? Explain.

Exercises 9–12, find the point(s) guaranteed by the Mean ne Theorem for the closed interval $[a, b]$.

$f(x) = x^{2/3}, \quad [1, 8]$

10. $f(x) = \dfrac{1}{x}, \quad [1, 4]$

$f(x) = x - \cos x, \quad \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

12. $f(x) = \sqrt{x} - 2x, \quad [0, 4]$

For the function $f(x) = Ax^2 + Bx + C$, determine the value of c guaranteed by the Mean Value Theorem on the interval $[x_1, x_2]$.

Demonstrate the result of Exercise 13 for $f(x) = 2x^2 - 3x + 1$ on the interval $[0, 4]$.

3.3 **In Exercises 15–18, find the critical numbers (if any) and the open intervals on which the function is increasing or decreasing.**

15. $f(x) = (x - 1)^2(x - 3)$

16. $g(x) = (x + 1)^3$

17. $h(x) = \sqrt{x}(x - 3), \quad x > 0$

18. $f(x) = \sin x + \cos x, \quad [0, 2\pi]$

In Exercises 19 and 20, use the First Derivative Test to find any relative extrema of the function. Use a graphing utility to verify your results.

19. $h(t) = \dfrac{1}{4}t^4 - 8t$

20. $g(x) = \dfrac{3}{2}\sin\left(\dfrac{\pi x}{2} - 1\right), \quad [0, 4]$

21. ***Harmonic Motion*** The height of an object attached to a spring is given by the harmonic equation

$$y = \tfrac{1}{3}\cos 12t - \tfrac{1}{4}\sin 12t$$

where y is measured in inches and t is measured in seconds.

(a) Calculate the height and velocity of the object when $t = \pi/8$ second.

(b) Show that the maximum displacement of the object is $\frac{5}{12}$ inch.

(c) Find the period P of y. Also, find the frequency f (number of oscillations per second) if $f = 1/P$.

22. ***Writing*** The general equation giving the height of an oscillating object attached to a spring is

$$y = A\sin\sqrt{\frac{k}{m}}\,t + B\cos\sqrt{\frac{k}{m}}\,t$$

where k is the spring constant and m is the mass of the object.

(a) Show that the maximum displacement of the object is $\sqrt{A^2 + B^2}$.

(b) Show that the object oscillates with a frequency of

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

3.4 **In Exercises 23 and 24, determine the points of inflection of the function.**

23. $f(x) = x + \cos x, \quad [0, 2\pi]$

24. $f(x) = (x + 2)^2(x - 4)$

In Exercises 25 and 26, use the Second Derivative Test to find all relative extrema.

25. $g(x) = 2x^2(1 - x^2)$

26. $h(t) = t - 4\sqrt{t + 1}$

Think About It **In Exercises 27 and 28, sketch the graph of a function f having the indicated characteristics.**

27. $f(0) = f(6) = 0$
$f'(3) = f'(5) = 0$
$f'(x) > 0$ if $x < 3$
$f'(x) > 0$ if $3 < x < 5$
$f'(x) < 0$ if $x > 5$
$f''(x) < 0$ if $x < 3$ and $x > 4$
$f''(x) > 0,\ 3 < x < 4$

28. $f(0) = 4,\ f(6) = 0$
$f'(x) < 0$ if $x < 2$ and $x > 4$
$f'(2)$ does not exist.
$f'(4) = 0$
$f'(x) > 0$ if $2 < x < 4$
$f''(x) < 0,\ x \neq 2$

29. ***Writing*** A newspaper headline states that "The rate of growth of the national deficit is decreasing." What does this mean? What does it imply about the graph of the deficit as a function of time?

30. ***Inventory Cost*** The cost of inventory depends on the ordering and storage costs according to the inventory model

$$C = \left(\frac{Q}{x}\right)s + \left(\frac{x}{2}\right)r.$$

Determine the order size that will minimize the cost, assuming that sales occur at a constant rate, Q is the number of units sold per year, r is the cost of storing one unit for 1 year, s is the cost of placing an order, and x is the number of units per order.

31. ***Modeling Data*** Outlays for national defense D (in billions of dollars) for selected years from 1970 through 1999 are shown in the table, where t is time in years, with $t = 0$ corresponding to 1970. *(Source: U.S. Office of Management and Budget)*

t	0	5	10	15	20	25	26	27	28	29
D	90.4	103.1	155.1	279.0	328.3	309.9	302.7	309.8	310.3	320.2

(a) Use the regression capabilities of a graphing utility to fit a model of the form $D = at^4 + bt^3 + ct^2 + dt + e$ to the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) For the years shown in the table, when does the model indicate that the outlay for national defense is at a maximum? When is it at a minimum?

(d) For the years shown in the table, when does the model indicate that the outlay for national defense is increasing at the greatest rate?

32. ***Modeling Data*** The manager of a store recorded the ann sales S (in thousands of dollars) of a product over a perio 7 years, as shown in the table, where t is the time in years, w $t = 1$ corresponding to 1991.

t	1	2	3	4	5	6	7
S	5.4	6.9	11.5	15.5	19.0	22.0	23.

(a) Use the regression capabilities of a graphing utility to a model of the form $S = at^3 + bt^2 + ct + d$ for the da

(b) Use a graphing utility to plot the data and graph the mo

(c) Use calculus to find the time t when sales were increas at the greatest rate.

(d) Do you think the model would be accurate for predic future sales? Explain.

3.5 **In Exercises 33–36, find the limit.**

33. $\displaystyle\lim_{x\to\infty} \frac{2x^2}{3x^2 + 5}$

34. $\displaystyle\lim_{x\to\infty} \frac{2x}{3x^2 + 5}$

35. $\displaystyle\lim_{x\to\infty} \frac{5\cos x}{x}$

36. $\displaystyle\lim_{x\to\infty} \frac{3x}{\sqrt{x^2 + 4}}$

In Exercises 37–40, find any vertical and horizontal asympt of the graph of the function. Use a graphing utility to ve your results.

37. $h(x) = \dfrac{2x + 3}{x - 4}$

38. $g(x) = \dfrac{5x^2}{x^2 + 2}$

39. $f(x) = \dfrac{3}{x} - 2$

40. $f(x) = \dfrac{3x}{\sqrt{x^2 + 2}}$

In Exercises 41–44, use a graphing utility to graph the functi Use the graph to approximate any relative extrema asymptotes.

41. $f(x) = x^3 + \dfrac{243}{x}$

42. $f(x) = |x^3 - 3x^2 + 2x|$

43. $f(x) = \dfrac{x - 1}{1 + 3x^2}$

44. $g(x) = \dfrac{\pi^2}{3} - 4\cos x + \cos$

3.6 **In Exercises 45–62, analyze and sketch the graph of function.**

45. $f(x) = 4x - x^2$

46. $f(x) = 4x^3 - x^4$

47. $f(x) = x\sqrt{16 - x^2}$

48. $f(x) = (x^2 - 4)^2$

49. $f(x) = (x - 1)^3(x - 3)^2$

50. $f(x) = (x - 3)(x + 2)^3$

51. $f(x) = x^{1/3}(x + 3)^{2/3}$

52. $f(x) = (x - 2)^{1/3}(x + 1)^{2}$

53. $f(x) = \dfrac{x + 1}{x - 1}$

54. $f(x) = \dfrac{2x}{1 + x^2}$

$f(x) = \dfrac{4}{1 + x^2}$

$f(x) = \dfrac{x^2}{1 + x^4}$

$f(x) = x^3 + x + \dfrac{4}{x}$

$f(x) = x^2 + \dfrac{1}{x}$

$f(x) = |x^2 - 9|$

$f(x) = |x - 1| + |x - 3|$

$f(x) = x + \cos x, \quad 0 \le x \le 2\pi$

$f(x) = \dfrac{1}{\pi}(2 \sin \pi x - \sin 2\pi x), \quad -1 \le x \le 1$

Find the maximum and minimum points on the graph of

$x^2 + 4y^2 - 2x - 16y + 13 = 0$

(a) without using calculus.

(b) using calculus.

Consider the function $f(x) = x^n$ for positive integer values of n.

(a) For what values of n does the function have a relative minimum at the origin?

(b) For what values of n does the function have a point of inflection at the origin?

.7

Minimum Distance At noon, ship A is 100 kilometers due east of ship B. Ship A is sailing west at 12 kilometers per hour, and ship B is sailing south at 10 kilometers per hour. At what time will the ships be nearest to each other, and what will this distance be?

Maximum Area Find the dimensions of the rectangle of maximum area, with sides parallel to the coordinate axes, that can be inscribed in the ellipse given by

$$\frac{x^2}{144} + \frac{y^2}{16} = 1.$$

Minimum Length A right triangle in the first quadrant has the coordinate axes as sides, and the hypotenuse passes through the point $(1, 8)$. Find the vertices of the triangle such that the length of the hypotenuse is minimum.

Minimum Length The wall of a building is to be braced by a beam that must pass over a parallel fence 5 feet high and 4 feet from the building. Find the length of the shortest beam that can be used.

Maximum Area Three sides of a trapezoid have the same length s. Of all such possible trapezoids, show that the one of maximum area has a fourth side of length $2s$.

Maximum Area Show that the greatest area of any rectangle inscribed in a triangle is one half that of the triangle.

Minimum Distance Find the length of the longest pipe that can be carried level around a right-angle corner at the intersection of two corridors of widths 4 feet and 6 feet. (Do not use trigonometry.)

72. ***Minimum Distance*** Rework Exercise 71, given corridors of widths a meters and b meters.

73. ***Minimum Distance*** A hallway of width 6 feet meets a hallway of width 9 feet at right angles. Find the length of the longest pipe that can be carried level around this corner. [*Hint:* If L is the length of the pipe, show that

$$L = 6 \csc \theta + 9 \csc\left(\frac{\pi}{2} - \theta\right)$$

where θ is the angle between the pipe and the wall of the narrower hallway.]

74. ***Minimum Distance*** Rework Exercise 73, given that one hallway is of width a meters and the other is of width b meters. Show that the result is the same as in Exercise 72.

Minimum Cost **In Exercises 75 and 76, find the speed v, in miles per hour, that will minimize costs on a 110-mile delivery trip. The cost per hour for fuel is C dollars, and the driver is paid W dollars per hour. (Assume there are no costs other than wages and fuel.)**

75. Fuel cost: $C = \dfrac{v^2}{600}$

Driver: $W = \$5$

76. Fuel cost: $C = \dfrac{v^2}{500}$

Driver: $W = \$7.50$

3.8 **In Exercises 77 and 78, use Newton's Method to approximate any real zeros of the function accurate to three decimal places. Use the root-finding capabilities of a graphing utility to verify your results.**

77. $f(x) = x^3 - 3x - 1$

78. $f(x) = x^3 + 2x + 1$

In Exercises 79 and 80, use Newton's Method to approximate, to three decimal places, the x-value of the points of intersection of the equations. Use a graphing utility to verify your results.

79. $y = x^4$

$y = x + 3$

80. $y = \sin \pi x$

$y = 1 - x$

3.9 **In Exercises 81 and 82, find the differential dy.**

81. $y = x(1 - \cos x)$

82. $y = \sqrt{36 - x^2}$

83. ***Surface Area and Volume*** The diameter of a sphere is measured to be 18 centimeters, with a maximum possible error of 0.05 centimeter. Use differentials to approximate the possible propagated error and percent error in calculating the surface area and the volume of the sphere.

84. ***Demand Function*** A company finds that the demand for its commodity is $p = 75 - \frac{1}{4}x$. If x changes from 7 to 8, find and compare the values of Δp and dp.

P.S. Problem Solving

1. Prove Darboux's Theorem: Let f be differentiable on the closed interval $[a, b]$ such that $f'(a) = y_1$ and $f'(b) = y_2$. If d lies between y_1 and y_2, then there exists c in (a, b) such that $f'(c) = d$.

2. (a) Let $V = x^3$. Find dV and ΔV. Show that for small values of x, the difference $\Delta V - dV$ is very small in the sense that there exists ε such that $\Delta V - dV = \varepsilon\Delta x$, where $\varepsilon \to 0$ as $\Delta x \to 0$.

(b) Generalize this result by showing that if $y = f(x)$ is a differentiable function, then $\Delta y - dy = \varepsilon\Delta x$, where $\varepsilon \to 0$ as $\Delta x \to 0$.

3. (a) Graph the fourth-degree polynomial $p(x) = ax^4 - 6x^2$ for $a = -3, -2, -1, 0, 1, 2,$ and 3. For what values of the constant a does p have a relative minimum or relative maximum?

(b) Show that p has a relative maximum for all values of the constant a.

(c) Determine analytically the values of a for which p has a relative minimum.

(d) Let $(x, y) = (x, p(x))$ be a relative extremum of p. Show that (x, y) lies on the graph of $y = -3x^2$. Verify this result graphically by graphing $y = -3x^2$ together with the seven curves from part (a).

4. Let f and g be continuous functions on $[a, b]$ and differentiable on (a, b). Prove that if $f(a) = g(a)$ and $g'(x) > f'(x)$ for all x in (a, b), then $g(b) > f(b)$.

5. Graph the fourth-degree polynomial $p(x) = x^4 + ax^2 + 1$ for various values of the constant a.

(a) Determine the values of a for which p has exactly one relative minimum.

(b) Determine the values of a for which p has exactly one relative maximum.

(c) Determine the values of a for which p has exactly two relative minima.

(d) Show that the graph of p cannot have exactly two relative extrema.

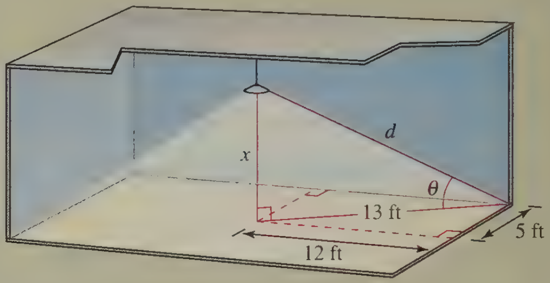
6. (a) Let $f(x) = ax^2 + bx + c$, $a \neq 0$ be a quadratic polynomial. How many points of inflection does the graph of f have?

(b) Let $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$ be a cubic polynomial. How many points of inflection does the graph of f have?

(c) Suppose the function $y = f(x)$ satisfies the equation $\dfrac{dy}{dx} = ky(L - y)$, where k and L are positive constants. Show that the graph of f has a point of inflection at the point where $y = \dfrac{L}{2}$. (This equation is called the **logistics differential equation.**)

7. Let $f(x) = \dfrac{c}{x} + x^2$. Determine all values of the constant c such that f has a relative minimum, but no relative maximum.

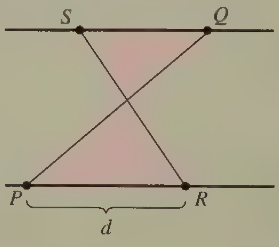
8. The amount of illumination of a surface is proportional to intensity of the light source, inversely proportional to square of the distance from the light source, and proportiona sin θ, where θ is the angle at which the light strikes the surfa A rectangular room measures 10 feet by 24 feet, with a 10-f ceiling. Determine the height at which the light should placed to allow the corners of the floor to receive as much li as possible.



9. Prove the following **Extended Mean Value Theorem.** If f a f' are continuous on the closed interval $[a, b]$, and if f'' exists the open interval (a, b), then there exists a number c in (a such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{1}{2}f''(c)(b - a)^2.$$

10. The line joining P and Q crosses the two parallel lines, shown in the figure. The point R is d units from P. How from Q should the point S be chosen so that the sum of areas of the two shaded triangles is a minimum? So that sum is a maximum?



11. The efficiency E of a screw with square threads is

$$E = \frac{\tan\phi(1 - \mu\tan\phi)}{\mu + \tan\phi}$$

where μ is the coefficient of sliding friction and ϕ is the an of inclination of the threads to a plane perpendicular to the a of the screw. Find the angle ϕ that yields maximum efficie when $\mu = 0.1$.

(a) Prove that $\lim_{x\to\infty} x^2 = \infty$.

(b) Prove that $\lim_{x\to\infty}\left(\frac{1}{x^2}\right) = 0$.

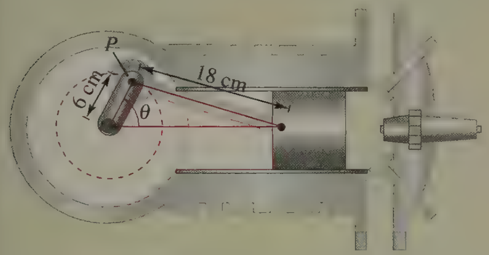
(c) Let L be a real number. Prove that if $\lim_{x\to\infty} f(x) = L$, then

$$\lim_{y\to 0^+} f\left(\frac{1}{y}\right) = L.$$

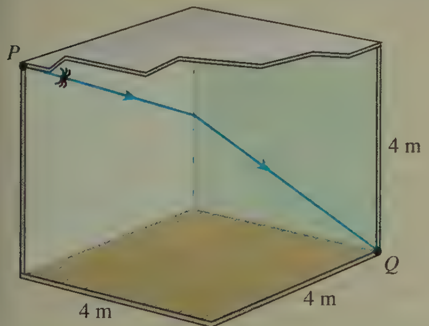
In the engine shown in the figure, a connecting rod 18 centimeters long is fastened to a crank of radius 6 centimeters at point P. The crankshaft rotates counterclockwise at a constant rate of 200 revolutions per minute. The horizontal velocity (cm/min) of point P is

$$v = -2400\pi \sin \theta$$

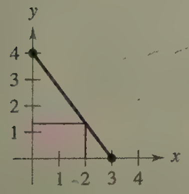
where θ is the central angle of the crankshaft. What values of θ produce a maximum horizontal velocity?

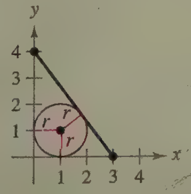


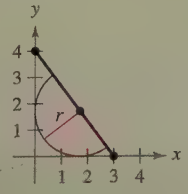
Consider a room in the shape of a cube, 4 meters on each side. A bug at point P wants to walk to point Q at the opposite corner, as indicated in the figure. Use calculus to determine the shortest path. Can you solve the problem without calculus?



The figures show a rectangle, a circle, and a semicircle inscribed in a triangle bounded by the coordinate axes and the first quadrant portion of the line with intercepts (3, 0) and (0, 4). Find the dimensions of each inscribed figure such that its area is maximum. State whether calculus was helpful in finding the required dimensions. Explain your reasoning.







16. The police department must determine the speed limit on a bridge such that the flow rate of cars is maximum per unit time. The greater the speed limit, the farther apart the cars must be in order to keep a safe stopping distance. Experimental data on the stopping distance d (in meters) for various velocities v (in kilometers per hour) are shown in the table.

v	20	40	60	80	100
d	5.1	13.7	27.2	44.2	66.4

(a) Convert the speeds v in the table to the speeds s in meters per second. Use the regression capabilities of a graphing utility to find a model of the form $d(s) = as^2 + bs + c$ for the data.

(b) Consider two consecutive vehicles of average length 5.5 meters, traveling at a safe speed on the bridge. Let T be the difference between the times (in seconds) when the front bumpers of the vehicles pass a given point on the bridge. Verify that this difference in times is given by

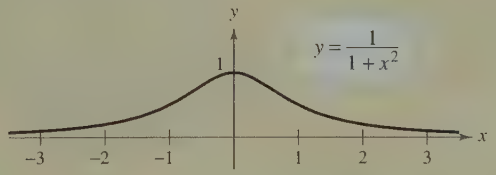
$$T = \frac{d(s)}{s} + \frac{5.5}{s}.$$

(c) Use a graphing utility to graph the function T and estimate the speed s that minimizes the time between vehicles.

(d) Use calculus to determine the speed that minimizes T. What is the minimum value of T? Convert the required speed to kilometers per hour.

(e) Find the optimal distance between vehicles for the posted speed limit determined in part (d).

17. Find the point on the graph of $y = \frac{1}{1 + x^2}$ where the tangent line has the greatest slope, and the point where the tangent line has the least slope.



18. (a) Let x be a positive number. Use the *table* feature of a graphing utility to verify that $\sqrt{1 + x} < \frac{1}{2}x + 1$.

(b) Use the Mean Value Theorem to prove that $\sqrt{1 + x} < \frac{1}{2}x + 1$ for all positive real numbers x.

19. (a) Let x be a positive number. Use the *table* feature of a graphing utility to verify that $\sin x < x$.

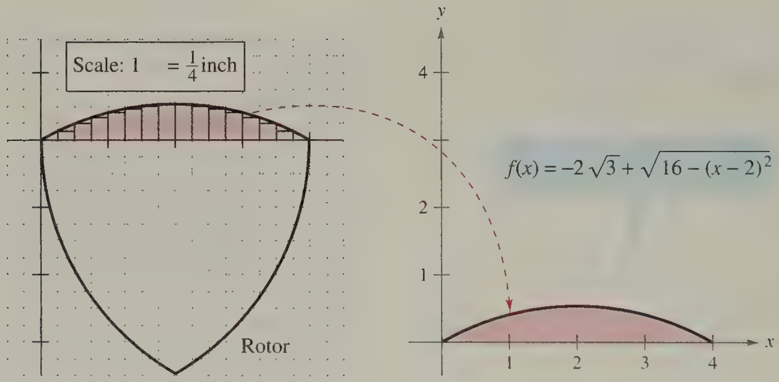
(b) Use the Mean Value Theorem to prove that $\sin x < x$ for all positive real numbers x.

The Wankel Rotary Engine and Area

Named for Felix Wankel, who developed its basic principles in the 1950s, the Wankel rotary engine presents an alternative to the piston engine commonly used in automobiles. Many auto makers, including Mercedes-Benz, Citroën, and Ford, have experimented with rotary engines. By far the greatest number of Wankel-powered vehicles have been put on the road by Mazda, whose current rotary engine design is the RX-7.

The Wankel rotary engine has several advantages over the piston engine. A rotary engine is approximately half the size and weight of a piston engine of equivalent power. Compared with about 97 major moving parts in a V-8 engine, the typical two-rotor rotary engine has only three major moving parts. As a result, the Wankel engine has lower labor and material costs and less internal energy waste.

Although many different designs are possible for the rotary engine, the most common configuration is a two-lobed housing containing a three-sided rotor. The size of the rotor in comparison with the size of the housing cavity is critical in determining the compression ratio and thus the combustion efficiency.


Scale: 1 = $\frac{1}{4}$ inch
Rotor
y
4
2
1
$f(x) = -2\sqrt{3} + \sqrt{16 - (x-2)^2}$
1
2
3
4
x

QUESTIONS

1. The region shown in the figure on the right above is bounded above by the graph of

$$f(x) = -2\sqrt{3} + \sqrt{16 - (x-2)^2}$$

and below by the x-axis. Describe different ways in which you might approximate the area of the region. Then choose one of the ways and use it to obtain an approximation. What type of accuracy do you think your approximation has?

2. Now that you have found one approximation for the area of the region, describe a way that you can improve your approximation. Does your strategy allow you to obtain an approximation that is arbitrarily close to the actual area? Explain.

3. Use your approximation to estimate the area of the "bulged triangle" shown in the figure on the left above.

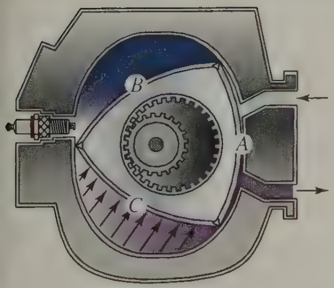
The concepts presented here will be explored further in this chapter. For an extension of this application, see Lab 6 in the lab series that accompanies this text at college.hmco.com.

4

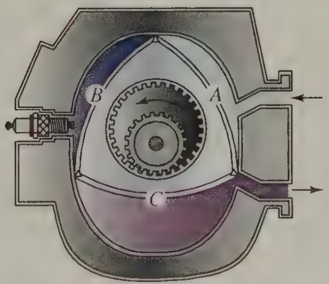
Integration

2001, Mazda Motor Corporation nveiled the new RX-8 concept car the North American International uto Show in Detroit. The RX-8 is owered by an iteration of the Vankel rotary engine. Although the st mass-produced car equipped ith a rotary engine sold worldwide as the RX-7, whose shipment to e United States ended in 1995, the lazda Corporation intends to establish an interest in rotary-owered cars through new designs ke the RX-8.

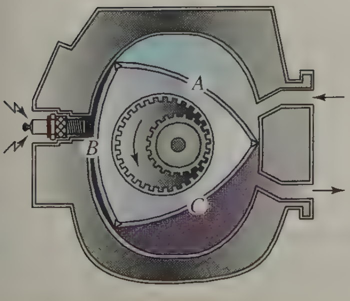
Reuters/Rebecca Cook/Archive Photos



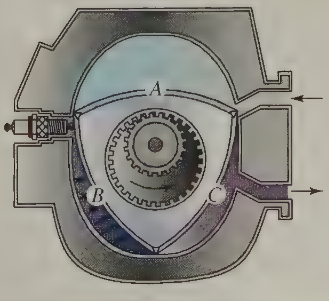
A sweeps out exhaust, *B* begins compression, and *C* is nearly finished with expansion.



A moves back to allow intake while *B* continues compression. *C* begins to push out exhaust.



Ignition takes place in *B*, *A* continues intake, and *C* continues exhaust.



Intake is nearly complete in *A*, *B* expands following ignition, and *C* is nearly finished with exhaust.

Section 4.1 Antiderivatives and Indefinite Integration

- Write the general solution of a differential equation.
- Use indefinite integral notation for antiderivatives.
- Use basic integration rules to find antiderivatives.
- Find a particular solution of a differential equation.

EXPLORATION

Finding Antiderivatives For each of the following derivatives, describe the original function F.

a. $F'(x) = 2x$

b. $F'(x) = x$

c. $F'(x) = x^2$

d. $F'(x) = \frac{1}{x^2}$

e. $F'(x) = \frac{1}{x^3}$

f. $F'(x) = \cos x$

What strategy did you use to find F?

Antiderivatives

Suppose you were asked to find a function F whose derivative is $f(x) = 3x^2$. Fro your knowledge of derivatives, you would probably say that

$$F(x) = x^3 \text{ because } \frac{d}{dx}[x^3] = 3x^2.$$

The function F is an *antiderivative* of f.

Definition of an Antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Note that F is called *an* antiderivative of f, rather than *the* antiderivative of f. T see why, observe that

$$F_1(x) = x^3,\quad F_2(x) = x^3 - 5,\quad \text{and} \quad F_3(x) = x^3 + 97$$

are all antiderivatives of $f(x) = 3x^2$. In fact, for any constant C, the function given b $F(x) = x^3 + C$ is an antiderivative of f.

THEOREM 4.1 Representation of Antiderivatives

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form

$$G(x) = F(x) + C, \text{ for all } x \text{ in } I$$

where C is a constant.

Proof The proof of one direction is straightforward. That is, if $G(x) = F(x) +$ c $F'(x) = f(x)$, and C is a constant, then

$$G'(x) = \frac{d}{dx}[F(x) + C] = F'(x) + 0 = f(x).$$

To prove the other direction, you can define a function H such that

$$H(x) = G(x) - F(x).$$

If H is not constant on the interval I, there must exist a and b $(a < b)$ in the interv such that $H(a) \neq H(b)$. Moreover, because H is differentiable on (a, b), you can app the Mean Value Theorem to conclude that there exists some c in (a, b) such that

$$H'(c) = \frac{H(b) - H(a)}{b - a}.$$

Because $H(b) \neq H(a)$, it follows that $H'(c) \neq 0$. However, because $G'(c) = F'(c)$, yo know that $H'(c) = G'(c) - F'(c) = 0$, which contradicts the fact that $H'(c) \neq$ Consequently, you can conclude that $H(x)$ is a constant, C. Therefore, $G(x) - F(x) =$ and it follows that $G(x) = F(x) + C$.

Using Theorem 4.1, you can represent the entire family of antiderivatives of a function by adding a constant to a *known* antiderivative. For example, knowing that $D_x[x^2] = 2x$, you can represent the family of *all* antiderivatives of $f(x) = 2x$ by

$$G(x) = x^2 + C \qquad \text{Family of all antiderivatives of } f(x) = 2x$$

where C is a constant. The constant C is called the **constant of integration**. The family of functions represented by G is the **general antiderivative** of f, and $G(x) = x^2 + C$ is the **general solution** of the *differential equation*

$$G'(x) = 2x. \qquad \text{Differential equation}$$

A **differential equation** in x and y is an equation that involves x, y, and derivatives of y. For instance, $y' = 3x$ and $y' = x^2 + 1$ are examples of differential equations.

Example 1 Solving a Differential Equation

Find the general solution of the differential equation $y' = 2$.

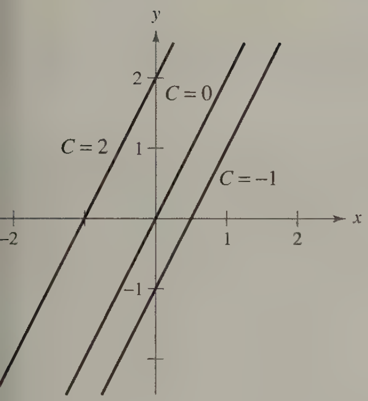
Solution To begin, you need to find a function whose derivative is 2. One such function is

$$y = 2x. \qquad 2x \text{ is } \textit{an} \text{ antiderivative of } 2.$$

Now, you can use Theorem 4.1 to conclude that the general solution of the differential equation is

$$y = 2x + C. \qquad \text{General solution}$$

The graphs of several functions of the form $y = 2x + C$ are shown in Figure 4.1.



nctions of the form $y = 2x + C$

gure 4.1

Notation for Antiderivatives

When solving a differential equation of the form

$$\frac{dy}{dx} = f(x)$$

it is convenient to write it in the equivalent differential form

$$dy = f(x)\,dx.$$

The operation of finding all solutions of this equation is called **antidifferentiation** (or **indefinite integration**) and is denoted by an integral sign $\int$. The general solution is denoted by

$$y = \int \underbrace{f(x)}_{\text{Integrand}} \overbrace{dx}^{\text{Variable of integration}} = F(x) + \overbrace{C}^{\text{Constant of integration}}.$$

The expression $\int f(x)\,dx$ is read as the *antiderivative of f with respect to x.* So, the differential dx serves to identify x as the variable of integration. The term **indefinite integral** is a synonym for antiderivative.

TE In this text, whenever we write $(x)\,dx = F(x) + C$, we mean that F an antiderivative of f *on an interval*.

Basic Integration Rules

The inverse nature of integration and differentiation can be verified by substitutin $F'(x)$ for $f(x)$ in the indefinite integration definition to obtain

$$\int F'(x)\,dx = F(x) + C.$$

Integration is the "inverse" of differentiation.

Moreover, if $\int f(x)\,dx = F(x) + C$, then

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

Differentiation is the "inverse" of integration.

These two equations allow you to obtain integration formulas directly from diffe entiation formulas, as shown in the following summary.

Basic Integration Rules

Differentiation Formula	*Integration Formula*	
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$	
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$	
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$	
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$	
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$	Power Rule
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$	
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$	
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$	
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$	
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$	
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$	

NOTE Note that the Power Rule for integration has the restriction that $n \neq -1$. Th evaluation of $\int 1/x\,dx$ must wait until the introduction of the natural logarithm function Chapter 5.

Example 2 Applying the Basic Integration Rules

Describe the antiderivatives of $3x$.

Solution

$$\begin{aligned}\int 3x\,dx &= 3\int x\,dx && \text{Constant Multiple Rule}\\ &= 3\int x^1\,dx && \text{Rewrite } (x = x^1).\\ &= 3\left(\frac{x^2}{2}\right) + C && \text{Power Rule } (n = 1)\\ &= \frac{3}{2}x^2 + C && \text{Simplify.}\end{aligned}$$

When indefinite integrals are evaluated, a strict application of the basic integration rules tends to produce complicated constants of integration. For instance, in Example 2, we could have written

$$\begin{aligned}\int 3x\,dx &= 3\int x\,dx\\ &= 3\left(\frac{x^2}{2} + C\right)\\ &= \frac{3}{2}x^2 + 3C.\end{aligned}$$

However, because C represents *any* constant, it is both cumbersome and unnecessary to write $3C$ as the constant of integration, and we choose the simpler form, $\frac{3}{2}x^2 + C$.

In Example 2, note that the general pattern of integration is similar to that of differentiation.

Original integral ⇨ Rewrite ⇨ Integrate ⇨ Simplify

Example 3 Rewriting Before Integrating

	Original Integral	*Rewrite*	*Integrate*	*Simplify*
a.	$\int \frac{1}{x^3}\,dx$	$\int x^{-3}\,dx$	$\frac{x^{-2}}{-2} + C$	$-\frac{1}{2x^2} + C$
b.	$\int \sqrt{x}\,dx$	$\int x^{1/2}\,dx$	$\frac{x^{3/2}}{3/2} + C$	$\frac{2}{3}x^{3/2} + C$
c.	$\int 2\sin x\,dx$	$2\int \sin x\,dx$	$2(-\cos x) + C$	$-2\cos x + C$

ECHNOLOGY Some software rograms, such as *Derive*, *Maple*, *lathcad*, *Mathematica*, and the *TI-89*, re capable of performing integration ymbolically. If you have access to uch a symbolic integration utility, try sing it to evaluate the indefinite ntegrals in Example 3.

Remember that you can check your answer to an antidifferentiation problem by differentiating. For instance, in Example 3b, you can check that $\frac{2}{3}x^{3/2} + C$ is the correct antiderivative by differentiating the answer to obtain

$$D_x\left[\frac{2}{3}x^{3/2} + C\right] = \left(\frac{2}{3}\right)\left(\frac{3}{2}\right)x^{1/2} = \sqrt{x}. \qquad \text{Use differentiation to check antiderivative.}$$

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indicates that in the Interactive 3.0 *CD-ROM and* Internet 3.0 *versions of this text (available at* college.hmco.com) *you will find an Open Exploration, which further explores this example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

The basic integration rules listed earlier in this section allow you to integrate c
polynomial function, as demonstrated in Example 4.

Example 4 Integrating Polynomial Functions

a. $$\int dx = \int 1\, dx$$

Integrand is understood to be

$$= x + C$$

Integrate.

b. $$\int (x+2)\, dx = \int x\, dx + \int 2\, dx$$

$$= \frac{x^2}{2} + C_1 + 2x + C_2$$

Integrate.

$$= \frac{x^2}{2} + 2x + C$$

$C = C_1 + C_2$

The second line in the solution is usually omitted.

c. $$\int (3x^4 - 5x^2 + x)\, dx = 3\left(\frac{x^5}{5}\right) - 5\left(\frac{x^3}{3}\right) + \frac{x^2}{2} + C$$

Integrate.

$$= \frac{3}{5}x^5 - \frac{5}{3}x^3 + \frac{1}{2}x^2 + C$$

Simplify.

Example 5 Rewriting Before Integrating

$$\int \frac{x+1}{\sqrt{x}}\, dx = \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx$$

Rewrite as two fractions.

$$= \int (x^{1/2} + x^{-1/2})\, dx$$

Rewrite with fractional exponents.

$$= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C$$

Integrate.

$$= \frac{2}{3}x^{3/2} + 2x^{1/2} + C$$

Simplify.

NOTE When integrating quotients, do not integrate the numerator and denomina separately. This is no more valid in integration than it is in differentiation. For instance, Example 5, be sure you understand that

$$\int \frac{x+1}{\sqrt{x}}\, dx \neq \frac{\int (x+1)\, dx}{\int \sqrt{x}\, dx}.$$

Example 6 Rewriting Before Integrating

$$\int \frac{\sin x}{\cos^2 x}\, dx = \int \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) dx$$

Rewrite as a product.

$$= \int \sec x \tan x\, dx$$

Rewrite using trigonometric identities.

$$= \sec x + C$$

Integrate.

Initial Conditions and Particular Solutions

You have already seen that the equation $y = \int f(x)\,dx$ has many solutions (each differing from the others by a constant). This means that the graphs of any two antiderivatives of f are vertical translations of each other. For example, Figure 4.2 shows the graphs of several antiderivatives of the form

$$y = \int (3x^2 - 1)\,dx = x^3 - x + C \qquad \text{General solution}$$

for various integer values of C. Each of these antiderivatives is a solution of the differential equation

$$\frac{dy}{dx} = 3x^2 - 1.$$

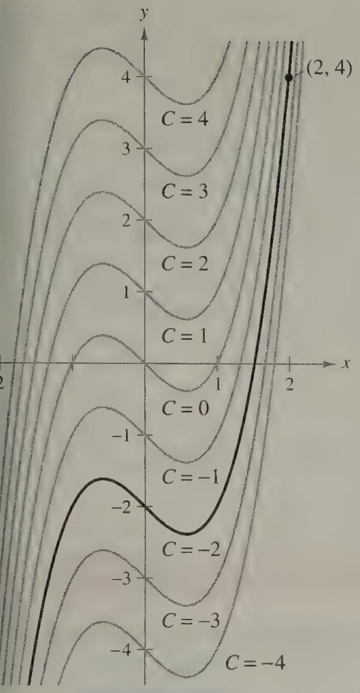
In many applications of integration, you are given enough information to determine a **particular solution**. To do this, you need only know the value of $y = F(x)$ for one value of x. (This information is called an **initial condition.**) For example, in Figure 4.2, only one curve passes through the point (2, 4). To find this curve, you can use the following information.

$$F(x) = x^3 - x + C \qquad \text{General solution}$$

$$F(2) = 4 \qquad \text{Initial condition}$$

By using the initial condition in the general solution, you can determine that $F(2) = 8 - 2 + C = 4$, which implies that $C = -2$. So, you obtain

$$F(x) = x^3 - x - 2. \qquad \text{Particular solution}$$



$F(x) = x^3 - x + C$

particular solution that satisfies initial condition $F(2) = 4$ is $) = x^3 - x - 2$.

ure 4.2

Example 7 Finding a Particular Solution

Find the general solution of

$$F'(x) = \frac{1}{x^2}, \quad x > 0$$

and find the particular solution that satisfies the initial condition $F(1) = 0$.

Solution To find the general solution, integrate to obtain

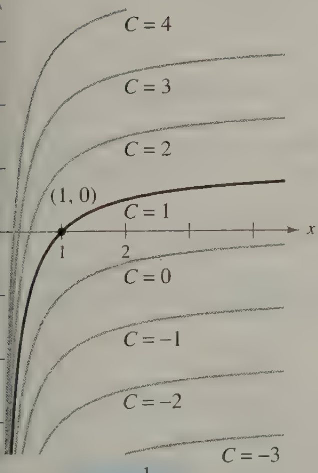
$$\begin{aligned} F(x) &= \int \frac{1}{x^2}\,dx && F(x) = \textstyle\int F'(x)dx \\ &= \int x^{-2}\,dx && \text{Rewrite as a power.} \\ &= \frac{x^{-1}}{-1} + C && \text{Integrate.} \\ &= -\frac{1}{x} + C, \quad x > 0. && \text{General solution} \end{aligned}$$

Using the initial condition $F(1) = 0$, you can solve for C as follows.

$$F(1) = -\frac{1}{1} + C = 0 \quad \Longrightarrow \quad C = 1$$

So, the particular solution, as shown in Figure 4.3, is

$$F(x) = -\frac{1}{x} + 1, \quad x > 0. \qquad \text{Particular solution}$$



$F(x) = -\frac{1}{x} + C$

particular solution that satisfies initial condition $F(1) = 0$ is $) = -(1/x) + 1, x > 0$.

ure 4.3

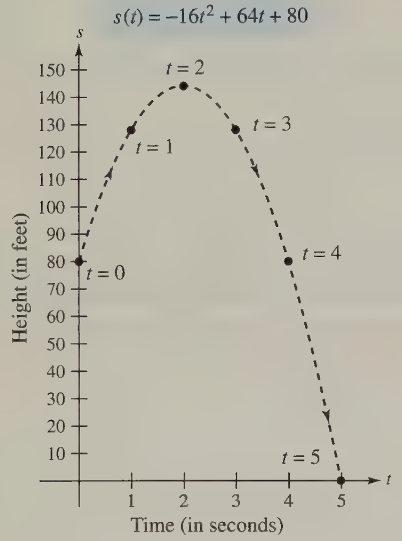
So far in this section we have been using x as the variable of integration. applications, it is often convenient to use a different variable. For instance, in t following example involving *time*, the variable of integration is t.

Example 8 Solving a Vertical Motion Problem

A ball is thrown upward with an initial velocity of 64 feet per second from an init height of 80 feet. See Figure 4.4.

a. Find the position function giving the height s as a function of the time t.

b. When does the ball hit the ground?



Height of a ball at time t
Figure 4.4

Solution

a. Let $t = 0$ represent the initial time. The two given initial conditions can be writt as follows.

$$s(0) = 80 \qquad \text{Initial height is 80 feet.}$$

$$s'(0) = 64 \qquad \text{Initial velocity is 64 feet per second.}$$

Using -32 feet per second per second as the acceleration due to gravity, you c write

$$s''(t) = -32$$

$$s'(t) = \int s''(t)\,dt = \int -32\,dt = -32t + C_1.$$

Using the initial velocity, you obtain $s'(0) = 64 = -32(0) + C_1$, which impli that $C_1 = 64$. Next, by integrating $s'(t)$, you obtain

$$s(t) = \int s'(t)\,dt = \int (-32t + 64)\,dt = -16t^2 + 64t + C_2.$$

Using the initial height, you obtain

$$s(0) = 80 = -16(0^2) + 64(0) + C_2$$

which implies that $C_2 = 80$. Therefore, the position function is

$$s(t) = -16t^2 + 64t + 80.$$

b. Using the position function found in part (a), you can find the time that the ball h the ground by solving the equation $s(t) = 0$.

$$s(t) = -16t^2 + 64t + 80 = 0$$

$$-16(t + 1)(t - 5) = 0$$

$$t = -1, 5$$

Because t must be positive, you can conclude that the ball hits the ground seconds after it was thrown.

Example 8 shows how to use calculus to analyze vertical motion problems which the acceleration is determined by a gravitational force. You can use a simi strategy to analyze other linear motion problems (vertical or horizontal) in which t acceleration (or deceleration) is the result of some other force, as you can see Exercises 77–88.

NOTE In Example 8, note that the position function has the form

$$s(t) = \tfrac{1}{2}gt^2 + v_0t + s_0,$$

where $g = -32$, v_0 is the initial velocity, and s_0 is the initial height, as presented in Section 2.2.

Before you begin the exercise set, be sure you realize that one of the most important steps in integration is *rewriting the integrand* in a form that fits the basic integration rules. To further illustrate this point, here are some additional examples.

Original Integral	*Rewrite*	*Integrate*	*Simplify*
$\int \frac{2}{\sqrt{x}}\,dx$	$2\int x^{-1/2}\,dx$	$2\left(\frac{x^{1/2}}{1/2}\right) + C$	$4x^{1/2} + C$
$\int (t^2 + 1)^2\,dt$	$\int (t^4 + 2t^2 + 1)\,dt$	$\frac{t^5}{5} + 2\left(\frac{t^3}{3}\right) + t + C$	$\frac{1}{5}t^5 + \frac{2}{3}t^3 + t + C$
$\int \frac{x^3 + 3}{x^2}\,dx$	$\int (x + 3x^{-2})\,dx$	$\frac{x^2}{2} + 3\left(\frac{x^{-1}}{-1}\right) + C$	$\frac{1}{2}x^2 - \frac{3}{x} + C$
$\int \sqrt[3]{x}(x - 4)\,dx$	$\int (x^{4/3} - 4x^{1/3})\,dx$	$\frac{x^{7/3}}{7/3} - 4\left(\frac{x^{4/3}}{4/3}\right) + C$	$\frac{3}{7}x^{4/3}(x - 7) + C$

XERCISES FOR SECTION 4.1

Exercises 1–4, verify the statement by showing that the ivative of the right side equals the integrand of the left side.

$\int \left(-\frac{9}{x^4}\right) dx = \frac{3}{x^3} + C$

$\int \left(4x^3 - \frac{1}{x^2}\right) dx = x^4 + \frac{1}{x} + C$

$\int (x - 2)(x + 2)\,dx = \frac{1}{3}x^3 - 4x + C$

$\int \frac{x^2 - 1}{x^{3/2}}\,dx = \frac{2(x^2 + 3)}{3\sqrt{x}} + C$

Exercises 5–8, find the general solution of the differential ation and check the result by differentiation.

$\frac{dy}{dt} = 3t^2$

6. $\frac{dr}{d\theta} = \pi$

$\frac{dy}{dx} = x^{3/2}$

8. $\frac{dy}{dx} = 2x^{-3}$

Exercises 9–14, complete the table using Example 3 and the mples at the top of this page as a model.

Original Integral	*Rewrite*	*Integrate*	*Simplify*
$\int \sqrt[3]{x}\,dx$			
$\int \frac{1}{x^2}\,dx$			
$\int \frac{1}{x\sqrt{x}}\,dx$			
$\int x(x^2 + 3)\,dx$			
$\int \frac{1}{2x^3}\,dx$			
$\int \frac{1}{(3x)^2}\,dx$			

In Exercises 15–34, find the indefinite integral and check the result by differentiation.

15. $\int (x + 3)\,dx$
16. $\int (5 - x)\,dx$
17. $\int (2x - 3x^2)\,dx$
18. $\int (4x^3 + 6x^2 - 1)\,dx$
19. $\int (x^3 + 2)\,dx$
20. $\int (x^3 - 4x + 2)\,dx$
21. $\int (x^{3/2} + 2x + 1)\,dx$
22. $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx$
23. $\int \sqrt[3]{x^2}\,dx$
24. $\int (\sqrt[4]{x^3} + 1)\,dx$
25. $\int \frac{1}{x^3}\,dx$
26. $\int \frac{1}{x^4}\,dx$
27. $\int \frac{x^2 + x + 1}{\sqrt{x}}\,dx$
28. $\int \frac{x^2 + 2x - 3}{x^4}\,dx$
29. $\int (x + 1)(3x - 2)\,dx$
30. $\int (2t^2 - 1)^2\,dt$
31. $\int y^2\sqrt{y}\,dy$
32. $\int (1 + 3t)t^2\,dt$
33. $\int dx$
34. $\int 3\,dt$

In Exercises 35–42, find the indefinite integral and check the result by differentiation.

35. $\int (2\sin x + 3\cos x)\,dx$
36. $\int (t^2 - \sin t)\,dt$
37. $\int (1 - \csc t \cot t)\,dt$
38. $\int (\theta^2 + \sec^2\theta)\,d\theta$
39. $\int (\sec^2\theta - \sin\theta)\,d\theta$
40. $\int \sec y(\tan y - \sec y)\,dy$
41. $\int (\tan^2 y + 1)\,dy$
42. $\int \frac{\cos x}{1 - \cos^2 x}\,dx$

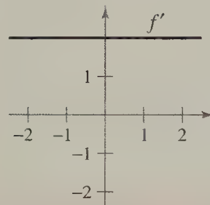
In Exercises 43 and 44, sketch the graphs of the function $g(x) = f(x) + C$ for $C = -2$, $C = 0$, and $C = 3$ on the same set of coordinate axes.

43. $f(x) = \cos x$

44. $f(x) = \sqrt{x}$

In Exercises 45–48, the graph of the derivative of a function is given. Sketch the graphs of *two* functions that have the given derivative. (There is more than one correct answer.) To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

45.



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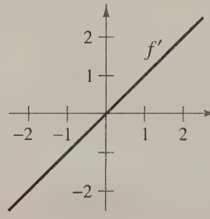

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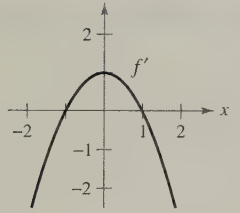

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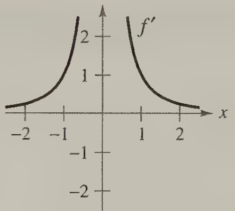
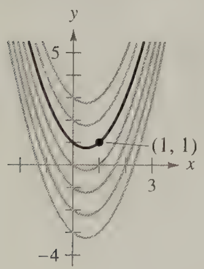
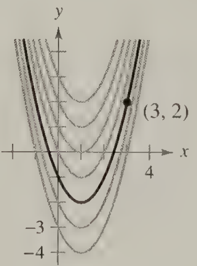


In Exercises 49–52, find the equation for y, given the derivative and the indicated point on the curve.

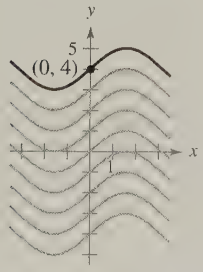
49. $\dfrac{dy}{dx} = 2x - 1$



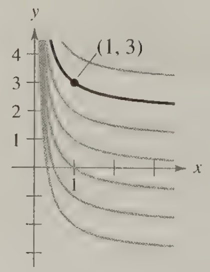
50. $\dfrac{dy}{dx} = 2(x - 1)$



51. $\dfrac{dy}{dx} = \cos x$

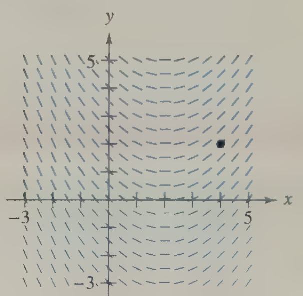


52. $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$, $x > 0$

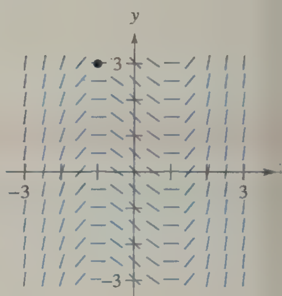


Slope Fields **In Exercises 53 and 54, a differential equatio point, and a slope field are given. A *slope field* (or *direction fi* consists of line segments with slopes given by the differen equation. These line segments give a visual perspective of slopes of the solutions of the differential equation. (a) Ske two approximate solutions of the differential equation on slope field, one of which passes through the indicated po (To print an enlarged copy of the graph, go to the web *www.mathgraphs.com*.) (b) Use integration to find the particu solution of the differential equation and use a graphing uti to graph the solution. Compare the result with the sketche part (a).**

53. $\dfrac{dy}{dx} = \dfrac{1}{2}x - 1$, $(4, 2)$



54. $\dfrac{dy}{dx} = x^2 - 1$, $(-1, 3)$



In Exercises 55–62, solve the differential equation.

55. $f'(x) = 4x, f(0) = 6$

56. $g'(x) = 6x^2, g(0) = -1$

57. $h'(t) = 8t^3 + 5, h(1) = -4$

58. $f'(s) = 6s - 8s^3, f(2) = 3$

59. $f''(x) = 2,\ f'(2) = 5,\ f(2) = 10$

60. $f''(x) = x^2,\ f'(0) = 6,\ f(0) = 3$

61. $f''(x) = x^{-3/2},\ f'(4) = 2,\ f(0) = 0$

62. $f''(x) = \sin x,\ f'(0) = 1,\ f(0) = 6$

63. ***Tree Growth*** An evergreen nursery usually sells a cert shrub after 6 years of growth and shaping. The growth r during those 6 years is approximated by

$$\frac{dh}{dt} = 1.5t + 5$$

where t is the time in years and h is the height in centimete The seedlings are 12 centimeters tall when planted ($t = 0$).

(a) Find the height after t years.

(b) How tall are the shrubs when they are sold?

64. ***Population Growth*** The rate of growth dP/dt of a populat of bacteria is proportional to the square root of t, where P is population size and t is the time in days ($0 \le t \le 10$). That

$$\frac{dP}{dt} = k\sqrt{t}.$$

The initial size of the population is 500. After 1 day the po lation has grown to 600. Estimate the population after 7 day

Getting at the Concept

5. Use the graph of f' in the figure to answer the following, given that $f(0) = -4$.

(a) Approximate the slope of f at $x = 4$. Explain.

(b) Is it possible that $f(2) = -1$? Explain.

(c) Is $f(5) - f(4) > 0$? Explain.

(d) Approximate the value of x where f is maximum. Explain.

(e) Approximate any intervals in which the graph of f is concave upward and any intervals in which it is concave downward. Approximate the x-coordinates of any points of inflection.

(f) Approximate the x-coordinate of the minimum of $f''(x)$.

(g) Sketch an approximate graph of f. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*

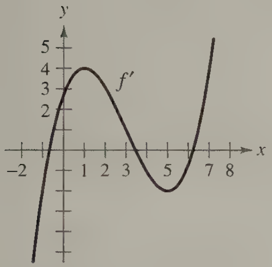


Figure for 65

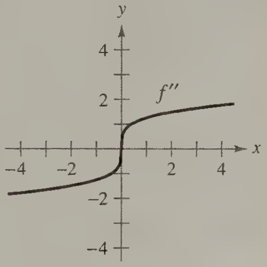


Figure for 66

6. The graphs of f and f' each pass through the origin. Use the graph of f'' shown in the figure to sketch the graphs of f and f'. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*

rtical Motion **In Exercises 67–70, use $a(t) = -32$ feet per cond per second as the acceleration due to gravity. (Neglect resistance.)**

A ball is thrown vertically upward from a height of 6 feet with an initial velocity of 60 feet per second. How high will the ball go?

Show that the height above the ground of an object thrown upward from a point s_0 feet above the ground with an initial velocity of v_0 feet per second is given by the function

$$f(t) = -16t^2 + v_0t + s_0.$$

With what initial velocity must an object be thrown upward (from ground level) to reach the top of the Washington Monument (approximately 550 feet)?

A balloon, rising vertically with a velocity of 16 feet per second, releases a sandbag at the instant it is 64 feet above the ground.

(a) How many seconds after its release will the bag strike the ground?

(b) At what velocity will it hit the ground?

Vertical Motion **In Exercises 71–74, use $a(t) = -9.8$ meters per second per second as the acceleration due to gravity. (Neglect air resistance.)**

71. Show that the height above the ground of an object thrown upward from a point s_0 meters above the ground with an initial velocity of v_0 meters per second is given by the function

$$f(t) = -4.9t^2 + v_0t + s_0.$$

72. The Grand Canyon is 1600 meters deep at its deepest point. A rock is dropped from the rim above this point. Express the height of the rock as a function of the time t in seconds. How long will it take the rock to hit the canyon floor?

73. A baseball is thrown upward from a height of 2 meters with a velocity of 10 meters per second. Determine its maximum height.

74. With what initial velocity must an object be thrown upward (from a height of 2 meters) to reach a maximum height of 200 meters?

75. ***Lunar Gravity*** On the moon, the acceleration due to gravity is -1.6 meters per second per second. A stone is dropped from a cliff on the moon and hits the surface of the moon 20 seconds later. How far did it fall? What was its velocity at impact?

76. ***Escape Velocity*** The minimum velocity required for an object to escape earth's gravitational pull is obtained from the solution of the equation

$$\int v\,dv = -GM\int \frac{1}{y^2}\,dy$$

where v is the velocity of the object projected from earth, y is the distance from the center of earth, G is the gravitational constant, and M is the mass of earth. Show that v and y are related by the equation

$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

where v_0 is the initial velocity of the object and R is the radius of earth.

Rectilinear Motion **In Exercises 77–80, consider a particle moving along the x-axis where $x(t)$ is the position of the particle at time t, $x'(t)$ is its velocity, and $x''(t)$ is its acceleration.**

77. $x(t) = t^3 - 6t^2 + 9t - 2, \quad 0 \le t \le 5$

(a) Find the velocity and acceleration of the particle.

(b) Find the open t-intervals on which the particle is moving to the right.

(c) Find the velocity of the particle when the acceleration is 0.

78. Repeat Exercise 77 for the position function

$$x(t) = (t-1)(t-3)^2, \quad 0 \le t \le 5.$$

79. A particle moves along the x-axis at a velocity of $v(t) = 1/\sqrt{t}$, $t > 0$. At time $t = 1$, its position is $x = 4$. Find the acceleration and position functions for the particle.

80. A particle, initially at rest, moves along the x-axis such that its acceleration at time $t > 0$ is given by $a(t) = \cos t$. At the time $t = 0$, its position is $x = 3$.

(a) Find the velocity and position functions for the particle.

(b) Find the values of t for which the particle is at rest.

81. ***Acceleration*** The maker of a certain automobile advertises that it takes 13 seconds to accelerate from 25 kilometers per hour to 80 kilometers per hour. Assuming constant acceleration, compute the following.

(a) The acceleration in meters per second per second

(b) The distance the car travels during the 13 seconds

82. ***Deceleration*** A car traveling at 45 miles per hour is brought to a stop, at constant deceleration, 132 feet from where the brakes are applied.

(a) How far has the car moved when its speed has been reduced to 30 miles per hour?

(b) How far has the car moved when its speed has been reduced to 15 miles per hour?

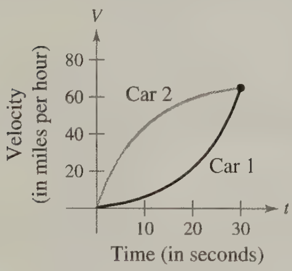
(c) Draw the real number line from 0 to 132, and plot the points found in parts (a) and (b). What can you conclude?

83. ***Acceleration*** At the instant the traffic light turns green, a car that has been waiting at an intersection starts with a constant acceleration of 6 feet per second per second. At the same instant, a truck traveling with a constant velocity of 30 feet per second passes the car.

(a) How far beyond its starting point will the car pass the truck?

(b) How fast will the car be traveling when it passes the truck?

84. ***Think About It*** Two cars starting from rest accelerate to 65 miles per hour in 30 seconds. The velocity of each car is shown in the figure. Are the cars side by side at the end of the 30-second time interval? Explain.



85. ***Data Analysis*** The table shows the velocities (in miles per hour) of two cars on the entrance ramp of an interstate highway. The time t is in seconds.

t	0	5	10	15	20	25	30
v_1	0	2.5	7	16	29	45	65
v_2	0	21	38	51	60	64	65

(a) Rewrite the table converting miles per hour to feet per second.

(b) Use the regression capabilities of a graphing utility to fit quadratic models to the data in part (a).

(c) Approximate the distance traveled by each car during the 30 seconds. Explain the difference in distances.

86. ***Data Analysis*** A vehicle slows to a stop from 45 miles per h in 6 seconds. The table shows the velocities in feet per second

t	0	1	2	3	4	5	6
v	66.0	61.1	48.9	33.0	17.1	4.8	0

(a) Use the regression capabilities of a graphing utility to fi cubic model to the data.

(b) Approximate the distance traveled by the car during the seconds.

87. ***Acceleration*** Assume that a fully loaded plane starting fro rest has a constant acceleration while moving down a runwa The plane requires 0.7 mile of runway and a speed of 160 mil per hour in order to lift off. What is the plane's acceleration?

88. ***Airplane Separation*** Two airplanes are in a straight-line lan ing pattern and, according to FAA regulations, must keep at lea a 3-mile separation. Airplane A is 10 miles from touchdown a is gradually slowing its speed from 150 miles per hour to a lan ing speed of 100 miles per hour. Airplane B is 17 miles fro touchdown and is gradually slowing its speed from 250 miles p hour to a landing speed of 115 miles per hour.

(a) Assuming the deceleration of each airplane is constant, fi the position functions s_1 and s_2 for Airplane A and Airpla B. Let $t = 0$ represent the times when the airplanes are and 17 miles from the airport.

(b) Use a graphing utility to graph the position functions.

(c) Find a formula for the magnitude of the distance d betwe the two airplanes as a function of t. Use a graphing utili to graph d. Is $d < 3$ for some time prior to the landing Airplane A?If so, find that time.

True or False? **In Exercises 89–94, determine whether t statement is true or false. If it is false, explain why or give a example that shows it is false.**

89. Each antiderivative of an nth-degree polynomial function is a $(n + 1)$th-degree polynomial function.

90. If $p(x)$ is a polynomial function, then p has exactly o antiderivative whose graph contains the origin.

91. If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, then $F(x)$ $G(x) + C$.

92. If $f'(x) = g(x)$, then $\int g(x)dx = f(x) + C$.

93. $\int f(x)g(x)dx = \int f(x)dx \int g(x)dx$

94. The antiderivative of $f(x)$ is unique.

95. If $f'(x) = \begin{cases} 1, & 0 \le x < 2 \\ 3x, & 2 \le x \le 5 \end{cases}$, f is continuous, and $f(1) = 3$, find f. Is f differentiable at $x = 2$?

96. Let $s(x)$ and $c(x)$ be two functions satisfying $s'(x) = c(x)$ a $c'(x) = -s(x)$ for all x. If $s(0) = 0$ and $c(0) = 1$, prove th $[s(x)]^2 + [c(x)]^2 = 1$.

Section 4.2 Area

- Use sigma notation to write and evaluate a sum.
- Understand the concept of area.
- Approximate the area of a plane region.
- Find the area of a plane region using limits.

Sigma Notation

In the preceding section, you studied antidifferentiation. In this section, you will look further into a problem introduced in Section 1.1—that of finding the area of a region in the plane. At first glance, these two ideas may seem unrelated, but you will discover in Section 4.4 that they are closely related by an extremely important theorem called the Fundamental Theorem of Calculus.

We begin this section by introducing a concise notation for sums. This notation is called **sigma notation** because it uses the uppercase Greek letter sigma, written as Σ.

Sigma Notation

The sum of n terms $a_1, a_2, a_3, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index of summation,** a_i is the ***i*th term** of the sum, and the **upper and lower bounds of summation** are n and 1.

NOTE The upper and lower bounds must be constant with respect to the index of summation. However, the lower bound doesn't have to be 1. Any integer less than or equal to the upper bound is legitimate.

Example 1 Examples of Sigma Notation

a. $\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6$

b. $\sum_{i=0}^{5} (i + 1) = 1 + 2 + 3 + 4 + 5 + 6$

c. $\sum_{j=3}^{7} j^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2$

d. $\sum_{k=1}^{n} \frac{1}{n}(k^2 + 1) = \frac{1}{n}(1^2 + 1) + \frac{1}{n}(2^2 + 1) + \cdots + \frac{1}{n}(n^2 + 1)$

e. $\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$

From parts (a) and (b), notice that the same sum can be represented in different ways using sigma notation.

R FURTHER INFORMATION For a ometric interpretation of summation mulas, see the article, "Looking at k and $\sum_{k=1}^{n} k^2$ Geometrically" by Eric gblom in *Mathematics Teacher.* To w this article, go to the website *w.matharticles.com.*

Although any variable can be used as the index of summation i, j, and k are often used. Notice in Example 1 that the index of summation does not appear in the terms of the expanded sum.

The Sum of the First 100 Integers

Carl Friedrich Gauss's (1777–1855) teacher asked him to add all the integers from 1 to 100. When Gauss returned with the correct answer after only a few moments, the teacher could only look at him in astounded silence. This is what Gauss did:

$$\begin{array}{rcrcrcccr} 1 & + & 2 & + & 3 & + & \cdots & + & 100 \\ 100 & + & 99 & + & 98 & + & \cdots & + & 1 \\ \hline 101 & + & 101 & + & 101 & + & \cdots & + & 101 \end{array}$$

$$\frac{100 \times 101}{2} = 5050$$

This is generalized by Theorem 4.2, where

$$\sum_{i=1}^{100} i = \frac{100(101)}{2} = 5050.$$

The following properties of summation can be derived using the associative a commutative properties of addition and the distributive property of addition o multiplication. (In the first property, k is a constant.)

1. $\displaystyle\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$

2. $\displaystyle\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

The next theorem lists some useful formulas for sums of powers. A proof of t theorem is given in Appendix B.

THEOREM 4.2 Summation Formulas

1. $\displaystyle\sum_{i=1}^{n} c = cn$

2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Example 2 Evaluating a Sum

Evaluate $\displaystyle\sum_{i=1}^{n} \frac{i+1}{n^2}$ for $n = 10, 100, 1000$, and $10{,}000$.

Solution Applying Theorem 4.2, you can write

$$\begin{aligned} \sum_{i=1}^{n} \frac{i+1}{n^2} &= \frac{1}{n^2}\sum_{i=1}^{n}(i+1) && \text{Factor constant } 1/n^2 \text{ out of sum.} \\ &= \frac{1}{n^2}\left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right) && \text{Write as two sums.} \\ &= \frac{1}{n^2}\left[\frac{n(n+1)}{2} + n\right] && \text{Apply Theorem 4.2.} \\ &= \frac{1}{n^2}\left[\frac{n^2+3n}{2}\right] && \text{Simplify.} \\ &= \frac{n+3}{2n}. && \text{Simplify.} \end{aligned}$$

Now you can evaluate the sum by substituting the appropriate values of n, as show in the table at the left.

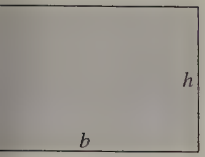
n	$\displaystyle\sum_{i=1}^{n} \frac{i+1}{n^2} = \frac{n+3}{2n}$
10	0.65000
100	0.51500
1,000	0.50150
10,000	0.50015

In the table, note that the sum appears to approach a limit as n increases. Althou the discussion of limits at infinity in Section 3.5 applies to a variable x, where x c be any real number, many of the same results hold true for limits involving t variable n, where n is restricted to positive integer values. So, to find the limit $(n+3)/2n$ as n approaches infinity, you can write

$$\lim_{n\to\infty} \frac{n+3}{2n} = \frac{1}{2}.$$

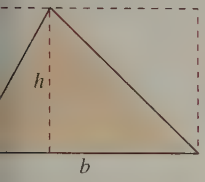
Area

In Euclidean geometry, the simplest type of plane region is a rectangle. Although people often say that the *formula* for the area of a rectangle is $A = bh$, as shown in Figure 4.5, it is actually more proper to say that this is the *definition* of the **area of a rectangle.**

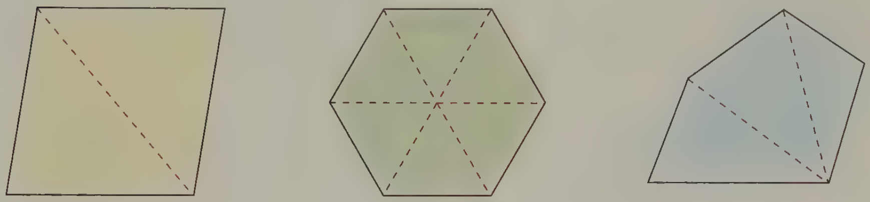


tangle: $A = bh$
ure 4.5

From this definition, you can develop formulas for the areas of many other plane regions. For example, to determine the area of a triangle, you can form a rectangle whose area is twice that of the triangle, as shown in Figure 4.6. Once you know how to find the area of a triangle, you can determine the area of any polygon by subdividing the polygon into triangular regions, as shown in Figure 4.7.



angle: $A = \frac{1}{2}bh$
ure 4.6


Parallelogram Hexagon Polygon
Figure 4.7

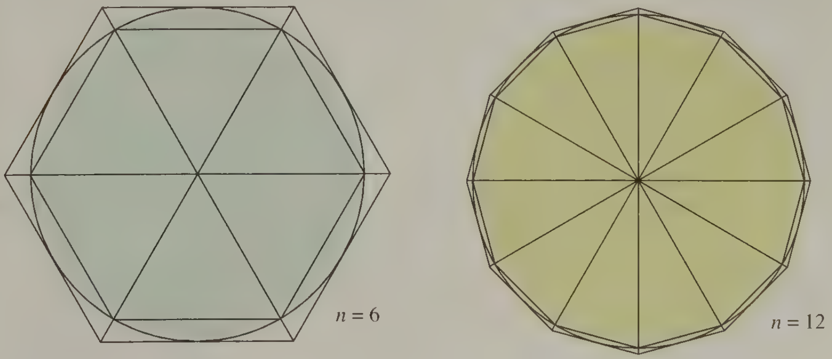
Finding the areas of regions other than polygons is more difficult. The ancient Greeks were able to determine formulas for the areas of some general regions (principally those bounded by conics) by the *exhaustion* method. The clearest description of this method was given by Archimedes. Essentially, the method is a limiting process in which the area is squeezed between two polygons—one inscribed in the region and one circumscribed about the region.

For instance, in Figure 4.8 the area of a circular region is approximated by an n-sided inscribed polygon and an n-sided circumscribed polygon. For each value of n the area of the inscribed polygon is less than the area of the circle, and the area of the circumscribed polygon is greater than the area of the circle. Moreover, as n increases, the areas of both polygons become better and better approximations of the area of the circle.

Mary Evans Picture Library

ARCHIMEDES (287–212 B.C.)

rchimedes used the method of exhaustion to
erive formulas for the areas of ellipses,
arabolic segments, and sectors of a spiral.
e is considered to have been the greatest
pplied mathematician of antiquity.



The exhaustion method for finding the area of a circular region
Figure 4.8

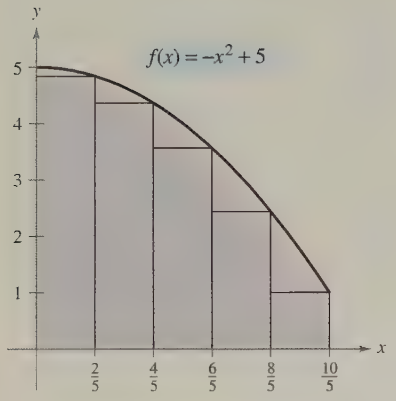
In the remaining examples in this section, we use a process that is similar to that used by Archimedes to determine the area of a plane region.

R FURTHER INFORMATION For an
rnative development of the formula for
area of a circle, see the article "Proof
hout Words: Area of a Disk is πR^2"
Russell Jay Hendel in *Mathematics*
gazine. To view this article, go to the
bsite *www.matharticles.com*.

The Area of a Plane Region

Recall from Section 1.1 that the origins of calculus are connected to two class problems: the tangent line problem and the area problem. We begin the investigati of the area problem with an example.

Example 3 Approximating the Area of a Plane Region



(a) The area of the parabolic region is greater than the area of the rectangles.

Use the five rectangles in Figure 4.9(a) and (b) to find *two* approximations of the ar of the region lying between the graph of

$$f(x) = -x^2 + 5$$

and the x-axis between $x = 0$ and $x = 2$.

Solution

a. The right endpoints of the five intervals are $\frac{2}{5}i$, where $i = 1, 2, 3, 4, 5$. The width each rectangle is $\frac{2}{5}$, and the height of each rectangle can be obtained by evaluati f at the right endpoint of each interval.

$$\left[0, \frac{2}{5}\right], \left[\frac{2}{5}, \frac{4}{5}\right], \left[\frac{4}{5}, \frac{6}{5}\right], \left[\frac{6}{5}, \frac{8}{5}\right], \left[\frac{8}{5}, \frac{10}{5}\right]$$

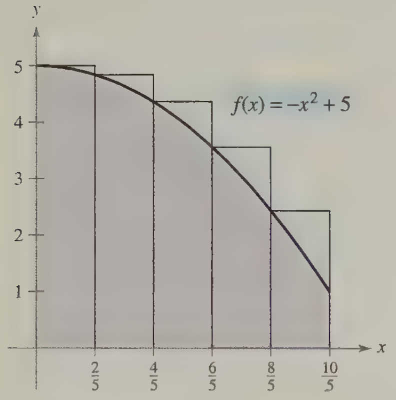
$$\uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow$$

Evaluate f at the right endpoints of these intervals.

The sum of the areas of the five rectangles is

$$\sum_{i=1}^{5} \overbrace{f\left(\frac{2i}{5}\right)}^{\text{Height}} \overbrace{\left(\frac{2}{5}\right)}^{\text{Width}} = \sum_{i=1}^{5} \left[-\left(\frac{2i}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{162}{25} = 6.48.$$

Because each of the five rectangles lies inside the parabolic region, you c conclude that the area of the parabolic region is greater than 6.48.



(b) The area of the parabolic region is less than the area of the rectangles.

Figure 4.9

b. The left endpoints of the five intervals are $\frac{2}{5}(i - 1)$, where $i = 1, 2, 3, 4, 5$. T width of each rectangle is $\frac{2}{5}$, and the height of each rectangle can be obtained evaluating f at the left endpoint of each interval.

$$\sum_{i=1}^{5} \overbrace{f\left(\frac{2i - 2}{5}\right)}^{\text{Height}} \overbrace{\left(\frac{2}{5}\right)}^{\text{Width}} = \sum_{i=1}^{5} \left[-\left(\frac{2i - 2}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{202}{25} = 8.08.$$

Because the parabolic region lies within the union of the five rectangular regior you can conclude that the area of the parabolic region is less than 8.08.

By combining the results in parts (a) and (b), you can conclude that

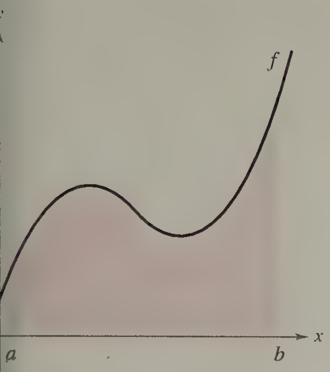
$$6.48 < (\text{Area of region}) < 8.08.$$

NOTE By increasing the number of rectangles used in Example 3, you can obtain closer a closer approximations of the area of the region. For instance, using 25 rectangles of width each, you can conclude that

$$7.17 < (\text{Area of region}) < 7.49.$$

Upper and Lower Sums

The procedure used in Example 3 can be generalized as follows. Consider a plane region bounded above by the graph of a nonnegative, continuous function $y = f(x)$, as shown in Figure 4.10. The region is bounded below by the x-axis, and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$.



region under a curve
ure 4.10

To approximate the area of the region, begin by subdividing the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$, as shown in Figure 4.11. The endpoints of the intervals are as follows.

$$\overbrace{a + 0(\Delta x)}^{a = x_0} < \overbrace{a + 1(\Delta x)}^{x_1} < \overbrace{a + 2(\Delta x)}^{x_2} < \cdot \cdot \cdot < \overbrace{a + n(\Delta x)}^{x_n = b}$$

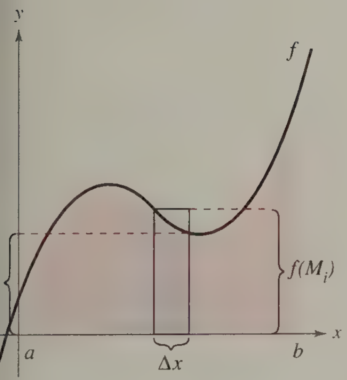
Because f is continuous, the Extreme Value Theorem guarantees the existence of a minimum and a maximum value of $f(x)$ in *each* subinterval.

$$f(m_i) = \text{Minimum value of } f(x) \text{ in } i\text{th subinterval}$$

$$f(M_i) = \text{Maximum value of } f(x) \text{ in } i\text{th subinterval}$$

Next, define an **inscribed rectangle** lying *inside* the ith subregion and a **circumscribed rectangle** extending *outside* the ith subregion. The height of the ith inscribed rectangle is $f(m_i)$ and the height of the ith circumscribed rectangle is $f(M_i)$. For *each* i, the area of the inscribed rectangle is less than or equal to the area of the circumscribed rectangle.

$$\begin{pmatrix}\text{Area of inscribed}\\ \text{rectangle}\end{pmatrix} = f(m_i)\,\Delta x \le f(M_i)\,\Delta x = \begin{pmatrix}\text{Area of circumscribed}\\ \text{rectangle}\end{pmatrix}$$



interval $[a, b]$ is divided into n
tervals of width $\Delta x = \dfrac{b - a}{n}$.
ure 4.11

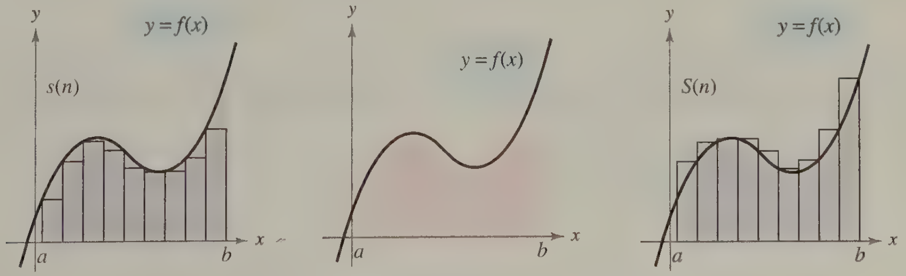
The sum of the areas of the inscribed rectangles is called a **lower sum,** and the sum of the areas of the circumscribed rectangles is called an **upper sum.**

$$\text{Lower sum} = s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x \qquad \text{Area of inscribed rectangles}$$

$$\text{Upper sum} = S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x \qquad \text{Area of circumscribed rectangles}$$

From Figure 4.12, you can see that the lower sum $s(n)$ is less than or equal to the upper sum $S(n)$. Moreover, the actual area of the region lies between these two sums.

$$s(n) \le (\text{Area of region}) \le S(n)$$

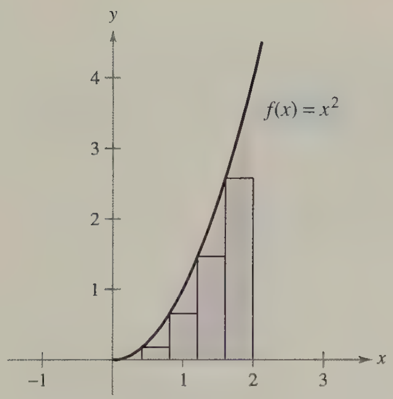


Area of inscribed rectangles is less than area of region.

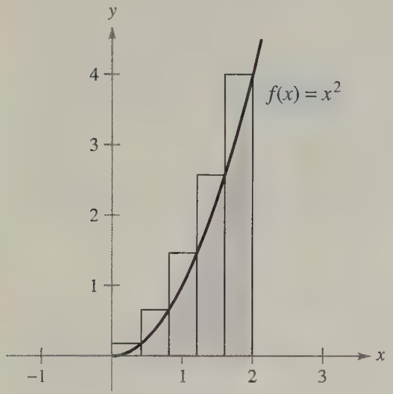
Area of region

Area of circumscribed rectangles is greater than area of region.

Figure 4.12



Inscribed rectangles



Circumscribed rectangles
Figure 4.13

Example 4 Finding Upper and Lower Sums for a Region

Find the upper and lower sums for the region bounded by the graph of $f(x) = x^2$ the x-axis between $x = 0$ and $x = 2$.

Solution To begin, partition the interval $[0, 2]$ into n subintervals, each of width.

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}.$$

Figure 4.13 shows the endpoints of the subintervals and several inscribed circumscribed rectangles. Because f is increasing on the interval $[0, 2]$, the minim value on each subinterval occurs at the left endpoint, and the maximum value occ at the right endpoint.

Left Endpoints	*Right Endpoints*
$m_i = 0 + (i-1)\left(\frac{2}{n}\right) = \frac{2(i-1)}{n}$	$M_i = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$

Using the left endpoints, the lower sum is

$$\begin{aligned}
s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x &= \sum_{i=1}^{n} f\left[\frac{2(i-1)}{n}\right]\left(\frac{2}{n}\right)\\
&= \sum_{i=1}^{n}\left[\frac{2(i-1)}{n}\right]^2\left(\frac{2}{n}\right)\\
&= \sum_{i=1}^{n}\left(\frac{8}{n^3}\right)(i^2 - 2i + 1)\\
&= \frac{8}{n^3}\left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right)\\
&= \frac{8}{n^3}\left\{\frac{n(n+1)(2n+1)}{6} - 2\left[\frac{n(n+1)}{2}\right] + n\right\}\\
&= \frac{4}{3n^3}(2n^3 - 3n^2 + n)\\
&= \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}. \qquad \text{Lower sum}
\end{aligned}$$

Using the right endpoints, the upper sum is

$$\begin{aligned}
S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x &= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)\\
&= \sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)\\
&= \sum_{i=1}^{n}\left(\frac{8}{n^3}\right)i^2\\
&= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\\
&= \frac{4}{3n^3}(2n^3 + 3n^2 + n)\\
&= \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}. \qquad \text{Upper sum}
\end{aligned}$$

EXPLORATION

the region given in Example 4, luate the lower sum

$$s(n) = \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}$$

the upper sum

$$S(n) = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

$n = 10, 100$, and 1000. Use your ults to determine the area of the ion.

Example 4 illustrates some important things about lower and upper sums. First, notice that for any value of n, the lower sum is less than (or equal to) the upper sum.

$$s(n) = \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2} < \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} = S(n)$$

Second, the difference between these two sums lessens as n increases. In fact, if you take the limit as $n \to \infty$, both the upper sum and the lower sum approach $\frac{8}{3}$.

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty}\left(\frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3}$$ Lower sum limit

$$\lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3}$$ Upper sum limit

The next theorem shows that the equivalence of the limits (as $n \to \infty$) of the upper and lower sums is not mere coincidence. It is true for all functions that are continuous and nonnegative on the closed interval $[a, b]$. The proof of this theorem is best left to a course in advanced calculus.

THEOREM 4.3 Limit of the Lower and Upper Sums

Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\begin{aligned}\lim_{n\to\infty} s(n) &= \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\,\Delta x\\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\,\Delta x\\ &= \lim_{n\to\infty} S(n)\end{aligned}$$

where $\Delta x = (b - a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

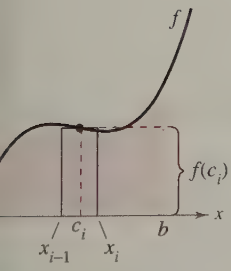
Because the same limit is attained for both the minimum value $f(m_i)$ and the maximum value $f(M_i)$, it follows from the Squeeze Theorem (Theorem 1.8) that the choice of x in the ith subinterval does not affect the limit. This means that you are free to choose an *arbitrary* x-value in the ith subinterval, as in the following *definition of the area of a region in the plane.*

Definition of the Area of a Region in the Plane

Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x, \qquad x_{i-1} \le c_i \le x_i$$

where $\Delta x = (b - a)/n$ (see Figure 4.14).

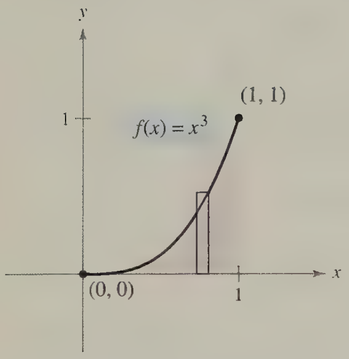


vidth of the ith subinterval is $= x_i - x_{i-1}$.

re 4.14

Example 5 Finding Area by the Limit Definition

Find the area of the region bounded by the graph $f(x) = x^3$, the x-axis, and the vert lines $x = 0$ and $x = 1$, as shown in Figure 4.15.



The area of the region bounded by the graph of f, the x-axis, $x = 0$, and $x = 1$ is $\frac{1}{4}$.
Figure 4.15

Solution Begin by noting that f is continuous and nonnegative on the interval [0, Next, partition the interval [0, 1] into n subintervals, each of width $\Delta x = 1$ According to the definition of area, you can choose any x-value in the ith subinter For this example, the right endpoints $c_i = i/n$ are convenient.

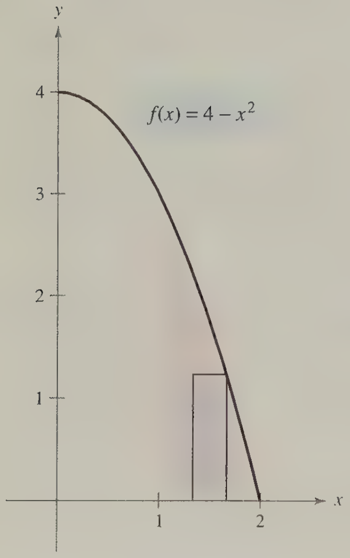
Right endpoints: $c_i = \frac{i}{n}$

$$\begin{aligned}\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\left(\frac{1}{n}\right)\\ &= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n} i^3\\ &= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]\\ &= \lim_{n\to\infty}\left(\frac{1}{4}+\frac{1}{2n}+\frac{1}{4n^2}\right)\\ &= \frac{1}{4}\end{aligned}$$

The area of the region is $\frac{1}{4}$.

Example 6 Finding Area by the Limit Definition

Find the area of the region bounded by the graph of $f(x) = 4 - x^2$, the x-axis, and vertical lines $x = 1$ and $x = 2$, as shown in Figure 4.16.



The area of the region bounded by the graph of f, the x-axis, $x = 1$, and $x = 2$ is $\frac{5}{3}$.
Figure 4.16

Solution The function f is continuous and nonnegative on the interval [1, 2], and you begin by partitioning the interval into n subintervals, each of width $\Delta x = 1$ Choosing the right endpoint,

$$c_i = a + i\Delta x = 1 + \frac{i}{n}$$

Right endpoints

of each subinterval, you obtain the following.

$$\begin{aligned}\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[4-\left(1+\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right)\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(3-\frac{2i}{n}-\frac{i^2}{n^2}\right)\left(\frac{1}{n}\right)\\ &= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}3-\frac{2}{n^2}\sum_{i=1}^{n}i-\frac{1}{n^3}\sum_{i=1}^{n}i^2\right)\\ &= \lim_{n\to\infty}\left[3-\left(1+\frac{1}{n}\right)-\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)\right]\\ &= 3-1-\frac{1}{3}\\ &= \frac{5}{3}\end{aligned}$$

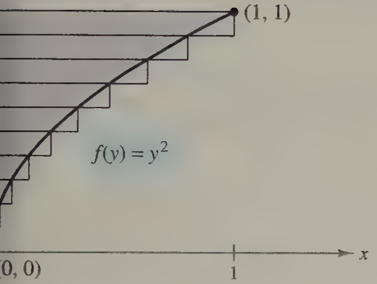
The area of the region is $\frac{5}{3}$.

The last example in this section looks at a region that is bounded by the y-axis (rather than by the x-axis).

Example 7 A Region Bounded by the y-Axis

Find the area of the region bounded by the graph of $f(y) = y^2$ and the y-axis for $0 \le y \le 1$, as shown in Figure 4.17.

Solution When f is a continuous, nonnegative function of y, you still can use the same basic procedure illustrated in Examples 5 and 6. Begin by partitioning the interval $[0, 1]$ into n subintervals, each of width $\Delta y = 1/n$. Then, using the upper endpoints $c_i = i/n$, you obtain the following.



area of the region bounded by the graph
f and the y-axis for $0 \le y \le 1$ is $\frac{1}{3}$.
ure 4.17

$$\begin{aligned}\text{Area} &= \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta y = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right) \qquad \text{Upper endpoints: } c_i = \frac{i}{n}\\ &= \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n} i^2\\ &= \lim_{n\to\infty}\frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\\ &= \lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)\\ &= \frac{1}{3}\end{aligned}$$

The area of the region is $\frac{1}{3}$.

XERCISES FOR SECTION 4.2

Exercises 1–6, find the sum. Use the summation capabilities graphing utility to verify your result.

$\sum_{i=1}^{5}(2i+1)$

2. $\sum_{k=3}^{6}k(k-2)$

$\sum_{k=0}^{4}\frac{1}{k^2+1}$

4. $\sum_{j=3}^{5}\frac{1}{j}$

$\sum_{k=1}^{4}c$

6. $\sum_{i=1}^{4}[(i-1)^2+(i+1)^3]$

Exercises 7–14, use sigma notation to write the sum.

$\frac{1}{3(1)}+\frac{1}{3(2)}+\frac{1}{3(3)}+\cdots+\frac{1}{3(9)}$

$\frac{5}{1+1}+\frac{5}{1+2}+\frac{5}{1+3}+\cdots+\frac{5}{1+15}$

$\left[5\left(\frac{1}{8}\right)+3\right]+\left[5\left(\frac{2}{8}\right)+3\right]+\cdots+\left[5\left(\frac{8}{8}\right)+3\right]$

$\left[1-\left(\frac{1}{4}\right)^2\right]+\left[1-\left(\frac{2}{4}\right)^2\right]+\cdots+\left[1-\left(\frac{4}{4}\right)^2\right]$

$\left[\left(\frac{2}{n}\right)^3-\frac{2}{n}\right]\left(\frac{2}{n}\right)+\cdots+\left[\left(\frac{2n}{n}\right)^3-\frac{2n}{n}\right]\left(\frac{2}{n}\right)$

12. $\left[1-\left(\frac{2}{n}-1\right)^2\right]\left(\frac{2}{n}\right)+\cdots+\left[1-\left(\frac{2n}{n}-1\right)^2\right]\left(\frac{2}{n}\right)$

13. $\left[2\left(1+\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right)+\cdots+\left[2\left(1+\frac{3n}{n}\right)^2\right]\left(\frac{3}{n}\right)$

14. $\left(\frac{1}{n}\right)\sqrt{1-\left(\frac{0}{n}\right)^2}+\cdots+\left(\frac{1}{n}\right)\sqrt{1-\left(\frac{n-1}{n}\right)^2}$

In Exercises 15–20, use the properties of summation and Theorem 4.2 to evaluate the sum. Use the summation capabilities of a graphing utility to verify your result.

15. $\sum_{i=1}^{20}2i$

16. $\sum_{i=1}^{15}(2i-3)$

17. $\sum_{i=1}^{20}(i-1)^2$

18. $\sum_{i=1}^{10}(i^2-1)$

19. $\sum_{i=1}^{15}i(i-1)^2$

20. $\sum_{i=1}^{10}i(i^2+1)$

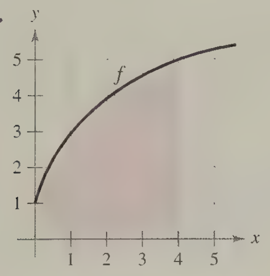
In Exercises 21 and 22, use the summation capabilities of a graphing utility to evaluate the sum. Then use the properties of summation and Theorem 4.2 to verify the sum.

21. $\sum_{i=1}^{20}(i^2+3)$

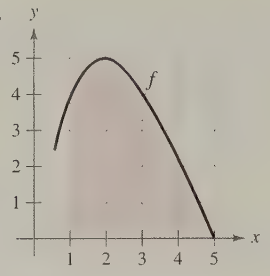
22. $\sum_{i=1}^{15}(i^3-2i)$

In Exercises 23–26, bound the area of the shaded region by approximating the upper and lower sums. Use rectangles of width 1.

23.



24.



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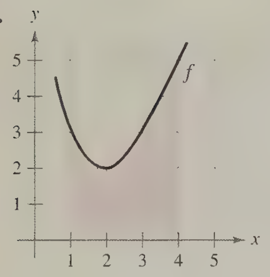
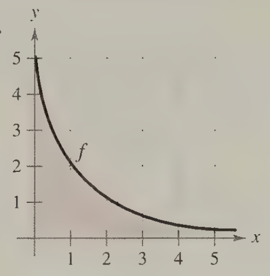
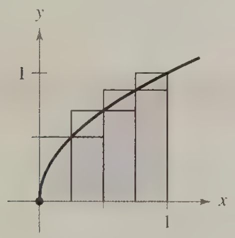


26.

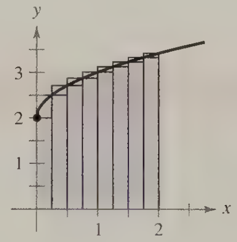


In Exercises 27–30, use upper and lower sums to approximate the area of the region using the indicated number of subintervals (of equal width).

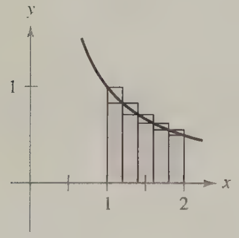
27. $y = \sqrt{x}$



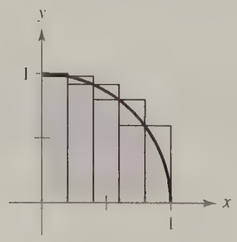
28. $y = \sqrt{x} + 2$



29. $y = \dfrac{1}{x}$



30. $y = \sqrt{1 - x^2}$



In Exercises 31–34, find the limit of $s(n)$ as $n \to \infty$.

31. $s(n) = \dfrac{81}{n^4}\left[\dfrac{n^2(n+1)^2}{4}\right]$

32. $s(n) = \dfrac{64}{n^3}\left[\dfrac{n(n+1)(2n+1)}{6}\right]$

33. $s(n) = \dfrac{18}{n^2}\left[\dfrac{n(n+1)}{2}\right]$

34. $s(n) = \dfrac{1}{n^2}\left[\dfrac{n(n+1)}{2}\right]$

In Exercises 35–38, use the summation formulas to rewrite t expression without the summation notation. Use the result find the sum for $n = 10, 100, 1000$, and $10{,}000$.

35. $\displaystyle\sum_{i=1}^{n} \frac{2i+1}{n^2}$

36. $\displaystyle\sum_{j=1}^{n} \frac{4j+3}{n^2}$

37. $\displaystyle\sum_{k=1}^{n} \frac{6k(k-1)}{n^3}$

38. $\displaystyle\sum_{i=1}^{n} \frac{4i^2(i-1)}{n^4}$

In Exercises 39–44, find a formula for the sum of n terms. U the formula to find the limit as $n \to \infty$.

39. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{16i}{n^2}$

40. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$

41. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}(i-1)^2$

42. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^2\Big($

43. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)\left(\frac{2}{n}\right)$

44. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^3\Big($

45. ***Numerical Reasoning*** Consider a triangle of area 2 bound by the graphs of $y = x$, $y = 0$, and $x = 2$.

(a) Sketch the region.

(b) Divide the interval $[0, 2]$ into n subintervals of equal wid and show that the endpoints are

$$0 < 1\left(\frac{2}{n}\right) < \cdots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right).$$

(c) Show that $s(n) = \displaystyle\sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(d) Show that $S(n) = \displaystyle\sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(e) Complete the table.

n	5	10	50	10
$s(n)$				
$S(n)$				

(f) Show that $\displaystyle\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n) = 2$.

46. ***Numerical Reasoning*** Consider a trapezoid of area 4 bound by the graphs of $y = x$, $y = 0$, $x = 1$, and $x = 3$.

(a) Sketch the region.

(b) Divide the interval $[1, 3]$ into n subintervals of equal wid and show that the endpoints are

$$1 < 1 + 1\left(\frac{2}{n}\right) < \cdots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right)$$

(c) Show that $s(n) = \displaystyle\sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(d) Show that $S(n) = \displaystyle\sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(e) Complete the table.

n	5	10	50	10
$s(n)$				
$S(n)$				

(f) Show that $\displaystyle\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n) = 4$.

Exercises 47–56, use the limit process to find the area of the ion between the graph of the function and the x-axis over the icated interval. Sketch the region.

Function	*Interval*	*Function*	*Interval*
$y = -2x + 3$	$[0, 1]$	**48.** $y = 3x - 4$	$[2, 5]$
$y = x^2 + 2$	$[0, 1]$	**50.** $y = x^2 + 1$	$[0, 3]$
$y = 16 - x^2$	$[1, 3]$	**52.** $y = 1 - x^2$	$[-1, 1]$
$y = 64 - x^3$	$[1, 4]$	**54.** $y = 2x - x^3$	$[0, 1]$
$y = x^2 - x^3$	$[-1, 1]$	**56.** $y = x^2 - x^3$	$[-1, 0]$

Exercises 57–62, use the limit process to find the area of the ion between the graph of the function and the y-axis over the icated y-interval. Sketch the region.

$f(y) = 3y, 0 \le y \le 2$ **58.** $g(y) = \frac{1}{2}y, 2 \le y \le 4$

$f(y) = y^2, 0 \le y \le 3$ **60.** $f(y) = 4y - y^2, 1 \le y \le 2$

$g(y) = 4y^2 - y^3, 1 \le y \le 3$

$h(y) = y^3 + 1, 1 \le y \le 2$

Exercises 63–66 use the *Midpoint Rule*

$$a \approx \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x$$

h $n = 4$ to approximate the area of the region bounded the graph of the function and the x-axis over the indicated rval.

Function	*Interval*
$f(x) = x^2 + 3$	$[0, 2]$
$f(x) = x^2 + 4x$	$[0, 4]$
$f(x) = \tan x$	$\left[0, \frac{\pi}{4}\right]$
$f(x) = \sin x$	$\left[0, \frac{\pi}{2}\right]$

ite a program for a graphing utility to approximate areas by ng the Midpoint Rule. Assume that the function is positive r the indicated interval and the subintervals are of equal th. In Exercises 67–70, use the program to approximate the a of the region between the graph of the function and the xis over the indicated interval, and complete the table.

	4	8	12	16	20
pproximate area					

Function	*Interval*	*Function*	*Interval*
$f(x) = \sqrt{x}$	$[0, 4]$	**68.** $f(x) = \frac{8}{x^2 + 1}$	$[2, 6]$
$f(x) = \tan\left(\frac{\pi x}{8}\right)$	$[1, 3]$	**70.** $f(x) = \cos\sqrt{x}$	$[0, 2]$

Getting at the Concept

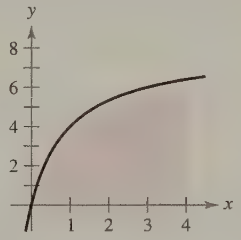
71. In your own words and using appropriate figures, describe the methods of upper sums and lower sums in approximating the area of a region.

72. Give the definition of the area of a region in the plane.

73. ***Graphical Reasoning*** Consider the region bounded by the graphs of

$$f(x) = \frac{8x}{x + 1},$$

$x = 0$, $x = 4$, and $y = 0$, as shown in the figure. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*



(a) Redraw the figure, and complete and shade the rectangles representing the lower sum when $n = 4$. Find this lower sum.

(b) Redraw the figure, and complete and shade the rectangles representing the upper sum when $n = 4$ Find this upper sum.

(c) Redraw the figure, and complete and shade the rectangles whose heights are determined by the functional values at the midpoint of each subinterval when $n = 4$. Find this sum using the Midpoint Rule.

(d) Verify the following formulas for approximating the area of the region using n subintervals of equal width.

Lower sum: $s(n) = \sum_{i=1}^{n} f\left[(i - 1)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

Upper sum: $S(n) = \sum_{i=1}^{n} f\left[(i)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

Midpoint Rule: $M(n) = \sum_{i=1}^{n} f\left[\left(i - \frac{1}{2}\right)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

(e) Use a graphing utility and the formulas in part (d) to complete the table.

n	4	8	20	100	200
$s(n)$					
$S(n)$					
$M(n)$					

(f) Explain why $s(n)$ increases and $S(n)$ decreases for increasing values of n, as shown in the table in part (e).

74. Use a graphing utility to complete the table for approximations of the area of the region bounded by the graphs of $f(x) = \sqrt[3]{x}$, $x = 0$, $x = 8$, and $y = 0$.

n	10	20	50	100	200
$s(n)$					
$S(n)$					
$M(n)$					

Approximation **In Exercises 75 and 76, determine which value best approximates the area of the region between the x-axis and the graph of the function over the indicated interval. (Make your selection on the basis of a sketch of the region and not by performing calculations.)**

75. $f(x) = 4 - x^2$, $[0, 2]$

(a) -2 (b) 6 (c) 10 (d) 3 (e) 8

76. $f(x) = \sin \dfrac{\pi x}{4}$, $[0, 4]$

(a) 3 (b) 1 (c) -2 (d) 8 (e) 6

True or False? **In Exercises 77 and 78, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

77. The sum of the first n positive integers is $n(n + 1)/2$.

78. If f is continuous and nonnegative on $[a, b]$, then the limits as $n \to \infty$ of its lower sum $s(n)$ and upper sum $S(n)$ both exist and are equal.

79. ***Monte Carlo Method*** The following computer program approximates the area of the region under the graph of a monotonic function and above the x-axis between $x = a$ and $x = b$. Run the program for $a = 0$ and $b = \pi/2$ for several values of N2. Explain why the Monte Carlo Method works. [*Adaptation of Monte Carlo Method program from James M. Sconyers, "Approximation of Area Under a Curve," MATHEMATICS TEACHER 77, no. 2 (February 1984). Copyright © 1984 by the National Council of Teachers of Mathematics. Reprinted with permission.*]

```
 10  DEF FNF(X)=SIN(X)
 20  A=0
 30  B=1.570796
 40  PRINT "Input Number of Random Points"
 50  INPUT N2
 60  N1=0
 70  IF FNF(A)>FNF(B) THEN YMAX=FNF(A) ELSE
     YMAX=FNF(B)
 80  FOR I=1 TO N2
 90  X=A+(B-A)*RND(1)
100  Y=YMAX*RND(1)
110  IF Y>=FNF(X) THEN GOTO 130
120  N1=N1+1
130  NEXT I
140  AREA=(N1/N2)*(B-A)*YMAX
150  PRINT "Approximate Area:"; AREA
160  END
```

80. ***Graphical Reasoning*** Consider an n-sided regular polyg inscribed in a circle of radius r. Join the vertices of the polygon to center of the circle, forming n congruent triangles (see figure).

(a) Determine the central angle θ in terms of n.

(b) Show that the area of each triangle is $\frac{1}{2}r^2 \sin \theta$.

(c) Let A_n be the sum of the areas of the n triangles. F $\lim\limits_{n\to\infty} A_n$.

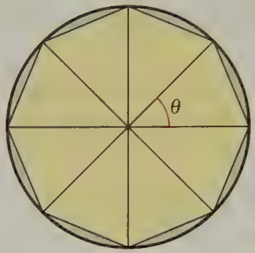


Figure for 80

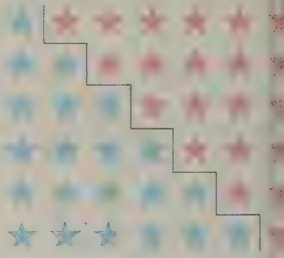


Figure for 81

81. ***Writing*** Use the figure to write a short paragraph explain why the formula

$$1 + 2 + \cdot \cdot \cdot + n = \tfrac{1}{2}n(n + 1)$$

is valid for all positive integers n.

82. Prove each of the formulas by mathematical induction. (Y may need to review the method of proof by induction fron precalculus text.)

(a) $\displaystyle\sum_{i=1}^{n} 2i = n(n + 1)$

(b) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n + 1)^2}{4}$

83. ***Modeling Data*** The table lists the measurements of a bounded by a stream and two straight roads that meet at ri angles, where x and y are measured in feet (see figure).

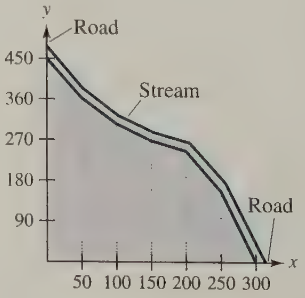
x	0	50	100	150	200	250	300
y	450	362	305	268	245	156	0

(a) Use the regression capabilities of a graphing utility to fi a model of the form

$$y = ax^3 + bx^2 + cx + d.$$

(b) Use a graphing utility to plot the data and graph the moc

(c) Use the model in part (a) to estimate the area of the lot.



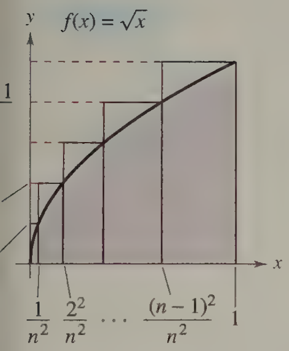
Section 4.3 Riemann Sums and Definite Integrals

- Understand the definition of a Riemann sum.
- Evaluate a definite integral using limits.
- Evaluate a definite integral using properties of definite integrals.

Riemann Sums

In the definition of area given in Section 4.2, the partitions have subintervals of *equal width*. This was done only for computational convenience. We begin this section with an example that shows that it is not necessary to have subintervals of equal width.

Example 1 A Partition with Subintervals of Unequal Widths



e subintervals do not have equal widths.
ure 4.18

Consider the region bounded by the graph of $f(x) = \sqrt{x}$ and the x-axis for $0 \le x \le 1$, as shown in Figure 4.18. Evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

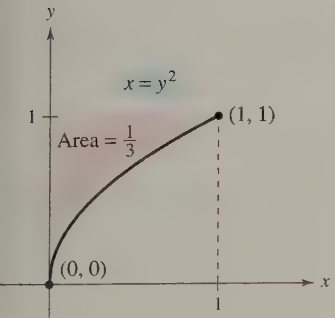
where c_i is the right endpoint of the partition given by $x_i = i^2/n^2$ and Δx_i is the width of the ith interval.

Solution The width of the ith interval is given by

$$\begin{aligned}\Delta x_i &= \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2}\\ &= \frac{i^2 - i^2 + 2i - 1}{n^2}\\ &= \frac{2i-1}{n^2}.\end{aligned}$$

So, the limit is

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{i^2}{n^2}}\left(\frac{2i-1}{n^2}\right)\\ &= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} (2i^2 - i)\\ &= \lim_{n\to\infty} \frac{1}{n^3}\left[2\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{n(n+1)}{2}\right]\\ &= \lim_{n\to\infty} \frac{4n^3 + 3n^2 - n}{6n^3}\\ &= \frac{2}{3}.\end{aligned}$$



area of the region bounded by
graph of $x = y^2$ and the y-axis for
$y \le 1$ is $\frac{1}{3}$.
ure 4.19

From Example 7 in Section 4.2, you know that the region shown in Figure 4.19 has an area of $\frac{1}{3}$. Because the square bounded by $0 \le x \le 1$ and $0 \le y \le 1$ has an area of 1, you can conclude that the area of the region shown in Figure 4.18 has an area of $\frac{2}{3}$. This agrees with the limit found in Example 1, even though that example used a partition having subintervals of unequal widths. The reason this particular partition gave the proper area is that as n increases, the *width of the largest subinterval approaches zero*. This is a key feature of the development of definite integrals.

The Granger Collection

Georg Friedrich Bernhard Riemann (1826–1866)

German mathematician Riemann did his most famous work in the areas of non-Euclidean geometry, differential equations, and number theory. It was Riemann's results in physics and mathematics that formed the structure on which Einstein's theory of general relativity is based.

In the preceding section, the limit of a sum was used to define the area of a regi in the plane. Finding area by this means is only one of *many* applications involvi the limit of a sum. A similar approach can be used to determine quantities as diver as arc length, average value, centroids, volumes, work, and surface areas. The follo ing definition is named after Georg Friedrich Bernhard Riemann. Although t definite integral had been defined and used long before the time of Riemann, generalized the concept to cover a broader category of functions.

In the following definition of a Riemann sum, note that the function f has restrictions other than being defined on the interval $[a, b]$. (In the preceding sectio the function f was assumed to be continuous and nonnegative because we we dealing with the area under a curve.)

Definition of a Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is *any* point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \qquad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ.

NOTE The sums in Section 4.2 are examples of Riemann sums, but there are more gene Riemann sums than those covered there.

The width of the largest subinterval of a partition Δ is the **norm** of the partiti and is denoted by $\|\Delta\|$. If every subinterval is of equal width, the partition is **regul** and the norm is denoted by

$$\|\Delta\| = \Delta x = \frac{b-a}{n}. \qquad \text{Regular partition}$$

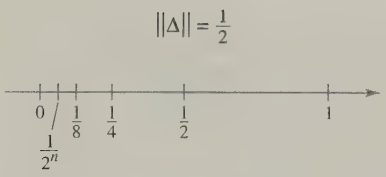
For a general partition, the norm is related to the number of subintervals of $[a, b]$ the following way.

$$\frac{b-a}{\|\Delta\|} \le n \qquad \text{General partition}$$

So, the number of subintervals in a partition approaches infinity as the norm of t partition approaches 0. That is, $\|\Delta\| \to 0$ implies that $n \to \infty$.

The converse of this statement is not true. For example, let Δ_n be the partition the interval $[0, 1]$ given by

$$0 < \frac{1}{2^n} < \frac{1}{2^{n-1}} < \cdot \cdot \cdot < \frac{1}{8} < \frac{1}{4} < \frac{1}{2} < 1.$$



$n \to \infty$ does not imply that $\|\Delta\| \to 0$.
Figure 4.20

As shown in Figure 4.20, for any positive value of n, the norm of the partition Δ_n is So, letting n approach infinity does not force $\|\Delta\|$ to approach 0. In a regular partitic however, the statements $\|\Delta\| \to 0$ and $n \to \infty$ are equivalent.

Definite Integrals

To define the definite integral, consider the following limit.

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = L$$

To say that this limit exists means that for $\varepsilon > 0$ there exists a $\delta > 0$ such that for every partition with $\|\Delta\| < \delta$ it follows that

$$\left| L - \sum_{i=1}^{n} f(c_i)\,\Delta x_i \right| < \varepsilon.$$

(This must be true for any choice of c_i in the ith subinterval of Δ.)

R FURTHER INFORMATION For ght into the history of the definite gral, see the article "The Evolution ntegration" by A. Shenitzer and J. rāns in *The American Mathematical nthly*. To view this article, go to the site *www.matharticles.com*.

Definition of a Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

exists (as described above), then f is **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

It is not a coincidence that the notation for definite integrals is similar to that used for indefinite integrals. You will see why in the next section when we discuss the Fundamental Theorem of Calculus. For now it is important to see that definite integrals and indefinite integrals are different identities. A definite integral is a *number*, whereas an indefinite integral is a *family of functions*.

A sufficient condition for a function f to be integrable on $[a, b]$ is that it is continuous on $[a, b]$. A proof of this theorem is beyond the scope of this text.

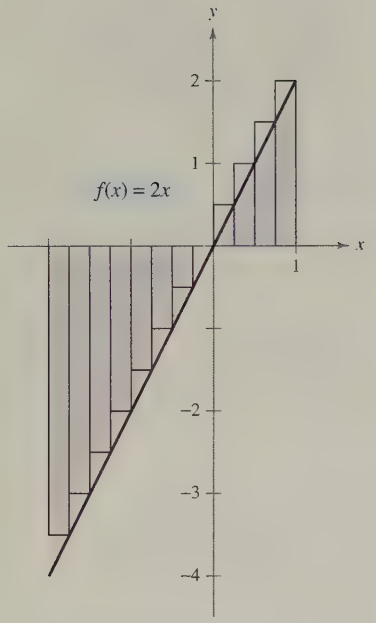
THEOREM 4.4 Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$.

EXPLORATION

The Converse of Theorem 4.4 Is the converse of Theorem 4.4 true? That is, if a function is integrable, does it have to be continuous? Explain your reasoning and give examples.

Describe the relationships among continuity, differentiability, and integrability. Which is the strongest condition? Which is the weakest? Which conditions imply other conditions?



Because the definite integral is negative, it does not represent the area of the region.
Figure 4.21

Example 2 Evaluating a Definite Integral as a Limit

Evaluate the definite integral $\int_{-2}^{1} 2x\,dx$.

Solution The function $f(x) = 2x$ is integrable on the interval $[-2, 1]$ because i continuous on $[-2, 1]$. Moreover, the definition of integrability implies that any p tition whose norm approaches 0 can be used to determine the limit. For computatio convenience, define Δ by subdividing $[-2, 1]$ into n subintervals of equal width

$$\Delta x_i = \Delta x = \frac{b-a}{n} = \frac{3}{n}.$$

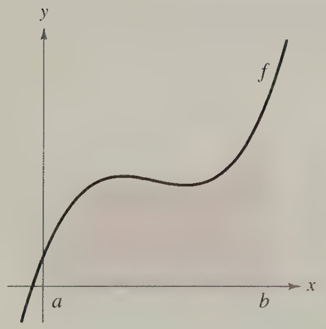
Choosing c_i as the right endpoint of each subinterval produces

$$c_i = a + i(\Delta x) = -2 + \frac{3i}{n}.$$

So, the definite integral is given by

$$\begin{aligned}
\int_{-2}^{1} 2x\,dx &= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\,\Delta x \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} 2\left(-2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \lim_{n \to \infty} \frac{6}{n} \sum_{i=1}^{n} \left(-2 + \frac{3i}{n}\right) \\
&= \lim_{n \to \infty} \frac{6}{n} \left\{-2n + \frac{3}{n}\left[\frac{n(n+1)}{2}\right]\right\} \\
&= \lim_{n \to \infty} \left(-12 + 9 + \frac{9}{n}\right) \\
&= -3.
\end{aligned}$$

Because the definite integral in Example 2 is negative, it *does not* represent area of the region shown in Figure 4.21. Definite integrals can be positive, negati or zero. For a definite integral to be interpreted as an area (as defined in Section 4 the function f must be continuous and nonnegative on $[a, b]$, as stated in the follo ing theorem. (The proof of this theorem is straightforward—you simply use the d nition of area given in Section 4.2.)



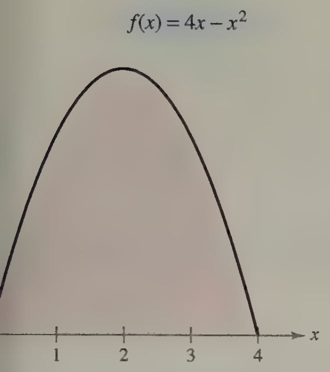
You can use a definite integral to find the area of the region bounded by the graph of f, the x-axis, $x = a$, and $x = b$.
Figure 4.22

THEOREM 4.5 The Definite Integral as the Area of a Region

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_{a}^{b} f(x)\,dx.$$

(See Figure 4.22.)



$= \int_0^4 (4x - x^2)\,dx$

re 4.23

As an example of Theorem 4.5, consider the region bounded by the graph of

$$f(x) = 4x - x^2$$

and the x-axis, as shown in Figure 4.23. Because f is continuous and nonnegative on the closed interval $[0, 4]$, the area of the region is

$$\text{Area} = \int_0^4 (4x - x^2)\,dx.$$

A straightforward technique for evaluating a definite integral such as this will be discussed in Section 4.4. For now, however, you can evaluate a definite integral in two ways—you can use the limit definition *or* you can check to see whether the definite integral represents the area of a common geometric region such as a rectangle, triangle, or semicircle.

Example 3 Areas of Common Geometric Figures

Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

a. $\int_1^3 4\,dx$ **b.** $\int_0^3 (x + 2)\,dx$ **c.** $\int_{-2}^2 \sqrt{4 - x^2}\,dx$

Solution A sketch of each region is shown in Figure 4.24.

a. This region is a rectangle of height 4 and width 2.

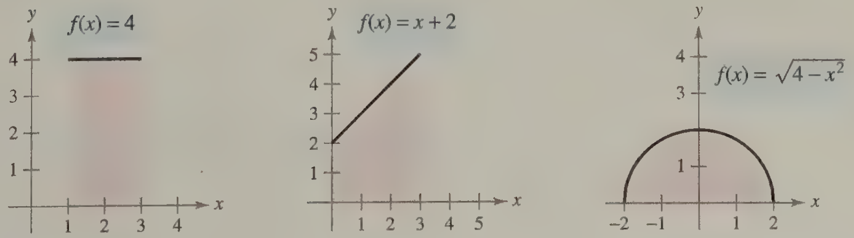
$$\int_1^3 4\,dx = (\text{Area of rectangle}) = 4(2) = 8$$

b. This region is a trapezoid with an altitude of 3 and parallel bases of lengths 2 and 5. The formula for the area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$.

$$\int_0^3 (x + 2)\,dx = (\text{Area of trapezoid}) = \frac{1}{2}(3)(2 + 5) = \frac{21}{2}$$

c. This region is a semicircle of radius 2. The formula for the area of a semicircle is $\frac{1}{2}\pi r^2$.

$$\int_{-2}^2 \sqrt{4 - x^2}\,dx = (\text{Area of semicircle}) = \frac{1}{2}\pi(2^2) = 2\pi$$



(a) (b) (c)

Figure 4.24

E The variable of integration in
inite integral is sometimes called
nmy variable because it can be
ced by any other variable without
ging the value of the integral. For
nce, the definite integrals

$$\int_0^3 (x + 2)\,dx$$

$$\int_0^3 (t + 2)\,dt$$

the same value.

Properties of Definite Integrals

The definition of the definite integral of f on the interval $[a, b]$ specifies that $a <$ Now, however, it is convenient to extend the definition to cover cases in which $a =$ or $a > b$. Geometrically, the following two definitions seem reasonable. For instan it makes sense to define the area of a region of zero width and finite height to be

Definition of Two Special Definite Integrals

1. If f is defined at $x = a$, then $\int_a^a f(x)\,dx = 0$.
2. If f is integrable on $[a, b]$, then $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

Example 4 Evaluating Definite Integrals

a. Because the sine function is defined at $x = \pi$, and the upper and lower limits integration are equal, you can write

$$\int_\pi^\pi \sin x\,dx = 0.$$

b. The integral $\int_3^0 (x + 2)\,dx$ is the same as that given in Example 3b except that upper and lower limits are interchanged. Because the integral in Example 3b ha value of $\frac{21}{2}$, you can write

$$\int_3^0 (x + 2)\,dx = -\int_0^3 (x + 2)\,dx = -\frac{21}{2}.$$

In Figure 4.25, the larger region can be divided at $x = c$ into two subregions wh intersection is a line segment. Because the line segment has zero area, it follows the area of the larger region is equal to the sum of the areas of the two smaller regio

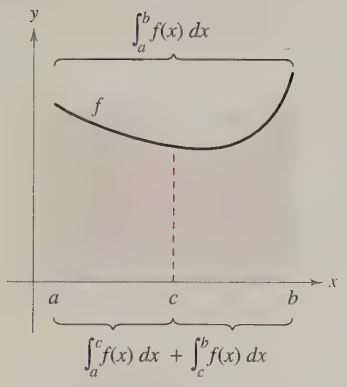


Figure 4.25

THEOREM 4.6 Additive Interval Property

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example 5 Using the Additive Interval Property

$$\int_{-1}^1 |x|\,dx = \int_{-1}^0 -x\,dx + \int_0^1 x\,dx \qquad \text{Theorem 4.6}$$

$$= \frac{1}{2} + \frac{1}{2} \qquad \text{Area of a triangle}$$

$$= 1$$

Because the definite integral is defined as the limit of a sum, it inherits the properties of summation given at the top of page 254.

THEOREM 4.7 Properties of Definite Integrals

If f and g are integrable on $[a, b]$ and k is a constant, then the functions of kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

2. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

Note that Property 2 of Theorem 4.7 can be extended to cover any finite number of functions. For example,

$$\int_a^b [f(x) + g(x) + h(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \int_a^b h(x)\,dx.$$

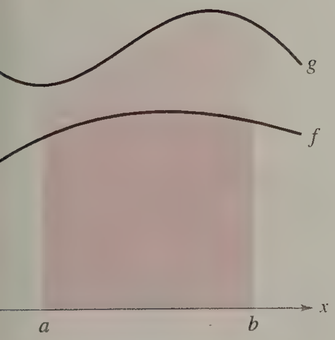
Example 6 Evaluation of a Definite Integral

Evaluate $\displaystyle\int_1^3 (-x^2 + 4x - 3)\,dx$ using each of the following values.

$$\int_1^3 x^2\,dx = \frac{26}{3}, \qquad \int_1^3 x\,dx = 4, \qquad \int_1^3 dx = 2$$

Solution

$$\begin{aligned}\int_1^3 (-x^2 + 4x - 3)\,dx &= \int_1^3 (-x^2)\,dx + \int_1^3 4x\,dx + \int_1^3 (-3)\,dx\\ &= -\int_1^3 x^2\,dx + 4\int_1^3 x\,dx - 3\int_1^3 dx\\ &= -\left(\frac{26}{3}\right) + 4(4) - 3(2)\\ &= \frac{4}{3}\end{aligned}$$



$f(x)\,dx \le \displaystyle\int_a^b g(x)\,dx$

ure 4.26

If f and g are continuous on the closed interval $[a, b]$ and

$0 \le f(x) \le g(x)$

for $a \le x \le b$, the following properties are true. First, the area of the region bounded by the graph of f and the x-axis (between a and b) must be nonnegative. Second, this area must be less than or equal to the area of the region bounded by the graph of g and the x-axis (between a and b), as shown in Figure 4.26. These two results are generalized in Theorem 4.8. (A proof of this theorem is given in Appendix B.)

THEOREM 4.8 Preservation of Inequality

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \leq \int_a^b f(x)\,dx.$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)\,dx \int_a^b g(x)\,dx.$$

EXERCISES FOR SECTION 4.3

In Exercises 1 and 2, use Example 1 as a model to evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

over the region bounded by the graphs of the equations.

1. $f(x) = \sqrt{x}, \quad y = 0, \quad x = 0, \quad x = 3$

(*Hint:* Let $c_i = 3i^2/n^2$.)

2. $f(x) = \sqrt[3]{x}, \quad y = 0, \quad x = 0, \quad x = 1$

(*Hint:* Let $c_i = i^3/n^3$.)

In Exercises 3–8, evaluate the definite integral by the limit definition.

3. $\displaystyle\int_4^{10} 6\,dx$ **4.** $\displaystyle\int_{-2}^{3} x\,dx$

5. $\displaystyle\int_1^{1} x^3\,dx$ **6.** $\displaystyle\int_1^{3} 3x^2\,dx$

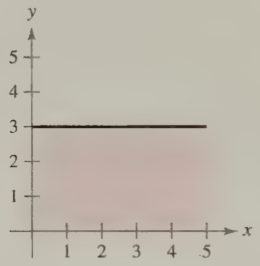
7. $\displaystyle\int_1^{2} (x^2 + 1)\,dx$ **8.** $\displaystyle\int_{-1}^{2} (3x^2 + 2)\,dx$

In Exercises 9–12, express the limit as a definite integral on the interval $[a, b]$, where c_i is any point in the ith subinterval.

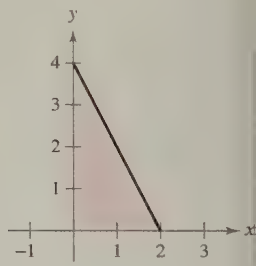
	Limit	*Interval*
9.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (3c_i + 10)\,\Delta x_i$	$[-1, 5]$
10.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 6c_i(4 - c_i)^2\,\Delta x_i$	$[0, 4]$
11.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{c_i^2 + 4}\,\Delta x_i$	$[0, 3]$
12.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \left(\frac{3}{c_i^2}\right)\Delta x_i$	$[1, 3]$

In Exercises 13–22, set up a definite integral that yields the ar of the region. (Do not evaluate the integral.)

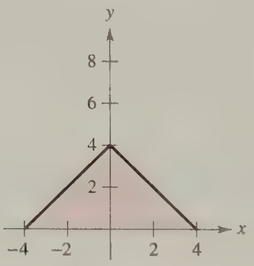
13. $f(x) = 3$



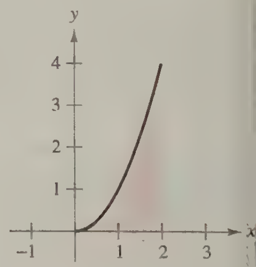
14. $f(x) = 4 - 2x$



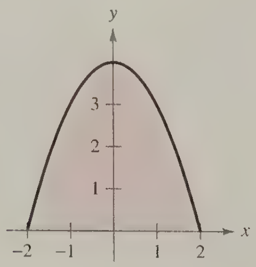
15. $f(x) = 4 - |x|$



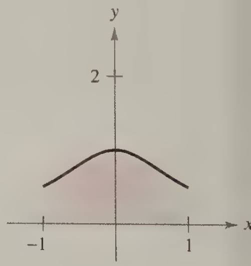
16. $f(x) = x^2$



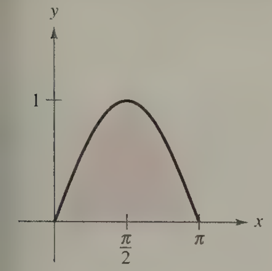
17. $f(x) = 4 - x^2$



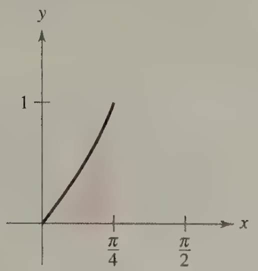
18. $f(x) = \dfrac{1}{x^2 + 1}$



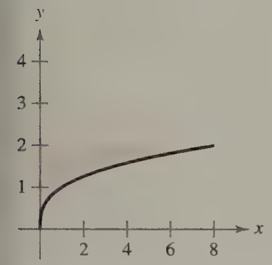
$f(x) = \sin x$



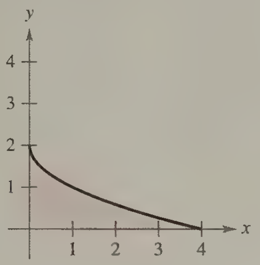
20. $f(x) = \tan x$



$g(y) = y^3$



22. $f(y) = (y - 2)^2$



Exercises 23–32, sketch the region whose area is given by the finite integral. Then use a geometric formula to evaluate the egral $(a > 0, r > 0)$.

$\int_0^3 4\,dx$

24. $\int_{-a}^{a} 4\,dx$

$\int_0^4 x\,dx$

26. $\int_0^4 \frac{x}{2}\,dx$

$\int_0^2 (2x + 5)\,dx$

28. $\int_0^8 (8 - x)\,dx$

$\int_{-1}^{1} (1 - |x|)\,dx$

30. $\int_{-a}^{a} (a - |x|)\,dx$

$\int_{-3}^{3} \sqrt{9 - x^2}\,dx$

32. $\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$

Exercises 33–40, evaluate the integral using the following ues.

$x^3\,dx = 60,$ $\quad \int_2^4 x\,dx = 6,$ $\quad \int_2^4 dx = 2$

$\int_4^2 x\,dx$

34. $\int_2^2 x^3\,dx$

$\int_2^4 4x\,dx$

36. $\int_2^4 15\,dx$

$\int_2^4 (x - 8)\,dx$

38. $\int_2^4 (x^3 + 4)\,dx$

$\int_2^4 \left(\frac{1}{2}x^3 - 3x + 2\right)dx$

40. $\int_2^4 (6 + 2x - x^3)\,dx$

41. Given $\int_0^5 f(x)\,dx = 10$ and $\int_5^7 f(x)\,dx = 3$, find

(a) $\int_0^7 f(x)\,dx.$ (b) $\int_5^0 f(x)\,dx.$

(c) $\int_5^5 f(x)\,dx.$ (d) $\int_0^5 3f(x)\,dx.$

42. Given $\int_0^3 f(x)\,dx = 4$ and $\int_3^6 f(x)\,dx = -1$, find

(a) $\int_0^6 f(x)\,dx.$ (b) $\int_6^3 f(x)\,dx.$

(c) $\int_3^3 f(x)\,dx.$ (d) $\int_3^6 -5f(x)\,dx.$

43. Given $\int_2^6 f(x)\,dx = 10$ and $\int_2^6 g(x)\,dx = -2$, find

(a) $\int_2^6 [f(x) + g(x)]\,dx.$ (b) $\int_2^6 [g(x) - f(x)]\,dx.$

(c) $\int_2^6 2g(x)\,dx.$ (d) $\int_2^6 3f(x)\,dx.$

44. Given $\int_{-1}^{1} f(x)\,dx = 0$ and $\int_0^1 f(x)\,dx = 5$, find

(a) $\int_{-1}^{0} f(x)\,dx.$ (b) $\int_0^1 f(x)\,dx - \int_{-1}^{0} f(x)\,dx.$

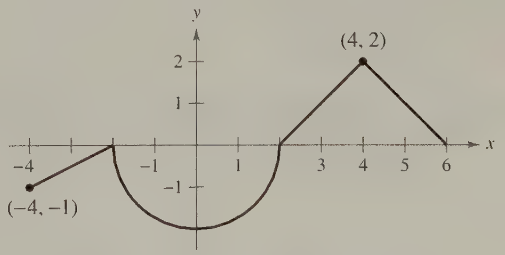
(c) $\int_{-1}^{1} 3f(x)\,dx.$ (d) $\int_0^1 3f(x)\,dx.$

45. ***Think About It*** The graph of f consists of line segments and a semicircle, as shown in the figure. Evaluate each definite integral by using geometric formulas.

(a) $\int_0^2 f(x)\,dx$ (b) $\int_2^6 f(x)\,dx$

(c) $\int_{-4}^{2} f(x)\,dx$ (d) $\int_{-4}^{6} f(x)\,dx$

(e) $\int_{-4}^{6} |f(x)|\,dx$ (f) $\int_{-4}^{6} [f(x) + 2]\,dx$



46. ***Think About It*** Consider the function f that is continuous on the interval $[-5, 5]$ and for which

$$\int_0^5 f(x)\,dx = 4.$$

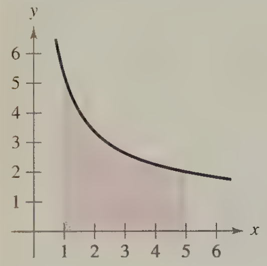
Evaluate each integral.

(a) $\int_0^5 [f(x) + 2]\,dx$ (b) $\int_{-2}^{3} f(x + 2)\,dx$

(c) $\int_{-5}^{5} f(x)\,dx$ (f is even.) (d) $\int_{-5}^{5} f(x)\,dx$ (f is odd.)

Getting at the Concept

In Exercises 47–50, use the figure to fill in the blank with the symbol <, >, or =.



47. The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the left endpoint of the ith subinterval.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \quad \square \quad \int_1^5 f(x)\,dx$$

48. The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the right endpoint of the ith subinterval.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \quad \square \quad \int_1^5 f(x)\,dx$$

49. The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the midpoint of the ith subinterval.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \quad \square \quad \int_1^5 f(x)\,dx$$

50. Let T be the average of the results of Exercises 47 and 48.

$$T \quad \square \quad \int_1^5 f(x)\,dx$$

51. Determine whether the function $f(x) = \dfrac{1}{x-4}$ is integrable on the interval $[3, 5]$. Explain.

52. Give an example of a function that is integrable on the interval $[-1, 1]$, but not continuous on $[-1, 1]$.

In Exercises 53–56, determine which value best approximates the definite integral. Make your selection on the basis of a sketch.

53. $\displaystyle\int_0^4 \sqrt{x}\,dx$

(a) 5 (b) -3 (c) 10 (d) 2 (e) 8

54. $\displaystyle\int_0^{1/2} 4\cos \pi x\,dx$

(a) 4 (b) $\frac{4}{3}$ (c) 16 (d) 2π (e) -6

55. $\displaystyle\int_0^1 2\sin \pi x\,dx$

(a) 6 (b) $\frac{1}{2}$ (c) 4 (d) $\frac{5}{4}$

56. $\displaystyle\int_0^9 \left(1+\sqrt{x}\right)dx$

(a) -3 (b) 9 (c) 27 (d) 3

Write a program for your graphing utility to approximate definite integral using the Riemann sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

where the subintervals are of equal width. The output shou give three approximations of the integral where c_i is the le hand endpoint $L(n)$, midpoint $M(n)$, and right-hand endpoi $R(n)$ of each subinterval. In Exercises 57–60, use the progra to approximate the definite integral and complete the table.

n	4	8	12	16	20
$L(n)$					
$M(n)$					
$R(n)$					

57. $\displaystyle\int_0^3 x\sqrt{3-x}\,dx$

58. $\displaystyle\int_0^3 \frac{5}{x^2+1}\,dx$

59. $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx$

60. $\displaystyle\int_0^3 x\sin x\,dx$

True or False? **In Exercises 61–66, determine whether t statement is true or false. If it is false, explain why or give example that shows it is false.**

61. $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

62. $\displaystyle\int_a^b f(x)g(x)\,dx = \left[\int_a^b f(x)\,dx\right]\left[\int_a^b g(x)\,dx\right]$

63. If the norm of a partition approaches zero, then the number subintervals approaches infinity.

64. If f is increasing on $[a, b]$, then the minimum value of $f(x)$ $[a, b]$ is $f(a)$.

65. The value of $\displaystyle\int_a^b f(x)\,dx$ must be positive.

66. If $\displaystyle\int_a^b f(x)\,dx > 0$, then f is nonnegative for all x in $[a, b]$.

67. Find the Riemann sum for $f(x) = x^2 + 3x$ over the inter $[0, 8]$, where $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 7$, and $x_4 = 8$, a where $c_1 = 1$, $c_2 = 2$, $c_3 = 5$, and $c_4 = 8$.

68. Find the Riemann sum for $f(x) = \sin x$ over the interval $[0, 2$ where $x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/3$, $x_3 = \pi$, and $x_4 = 2\pi$, a where $c_1 = \pi/6$, $c_2 = \pi/3$, $c_3 = 2\pi/3$, and $c_4 = 3\pi/2$.

69. ***Think About It*** Determine whether the Dirichlet function

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

is integrable on the interval $[0, 1]$. Explain.

70. Evaluate, if possible, the integral $\displaystyle\int_0^2 [\![x]\!]\,dx$.

71. Determine $\displaystyle\lim_{n\to\infty} \frac{1}{n^3}\left[1^2 + 2^2 + 3^2 + \cdots + n^2\right]$ by using appropriate Riemann sum.

Section 4.4 The Fundamental Theorem of Calculus

- Evaluate a definite integral using the Fundamental Theorem of Calculus.
- Understand and use the Mean Value Theorem for Integrals.
- Find the average value of a function over a closed interval.
- Understand and use the Second Fundamental Theorem of Calculus.

EXPLORATION

ntegration and Antidifferentiation
hroughout this chapter, we have been sing the integral sign to denote an ntiderivative (a family of functions) nd a definite integral (a number).

ntidifferentiation: $\int f(x)\,dx$

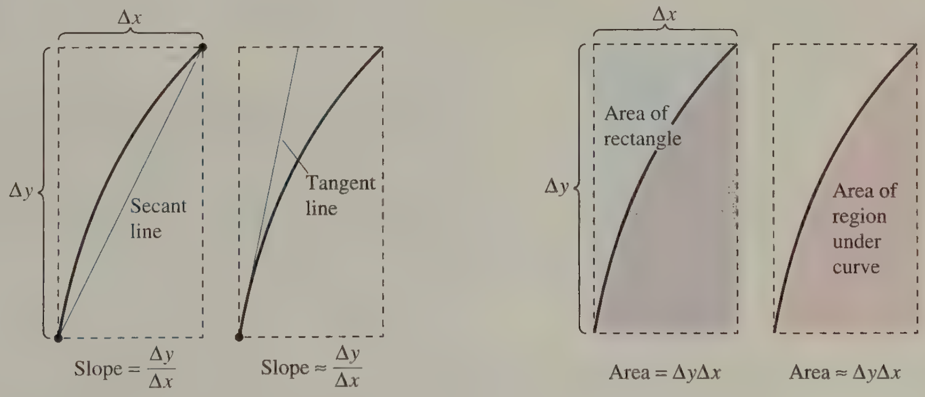
efinite integration: $\int_a^b f(x)\,dx$

he use of this same symbol for both perations makes it appear that they re related. In the early work with alculus, however, it was not known at the two operations were related. o you think the symbol ∫ was first pplied to antidifferentiation or to efinite integration? Explain your easoning. (*Hint:* The symbol was rst used by Leibniz and was derived om the letter *S*.)

The Fundamental Theorem of Calculus

You have now been introduced to the two major branches of calculus: differential calculus (introduced with the tangent line problem) and integral calculus (introduced with the area problem). At this point, these two problems might seem unrelated—but there is a very close connection. The connection was discovered independently by Isaac Newton and Gottfried Leibniz and is stated in a theorem that is appropriately called the **Fundamental Theorem of Calculus.**

Informally, the theorem states that differentiation and (definite) integration are inverse operations, in the same sense that division and multiplication are inverse operations. To see how Newton and Leibniz might have anticipated this relationship, consider the approximations shown in Figure 4.27. When we defined the slope of the tangent line, we used the *quotient* $\Delta y/\Delta x$ (the slope of the secant line). Similarly, when we defined the area of a region under a curve, we used the *product* $\Delta y \Delta x$ (the area of a rectangle). So, at least in the primitive approximation stage, the operations of differentiation and definite integration appear to have an inverse relationship in the same sense that division and multiplication are inverse operations. The Fundamental Theorem of Calculus states that the limit processes (used to define the derivative and definite integral) preserve this inverse relationship.



(a) Differentiation **(b)** Definite integration

Differentiation and definite integration have an "inverse" relationship.
Figure 4.27

THEOREM 4.9 The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof The key to the proof is in writing the difference $F(b) - F(a)$ in a convenie form. Let Δ be the following partition of $[a, b]$.

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

By pairwise subtraction and addition of like terms, you can write

$$F(b) - F(a) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \cdots - F(x_1) + F(x_1) - F(x_0)$$

$$= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})].$$

By the Mean Value Theorem, you know that there exists a number c_i in the ith subi terval such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}.$$

Because $F'(c_i) = f(c_i)$, you can let $\Delta x_i = x_i - x_{i-1}$ and obtain

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

This important equation tells you that by applying the Mean Value Theorem you c always find a collection of c_i's such that the *constant* $F(b) - F(a)$ is a Riemann su of f on $[a, b]$. Taking the limit (as $\|\Delta\| \to 0$) produces

$$F(b) - F(a) = \int_a^b f(x)\, dx.$$

The following guidelines can help you understand the use of the Fundament Theorem of Calculus.

Guidelines for Using the Fundamental Theorem of Calculus

1. *Provided you can find* an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\int_a^b f(x)\, dx = F(x)\Big]_a^b$$
$$= F(b) - F(a)$$

For instance, to evaluate $\int_1^3 x^3\, dx$, you can write

$$\int_1^3 x^3\, dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\int_a^b f(x)\, dx = \Big[F(x) + C\Big]_a^b$$
$$= [F(b) + C] - [F(a) + C]$$
$$= F(b) - F(a).$$

Example 1 Evaluating a Definite Integral

Evaluate each definite integral.

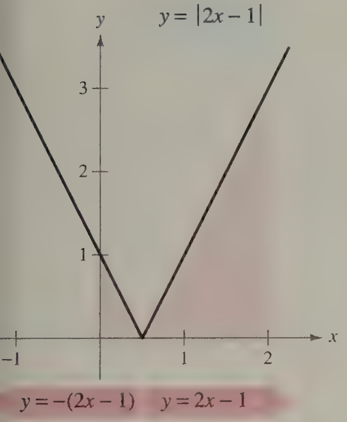
a. $\displaystyle\int_1^2 (x^2 - 3)\,dx$ **b.** $\displaystyle\int_1^4 3\sqrt{x}\,dx$ **c.** $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx$

Solution

a. $\displaystyle\int_1^2 (x^2 - 3)\,dx = \left[\frac{x^3}{3} - 3x\right]_1^2 = \left(\frac{8}{3} - 6\right) - \left(\frac{1}{3} - 3\right) = -\frac{2}{3}$

b. $\displaystyle\int_1^4 3\sqrt{x}\,dx = 3\int_1^4 x^{1/2}\,dx = 3\left[\frac{x^{3/2}}{3/2}\right]_1^4 = 2(4)^{3/2} - 2(1)^{3/2} = 14$

c. $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big]_0^{\pi/4} = 1 - 0 = 1$



e definite integral of y on $[0, 2]$ is $\frac{5}{2}$.
gure 4.28

Example 2 A Definite Integral Involving Absolute Value

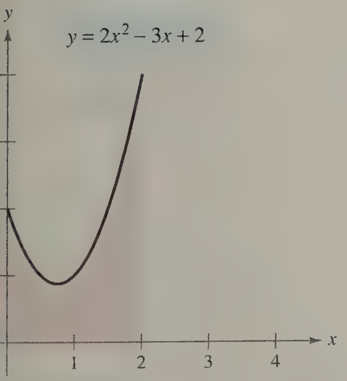
Evaluate $\displaystyle\int_0^2 |2x - 1|\,dx$.

Solution Using Figure 4.28 and the definition of absolute value, you can rewrite the integrand as follows.

$$|2x - 1| = \begin{cases} -(2x - 1), & x < \frac{1}{2} \\ 2x - 1, & x \ge \frac{1}{2} \end{cases}$$

From this, you can rewrite the integral in two parts.

$$\begin{aligned}\int_0^2 |2x - 1|\,dx &= \int_0^{1/2} -(2x - 1)\,dx + \int_{1/2}^2 (2x - 1)\,dx \\ &= \Big[-x^2 + x\Big]_0^{1/2} + \Big[x^2 - x\Big]_{1/2}^2 \\ &= \left(-\frac{1}{4} + \frac{1}{2}\right) - (0 + 0) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right) \\ &= \frac{5}{2}\end{aligned}$$



e area of the region bounded by the graph
y, the x-axis, $x = 0$, and $x = 2$ is $\frac{10}{3}$.
gure 4.29

Example 3 Using the Fundamental Theorem to Find Area

Find the area of the region bounded by the graph of $y = 2x^2 - 3x + 2$, the x-axis, and the vertical lines $x = 0$ and $x = 2$, as shown in Figure 4.29.

Solution Note that $y > 0$ on the interval $[0, 2]$.

$$\text{Area} = \int_0^2 (2x^2 - 3x + 2)\,dx \qquad \text{Integrate between } x = 0 \text{ and } x = 2.$$

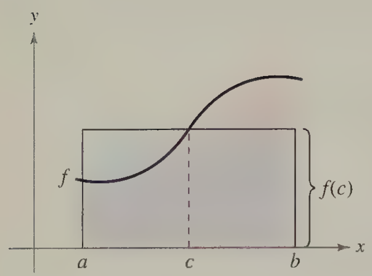
$$= \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^2 \qquad \text{Find antiderivative.}$$

$$= \left(\frac{16}{3} - 6 + 4\right) - (0 - 0 + 0) \qquad \text{Apply Fundamental Theorem.}$$

$$= \frac{10}{3} \qquad \text{Simplify.}$$

The Mean Value Theorem for Integrals

In Section 4.2, you saw that the area of a region under a curve is greater than the ar of an inscribed rectangle and less than the area of a circumscribed rectangle. T Mean Value Theorem for Integrals states that somewhere "between" the inscribed ar circumscribed rectangles there is a rectangle whose area is precisely equal to the ar of the region under the curve, as shown in Figure 4.30.



Mean value rectangle:

$$f(c)(b-a)=\int_a^b f(x)\,dx$$

Figure 4.30

THEOREM 4.10 Mean Value Theorem for Integrals

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Proof

Case 1: If f is constant on the interval $[a, b]$, the theorem is clearly valid because can be any point in $[a, b]$.

Case 2: If f is not constant on $[a, b]$, then, by the Extreme Value Theorem, you ca choose $f(m)$ and $f(M)$ to be the minimum and maximum values of f on $[a, b]$. Becau $f(m) \le f(x) \le f(M)$ for all x in $[a, b]$, you can apply Theorem 4.8 to write t following.

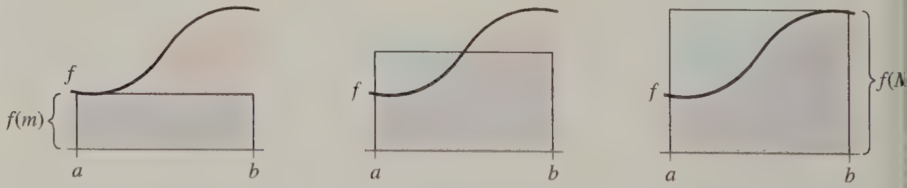
$$\int_a^b f(m)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(M)\,dx \qquad \text{See Figure 4.31.}$$

$$f(m)(b-a) \le \int_a^b f(x)\,dx \le f(M)(b-a)$$

$$f(m) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(M)$$

From the third inequality, you can apply the Intermediate Value Theorem to conclu that there exists some c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \qquad \text{or} \qquad f(c)(b-a) = \int_a^b f(x)\,dx.$$



Inscribed rectangle (less than actual area)

$\int_a^b f(m)\,dx = f(m)(b-a)$

Mean value rectangle (equal to actual area)

$\int_a^b f(x)\,dx$

Circumscribed rectangle (greater than actual area)

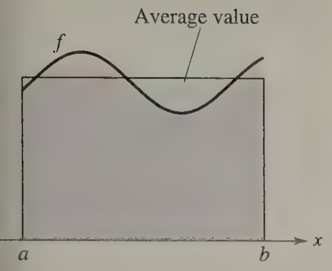
$\int_a^b f(M)\,dx = f(M)(b-a)$

Figure 4.31

NOTE Notice that Theorem 4.10 does not specify how to determine c. It merely guarante the existence of at least one number c in the interval.

Average Value of a Function

The value of $f(c)$ given in the Mean Value Theorem for Integrals is called the **average value** of f on the interval $[a, b]$.



age value $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

re 4.32

Definition of the Average Value of a Function on an Interval

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

NOTE Notice in Figure 4.32 that the area of the region under the graph of f is equal to the area of the rectangle whose height is the average value.

To see why the average value of f is defined in this way, suppose that you partition $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. If c_i is any point in the ith subinterval, the arithmetic average (or mean) of the function values at the c_i's is given by

$$a_n = \frac{1}{n}[f(c_1) + f(c_2) + \cdot \cdot \cdot + f(c_n)]. \qquad \text{Average of } f(c_1), . . . , f(c_n)$$

By multiplying and dividing by $(b-a)$, you can write the average as

$$\begin{aligned} a_n &= \frac{1}{n}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{b-a}\right) = \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{n}\right) \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\,\Delta x. \end{aligned}$$

Finally, taking the limit as $n \to \infty$ produces the average value of f on the interval $[a, b]$, as given in the definition above.

This development of the average value of a function on an interval is only one of many practical uses of definite integrals to represent summation processes. In Chapter 6, you will study other applications, such as volume, arc length, centers of mass, and work.

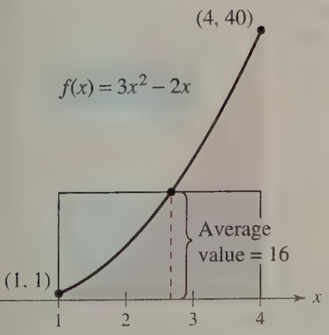
Example 4 Finding the Average Value of a Function

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

Solution The average value is given by

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,dx &= \frac{1}{3}\int_1^4 (3x^2 - 2x)\,dx \\ &= \frac{1}{3}\Big[x^3 - x^2\Big]_1^4 \\ &= \frac{1}{3}[64 - 16 - (1 - 1)] = \frac{48}{3} = 16. \end{aligned}$$

(See Figure 4.33.)



re 4.33

Archive Photos

The first person to fly at a speed greater than the speed of sound was Charles Yeager. On October 14, 1947, flying in an *X-1* rocket plane at an altitude of 12.8 kilometers, Yeager was clocked at 299.5 meters per second. If Yeager had been flying at an altitude under 10.375 kilometers, his speed of 299.5 meters per second would not have "broken the sound barrier." The photo above shows the *X-1* and its *B-29* mother plane.

Example 5 The Speed of Sound

At different altitudes in earth's atmosphere, sound travels at different speeds. ' speed of sound $s(x)$ (in meters per second) can be modeled by

$$s(x) = \begin{cases} -4x + 341, & 0 \le x < 11.5 \\ 295, & 11.5 \le x < 22 \\ \frac{3}{4}x + 278.5, & 22 \le x < 32 \\ \frac{3}{2}x + 254.5, & 32 \le x < 50 \\ -\frac{3}{2}x + 404.5, & 50 \le x \le 80 \end{cases}$$

where x is the altitude in kilometers (see Figure 4.34). What is the average spee sound over the interval $[0, 80]$?

Solution Begin by integrating $s(x)$ over the interval $[0, 80]$. To do this, you can br the integral into five parts.

$$\int_0^{11.5} s(x)\,dx = \int_0^{11.5} (-4x + 341)\,dx = \Big[-2x^2 + 341x\Big]_0^{11.5} = 3657$$

$$\int_{11.5}^{22} s(x)\,dx = \int_{11.5}^{22} (295)\,dx = \Big[295x\Big]_{11.5}^{22} = 3097.5$$

$$\int_{22}^{32} s(x)\,dx = \int_{22}^{32} \left(\tfrac{3}{4}x + 278.5\right)dx = \Big[\tfrac{3}{8}x^2 + 278.5x\Big]_{22}^{32} = 2987.5$$

$$\int_{32}^{50} s(x)\,dx = \int_{32}^{50} \left(\tfrac{3}{2}x + 254.5\right)dx = \Big[\tfrac{3}{4}x^2 + 254.5x\Big]_{32}^{50} = 5688$$

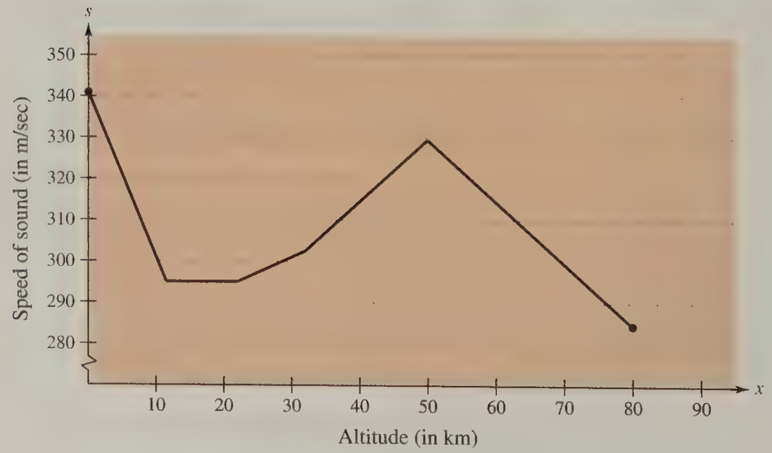
$$\int_{50}^{80} s(x)\,dx = \int_{50}^{80} \left(-\tfrac{3}{2}x + 404.5\right)dx = \Big[-\tfrac{3}{4}x^2 + 404.5x\Big]_{50}^{80} = 9210$$

By adding the values of the five integrals, you have

$$\int_0^{80} s(x)\,dx = 24{,}640.$$

Therefore, the average speed of sound from an altitude of 0 kilometers to an altit of 80 kilometers is

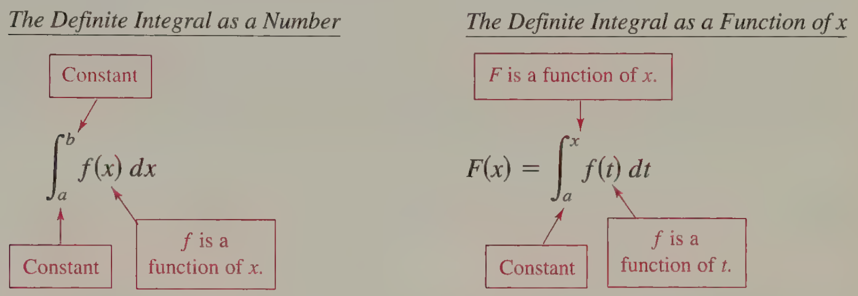
$$\text{Average speed} = \frac{1}{80}\int_0^{80} s(x)\,dx = \frac{24{,}640}{80} = 308 \text{ meters per second.}$$



Speed of sound depends on altitude.
Figure 4.34

The Second Fundamental Theorem of Calculus

When we defined the definite integral of f on the interval $[a, b]$, we used the constant b as the upper limit of integration and x as the variable of integration. We now look at a slightly different situation in which the variable x is used as the upper limit of integration. To avoid the confusion of using x in two different ways, we temporarily switch to using t as the variable of integration. (Remember that the definite integral is *not* a function of its variable of integration.)


The Definite Integral as a Number
Constant
$\int_a^b f(x)\,dx$
Constant
f is a function of x.
The Definite Integral as a Function of x
F is a function of x.
$F(x) = \int_a^x f(t)\,dt$
Constant
f is a function of t.

EXPLORATION

se a graphing utility to graph e function

$$F(x) = \int_0^x \cos t\,dt$$

r $0 \le x \le \pi$. Do you recognize is graph? Explain.

Example 6 The Definite Integral as a Function

Evaluate the function

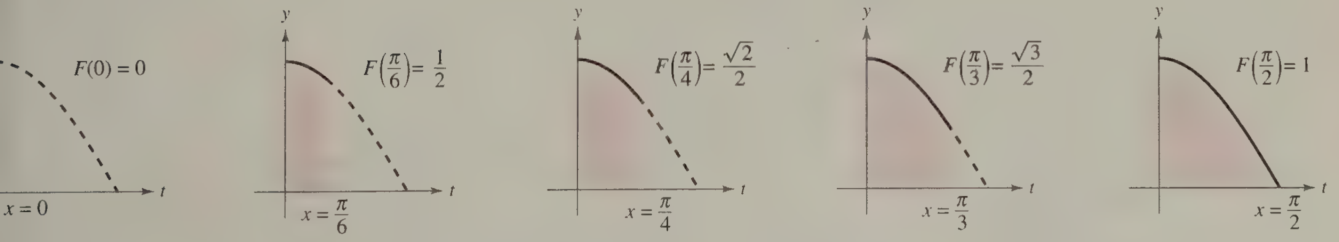
$$F(x) = \int_0^x \cos t\,dt$$

at $x = 0, \pi/6, \pi/4, \pi/3$, and $\pi/2$.

Solution You could evaluate five different definite integrals, one for each of the given upper limits. However, it is much simpler to fix x (as a constant) temporarily and apply the Fundamental Theorem once, to obtain

$$\int_0^x \cos t\,dt = \sin t\Big]_0^x = \sin x - \sin 0 = \sin x.$$

Now, using $F(x) = \sin x$, you can obtain the results shown in Figure 4.35.



$= \int_0^x \cos t\,dt$ is the area under the curve $f(t) = \cos t$ from 0 to x.

ure 4.35

You can think of the function $F(x)$ as *accumulating* the area under the curve $f(t) = \cos t$ from $t = 0$ to $t = x$. For $x = 0$, the area is 0 and $F(0) = 0$. For $x = \pi/2$, $F(\pi/2) = 1$ gives the accumulated area under the cosine curve on the entire interval $[0, \pi/2]$. This interpretation of an integral as an **accumulation function** is used often in applications of integration.

In Example 6, note that the derivative of F is the original integrand (with only variable changed). That is,

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}[\sin x] = \frac{d}{dx}\left[\int_0^x \cos t\, dt\right] = \cos x.$$

We generalize this result in the following theorem, called the **Second Fundamen Theorem of Calculus.**

THEOREM 4.11 The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\, dt\right] = f(x).$$

Proof Begin by defining F as

$$F(x) = \int_a^x f(t)\, dt.$$

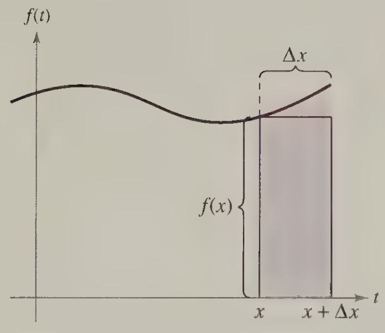
Then, by the definition of the derivative, you can write

$$\begin{aligned} F'(x) &= \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)\, dt - \int_a^x f(t)\, dt\right] \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)\, dt + \int_x^a f(t)\, dt\right] \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_x^{x+\Delta x} f(t)\, dt\right]. \end{aligned}$$

From the Mean Value Theorem for Integrals (assuming $\Delta x > 0$), you know th exists a number c in the interval $[x, x + \Delta x]$ such that the integral in the express above is equal to $f(c)\, \Delta x$. Moreover, because $x \le c \le x + \Delta x$, it follows that c as $\Delta x \to 0$. So, you obtain

$$\begin{aligned} F'(x) &= \lim_{\Delta x \to 0}\left[\frac{1}{\Delta x} f(c)\, \Delta x\right] \\ &= \lim_{\Delta x \to 0} f(c) \\ &= f(x). \end{aligned}$$

A similar argument can be made for $\Delta x < 0$.



$$f(x)\, \Delta x \approx \int_x^{x+\Delta x} f(t)\, dt$$

Figure 4.36

NOTE Using the area model for definite integrals, you can view the approximation

$$f(x)\, \Delta x \approx \int_x^{x+\Delta x} f(t)\, dt$$

as saying that the area of the rectangle of height $f(x)$ and width Δx is approximately equa the area of the region lying between the graph of f and the x-axis on the interval $[x, x +$ as shown in Figure 4.36.

Note that the Second Fundamental Theorem of Calculus tells you that if a function is continuous, you can be sure that it has an antiderivative. This antiderivative need not, however, be an elementary function. (Recall the discussion of elementary functions in Section P.3.)

Example 7 Using the Second Fundamental Theorem of Calculus

Evaluate $\dfrac{d}{dx}\left[\displaystyle\int_0^x \sqrt{t^2+1}\,dt\right]$.

Solution Note that $f(t) = \sqrt{t^2+1}$ is continuous on the entire real line. So, using the Second Fundamental Theorem of Calculus, you can write

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right] = \sqrt{x^2+1}.$$

The differentiation shown in Example 7 is a straightforward application of the Second Fundamental Theorem of Calculus. The next example shows how this theorem can be combined with the Chain Rule to find the derivative of a function.

Example 8 Using the Second Fundamental Theorem of Calculus

Find the derivative of $F(x) = \displaystyle\int_{\pi/2}^{x^3} \cos t\,dt$.

Solution Using $u = x^3$, you can apply the Second Fundamental Theorem of Calculus with the Chain Rule as follows.

$$\begin{aligned}
F'(x) &= \frac{dF}{du}\frac{du}{dx} && \text{Chain Rule}\\
&= \frac{d}{du}[F(x)]\frac{du}{dx} && \text{Definition of } \frac{dF}{du}\\
&= \frac{d}{du}\left[\int_{\pi/2}^{x^3} \cos t\,dt\right]\frac{du}{dx} && \text{Substitute } \int_{\pi/2}^{x^3} \cos t\,dt \text{ for } F(x).\\
&= \frac{d}{du}\left[\int_{\pi/2}^{u} \cos t\,dt\right]\frac{du}{dx} && \text{Substitute } u \text{ for } x^3.\\
&= (\cos u)(3x^2) && \text{Apply Second Fundamental Theorem of Calculus.}\\
&= (\cos x^3)(3x^2) && \text{Rewrite as function of } x.
\end{aligned}$$

Because the integrand in Example 8 is easily integrated, you can verify the derivative as follows.

$$F(x) = \int_{\pi/2}^{x^3} \cos t\,dt = \sin t\Big]_{\pi/2}^{x^3} = \sin x^3 - \sin\frac{\pi}{2} = (\sin x^3) - 1$$

In this form, you can apply the Power Rule to verify that the derivative is the same as that obtained in Example 8.

$$F'(x) = (\cos x^3)(3x^2)$$

EXERCISES FOR SECTION 4.4

Graphical Reasoning **In Exercises 1–4, use a graphing utility to graph the integrand. Use the graph to determine whether the definite integral is positive, negative, or zero.**

1. $\displaystyle\int_0^{\pi} \frac{4}{x^2+1}\,dx$

2. $\displaystyle\int_0^{\pi} \cos x\,dx$

3. $\displaystyle\int_{-2}^{2} x\sqrt{x^2+1}\,dx$

4. $\displaystyle\int_{-2}^{2} x\sqrt{2-x}\,dx$

In Exercises 5–26, evaluate the definite integral of the algebraic function. Use a graphing utility to verify your result.

5. $\displaystyle\int_0^1 2x\,dx$

6. $\displaystyle\int_2^7 3\,dv$

7. $\displaystyle\int_{-1}^0 (x-2)\,dx$

8. $\displaystyle\int_2^5 (-3v+4)\,dv$

9. $\displaystyle\int_{-1}^1 (t^2-2)\,dt$

10. $\displaystyle\int_1^3 (3x^2+5x-4)\,dx$

11. $\displaystyle\int_0^1 (2t-1)^2\,dt$

12. $\displaystyle\int_{-1}^1 (t^3-9t)\,dt$

13. $\displaystyle\int_1^2 \left(\frac{3}{x^2}-1\right)dx$

14. $\displaystyle\int_{-2}^{-1} \left(u-\frac{1}{u^2}\right)du$

15. $\displaystyle\int_1^4 \frac{u-2}{\sqrt{u}}\,du$

16. $\displaystyle\int_{-3}^3 v^{1/3}\,dv$

17. $\displaystyle\int_{-1}^1 \left(\sqrt[3]{t}-2\right)dt$

18. $\displaystyle\int_1^8 \sqrt{\frac{2}{x}}\,dx$

19. $\displaystyle\int_0^1 \frac{x-\sqrt{x}}{3}\,dx$

20. $\displaystyle\int_0^2 (2-t)\sqrt{t}\,dt$

21. $\displaystyle\int_{-1}^0 \left(t^{1/3}-t^{2/3}\right)dt$

22. $\displaystyle\int_{-8}^{-1} \frac{x-x^2}{2\sqrt[3]{x}}\,dx$

23. $\displaystyle\int_0^3 |2x-3|\,dx$

24. $\displaystyle\int_1^4 (3-|x-3|)\,dx$

25. $\displaystyle\int_0^3 |x^2-4|\,dx$

26. $\displaystyle\int_0^4 |x^2-4x+3|\,dx$

In Exercises 27–32, evaluate the definite integral of the trigonometric function. Use a graphing utility to verify your result.

27. $\displaystyle\int_0^{\pi} (1+\sin x)\,dx$

28. $\displaystyle\int_0^{\pi/4} \frac{1-\sin^2\theta}{\cos^2\theta}\,d\theta$

29. $\displaystyle\int_{-\pi/6}^{\pi/6} \sec^2 x\,dx$

30. $\displaystyle\int_{\pi/4}^{\pi/2} (2-\csc^2 x)\,dx$

31. $\displaystyle\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,d\theta$

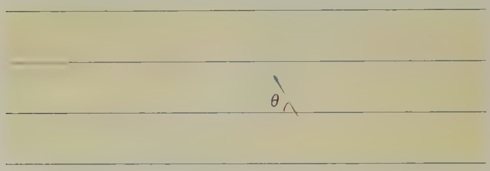
32. $\displaystyle\int_{-\pi/2}^{\pi/2} (2t+\cos t)\,dt$

33. ***Depreciation*** A company purchases a new machine for which the rate of depreciation is $dV/dt = 10{,}000(t-6)$, $0 \le t \le 5$, where V is the value of the machine after t years. Set up and evaluate the definite integral that yields the total loss of value of the machine over the first 3 years.

34. ***Buffon's Needle Experiment*** A horizontal plane is ruled w parallel lines 2 inches apart. If a 2-inch needle is tossed rando onto the plane, the probability that the needle will touch a lin

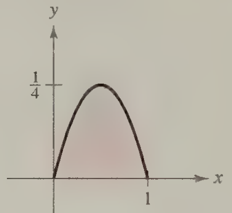
$$P = \frac{2}{\pi}\int_0^{\pi/2} \sin\theta\,d\theta$$

where θ is the acute angle between the needle and any one the parallel lines. Find this probability.

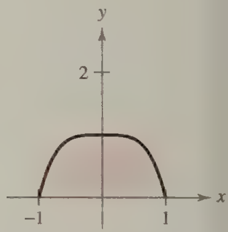


In Exercises 35–40, determine the area of the indicated regi

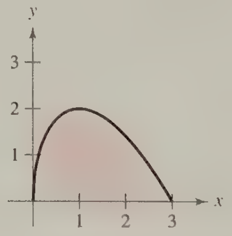
35. $y = x - x^2$



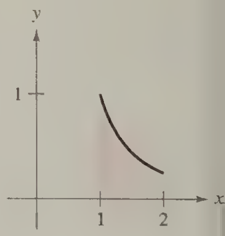
36. $y = 1 - x^4$



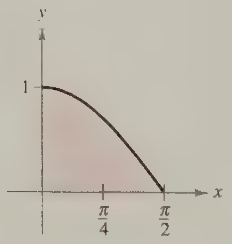
37. $y = (3-x)\sqrt{x}$



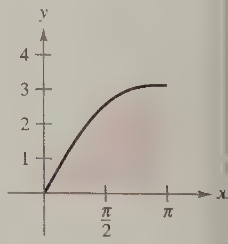
38. $y = \dfrac{1}{x^2}$



39. $y = \cos x$



40. $y = x + \sin x$



In Exercises 41–44, find the area of the region bounded by graphs of the equations.

41. $y = 3x^2 + 1,\quad x = 0,\quad x = 2,\quad y = 0$

42. $y = 1 + \sqrt[3]{x},\quad x = 0,\quad x = 8,\quad y = 0$

43. $y = x^3 + x,\quad x = 2,\quad y = 0$

44. $y = -x^2 + 3x,\quad y = 0$

Exercises 45–48, find the value(s) of c guaranteed by the an Value Theorem for Integrals for the function over the icated interval.

Function	*Interval*
$f(x) = x - 2\sqrt{x}$	$[0, 2]$
$f(x) = \dfrac{9}{x^3}$	$[1, 3]$
$f(x) = 2\sec^2 x$	$[-\pi/4, \pi/4]$
$f(x) = \cos x$	$[-\pi/3, \pi/3]$

Exercises 49–52, find the average value of the function over interval and all values of x in the interval for which the func- equals its average value.

Function	*Interval*
$f(x) = 4 - x^2$	$[-2, 2]$
$f(x) = \dfrac{4(x^2+1)}{x^2}$	$[1, 3]$
$f(x) = \sin x$	$[0, \pi]$
$f(x) = \cos x$	$[0, \pi/2]$

etting at the Concept

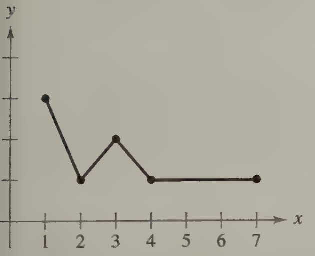
3. State the Fundamental Theorem of Calculus.

4. The graph of f is given in the figure.

(a) Evaluate $\displaystyle\int_1^7 f(x)\,dx$.

(b) Determine the average value of f on the interval $[1, 7]$.

(c) Determine the answers to parts (a) and (b) if the graph is translated two units upward.



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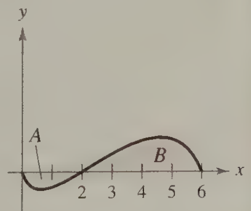


Figure for 55–60

Exercises 55–60, use the graph of f shown in the figure. he shaded region A has an area of 1.5, and $\int_0^6 f(x)\,dx = 3.5$. se this information to fill in the blanks.

5. $\displaystyle\int_0^2 f(x)\,dx =$ 56. $\displaystyle\int_2^6 f(x)\,dx =$

7. $\displaystyle\int_0^6 |f(x)|\,dx =$ 58. $\displaystyle\int_0^2 -2f(x)\,dx =$

9. $\displaystyle\int_0^6 [2 + f(x)]\,dx =$

0. The average value of f over the interval $[0, 6]$ is .

61. ***Force*** The force F (in newtons) of a hydraulic cylinder in a press is proportional to the square of $\sec x$, where x is the distance (in meters) that the cylinder is extended in its cycle. The domain of F is $[0, \pi/3]$, and $F(0) = 500$.

(a) Find F as a function of x.

(b) Find the average force exerted by the press over the interval $[0, \pi/3]$.

62. ***Blood Flow*** The velocity v of the flow of blood at a distance r from the central axis of an artery of radius R is

$$v = k(R^2 - r^2)$$

where k is the constant of proportionality. Find the average rate of flow of blood along a radius of the artery. (Use 0 and R as the limits of integration.)

63. ***Respiratory Cycle*** The volume V in liters of air in the lungs during a 5-second respiratory cycle is approximated by the model

$$V = 0.1729t + 0.1522t^2 - 0.0374t^3$$

where t is the time in seconds. Approximate the average volume of air in the lungs during one cycle.

64. ***Average Profit*** A company introduces a new product, and the profit in thousands of dollars over the first 6 months is approximated by the model

$$P = 5(\sqrt{t} + 30), \qquad t = 1, 2, 3, 4, 5, 6.$$

(a) Use the model to complete the table and use the entries to calculate (arithmetically) the average profit over the first 6 months.

t	1	2	3	4	5	6
P						

(b) Find the average value of the profit function by integration and compare the result with that in part (a). (Integrate over the interval $[0.5, 6.5]$.)

(c) What, if any, is the advantage of using the approximation of the average given by the definite integral? (Note that the integral approximation utilizes all real values of t in the interval rather than just integers.)

65. ***Average Sales*** A company fit a model to the monthly sales data of a seasonal product. The model is

$$S(t) = \frac{t}{4} + 1.8 + 0.5\sin\left(\frac{\pi t}{6}\right), \qquad 0 \le t \le 24$$

where S is sales (in thousands) and t is time in months.

(a) Use a graphing utility to graph $f(t) = 0.5\sin(\pi t/6)$ for $0 \le t \le 24$. Use the graph to explain why the average value of $f(t)$ is 0 over the interval.

(b) Use a graphing utility to graph $S(t)$ and the line $g(t) = t/4 + 1.8$ in the same viewing window. Use the graph and the result of part (a) to explain why g is called the *trend line*.

66. ***Modeling Data*** In the manufacturing process of a product, there is a repetitive heating cycle of 4 minutes. During a review of the process, the flow R (cubic feet per minute) of natural gas was measured in 1-minute intervals and the results were recorded in the table.

t	0	1	2	3	4
R	0	62	76	38	0

(a) Use a graphing utility to find a model of the form $R = at^4 + bt^3 + ct^2 + dt + e$ for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the Fundamental Theorem of Calculus to approximate the number of cubic feet of natural gas used in one heating cycle.

67. ***Modeling Data*** A radio-controlled experimental vehicle is tested on a straight track. It starts from rest, and its velocity v (meters per second) is recorded in the table every 10 seconds for 1 minute.

t	0	10	20	30	40	50	60
v	0	5	21	40	62	78	83

(a) Use a graphing utility to find a model of the form $v = at^3 + bt^2 + ct + d$ for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the Fundamental Theorem of Calculus to approximate the distance traveled by the vehicle during the test.

68. ***Modeling Data*** A department store manager wants to estimate the number of customers that enter the store from noon until closing at 9 P.M. The table shows the number of customers N entering the store during a randomly selected minute each hour from $t - 1$ to t, with $t = 0$ corresponding to noon.

t	1	2	3	4	5	6	7	8	9
N	6	7	9	12	15	14	11	7	2

(a) Draw a histogram of the data.

(b) Estimate the total number of customers entering the store between noon and 9 P.M.

(c) Use the regression capabilities of a graphing utility to find a model of the form

$$N(t) = at^3 + bt^2 + ct + d$$

for the data.

(d) Use a graphing utility to plot the data and graph the model.

(e) Use a graphing utility to evaluate $\int_0^9 N(t)\,dt$, and use the result to estimate the number of customers entering the store between noon and 9 P.M. Compare this with your answer in part (b).

(f) Estimate the average number of customers entering the store per minute between 3 P.M. and 7 P.M.

In Exercises 69–74, find F as a function of x and evaluate i
$x = 2$, $x = 5$, and $x = 8$.

69. $F(x) = \displaystyle\int_0^x (t - 5)\,dt$

70. $F(x) = \displaystyle\int_2^x (t^3 + 2t - 2$

71. $F(x) = \displaystyle\int_1^x \frac{10}{v^2}\,dv$

72. $F(x) = \displaystyle\int_2^x -\frac{2}{t^3}\,dt$

73. $F(x) = \displaystyle\int_1^x \cos\theta\,d\theta$

74. $F(x) = \displaystyle\int_0^x \sin\theta\,d\theta$

In Exercises 75–80, (a) integrate to find F as a function of x a
(b) demonstrate the Second Fundamental Theorem of Calcu
by differentiating the result in part (a).

75. $F(x) = \displaystyle\int_0^x (t + 2)\,dt$

76. $F(x) = \displaystyle\int_0^x t(t^2 + 1)\,dt$

77. $F(x) = \displaystyle\int_8^x \sqrt[3]{t}\,dt$

78. $F(x) = \displaystyle\int_4^x \sqrt{t}\,dt$

79. $F(x) = \displaystyle\int_{\pi/4}^x \sec^2 t\,dt$

80. $F(x) = \displaystyle\int_{\pi/3}^x \sec t\tan t\,d$

In Exercises 81–86, use the Second Fundamental Theorem
Calculus to find $F'(x)$.

81. $F(x) = \displaystyle\int_{-2}^x (t^2 - 2t)\,dt$

82. $F(x) = \displaystyle\int_1^x \frac{t^2}{t^2 + 1}\,dt$

83. $F(x) = \displaystyle\int_{-1}^x \sqrt{t^4 + 1}\,dt$

84. $F(x) = \displaystyle\int_1^x \sqrt[4]{t}\,dt$

85. $F(x) = \displaystyle\int_0^x t\cos t\,dt$

86. $F(x) = \displaystyle\int_0^x \sec^3 t\,dt$

In Exercises 87–92, find $F'(x)$.

87. $F(x) = \displaystyle\int_x^{x+2} (4t + 1)\,dt$

88. $F(x) = \displaystyle\int_{-x}^x t^3\,dt$

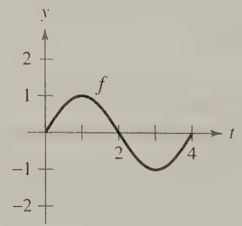
89. $F(x) = \displaystyle\int_0^{\sin x} \sqrt{t}\,dt$

90. $F(x) = \displaystyle\int_2^{x^2} \frac{1}{t^3}\,dt$

91. $F(x) = \displaystyle\int_0^{x^3} \sin t^2\,dt$

92. $F(x) = \displaystyle\int_0^{x^2} \sin\theta^2\,d\theta$

93. ***Graphical Analysis*** Approximate the graph of g on interval $0 \le x \le 4$ where $g(x) = \int_0^x f(t)\,dt$. Identify x-coordinate of an extremum of g. To print an enlarged cop the graph, go to the website *www.mathgraphs.com.*



Jse the function f in the figure and the function g defined by

$$(x) = \int_0^x f(t)\,dt.$$

a) Complete the table.

	1	2	3	4	5	6	7	8	9	10
$x)$										

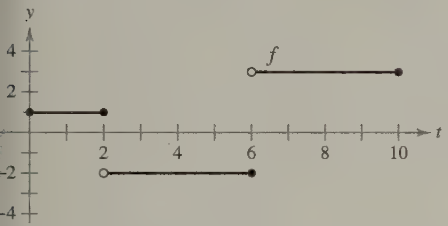
b) Plot the points from the table in part (a) and graph g.

c) Where does g have its minimum? Explain.

d) Where does g have a maximum? Explain.

e) On what interval does g increase at the greatest rate? Explain.

f) Identify the zeros of g.



Cost The total cost of purchasing and maintaining a piece of equipment for x years is

$$C(x) = 5000\left(25 + 3\int_0^x t^{1/4}\,dt\right).$$

a) Perform the integration to write C as a function of x.

b) Find $C(1)$, $C(5)$, and $C(10)$.

rea The area A between the graph of the function $g(t) =$ $- 4/t^2$ and the t-axis over the interval $[1, x]$ is

$$(x) = \int_1^x \left(4 - \frac{4}{t^2}\right) dt.$$

a) Find the horizontal asymptote of the graph of g.

b) Integrate to find A as a function of x. Does the graph of A have a horizontal asymptote? Explain.

True or False? **In Exercises 97–99, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

97. If $F'(x) = G'(x)$ on the interval $[a, b]$, then $F(b) - F(a) = G(b) - G(a)$.

98. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

99. $\displaystyle\int_{-1}^{1} x^{-2}\,dx = \Big[-x^{-1}\Big]_{-1}^{1} = (-1) - 1 = -2$

100. Prove: $\displaystyle\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] = f(v(x))v'(x) - f(u(x))u'(x).$

101. Show that the function

$$f(x) = \int_0^{1/x} \frac{1}{t^2 + 1}\,dt + \int_0^x \frac{1}{t^2 + 1}\,dt$$

is constant for $x > 0$.

102. Let $G(x) = \displaystyle\int_0^x \left[s\int_0^s f(t)dt\right] ds$, where f is continuous for all real t. Find

(a) $G(0)$. (b) $G'(0)$.

(c) $G''(x)$. (d) $G''(0)$.

Rectilinear Motion **In Exercises 103–105, consider a particle moving along the x-axis where $x(t)$ is the position of the particle at time t, $x'(t)$ is its velocity, and $\int_a^b |x'(t)|\,dt$ is the distance the particle travels in the interval of time.**

103. The position function is

$$x(t) = t^3 - 6t^2 + 9t - 2, \quad 0 \le t \le 5.$$

Find the total distance the particle travels in 5 units of time.

104. Repeat Exercise 103 for the position function given by $x(t) = (t - 1)(t - 3)^2$, $0 \le t \le 5$.

105. A particle moves along the x-axis with velocity $v(t) = 1/\sqrt{t}$, $t > 0$. At time $t = 1$, its position is $x = 4$. Find the total distance traveled by the particle on the interval $1 \le t \le 4$.

ECTION PROJECT DEMONSTRATING THE FUNDAMENTAL THEOREM

e a graphing utility to graph the function $y_1 = \sin^2 t$ on the erval $0 \le t \le \pi$. Let $F(x)$ be the following function of x.

$$) = \int_0^x \sin^2 t\,dt$$

Complete the table and explain why the values of F are increasing.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$F(x)$							

(b) Use the integration capabilities of a graphing utility to graph F.

(c) Use the differentiation capabilities of a graphing utility to graph $F'(x)$. How is this graph related to the graph in part (b)?

(d) Verify that the derivative of $y = (1/2)t - (\sin 2t)/4$ is $\sin^2 t$. Graph y and write a short paragraph about how this graph is related to those in parts (b) and (c).

Section 4.5 Integration by Substitution

- Use pattern recognition to evaluate an indefinite integral.
- Use a change of variables to evaluate an indefinite integral.
- Use the General Power Rule for Integration to evaluate an indefinite integral.
- Use a change of variables to evaluate a definite integral.
- Evaluate a definite integral involving an even or odd function.

Pattern Recognition

In this section you will study techniques for integrating composite functions. discussion is split into two parts—*pattern recognition* and *change of variables.* techniques involve a ***u*-substitution.** With pattern recognition you perform the su tution mentally, and with change of variables you write the substitution steps.

The role of substitution in integration is comparable to the role of the Chain in differentiation. Recall that for differentiable functions given by $y = F(u)$ $u = g(x)$, the Chain Rule states that

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x).$$

From the definition of an antiderivative, it follows that

$$\begin{aligned}\int F'(g(x))g'(x)\,dx &= F(g(x)) + C\\ &= F(u) + C.\end{aligned}$$

These results are summarized in the following theorem.

THEOREM 4.12 Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

If $u = g(x)$, then $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C.$$

NOTE The statement of Theorem 4.12 doesn't tell how to distinguish between $f(g(x))$ and $g'(x)$ in the integrand. As you become more experienced at integration, your skill in doing this will increase. Of course, part of the key is familiarity with derivatives.

EXPLORATION

Recognizing Patterns The integrand in each of the following integrals fits th pattern $f(g(x))g'(x)$. Identify the pattern and use the result to evaluate the integ

a. $\int 2x(x^2 + 1)^4\,dx$ **b.** $\int 3x^2\sqrt{x^3 + 1}\,dx$ **c.** $\int \sec^2 x(\tan x + 3)\,dx$

The next three integrals are similar to the first three. Show how you can multi and divide by a constant to evaluate these integrals.

d. $\int x(x^2 + 1)^4\,dx$ **e.** $\int x^2\sqrt{x^3 + 1}\,dx$ **f.** $\int 2\sec^2 x(\tan x + 3)\,d$

STUDY TIP There are several techniques for applying substitution, each differing slightly from the others. However, you should remember that the goal is the same with every technique—*you are trying to find an antiderivative of the integrand.*

Examples 1 and 2 show how to apply Theorem 4.12 *directly*, by recognizing the presence of $f(g(x))$ and $g'(x)$. Note that the composite function in the integrand has an *outside function* f and an *inside function* g. Moreover, the derivative $g'(x)$ is present as a factor of the integrand.

$$\int \overset{\boxed{\text{Outside function}}}{f}(\underset{\boxed{\text{Inside function}}}{g(x)})\underbrace{g'(x)}_{\boxed{\text{Derivative of inside function}}}\,dx = F(g(x)) + C$$

Example 1 Recognizing the $f(g(x))g'(x)$ Pattern

Find $\int (x^2 + 1)^2(2x)\,dx$.

Solution Letting $g(x) = x^2 + 1$, you obtain

$$g'(x) = 2x$$

and

$$f(g(x)) = f(x^2 + 1) = (x^2 + 1)^2.$$

CHNOLOGY Try using a com-
ter algebra system, such as *Maple*,
rive, *Mathematica*, *Mathcad*, or the
89, to solve the integrals given in
amples 1 and 2. Do you obtain the
ne antiderivatives that are listed in
examples?

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Using the Power Rule for integration and Theorem 4.12, you can write

$$\int \overbrace{(x^2 + 1)^2}^{f(g(x))}\overbrace{(2x)}^{g'(x)}\,dx = \frac{1}{3}(x^2 + 1)^3 + C.$$

Try using the Chain Rule to check that the derivative of $\frac{1}{3}(x^2 + 1)^3 + C$ is the integrand of the original integral.

Example 2 Recognizing the $f(g(x))g'(x)$ Pattern

Find $\int 5 \cos 5x\,dx$.

Solution Letting $g(x) = 5x$, you obtain

$$g'(x) = 5$$

and

$$f(g(x)) = f(5x) = \cos 5x.$$

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Using the Cosine Rule for integration and Theorem 4.12, you can write

$$\int \overbrace{(\cos 5x)}^{f(g(x))}\overbrace{(5)}^{g'(x)}\,dx = \sin 5x + C.$$

You can check this by differentiating $\sin 5x + C$ to obtain the original integrand.

The integrands in Examples 1 and 2 fit the $f(g(x))g'(x)$ pattern exactly—you had to recognize the pattern. You can extend this technique considerably with Constant Multiple Rule

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

Many integrands contain the essential part (the variable part) of $g'(x)$ but are mis a constant multiple. In such cases, you can multiply and divide by the necessary stant multiple, as demonstrated in Example 3.

Example 3 Multiplying and Dividing by a Constant

Find $\int x(x^2+1)^2\,dx$.

Solution This is similar to the integral given in Example 1, except that the integ is missing a factor of 2. Recognizing that $2x$ is the derivative of $x^2 + 1$, you ca $g(x) = x^2 + 1$ and supply the $2x$ as follows.

$$\begin{aligned}
\int x(x^2+1)^2\,dx &= \int (x^2+1)^2\left(\frac{1}{2}\right)(2x)\,dx && \text{Multiply and divide by 2.}\\
&= \frac{1}{2}\int \overbrace{(x^2+1)^2}^{f(g(x))}\,\overbrace{(2x)}^{g'(x)}\,dx && \text{Constant Multiple Rule}\\
&= \frac{1}{2}\left[\frac{(x^2+1)^3}{3}\right] + C && \text{Integrate.}\\
&= \frac{1}{6}(x^2+1)^3 + C
\end{aligned}$$

In practice, most people would not write as many steps as are shown in Exa 3. For instance, you could evaluate the integral by simply writing

$$\begin{aligned}
\int x(x^2+1)^2\,dx &= \frac{1}{2}\int (x^2+1)^2\,2x\,dx\\
&= \frac{1}{2}\left[\frac{(x^2+1)^3}{3}\right] + C\\
&= \frac{1}{6}(x^2+1)^3 + C.
\end{aligned}$$

NOTE Be sure you see that the *Constant* Multiple Rule applies only to *constants*. You c multiply and divide by a variable and then move the variable outside the integral sign instance,

$$\int (x^2+1)^2\,dx \neq \frac{1}{2x}\int (x^2+1)^2\,(2x)\,dx.$$

After all, if it were legitimate to move variable quantities outside the integral sign, you move the entire integrand out and simplify the whole process. But the result would be inc

Change of Variables

With a formal **change of variables,** you completely rewrite the integral in terms of u and du (or any other convenient variable). Although this procedure can involve more written steps than the pattern recognition illustrated in Examples 1 to 3, it is useful for complicated integrands. The change of variable technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $du = g'(x)\,dx$, and the integral in Theorem 4.12 takes the form

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C.$$

Example 4 Change of Variables

Find $\int \sqrt{2x-1}\,dx$.

Solution First, let u be the inner function, $u = 2x - 1$. Then calculate the differential du to be $du = 2\,dx$. Now, using $\sqrt{2x-1} = \sqrt{u}$ and $dx = du/2$, substitute to obtain the following.

$$\int \sqrt{2x-1}\,dx = \int \sqrt{u}\left(\frac{du}{2}\right) \qquad \text{Integral in terms of } u$$

$$= \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C \qquad \text{Antiderivative in terms of } u$$

$$= \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{3}(2x-1)^{3/2} + C \qquad \text{Antiderivative in terms of } x$$

DY TIP Because integration is ly more difficult than differentiation, hould always check your answer to tegration problem by differentiating. nstance, in Example 4 you should entiate $\frac{1}{3}(2x-1)^{3/2} + C$ to verify ou obtain the original integrand.

Example 5 Change of Variables

Find $\int x\sqrt{2x-1}\,dx$.

Solution As in the previous example, let $u = 2x - 1$ and obtain $dx = du/2$. Because the integrand contains a factor of x, you must also solve for x in terms of u, as follows.

$$u = 2x - 1 \quad \Rightarrow \quad x = (u+1)/2 \qquad \text{Solve for } x \text{ in terms of } u.$$

Now, using substitution, you obtain the following.

$$\int x\sqrt{2x-1}\,dx = \int \left(\frac{u+1}{2}\right)u^{1/2}\left(\frac{du}{2}\right)$$

$$= \frac{1}{4}\int (u^{3/2} + u^{1/2})\,du$$

$$= \frac{1}{4}\left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C$$

$$= \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C$$

To complete the change of variables in Example 5, we solved for x in terms Sometimes this is very difficult. Fortunately it is not always necessary, as shown i next example.

Example 6 Change of Variables

Find $\int \sin^2 3x \cos 3x \, dx$.

Solution Because $\sin^2 3x = (\sin 3x)^2$, you can let $u = \sin 3x$. Then

$$du = (\cos 3x)(3)\, dx.$$

Now, because $\cos 3x \, dx$ is part of the given integral, you can write

$$\frac{du}{3} = \cos 3x \, dx.$$

Substituting u and $du/3$ in the given integral yields the following.

$$\begin{aligned}\int \sin^2 3x \cos 3x \, dx &= \int u^2 \frac{du}{3} \\ &= \frac{1}{3}\int u^2 \, du \\ &= \frac{1}{3}\left(\frac{u^3}{3}\right) + C \\ &= \frac{1}{9}\sin^3 3x + C\end{aligned}$$

You can check this by differentiating.

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{9}\sin^3 3x\right] &= \left(\frac{1}{9}\right)(3)(\sin 3x)^2(\cos 3x)(3) \\ &= \sin^2 3x \cos 3x\end{aligned}$$

Because differentiation produces the original integrand, you know that you obtained the correct antiderivative.

STUDY TIP When making a change of variables, be sure that your answer is written using the same variables as in the original integrand. For instance, in Example 6, you should not leave your answer as

$$\tfrac{1}{9}u^3 + C$$

but rather, replace u by $\sin 3x$.

We summarize the steps used for integration by substitution in the follo guidelines.

Guidelines for Making a Change of Variables

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\, dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

The General Power Rule for Integration

One of the most common u-substitutions involves quantities in the integrand that are raised to a power. Because of the importance of this type of substitution, it is given a special name—the **General Power Rule** for integration. A proof of this rule follows directly from the (simple) Power Rule for integration, together with Theorem 4.12.

THEOREM 4.13 The General Power Rule for Integration

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C, \qquad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1.$$

Example 7 Substitution and the General Power Rule

a. $$\int 3(3x-1)^4\,dx = \int \overbrace{(3x-1)^4}^{u^4}\overbrace{(3)\,dx}^{du} = \frac{\overbrace{(3x-1)^5}^{u^5/5}}{5} + C$$

b. $$\int (2x+1)(x^2+x)dx = \int \overbrace{(x^2+x)^1}^{u^1}\overbrace{(2x+1)\,dx}^{du} = \frac{\overbrace{(x^2+x)^2}^{u^2/2}}{2} + C$$

c. $$\int 3x^2\sqrt{x^3-2}\,dx = \int \overbrace{(x^3-2)^{1/2}}^{u^{1/2}}\overbrace{(3x^2)\,dx}^{du} = \overbrace{\frac{(x^3-2)^{3/2}}{3/2}}^{u^{3/2}/(3/2)} + C = \frac{2}{3}(x^3-2)^{3/2} + C$$

d. $$\int \frac{-4x}{(1-2x^2)^2}\,dx = \int \overbrace{(1-2x^2)^{-2}}^{u^{-2}}\overbrace{(-4x)\,dx}^{du} = \overbrace{\frac{(1-2x^2)^{-1}}{-1}}^{u^{-1}/(-1)} + C$$

e. $$\int \cos^2 x \sin x\,dx = -\int \overbrace{(\cos x)^2}^{u^2}\overbrace{(-\sin x)\,dx}^{du} = -\frac{\overbrace{(\cos x)^3}^{u^3/3}}{3} + C$$

EXPLORATION

pose you were asked to find one
ne following integrals. Which one
ld you choose? Explain your
oning.

$$\int \sqrt{x^3+1}\,dx \quad \text{or}$$

$$\int x^2\sqrt{x^3+1}\,dx$$

$$\int \tan(3x)\sec^2(3x)\,dx \quad \text{or}$$

$$\int \tan(3x)\,dx$$

Some integrals whose integrands involve a quantity raised to a power cannot be found by the General Power Rule. Consider the two integrals

$$\int x(x^2+1)^2\,dx \quad \text{and} \quad \int (x^2+1)^2\,dx.$$

The substitution $u = x^2 + 1$ works in the first integral but not in the second. (In the second, the substitution fails because the integrand lacks the factor x needed for du.) Fortunately, *for this particular integral,* you can expand the integrand as $(x^2+1)^2 = x^4 + 2x^2 + 1$ and use the (simple) Power Rule to integrate each term.

Change of Variables for Definite Integrals

When using u-substitution with a definite integral, it is often convenient to deter the limits of integration for the variable u rather than to convert the antiderivative to the variable x and evaluate at the original limits. This change of variables is s explicitly in the next theorem. The proof follows from Theorem 4.12 combined the Fundamental Theorem of Calculus.

THEOREM 4.14 Change of Variables for Definite Integrals

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a,$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 8 Change of Variables

Evaluate $\displaystyle\int_0^1 x(x^2+1)^3\,dx.$

Solution To evaluate this integral, let $u = x^2 + 1$. Then, you obtain

$$u = x^2 + 1 \Rightarrow du = 2x\,dx.$$

Before substituting, determine the new upper and lower limits of integration.

Lower Limit	*Upper Limit*
When $x = 0$, $u = 0^2 + 1 = 1$.	When $x = 1$, $u = 1^2 + 1 = 2$.

Now, you can substitute to obtain

$$\int_0^1 x(x^2+1)^3\,dx = \frac{1}{2}\int_0^1 (x^2+1)^3(2x)\,dx \qquad \text{Integration limits for } x$$

$$= \frac{1}{2}\int_1^2 u^3\,du \qquad \text{Integration limits for } u$$

$$= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left(4 - \frac{1}{4}\right)$$

$$= \frac{15}{8}.$$

Try converting the antiderivative $\frac{1}{2}(u^4/4)$ back to the variable x and evaluate the nite integral at the original limits of integration, as follows.

$$\frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 = \frac{1}{2}\left[\frac{(x^2+1)^4}{4}\right]_0^1 = \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}$$

Notice that you obtain the same result.

Example 9 Change of Variables

Evaluate $A = \int_1^5 \frac{x}{\sqrt{2x-1}}\,dx.$

Solution To evaluate this integral, let $u = \sqrt{2x-1}$. Then, you obtain

$$\begin{aligned} u^2 &= 2x - 1 \\ u^2 + 1 &= 2x \\ \frac{u^2+1}{2} &= x \\ u\,du &= dx. \end{aligned}$$

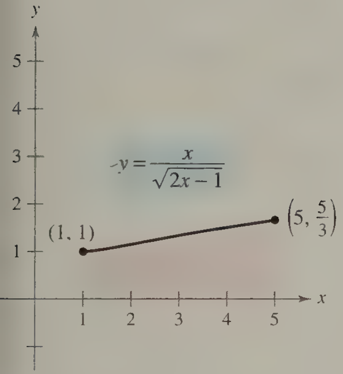
Differentiate both sides.

Before substituting, determine the new upper and lower limits of integration.

Lower Limit	*Upper Limit*
When $x = 1$, $u = \sqrt{2-1} = 1$.	When $x = 5$, $u = \sqrt{10-1} = 3$.

Now, substitute to obtain

$$\begin{aligned} \int_1^5 \frac{x}{\sqrt{2x-1}}\,dx &= \int_1^3 \frac{1}{u}\left(\frac{u^2+1}{2}\right) u\,du \\ &= \frac{1}{2}\int_1^3 (u^2+1)\,du \\ &= \frac{1}{2}\left[\frac{u^3}{3} + u\right]_1^3 \\ &= \frac{1}{2}\left(9 + 3 - \frac{1}{3} - 1\right) \\ &= \frac{16}{3}. \end{aligned}$$



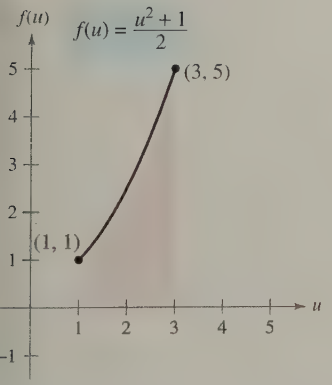
region before substitution has an area

ure 4.37

Geometrically, you can interpret the equation

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^3 \frac{u^2+1}{2}\,du$$

to mean that the two *different* regions shown in Figures 4.37 and 4.38 have the *same* area.



region after substitution has an area

ure 4.38

When evaluating definite integrals by substitution, it is possible for the upper limit of integration of the u-variable form to be smaller than the lower limit. If this happens, don't rearrange the limits. Simply evaluate as usual. For example, after substituting $u = \sqrt{1-x}$ in the integral

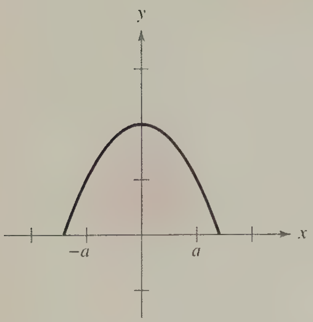
$$\int_0^1 x^2(1-x)^{1/2}\,dx$$

you obtain $u = \sqrt{1-1} = 0$ when $x = 1$, and $u = \sqrt{1-0} = 1$ when $x = 0$. So, the correct u-variable form of this integral is

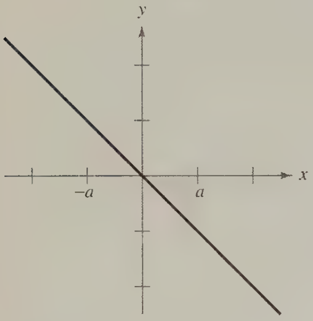
$$-2\int_1^0 (1-u^2)^2 u^2\,du.$$

Integration of Even and Odd Functions

Even with a change of variables, integration can be difficult. Occasionally, you c simplify the evaluation of a definite integral (over an interval that is symmetric abo the y-axis or about the origin) by recognizing the integrand to be an even or o function (see Figure 4.39).



Even function



Odd function
Figure 4.39

THEOREM 4.15 Integration of Even and Odd Functions

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an *even* function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$
2. If f is an *odd* function, then $\int_{-a}^{a} f(x)\,dx = 0.$

Proof Because f is even, you know that $f(x) = f(-x)$. Using Theorem 4.12 with t substitution $u = -x$ produces

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)(-du) = -\int_{a}^{0} f(u)\,du = \int_{0}^{a} f(u)\,du = \int_{0}^{a} f(x)\,dx.$$

Finally, using Theorem 4.6, you obtain

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx\\ &= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.\end{aligned}$$

This proves the first property. The proof of the second property is left to you (s Exercise 116).

Example 10 Integration of an Odd Function

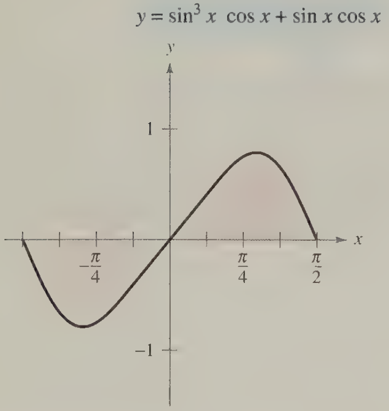
Evaluate $\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\,dx.$

Solution Letting $f(x) = \sin^3 x \cos x + \sin x \cos x$ produces

$$\begin{aligned}f(-x) &= \sin^3(-x)\cos(-x) + \sin(-x)\cos(-x)\\ &= -\sin^3 x \cos x - \sin x \cos x = -f(x).\end{aligned}$$

So, f is an odd function, and because f is symmetric about the origin ov $[-\pi/2, \pi/2]$, you can apply Theorem 4.15 to conclude that

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\,dx = 0.$$



Because f is an odd function,

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx = 0.$$

Figure 4.40

NOTE From Figure 4.40 you can see that the two regions on either side of the y-axis have same area. However, because one lies below the x-axis and one lies above it, integration p duces a cancellation effect. (We will say more about this in Section 6.1.)

XERCISES FOR SECTION 4.5

xercises 1–6, complete the table by identifying u and du for integral.

$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
$\int (5x^2+1)^2(10x)\,dx$		
$\int x^2\sqrt{x^3+1}\,dx$		
$\int \frac{x}{\sqrt{x^2+1}}\,dx$		
$\int \sec 2x \tan 2x\,dx$		
$\int \tan^2 x \sec^2 x\,dx$		
$\int \frac{\cos x}{\sin^2 x}\,dx$		

xercises 7–34, find the indefinite integral and check the lt by differentiation.

$\int (1+2x)^4(2)\,dx$ **8.** $\int (x^2-9)^3(2x)\,dx$

$\int \sqrt{9-x^2}\,(-2x)\,dx$ **10.** $\int \sqrt[3]{(1-2x^2)}(-4x)\,dx$

$\int x^3(x^4+3)^2\,dx$ **12.** $\int x^2(x^3+5)^4\,dx$

$\int x^2(x^3-1)^4\,dx$ **14.** $\int x(4x^2+3)^3\,dx$

$\int t\sqrt{t^2+2}\,dt$ **16.** $\int t^3\sqrt{t^4+5}\,dt$

$\int 5x\sqrt[3]{1-x^2}\,dx$ **18.** $\int u^2\sqrt{u^3+2}\,du$

$\int \frac{x}{(1-x^2)^3}\,dx$ **20.** $\int \frac{x^3}{(1+x^4)^2}\,dx$

$\int \frac{x^2}{(1+x^3)^2}\,dx$ **22.** $\int \frac{x^2}{(16-x^3)^2}\,dx$

$\int \frac{x}{\sqrt{1-x^2}}\,dx$ **24.** $\int \frac{x^3}{\sqrt{1+x^4}}\,dx$

$\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt$ **26.** $\int \left[x^2+\frac{1}{(3x)^2}\right]dx$

$\int \frac{1}{\sqrt{2x}}\,dx$ **28.** $\int \frac{1}{2\sqrt{x}}\,dx$

$\int \frac{x^2+3x+7}{\sqrt{x}}\,dx$ **30.** $\int \frac{t+2t^2}{\sqrt{t}}\,dt$

$\int t^2\left(t-\frac{2}{t}\right)dt$ **32.** $\int \left(\frac{t^3}{3}+\frac{1}{4t^2}\right)dt$

$\int (9-y)\sqrt{y}\,dy$ **34.** $\int 2\pi y(8-y^{3/2})\,dy$

In Exercises 35–38, solve the differential equation.

35. $\frac{dy}{dx} = 4x + \frac{4x}{\sqrt{16-x^2}}$

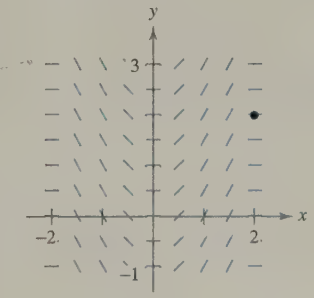
36. $\frac{dy}{dx} = \frac{10x^2}{\sqrt{1+x^3}}$

37. $\frac{dy}{dx} = \frac{x+1}{(x^2+2x-3)^2}$

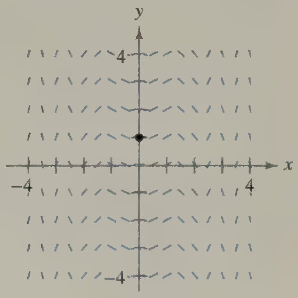
38. $\frac{dy}{dx} = \frac{x-4}{\sqrt{x^2-8x+1}}$

Slope Fields **In Exercises 39 and 40, a differential equation, a point, and a slope field are given. A *slope field* consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the directions of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the indicated point. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).**

39. $\frac{dy}{dx} = x\sqrt{4-x^2}, \quad (2, 2)$



40. $\frac{dy}{dx} = x\cos x^2, \quad (0, 1)$



In Exercises 41–54, find the indefinite integral.

41. $\int \pi \sin \pi x\,dx$ **42.** $\int 4x^3 \sin x^4\,dx$

43. $\int \sin 2x\,dx$ **44.** $\int \cos 6x\,dx$

45. $\int \frac{1}{\theta^2}\cos\frac{1}{\theta}\,d\theta$ **46.** $\int x \sin x^2\,dx$

47. $\int \sin 2x \cos 2x\,dx$ **48.** $\int \sec(1-x)\tan(1-x)\,dx$

49. $\int \tan^4 x \sec^2 x\,dx$ **50.** $\int \sqrt{\tan x}\,\sec^2 x\,dx$

51. $\int \frac{\csc^2 x}{\cot^3 x}\,dx$ **52.** $\int \frac{\sin x}{\cos^3 x}\,dx$

53. $\int \cot^2 x\,dx$ **54.** $\int \csc^2\left(\frac{x}{2}\right)dx$

In Exercises 55 and 56, find an equation for the function f that has the indicated derivative and whose graph passes through the given point.

Derivative	*Point*
55. $f'(x) = \cos\frac{x}{2}$	$(0, 3)$
56. $f'(x) = \pi\sec\pi x\tan\pi x$	$\left(\frac{1}{3}, 1\right)$

In Exercises 57–64, find the indefinite integral by the method shown in Example 5.

57. $\displaystyle\int x\sqrt{x+2}\,dx$

58. $\displaystyle\int x\sqrt{2x+1}\,dx$

59. $\displaystyle\int x^2\sqrt{1-x}\,dx$

60. $\displaystyle\int (x+1)\sqrt{2-x}\,dx$

61. $\displaystyle\int \frac{x^2-1}{\sqrt{2x-1}}\,dx$

62. $\displaystyle\int \frac{2x+1}{\sqrt{x+4}}\,dx$

63. $\displaystyle\int \frac{-x}{(x+1)-\sqrt{x+1}}\,dx$

64. $\displaystyle\int t\sqrt[3]{t-4}\,dt$

In Exercises 65–76, evaluate the definite integral. Use a graphing utility to verify your result.

65. $\displaystyle\int_{-1}^{1} x(x^2+1)^3\,dx$

66. $\displaystyle\int_{-2}^{4} x^2(x^3+8)^2\,dx$

67. $\displaystyle\int_{1}^{2} 2x^2\sqrt{x^3+1}\,dx$

68. $\displaystyle\int_{0}^{1} x\sqrt{1-x^2}\,dx$

69. $\displaystyle\int_{0}^{4} \frac{1}{\sqrt{2x+1}}\,dx$

70. $\displaystyle\int_{0}^{2} \frac{x}{\sqrt{1+2x^2}}\,dx$

71. $\displaystyle\int_{1}^{9} \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}\,dx$

72. $\displaystyle\int_{0}^{2} x\sqrt[3]{4+x^2}\,dx$

73. $\displaystyle\int_{1}^{2} (x-1)\sqrt{2-x}\,dx$

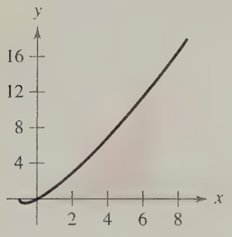
74. $\displaystyle\int_{1}^{5} \frac{x}{\sqrt{2x-1}}\,dx$

75. $\displaystyle\int_{0}^{\pi/2} \cos\left(\frac{2x}{3}\right)dx$

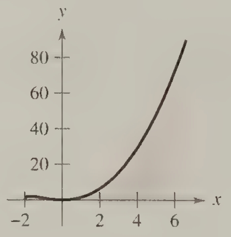
76. $\displaystyle\int_{\pi/3}^{\pi/2} (x+\cos x)\,dx$

In Exercises 77–82, find the area of the region. Use a graphing utility to verify your result.

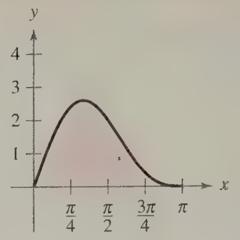
77. $\displaystyle\int_{0}^{7} x\sqrt[3]{x+1}\,dx$



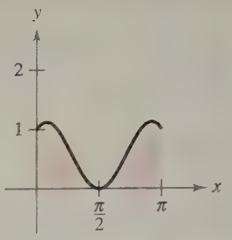
78. $\displaystyle\int_{-2}^{6} x^2\sqrt[3]{x+2}\,dx$



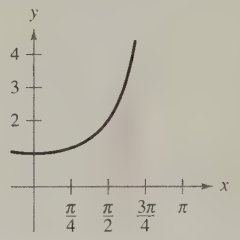
79. $y = 2\sin x + \sin 2x$



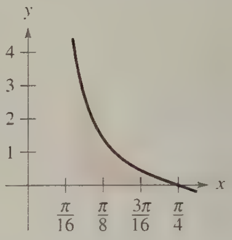
80. $y = \sin x + \cos 2x$



81. $\displaystyle\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)dx$



82. $\displaystyle\int_{\pi/12}^{\pi/4} \csc 2x\cot 2x\,dx$



In Exercises 83–88, use a graphing utility to evaluate integral. Graph the region whose area is given by the defin integral.

83. $\displaystyle\int_{0}^{4} \frac{x}{\sqrt{2x+1}}\,dx$

84. $\displaystyle\int_{0}^{2} x^3\sqrt{x+2}\,dx$

85. $\displaystyle\int_{3}^{7} x\sqrt{x-3}\,dx$

86. $\displaystyle\int_{1}^{5} x^2\sqrt{x-1}\,dx$

87. $\displaystyle\int_{0}^{3} \left(\theta+\cos\frac{\theta}{6}\right)d\theta$

88. $\displaystyle\int_{0}^{\pi/2} \sin 2x\,dx$

Writing **In Exercises 89 and 90, find the indefinite integral two ways. Explain any difference in the forms of the answer**

89. $\displaystyle\int (2x-1)^2\,dx$

90. $\displaystyle\int \sin x\cos x\,dx$

In Exercises 91–94, evaluate the integral using the propertie even and odd functions as an aid.

91. $\displaystyle\int_{-2}^{2} x^2(x^2+1)\,dx$

92. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^2 x\cos x\,dx$

93. $\displaystyle\int_{-2}^{2} x(x^2+1)^3\,dx$

94. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x\cos x\,dx$

95. Use $\int_0^2 x^2\,dx = \frac{8}{3}$ to evaluate the definite integrals without us the Fundamental Theorem of Calculus.

(a) $\displaystyle\int_{-2}^{0} x^2\,dx$

(b) $\displaystyle\int_{-2}^{2} x^2\,dx$

(c) $\displaystyle\int_{0}^{2} -x^2\,dx$

(d) $\displaystyle\int_{-2}^{0} 3x^2\,dx$

Use the symmetry of the graphs of the sine and cosine functions as an aid in evaluating each of the integrals.

(a) $\int_{-\pi/4}^{\pi/4} \sin x\,dx$ (b) $\int_{-\pi/4}^{\pi/4} \cos x\,dx$

(c) $\int_{-\pi/2}^{\pi/2} \cos x\,dx$ (d) $\int_{-\pi/2}^{\pi/2} \sin x \cos x\,dx$

xercises 97 and 98, write the integral as the sum of the gral of an odd function and the integral of an even function. this simplification to evaluate the integral.

$\int_{-4}^{4} (x^3 + 6x^2 - 2x - 3)\,dx$ **98.** $\int_{-\pi}^{\pi} (\sin 3x + \cos 3x)\,dx$

etting at the Concept

9. In your own words, state the guidelines for making a change of variables when integrating.

0. Describe why $\int x(5 - x^2)^3\,dx \neq \int u^3\,du$ where $u = 5 - x^2$.

1. Without integrating, explain why $\int_{-2}^{2} x(x^2 + 1)^2\,dx = 0$.

Cash Flow The rate of disbursement dQ/dt of a 2 million dollar federal grant is proportional to the square of $100 - t$. Time t is measured in days $(0 \le t \le 100)$, and Q is the amount that remains to be disbursed. Find the amount that remains to be disbursed after 50 days. Assume that all the money will be disbursed in 100 days.

Depreciation The rate of depreciation dV/dt of a machine is inversely proportional to the square of $t + 1$, where V is the value of the machine t years after it was purchased. If the initial value of the machine was \$500,000, and its value decreased \$100,000 in the first year, estimate its value after 4 years.

Rainfall The normal monthly rainfall at the Seattle-Tacoma airport can be approximated by the model

$$R = 3.121 + 2.399 \sin(0.524t + 1.377)$$

where R is measured in inches and t is the time in months, with $t = 1$ corresponding to January. *(Source: U.S. National Oceanic and Atmospheric Administration)*

(a) Determine the extrema of the function over a 1-year period.

(b) Use integration to approximate the normal annual rainfall. (*Hint:* Integrate over the interval [0, 12].)

(c) Approximate the average monthly rainfall during the months of October, November, and December.

Sales The sales of a seasonal product are given by the model

$$S = 74.50 + 43.75 \sin\frac{\pi t}{6}$$

where S is measured in thousands of units and t is the time in months, with $t = 1$ corresponding to January. Find the average sales for the following periods.

(a) The first quarter $(0 \le t \le 3)$

(b) The second quarter $(3 \le t \le 6)$

(c) The entire year $(0 \le t \le 12)$

106. ***Water Supply*** A model for the flow rate of water at a pumping station on a given day is

$$R(t) = 53 + 7\sin\left(\frac{\pi t}{6} + 3.6\right) + 9\cos\left(\frac{\pi t}{12} + 8.9\right)$$

where $0 \le t \le 24$. R is the flow rate in thousands of gallons per hour, and t is the time in hours.

(a) Use a graphing utility to graph the rate function and approximate the maximum flow rate at the pumping station.

(b) Approximate the total volume of water pumped in 1 day.

107. ***Electricity*** The oscillating current in an electrical circuit is

$$I = 2\sin(60\pi t) + \cos(120\pi t)$$

where I is measured in amperes and t is measured in seconds. Find the average current for each time interval.

(a) $0 \le t \le \frac{1}{60}$ (b) $0 \le t \le \frac{1}{240}$ (c) $0 \le t \le \frac{1}{30}$

108. ***Graphical Analysis*** Consider the functions f and g, where

$$f(x) = 6 \sin x \cos^2 x \quad \text{and} \quad g(t) = \int_0^t f(x)\,dx.$$

(a) Use a graphing utility to graph f and g in the same viewing window.

(b) Explain why g is nonnegative.

(c) Identify the points on the graph of g that correspond to the extrema of f.

(d) Does each of the zeros of f correspond to an extremum of g? Explain.

(e) Consider the function $h(t) = \int_{\pi/2}^{t} f(x)\,dx$. Use a graphing utility to graph h. What is the relationship between g and h? Verify your conjecture.

True or False? **In Exercises 109–114, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

109. $\int (2x + 1)^2\,dx = \frac{1}{3}(2x + 1)^3 + C$

110. $\int x(x^2 + 1)\,dx = \frac{1}{2}x^2\left(\frac{1}{3}x^3 + x\right) + C$

111. $\int_{-10}^{10} (ax^3 + bx^2 + cx + d)\,dx = 2\int_0^{10} (bx^2 + d)\,dx$

112. $\int_a^b \sin x\,dx = \int_a^{b+2\pi} \sin x\,dx$

113. $4\int \sin x \cos x\,dx = -\cos 2x + C$

114. $\int \sin^2 2x \cos 2x\,dx = \frac{1}{3}\sin^3 2x + C$

115. Show that if f is continuous on the entire real line, then

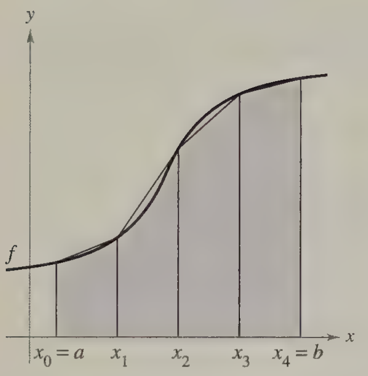
$$\int_a^b f(x + h)\,dx = \int_{a+h}^{b+h} f(x)\,dx.$$

116. Complete the proof of Theorem 4.15.

Section 4.6 Numerical Integration

- Approximate a definite integral using the Trapezoidal Rule.
- Approximate a definite integral using Simpson's Rule.
- Analyze the approximate error in the Trapezoidal Rule and in Simpson's Rule.

The Trapezoidal Rule



The area of the region can be approximated using four trapezoids.
Figure 4.41

Some elementary functions simply do not have antiderivatives that are element functions. For example, there is no elementary function that has any of the follow functions as its derivative.

$$\sqrt[3]{x}\sqrt{1-x}, \qquad \sqrt{x}\cos x, \qquad \frac{\cos x}{x}, \qquad \sqrt{1-x^3}, \qquad \sin x^2$$

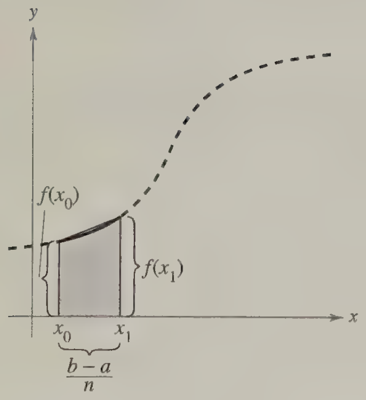
If you need to evaluate a definite integral involving a function whose antideriva cannot be found, the Fundamental Theorem of Calculus cannot be applied, and must resort to an approximation technique. Two such techniques are described in section.

One way to approximate a definite integral is to use n trapezoids, as show Figure 4.41. In the development of this method, assume that f is continuous positive on the interval $[a, b]$. So, the definite integral

$$\int_a^b f(x)\, dx$$

represents the area of the region bounded by the graph of f and the x-axis, fr $x = a$ to $x = b$. First, partition the interval $[a, b]$ into n subintervals, each of wi $\Delta x = (b - a)/n$, such that

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_n = b.$$



The area of the first trapezoid is $\left[\frac{f(x_0) + f(x_1)}{2}\right]\left(\frac{b-a}{n}\right)$.
Figure 4.42

Then form a trapezoid for each subinterval, as shown in Figure 4.42. The area of ith trapezoid is

$$\text{Area of } i\text{th trapezoid} = \left[\frac{f(x_{i-1}) + f(x_i)}{2}\right]\left(\frac{b-a}{n}\right).$$

This implies that the sum of the areas of the n trapezoids is

$$\begin{aligned}\text{Area} &= \left(\frac{b-a}{n}\right)\left[\frac{f(x_0) + f(x_1)}{2} + \cdot \cdot \cdot + \frac{f(x_{n-1}) + f(x_n)}{2}\right]\\ &= \left(\frac{b-a}{2n}\right)[f(x_0) + f(x_1) + f(x_1) + f(x_2) + \cdot \cdot \cdot + f(x_{n-1}) + f(x_n)]\\ &= \left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_1) + 2f(x_2) + \cdot \cdot \cdot + 2f(x_{n-1}) + f(x_n)].\end{aligned}$$

Letting $\Delta x = (b - a)/n$, you can take the limit as $n \to \infty$ to obtain

$$\begin{aligned}&\lim_{n\to\infty}\left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_1) + \cdot \cdot \cdot + 2f(x_{n-1}) + f(x_n)]\\ &= \lim_{n\to\infty}\left[\frac{[f(a) - f(b)]\,\Delta x}{2} + \sum_{i=1}^{n} f(x_i)\,\Delta x\right]\\ &= \lim_{n\to\infty}\frac{[f(a) - f(b)](b-a)}{2n} + \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x\\ &= 0 + \int_a^b f(x)\, dx.\end{aligned}$$

The result is summarized in the following theorem.

THEOREM 4.16 The Trapezoidal Rule

Let f be continuous on $[a, b]$. The Trapezoidal Rule for approximating $\int_a^b f(x)\,dx$ is given by

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

Moreover, as $n \to \infty$, the right-hand side approaches $\int_a^b f(x)\,dx$.

NOTE Observe that the coefficients in the Trapezoidal Rule have the following pattern.

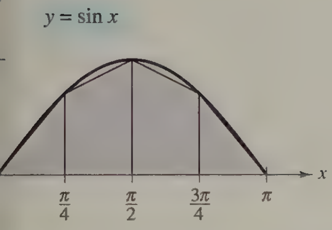
$$1 \quad 2 \quad 2 \quad 2 \quad \ldots \quad 2 \quad 2 \quad 1$$

Example 1 Approximation with the Trapezoidal Rule

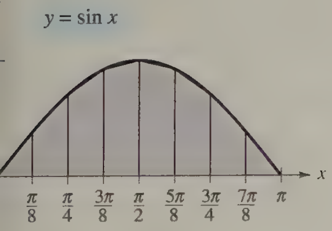
Use the Trapezoidal Rule to approximate

$$\int_0^{\pi} \sin x\,dx.$$

Compare the results for $n = 4$ and $n = 8$, as shown in Figure 4.43.



Four subintervals



Eight subintervals

ezoidal approximations
re 4.43

Solution When $n = 4$, $\Delta x = \pi/4$, and you obtain

$$\begin{aligned}\int_0^{\pi} \sin x\,dx &\approx \frac{\pi}{8}\left(\sin 0 + 2\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 2\sin\frac{3\pi}{4} + \sin\pi\right)\\ &= \frac{\pi}{8}\left(0 + \sqrt{2} + 2 + \sqrt{2} + 0\right) = \frac{\pi\left(1+\sqrt{2}\right)}{4} \approx 1.896.\end{aligned}$$

When $n = 8$, $\Delta x = \pi/8$, and you obtain

$$\begin{aligned}\int_0^{\pi} \sin x\,dx &\approx \frac{\pi}{16}\left(\sin 0 + 2\sin\frac{\pi}{8} + 2\sin\frac{\pi}{4} + 2\sin\frac{3\pi}{8} + 2\sin\frac{\pi}{2}\right.\\ &\qquad \left. + 2\sin\frac{5\pi}{8} + 2\sin\frac{3\pi}{4} + 2\sin\frac{7\pi}{8} + \sin\pi\right)\\ &= \frac{\pi}{16}\left(2 + 2\sqrt{2} + 4\sin\frac{\pi}{8} + 4\sin\frac{3\pi}{8}\right) \approx 1.974.\end{aligned}$$

For this particular integral, you could have found an antiderivative and determined that the exact area of the region is 2.

TECHNOLOGY Most graphing utilities and computer algebra systems have built-in programs that can be used to approximate the value of a definite integral. Try using such a program to approximate the integral in Example 1. How close is your approximation?

When you use such a program, you need to be aware of its limitations. Often, you are given no indication of the degree of accuracy of the approximation. Other times, you may be given an approximation that is completely wrong. For instance, try using a built-in numerical integration program to evaluate

$$\int_{-1}^{2} \frac{1}{x}\,dx.$$

Your calculator should give an error message. Does yours?

It is interesting to compare the Trapezoidal Rule with the Midpoint Rule given Section 4.2 (Exercises 63–66). For the Trapezoidal Rule, you average the funct values at the endpoints of the subintervals, but for the Midpoint Rule you take function values of the subinterval midpoints.

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x \qquad \text{Midpoint Rule}$$

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \left(\frac{f(x_i) + f(x_{i-1})}{2}\right)\Delta x \qquad \text{Trapezoidal Rule}$$

NOTE There are two important points that should be made concerning the Trapezoidal R (or the Midpoint Rule). First, the approximation tends to become more accurate as n increa For instance, in Example 1, if $n = 16$, the Trapezoidal Rule yields an approximation of 1.9 Second, although you could have used the Fundamental Theorem to evaluate the integra Example 1, this theorem cannot be used to evaluate an integral as simple as $\int_0^{\pi} \sin x^2\,dx$ beca $\sin x^2$ has no elementary antiderivative. Yet, the Trapezoidal Rule can be applied easily to integral.

Simpson's Rule

One way to view the trapezoidal approximation of a definite integral is to say that each subinterval you approximate f by a *first*-degree polynomial. In Simpson's Ru named after the English mathematician Thomas Simpson (1710–1761), you take t procedure one step further and approximate f by *second*-degree polynomials.

Before presenting Simpson's Rule, we list a theorem for evaluating integrals polynomials of degree 2 (or less).

THEOREM 4.17 Integral of $p(x) = Ax^2 + Bx + C$

If $p(x) = Ax^2 + Bx + C$, then

$$\int_a^b p(x)\,dx = \left(\frac{b-a}{6}\right)\left[p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right].$$

Proof

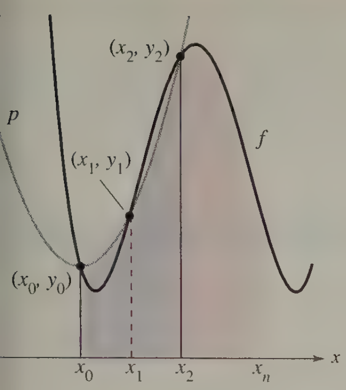
$$\begin{aligned}\int_a^b p(x)\,dx &= \int_a^b (Ax^2 + Bx + C)\,dx \\ &= \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_a^b \\ &= \frac{A(b^3 - a^3)}{3} + \frac{B(b^2 - a^2)}{2} + C(b - a) \\ &= \left(\frac{b-a}{6}\right)[2A(a^2 + ab + b^2) + 3B(b + a) + 6C]\end{aligned}$$

By expansion and collection of terms, the expression inside the brackets becomes

$$\underbrace{(Aa^2 + Ba + C)}_{p(a)} + \underbrace{4\left[A\left(\frac{b+a}{2}\right)^2 + B\left(\frac{b+a}{2}\right) + C\right]}_{4p\left(\frac{a+b}{2}\right)} + \underbrace{(Ab^2 + Bb + C)}_{p(b)}$$

and you can write

$$\int_a^b p(x)\,dx = \left(\frac{b-a}{6}\right)\left[p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right].$$



$(x)\,dx \approx \int_{x_0}^{x_2} f(x)\,dx$

re 4.44

To develop Simpson's Rule for approximating a definite integral, you again partition the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$. This time, however, n is required to be even, and the subintervals are grouped in pairs such that

$$a = \underbrace{x_0 < x_1 < x_2}_{[x_0,\, x_2]} \underbrace{< x_3 < x_4}_{[x_2,\, x_4]} < \cdot \cdot \cdot < \underbrace{x_{n-2} < x_{n-1} < x_n}_{[x_{n-2},\, x_n]} = b.$$

On each (double) subinterval $[x_{i-2}, x_i]$, you can approximate f by a polynomial p of degree less than or equal to 2. (See Exercise 46.) For example, on the subinterval $[x_0, x_2]$, choose the polynomial of least degree passing through the points (x_0, y_0), (x_1, y_1), and (x_2, y_2), as shown in Figure 4.44. Now, using p as an approximation of f on this subinterval, you have, by Theorem 4.17,

$$\begin{aligned}\int_{x_0}^{x_2} f(x)\,dx \approx \int_{x_0}^{x_2} p(x)\,dx &= \frac{x_2 - x_0}{6}\left[p(x_0) + 4p\left(\frac{x_0 + x_2}{2}\right) + p(x_2)\right] \\ &= \frac{2[(b-a)/n]}{6}[p(x_0) + 4p(x_1) + p(x_2)] \\ &= \frac{b-a}{3n}[f(x_0) + 4f(x_1) + f(x_2)].\end{aligned}$$

Repeating this procedure on the entire interval $[a, b]$ produces the following theorem.

THEOREM 4.18 Simpson's Rule (n is even)

Let f be continuous on $[a, b]$. Simpson's Rule for approximating $\int_a^b f(x)\,dx$ is

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdot \cdot \cdot + 4f(x_{n-1}) + f(x_n)].$$

Moreover, as $n \to \infty$, the right-hand side approaches $\int_a^b f(x)\,dx$.

E Observe that the coefficients mpson's Rule have the following rn.

1 4 2 4 2 4 . . . 4 2 4 1

In Example 1, the Trapezoidal Rule was used to estimate $\int_0^\pi \sin x\,dx$. In the next example, Simpson's Rule is applied to the same integral.

Example 2 Approximation with Simpson's Rule

Use Simpson's Rule to approximate

$$\int_0^\pi \sin x\,dx.$$

Compare the results for $n = 4$ and $n = 8$.

Solution When $n = 4$, you have

$$\int_0^\pi \sin x\,dx \approx \frac{\pi}{12}\left(\sin 0 + 4 \sin\frac{\pi}{4} + 2 \sin\frac{\pi}{2} + 4 \sin\frac{3\pi}{4} + \sin \pi\right) \approx 2.005.$$

When $n = 8$, you have $\int_0^\pi \sin x\,dx \approx 2.0003$.

E In Example 1, the Trapezoidal with $n = 8$ approximated n $x\,dx$ as 1.974. In Example 2, son's Rule with $n = 8$ gave an oximation of 2.0003. The antideriva- would produce the true value of 2.

Error Analysis

If you must use an approximation technique, it is important to know how accurate
can expect the approximation to be. The following theorem, which we list wit
proof, gives the formulas for estimating the errors involved in the use of Simps
Rule and the Trapezoidal Rule.

THEOREM 4.19 Errors in the Trapezoidal Rule and Simpson's Rule

If f has a continuous second derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)\,dx$ by the Trapezoidal Rule is

$$E \le \frac{(b-a)^3}{12n^2}\left[\max|f''(x)|\right], \quad a \le x \le b. \qquad \text{Trapezoidal Rule}$$

Moreover, if f has a continuous fourth derivative on $[a, b]$, then the error E in
approximating $\int_a^b f(x)\,dx$ by Simpson's Rule is

$$E \le \frac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right], \quad a \le x \le b. \qquad \text{Simpson's Rule}$$

Theorem 4.19 states that the errors generated by the Trapezoidal Rule
Simpson's Rule have upper bounds dependent on the extreme values of $f''(x)$
$f^{(4)}(x)$ in the interval $[a, b]$. Furthermore, these errors can be made arbitrarily smal
increasing n, provided that f'' and $f^{(4)}$ are continuous and therefore bounded in $[a$

TECHNOLOGY If you have access to a computer algebra system, try using it to evaluate the definite integral in Example 3. You should obtain a value of

$$\int_0^1 \sqrt{1+x^2}\,dx = \tfrac{1}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$$

$$\approx 1.14779.$$

("ln" represents the natural logarithmic function, which you will study in Section 5.1.)

Example 3 The Approximate Error in the Trapezoidal Rule

Determine a value of n such that the Trapezoidal Rule will approximate the valu
$\int_0^1 \sqrt{1+x^2}\,dx$ with an error that is less than 0.01.

Solution Begin by letting $f(x) = \sqrt{1+x^2}$ and finding the second derivative o

$$f'(x) = x(1+x^2)^{-1/2} \quad \text{and} \quad f''(x) = (1+x^2)^{-3/2}$$

The maximum value of $|f''(x)|$ on the interval $[0, 1]$ is $|f''(0)| = 1$. So, by Theo
4.19, you can write

$$E \le \frac{(b-a)^3}{12n^2}|f''(0)| = \frac{1}{12n^2}(1) = \frac{1}{12n^2}.$$

To obtain an error E that is less than 0.01, you must choose n such
$1/(12n^2) \le 1/100$.

$$100 \le 12n^2 \quad ⇨ \quad n \ge \sqrt{\frac{100}{12}} \approx 2.89$$

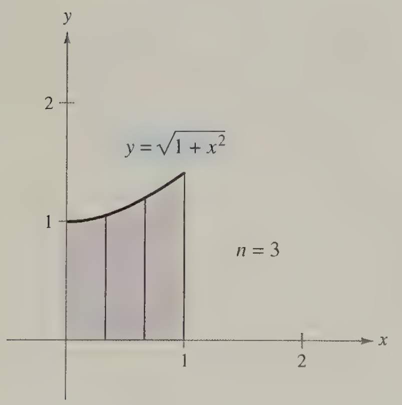
Therefore, you can choose $n = 3$ (because n must be greater than or equal to
and apply the Trapezoidal Rule, as shown in Figure 4.45, to obtain

$$\int_0^1 \sqrt{1+x^2}\,dx \approx \frac{1}{6}\sqrt{1+0^2} + 2\sqrt{1+\left(\tfrac{1}{3}\right)^2} + 2\sqrt{1+\left(\tfrac{2}{3}\right)^2} + \sqrt{1+1^2}$$

$$\approx 1.154.$$

So, with an error no larger than 0.01, you know that

$$1.144 \le \int_0^1 \sqrt{1+x^2}\,dx \le 1.164.$$



$$1.144 \le \int_0^1 \sqrt{1+x^2}\,dx \le 1.164$$

Figure 4.45

Lab Series LAB 6

XERCISES FOR SECTION 4.6

xercises 1–10, use the Trapezoidal Rule and Simpson's Rule pproximate the value of the definite integral for the indi-d value of n. Round your answer to four decimal places and pare the results with the exact value of the definite integral.

$\int_0^2 x^2\,dx, \quad n = 4$

2. $\int_0^1 \left(\frac{x^2}{2} + 1\right) dx, \quad n = 4$

$\int_0^2 x^3\,dx, \quad n = 4$

4. $\int_1^2 \frac{1}{x^2}\,dx, \quad n = 4$

$\int_0^2 x^3\,dx, \quad n = 8$

6. $\int_0^8 \sqrt[3]{x}\,dx, \quad n = 8$

$\int_4^9 \sqrt{x}\,dx, \quad n = 8$

8. $\int_1^3 (4 - x^2)\,dx, \quad n = 4$

$\int_1^2 \frac{1}{(x+1)^2}\,dx, \quad n = 4$

10. $\int_0^2 x\sqrt{x^2+1}\,dx, \quad n = 4$

xercises 11–20, approximate the definite integral using the pezoidal Rule and Simpson's Rule with $n = 4$. Compare e results with the approximation of the integral using a hing utility.

$\int_0^2 \sqrt{1 + x^3}\,dx$

12. $\int_0^2 \frac{1}{\sqrt{1 + x^3}}\,dx$

$\int_0^1 \sqrt{x}\,\sqrt{1 - x}\,dx$

14. $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx$

$\int_0^{\sqrt{\pi/2}} \cos x^2\,dx$

16. $\int_0^{\sqrt{\pi/4}} \tan x^2\,dx$

$\int_1^{1.1} \sin x^2\,dx$

18. $\int_0^{\pi/2} \sqrt{1 + \cos^2 x}\,dx$

$\int_0^{\pi/4} x\tan x\,dx$

$\int_0^{\pi} f(x)\,dx, \quad f(x) = \begin{cases} \frac{\sin x}{x}, & x > 0 \\ 1, & x = 0 \end{cases}$

etting at the Concept

. If the function f is concave upward on the interval $[a, b]$, will the Trapezoidal Rule yield a result greater than or less than $\int_a^b f(x)\,dx$? Explain.

. The Trapezoidal Rule and Simpson's Rule yield approximations of a definite integral $\int_a^b f(x)\,dx$ based on polynomial approximations of f. What degree polynomial is used for each?

xercises 23 and 24, use the error formulas in Theorem 4.19 timate the error in approximating the integral, with $n = 4$, g (a) the Trapezoidal Rule and (b) Simpson's Rule.

$\int_0^2 x^3\,dx$

24. $\int_0^1 \frac{1}{x+1}\,dx$

In Exercises 25 and 26, use the error formulas in Theorem 4.19 to find n such that the error in the approximation of the definite integral is less than 0.00001 using (a) the Trapezoidal Rule and (b) Simpson's Rule.

25. $\int_1^3 \frac{1}{x}\,dx$

26. $\int_0^1 \frac{1}{1+x}\,dx$

In Exercises 27–30, use a computer algebra system and the error formulas to find n such that the error in the approximation of the definite integral is less than 0.00001 using (a) the Trapezoidal Rule and (b) Simpson's Rule.

27. $\int_0^2 \sqrt{1+x}\,dx$

28. $\int_0^2 (x+1)^{2/3}\,dx$

29. $\int_0^1 \tan x^2\,dx$

30. $\int_0^1 \sin x^2\,dx$

31. Prove that Simpson's Rule is exact when approximating the integral of a cubic polynomial function, and demonstrate the result for

$$\int_0^1 x^3\,dx, \qquad n = 2.$$

32. Write a program for a graphing utility to approximate a definite integral using the Trapezoidal Rule and Simpson's Rule. Start with the program written in Section 4.3, Exercises 57–60, and note that the Trapezoidal Rule can be written as

$$T(n) = \tfrac{1}{2}[L(n) + R(n)]$$

and Simpson's Rule can be written as

$$S(n) = \tfrac{1}{3}[T(n/2) + 2M(n/2)].$$

[Recall that $L(n)$, $M(n)$, and $R(n)$ represent the Riemann sums using the left-hand endpoint, midpoint, and right-hand endpoint of subintervals of equal width.]

In Exercises 33–36, use the program in Exercise 32 to approximate the definite integral and complete the table.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4					
8					
10					
12					
16					
20					

33. $\int_0^4 \sqrt{2 + 3x^2}\,dx$

34. $\int_0^1 \sqrt{1 - x^2}\,dx$

35. $\int_0^4 \sin\sqrt{x}\,dx$

36. $\int_1^2 \frac{\sin x}{x}\,dx$

37. ***Area*** Use Simpson's Rule with $n = 14$ to approximate the area of the region bounded by the graphs of $y = \sqrt{x}\cos x$, $y = 0$, $x = 0$, and $x = \pi/2$.

38. ***Circumference*** The **elliptic integral**

$$8\sqrt{3}\int_0^{\pi/2} \sqrt{1 - \tfrac{2}{3}\sin^2\theta}\, d\theta$$

gives the circumference of an ellipse. Use Simpson's Rule with $n = 8$ to approximate the circumference.

39. ***Work*** To determine the size of the motor required to operate a press, a company must know the amount of work done when the press moves an object linearly 5 feet. The variable force to move the object is

$$F(x) = 100x\sqrt{125 - x^3}$$

where F is given in pounds and x gives the position of the unit in feet. Use Simpson's Rule with $n = 12$ to approximate the work W (in foot-pounds) done through one cycle if

$$W = \int_0^5 F(x)\, dx.$$

40. The table lists several measurements gathered in an experiment to approximate an unknown continuous function $y = f(x)$.

(a) Approximate the integral $\int_0^2 f(x)\, dx$ using the Trapezoidal Rule and Simpson's Rule.

x	0.00	0.25	0.50	0.75	1.00
y	4.32	4.36	4.58	5.79	6.14

x	1.25	1.50	1.75	2.00
y	7.25	7.64	8.08	8.14

(b) Use a graphing utility to find a model of the form $y = ax^3 + bx^2 + cx + d$ for the data. Integrate the resulting polynomial over $[0, 2]$ and compare the result with part (a).

Approximation of Pi **In Exercises 41 and 42, use Simpson's Rule with $n = 6$ to approximate π using the given equation. (In Section 5.9, you will be able to evaluate the integral using inverse trigonometric functions.)**

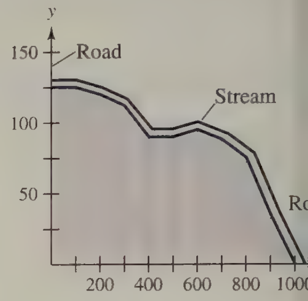
41. $\pi = \displaystyle\int_0^{1/2} \frac{6}{\sqrt{1 - x^2}}\, dx$

42. $\pi = \displaystyle\int_0^{1} \frac{4}{1 + x^2}\, dx$

Area **In Exercises 43 and 44, use the Trapezoidal Rule estimate the number of square meters of land in a lot wher and y are measured in meters, as shown in the figures. The la is bounded by a stream and two straight roads that meet at ri angles.**

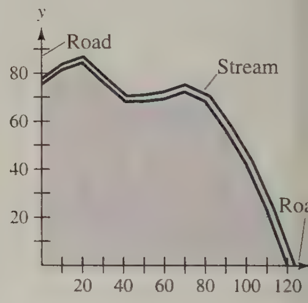
43.

x	y
0	125
100	125
200	120
300	112
400	90
500	90
600	95
700	88
800	75
900	35
1000	0



44.

x	y
0	75
10	81
20	84
30	76
40	67
50	68
60	69
70	72
80	68
90	56
100	42
110	23
120	0



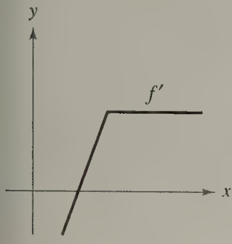
45. Use Simpson's Rule with $n = 10$ and a computer alge system to approximate t in the integral equation

$$\int_0^t \sin\sqrt{x}\, dx = 2.$$

46. Prove that you can find a polynomial $p(x) = Ax^2 + Bx$ that passes through any three points (x_1, y_1), (x_2, y_2), (x_3, y_3), where the x_i are distinct.

EVIEW EXERCISES FOR CHAPTER 4

In Exercises 1 and 2, use the graph of f' to sketch a ph of f. To print an enlarged copy of the graph, go to the site *www.mathgraphs.com*.



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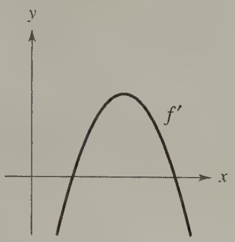


xercises 3–8, find the indefinite integral.

$\int (2x^2 + x - 1)\,dx$

$\int \frac{2}{\sqrt[3]{3x}}\,dx$

$\int \frac{x^3 + 1}{x^2}\,dx$

$\int \frac{x^3 - 2x^2 + 1}{x^2}\,dx$

$\int (4x - 3\sin x)\,dx$

$\int (5\cos x - 2\sec^2 x)\,dx$

Find the particular solution of the differential equation $f'(x) = -2x$ whose graph passes through the point $(-1, 1)$.

Find the particular solution of the differential equation $f''(x) = 6(x - 1)$ whose graph passes through the point $(2, 1)$ and is tangent to the line $3x - y - 5 = 0$ at that point.

Velocity and Acceleration An airplane taking off from a runway travels 3600 feet before lifting off. If it starts from rest, moves with constant acceleration, and makes the run in 30 seconds, with what speed does it lift off?

Velocity and Acceleration The speed of a car traveling in a straight line is reduced from 45 to 30 miles per hour in a distance of 264 feet. Find the distance in which the car can be brought to rest from 30 miles per hour, assuming the same constant deceleration.

Velocity and Acceleration A ball is thrown vertically upward from ground level with an initial velocity of 96 feet per second.

a) How long will it take the ball to rise to its maximum height?

b) What is the maximum height?

c) When is the velocity of the ball one-half the initial velocity?

d) What is the height of the ball when its velocity is one-half the initial velocity?

elocity and Acceleration Repeat Exercise 13 for an initial elocity of 40 meters per second.

4.2

15. Write in sigma notation (a) the sum of the first ten positive odd integers, (b) the sum of the cubes of the first n positive integers, and (c) $6 + 10 + 14 + 18 + \cdots + 42$.

16. Evaluate each sum for $x_1 = 2$, $x_2 = -1$, $x_3 = 5$, $x_4 = 3$, and $x_5 = 7$.

(a) $\frac{1}{5}\sum_{i=1}^{5} x_i$

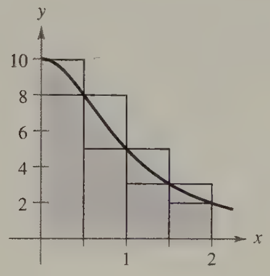
(b) $\sum_{i=1}^{5} \frac{1}{x_i}$

(c) $\sum_{i=1}^{5} (2x_i - x_i^2)$

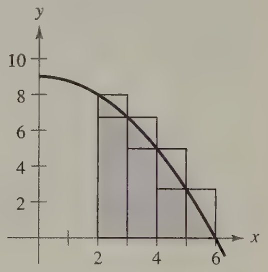
(d) $\sum_{i=2}^{5} (x_i - x_{i-1})$

In Exercises 17 and 18, use upper and lower sums to approximate the area of the region using the indicated number of subintervals of equal width.

17. $y = \frac{10}{x^2 + 1}$



18. $y = 9 - \frac{1}{4}x^2$



In Exercises 19–22, use the limit process to find the area of the region between the graph of the function and the x-axis over the indicated interval. Sketch the region.

	Function	*Interval*
19.	$y = 6 - x$	$[0, 4]$
20.	$y = x^2 + 3$	$[0, 2]$
21.	$y = 5 - x^2$	$[-2, 1]$
22.	$y = \frac{1}{4}x^3$	$[2, 4]$

23. Use the limit process to find the area of the region bounded by $x = 5y - y^2$, $x = 0$, $y = 2$, and $y = 5$.

24. Consider the region bounded by $y = mx$, $y = 0$, $x = 0$, and $x = b$.

(a) Find the upper and lower sums to approximate the area of the region when $\Delta x = b/4$.

(b) Find the upper and lower sums to approximate the area of the region when $\Delta x = b/n$.

(c) Find the area of the region by letting n approach infinity in both sums in part (b). Show that in each case you obtain the formula for the area of a triangle.

4.3 In Exercises 25 and 26, express the limit as a definite integral on the interval $[a, b]$, where c_i is any point in the ith subinterval.

Limit	*Interval*
25. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (2c_i - 3)\Delta x_i$	$[4, 6]$
26. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 3c_i(9 - c_i^2)\Delta x_i$	$[1, 3]$

In Exercises 27 and 28, sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral.

27. $\int_0^5 (5 - |x - 5|)\,dx$

28. $\int_{-4}^4 \sqrt{16 - x^2}\,dx$

In Exercises 29 and 30, use the given values to evaluate each definite integral.

29. If $\int_2^6 f(x)\,dx = 10$ and $\int_2^6 g(x)\,dx = 3$, find

(a) $\int_2^6 [f(x) + g(x)]\,dx$.
(b) $\int_2^6 [f(x) - g(x)]\,dx$.
(c) $\int_2^6 [2f(x) - 3g(x)]\,dx$.
(d) $\int_2^6 5f(x)\,dx$.

30. If $\int_0^3 f(x)\,dx = 4$ and $\int_3^6 f(x)\,dx = -1$, find

(a) $\int_0^6 f(x)\,dx$.
(b) $\int_6^3 f(x)\,dx$.
(c) $\int_4^4 f(x)\,dx$.
(d) $\int_3^6 -10f(x)\,dx$.

4.4 In Exercises 31 and 32, select the correct value of the definite integral.

31. $\int_1^8 (\sqrt[3]{x} + 1)\,dx$

(a) $\frac{81}{4}$ (b) $\frac{331}{12}$
(c) $\frac{73}{4}$ (d) $\frac{355}{12}$

32. $\int_1^3 \frac{12}{x^3}\,dx$

(a) $\frac{320}{9}$ (b) $-\frac{16}{3}$
(c) $-\frac{5}{9}$ (d) $\frac{16}{3}$

In Exercises 33–40, use the Fundamental Theorem of Calculus to evaluate the definite integral.

33. $\int_0^4 (2 + x)\,dx$

34. $\int_{-1}^1 (t^2 + 2)\,dt$

35. $\int_{-1}^1 (4t^3 - 2t)\,dt$

36. $\int_{-2}^2 (x^4 + 2x^2 - 5)\,dx$

37. $\int_4^9 x\sqrt{x}\,dx$

38. $\int_1^2 \left(\frac{1}{x^2} - \frac{1}{x^3}\right)dx$

39. $\int_0^{3\pi/4} \sin\theta\,d\theta$

40. $\int_{-\pi/4}^{\pi/4} \sec^2 t\,dt$

In Exercises 41–46, sketch the graph of the region whose are given by the integral, and find the area.

41. $\int_1^3 (2x - 1)\,dx$

42. $\int_0^2 (x + 4)\,dx$

43. $\int_3^4 (x^2 - 9)\,dx$

44. $\int_{-1}^2 (-x^2 + x + 2)$

45. $\int_0^1 (x - x^3)\,dx$

46. $\int_0^1 \sqrt{x}(1 - x)\,dx$

In Exercises 47 and 48, sketch the region bounded by the gra of the equations, and determine its area.

47. $y = \frac{4}{\sqrt{x}}, \quad y = 0, \quad x = 1, \quad x = 9$

48. $y = \sec^2 x, \quad y = 0, \quad x = 0, \quad x = \frac{\pi}{3}$

In Exercises 49 and 50, find the average value of the funct over the interval. Find the values of x at which the funct assumes its average value, and graph the function.

Function	*Interval*
49. $f(x) = \frac{1}{\sqrt{x}}$	$[4, 9]$
50. $f(x) = x^3$	$[0, 2]$

In Exercises 51–54, use the Second Fundamental Theorem Calculus to find $F'(x)$.

51. $F(x) = \int_0^x t^2\sqrt{1 + t^3}\,dt$

52. $F(x) = \int_1^x \frac{1}{t^2}\,dt$

53. $F(x) = \int_{-3}^x (t^2 + 3t + 2)\,dt$

54. $F(x) = \int_0^x \csc^2 t\,dt$

4.5 In Exercises 55–68, find the indefinite integral.

55. $\int (x^2 + 1)^3\,dx$

56. $\int \left(x + \frac{1}{x}\right)^2 dx$

57. $\int \frac{x^2}{\sqrt{x^3 + 3}}\,dx$

58. $\int x^2\sqrt{x^3 + 3}\,dx$

59. $\int x(1 - 3x^2)^4\,dx$

60. $\int \frac{x + 3}{(x^2 + 6x - 5)^2}\,d$

61. $\int \sin^3 x \cos x\,dx$

62. $\int x \sin 3x^2\,dx$

63. $\int \frac{\sin\theta}{\sqrt{1 - \cos\theta}}\,d\theta$

64. $\int \frac{\cos x}{\sqrt{\sin x}}\,dx$

65. $\int \tan^n x \sec^2 x\,dx, \quad n \neq -1$

66. $\int \sec 2x \tan 2x\,dx$

67. $\int (1 + \sec \pi x)^2 \sec \pi x \tan \pi x\,dx$

68. $\int \cot^4 \alpha \csc^2 \alpha\,d\alpha$

xercises 69–76, evaluate the definite integral. Use a graph-tility to verify your result.

$\int_{-1}^{2} x(x^2 - 4)\,dx$

70. $\int_0^1 x^2(x^3 + 1)^3\,dx$

$\int_0^3 \frac{1}{\sqrt{1+x}}\,dx$

72. $\int_3^6 \frac{x}{3\sqrt{x^2 - 8}}\,dx$

$2\pi\int_0^1 (y + 1)\sqrt{1 - y}\,dy$

74. $2\pi\int_{-1}^0 x^2\sqrt{x + 1}\,dx$

$\int_0^{\pi} \cos\frac{x}{2}\,dx$

76. $\int_{-\pi/4}^{\pi/4} \sin 2x\,dx$

ability **In Exercises 77 and 78, the function**

$$= kx^n(1 - x)^m, \qquad 0 \le x \le 1$$

e $n > 0$, $m > 0$, and k is a constant, can be used to repre-various probability distributions. If k is chosen such that

$$x)\,dx = 1$$

robability that x will fall between a and b $(0 \le a \le b \le 1)$ is

$$= \int_a^b f(x)\,dx.$$

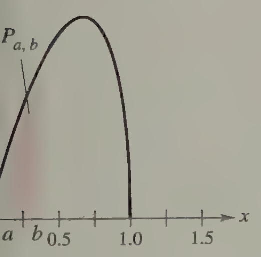
he probability that a person will remember between $a\%$ and % of material learned in a certain experiment is

$$_{a,b} = \int_a^b \frac{15}{4} x\sqrt{1 - x}\,dx$$

here x represents the percent remembered. (See figure.)

a) For a randomly chosen individual, what is the probability that he or she will recall between 50% and 75% of the material?

b) What is the median percent recall? That is, for what value of b is it true that the probability from 0 to b is 0.5?



re for 77

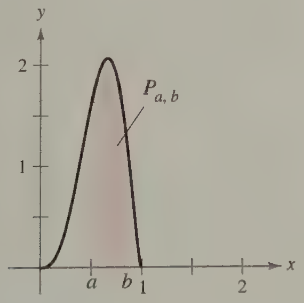


Figure for 78

he probability that ore samples taken from a certain region ontain between $a\%$ and $b\%$ iron is

$$_{a,b} = \int_a^b \frac{1155}{32} x^3(1 - x)^{3/2}\,dx$$

here x represents the percent of iron. (See figure.) What is the robability that a sample will contain between

a) 0% and 25% iron?

b) 50% and 100% iron?

79. Suppose that gasoline is increasing in price according to the equation

$$p = 1.20 + 0.04t$$

where p is the dollar price per gallon and t is the time in years, with $t = 0$ representing 1990. If an automobile is driven 15,000 miles a year and gets M miles per gallon, the annual fuel cost is

$$C = \frac{15{,}000}{M}\int_t^{t+1} p\,dt.$$

Estimate the annual fuel cost for the year (a) 2000 and (b) 2005.

80. ***Respiratory Cycle*** After exercising for a few minutes, a person has a respiratory cycle for which the rate of air intake is

$$v = 1.75 \sin\frac{\pi t}{2}.$$

Find the volume, in liters, of air inhaled during one cycle by integrating the function over the interval $[0, 2]$.

4.6 **In Exercises 81–84, use the Trapezoidal Rule and Simpson's Rule with $n = 4$, and use the integration capabilities of a graphing utility, to approximate the definite integral. Compare the results.**

81. $\int_1^2 \frac{1}{1 + x^3}\,dx$

82. $\int_0^1 \frac{x^{3/2}}{3 - x^2}\,dx$

83. $\int_0^{\pi/2} \sqrt{x}\cos x\,dx$

84. $\int_0^{\pi} \sqrt{1 + \sin^2 x}\,dx$

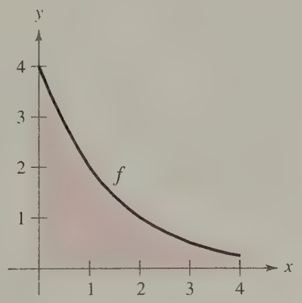
85. Let

$$I = \int_0^4 f(x)\,dx$$

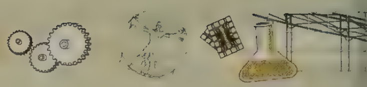
where f is shown in the figure. Let $L(n)$ and $R(n)$ represent the Riemann sums using the left-hand endpoint and right-hand endpoint of n subintervals of equal width. (Assume n is even.) Let $T(n)$ and $S(n)$ be the corresponding values of the Trapezoidal Rule and Simpson's Rule.

(a) For any n, list $L(n)$, $R(n)$, $T(n)$, and I in increasing order.

(b) Approximate $S(4)$.



P.S. Problem Solving



1. Let $L(x) = \int_1^x \frac{1}{t}\,dt,\ x > 0.$

(a) Find $L(1)$.

(b) Find $L'(x)$ and $L'(1)$.

(c) Use a graphing utility to approximate the value of x (to three decimal places) for which $L(x) = 1$.

(d) Prove that $L(x_1x_2) = L(x_1) + L(x_2)$ for all positive values of x_1 and x_2.

2. Let $F(x) = \int_2^x \sin t^2\,dt.$

(a) Use a graphing utility to complete the table.

x	0	1.0	1.5	1.9	2.0
$F(x)$					
x	2.1	2.5	3.0	4.0	5.0
$F(x)$					

(b) Let $G(x) = \frac{1}{x-2}F(x) = \frac{1}{x-2}\int_2^x \sin t^2\,dt$. Use a graphing utility to complete the table and estimate $\lim_{x\to 2} G(x)$.

x	1.9	1.95	1.99	2.01	2.1
$G(x)$					

(c) Use the definition of the derivative to find the exact value of the limit $\lim_{x\to 2} G(x)$.

3. The **Fresnel function** S is defined by the integral

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt.$$

(a) Graph the function $y = \sin\left(\frac{\pi x^2}{2}\right)$ on the interval $[0, 3]$.

(b) Use the graph in part (a) to sketch the graph of S on the interval $[0, 3]$.

(c) Locate all relative extrema of S on the interval $(0, 3)$.

(d) Locate all points of inflection of S on the interval $(0, 3)$.

4. Galileo Galilei (1564–1642) stated the following proposition concerning falling objects:

> *The time in which any space is traversed by a uniformly accelerating body is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed of the accelerating body and the speed just before acceleration began.*

Use the techniques of this section to verify this proposition.

5. The graph of the function f consists of the three line segm joining the points $(0, 0)$, $(2, -2)$, $(6, 2)$, and $(8, 3)$. The func F is defined by the integral

$$F(x) = \int_0^x f(t)\,dt.$$

(a) Sketch the graph of f.

(b) Complete the table of values.

x	0	1	2	3	4	5	6	7
$F(x)$								

(c) Find the extrema of F on the interval $[0, 8]$.

(d) Determine all points on inflection of inflection of F on interval $(0, 8)$.

6. A car is traveling in a straight line for one hour. Its velocity miles per hour at six-minute intervals is shown in the table.

t (hours)	0	0.1	0.2	0.3	0.4	0.5
v (mi/hr)	0	10	20	40	60	50

t (hours)	0.6	0.7	0.8	0.9	1.0
v (mi/hr)	40	35	40	50	65

(a) Produce a reasonable graph of the velocity function graphing these points and connecting them with a sm curve.

(b) Find the open intervals over which the acceleration positive.

(c) Find the average acceleration of the car (in miles per squared) over the interval $[0, 0.4]$.

(d) What does the integral $\int_0^1 v(t)\,dt$ signify? Approximate integral using the Trapezoidal Rule with five subinterva

(e) Approximate the acceleration at $t = 0.8$.

7. The **Two-Point Gaussian Quadrature Approximation** for

$$\int_{-1}^1 f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

(a) Use this formula to approximate

$$\int_{-1}^1 \cos x\,dx.$$

Find the error of the approximation.

(b) Use this formula to approximate

$$\int_{-1}^1 \frac{1}{1+x^2}\,dx.$$

(c) Prove that the Two-Point Gaussian Quadrature App mation is exact for all polynomials of degree 3 or less.

P.S.

Prove $\int_0^x f(t)(x-t)\,dt = \int_0^x \left(\int_0^t f(v)\,dv\right) dt.$

Prove $\int_a^b f(x)f'(x)\,dx = \frac{1}{2}([f(b)]^2 - [f(a)]^2).$

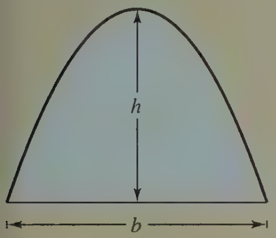
Use an appropriate Riemann sum to evaluate the limit

$$\lim_{n\to\infty} \frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdot\cdot\cdot + \sqrt{n}}{n^{3/2}}.$$

Use an appropriate Riemann sum to evaluate the limit

$$\lim_{n\to\infty} \frac{1^5 + 2^5 + 3^5 + \cdot\cdot\cdot + n^5}{n^6}.$$

Archimedes showed that the area of a parabolic arch is equal to $\frac{2}{3}$ the product of the base and the height, as indicated in the figure.



(a) Graph the parabolic arch bounded by $y = 9 - x^2$ and the x-axis. Use an appropriate integral to find the area A.

(b) Find the base and height of the arch and verify Archimedes' formula.

(c) Prove Archimedes' formula for a general parabola.

Suppose that f is integrable on $[a, b]$ and $0 < m \le f(x) \le M$ for all x in the interval $[a, b]$. Prove that

$$m(a-b) \le \int_a^b f(x)\,dx \le M(b-a).$$

Use this result to estimate $\int_0^1 \sqrt{1 + x^4}\,dx.$

Verify that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

by showing the following.

(a) $(1 + i)^3 - i^3 = 3i^2 + 3i + 1$

(b) $(n + 1)^3 = \sum_{i=1}^{n} (3i^2 + 3i + 1) + 1$

(c) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Prove that if f is a continuous function on a closed interval $[a, b]$, then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

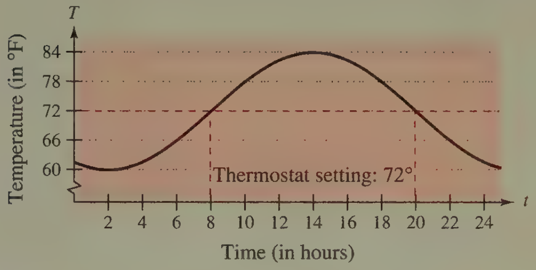
16. The temperature in degrees Fahrenheit is

$$T = 72 + 12 \sin\left[\frac{\pi(t-8)}{12}\right]$$

where t is time in hours, with $t = 0$ representing midnight. Suppose the hourly cost of cooling a house is \$0.10 per degree.

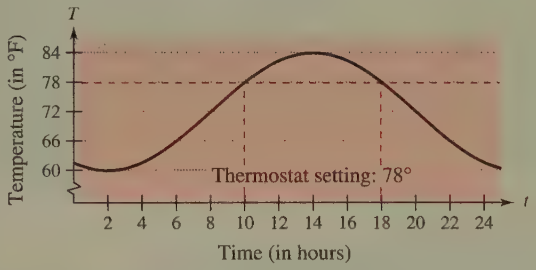
(a) Find the cost C of cooling the house if its thermostat is set at 72°F by evaluating the integral

$$C = 0.1\int_8^{20} \left[72 + 12 \sin \frac{\pi(t-8)}{12} - 72\right] dt. \text{ (See figure.)}$$



(b) Find the savings from resetting the thermostat to 78°F by evaluating the integral

$$C = 0.1\int_{10}^{18} \left[72 + 12 \sin \frac{\pi(t-8)}{12} - 78\right] dt. \text{ (See figure.)}$$



17. A manufacturer of fertilizer finds that national sales of fertilizer follow the seasonal pattern

$$F = 100{,}000\left[1 + \sin \frac{2\pi(t-60)}{365}\right]$$

where F is measured in pounds and t is time in days, with $t = 1$ representing January 1. The manufacturer wants to set up a schedule to produce a uniform amount of fertilizer each day. What should this amount be?

18. Let f be continuous on the interval $[0, b]$. Show that

$$\int_0^b \frac{f(x)}{f(x) + f(b-x)}\,dx = \frac{b}{2}.$$

Use this result to evaluate

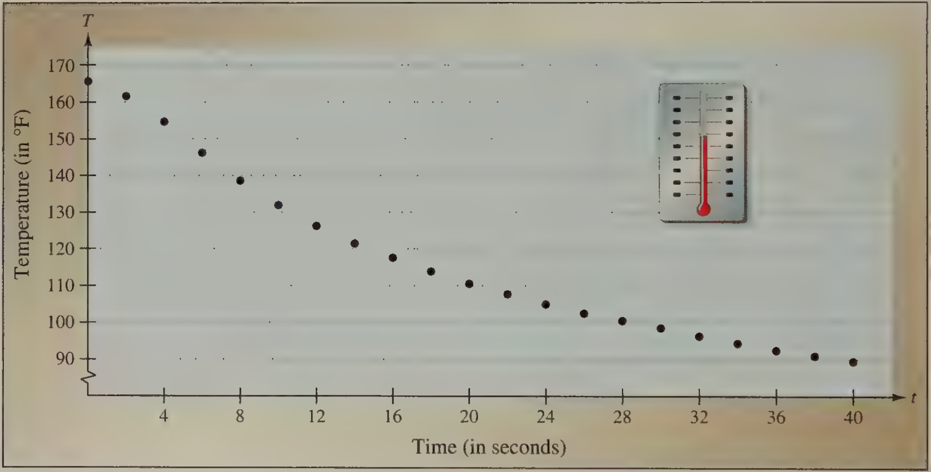
$$\int_0^1 \frac{\sin x}{\sin(1-x) + \sin x}\,dx.$$

Plastics and Cooling

What do Corvette fenders, panty hose, and garbage bags have in common? They are all made of plastic. The Greek word *plastikos*, meaning "able to be shaped," was modified to name the most versatile family of materials ever created. Since Bakelite was introduced in 1909, the plastics industry has steadily expanded to the point where today, plastics are used in nearly every aspect of our daily lives.

Several methods are used to shape plastic products, one of the most common being to pour hot, syrupy *plastic resin* into a mold or cast. The temperature of the molten resin is over 300°F. The mold is then cooled in a chiller system that is kept at 58°F before the part is ejected from the mold. To minimize the cost, it helps to eject the parts quickly, allowing the mold to be reused as soon as possible. Yet ejecting the part when it is too hot can cause warping or punctures. The rate at which objects cool is therefore of great interest.

To illustrate the rate of cooling, the *Texas Instruments Calculator-Based Laboratory (CBL) System* was used to measure the temperature of a cup of water over a 40-second period. The room temperature was measured at 69.55°F, and the water temperature at time $t = 0$ was measured at 165.58°F. The results are shown in the following scatter plot.



QUESTIONS

1. Describe the pattern of the temperature points over time. Does the *rate* at which the temperature changes seem to increase, decrease, or remain constant?
2. Imagine a curve running through the data points. How would you expect the curve to behave as the value of t increases? Would you expect the curve to intersect the line $T = 69.55$? Explain your reasoning.
3. Would the derivative of a function modeling the data points be increasing, decreasing, or constant? Explain your reasoning.
4. The data in the scatter plot can be modeled using a function of the form

   $$T = a \cdot b^t + c.$$

   Find values of a, b, and c that produce a reasonable model.

The concepts presented here will be explored further in this chapter. For an extension of this application, see Lab 7 in the lab series that accompanies this text at college.hmco.com.

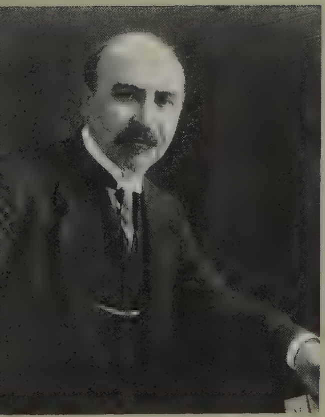
Logarithmic, Exponential, and Other Transcendental Functions

5

Plastic resin is produced in very small pieces that are easy to heat to a liquid state. The resin can then go through the injection molding process described on the facing page.

Ron Kimball

In 1953, Chevrolet introduced the Corvette, the first mass-produced automobile with a plastic body.



Leo Hendrik Baekeland attempted to create a shellac by combining phenol and formaldehyde. The experiment "failed" in that it did not result in shellac, but it did form the first completely synthetic plastic resin. Bakelite is still used today in the automotive and electronics industries.

Section 5.1 The Natural Logarithmic Function: Differentiation

- Develop and use properties of the natural logarithmic function.
- Understand the definition of the number e.
- Find derivatives of functions involving the natural logarithmic function.

The Granger Collection

John Napier (1550–1617)

Logarithms were invented by the Scottish mathematician John Napier. Although he did not introduce the *natural* logarithmic function, it is sometimes called the *Napierian* logarithm.

The Natural Logarithmic Function

Recall that the General Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1 \qquad \text{General Power Rule}$$

has an important disclaimer—it doesn't apply when $n = -1$. Consequently, we h not yet found an antiderivative for the function $f(x) = 1/x$. In this section, we will the Second Fundamental Theorem of Calculus to *define* such a function. T antiderivative is a function that we have not encountered previously in the text. I neither algebraic nor trigonometric, but falls into a new class of functions cal *logarithmic functions.* This particular function is the **natural logarithmic functio**

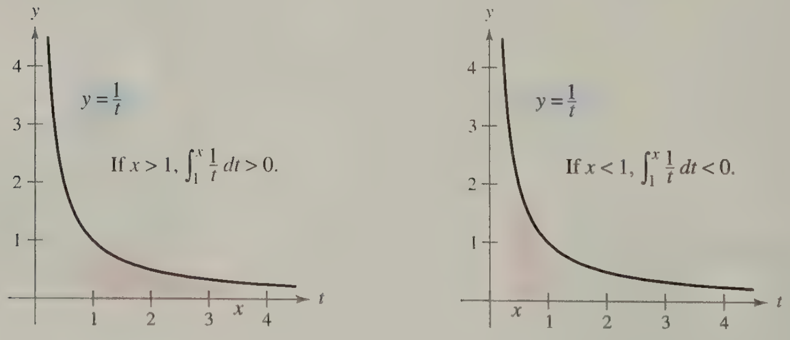
Definition of the Natural Logarithmic Function

The **natural logarithmic function** is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$

The domain of the natural logarithmic function is the set of all positive real numbers.

From the definition, you can see that $\ln x$ is positive for $x > 1$ and negative $0 < x < 1$, as shown in Figure 5.1. Moreover, $\ln(1) = 0$, because the upper and lo limits of integration are equal when $x = 1$.



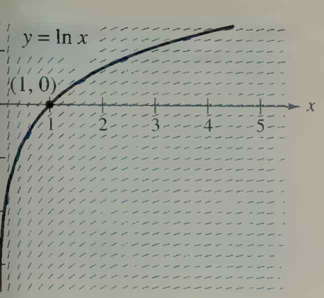
If $x > 1$, then $\ln x > 0$.

If $0 < x < 1$, then $\ln x < 0$.

Figure 5.1

EXPLORATION

Graphing the Natural Logarithmic Function Using *only* the definition of the natural logarithmic function, sketch a graph of the function. Explain your reasoning.



n small line segment has a slope of $\frac{1}{x}$.

ure 5.2

TE Slope fields can be helpful etting a visual perspective of the ctions of the solutions of a differen- equation.

To sketch the graph of $y = \ln x$, you can think of the natural logarithmic function as an *antiderivative* given by the differential equation

$$\frac{dy}{dx} = \frac{1}{x}.$$

Figure 5.2 is a computer-generated graph, called a *slope* (or *direction*) *field*, showing small line segments of slope $1/x$. The graph of $y = \ln x$ is the solution that passes through the point $(1, 0)$. For a complete discussion of slope fields, see Appendix A.

The following theorem lists some basic properties of the natural logarithmic function.

THEOREM 5.1 Properties of the Natural Logarithmic Function

The natural logarithmic function has the following properties.

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

Proof The domain of $f(x) = \ln x$ is $(0, \infty)$ by definition. Moreover, the function is continuous because it is differentiable. It is increasing because its derivative

$$f'(x) = \frac{1}{x} \qquad \text{First derivative}$$

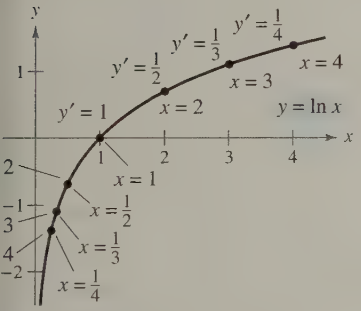
is positive for $x > 0$, as shown in Figure 5.3. It is concave downward because

$$f''(x) = -\frac{1}{x^2} \qquad \text{Second derivative}$$

is negative for $x > 0$. We leave the proof that f is one-to-one as an exercise (see Exercise 104). The following limits imply that its range is the entire real line.

$$\lim_{x\to 0^+} \ln x = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} \ln x = \infty$$

Verification of these two limits is given in Appendix B.



natural logarithmic function is increas- and its graph is concave downward.

ure 5.3

Using the definition of the natural logarithmic function, you can prove several important properties involving operations with natural logarithms. If you are already familiar with logarithms, you will recognize that these properties are characteristic of all logarithms.

THEOREM 5.2 Logarithmic Properties

If a and b are positive numbers and n is rational, then the following properties are true.

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

LOGARITHMS

pier coined the term *logarithm*, from the o Greek words *logos* (or ratio) and *arithmos* number), to describe the theory that he ent 20 years developing and that first peared in the book *Mirifici Logarithmorum nonis descriptio* (A Description of the arvelous Rule of Logarithms).

Proof We have already discussed the first property. The proof of the second prope follows from the fact that two antiderivatives of the same function differ at most b constant. From the Second Fundamental Theorem of Calculus and the definition of natural logarithmic function, you know that

$$\frac{d}{dx}[\ln x] = \frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right] = \frac{1}{x}.$$

So, consider the two derivatives

$$\frac{d}{dx}[\ln(ax)] = \frac{a}{ax} = \frac{1}{x}$$

and

$$\frac{d}{dx}[\ln a + \ln x] = 0 + \frac{1}{x} = \frac{1}{x}.$$

Because $\ln(ax)$ and $(\ln a + \ln x)$ are both antiderivatives of $1/x$, they must differ most by a constant.

$$\ln(ax) = \ln a + \ln x + C$$

By letting $x = 1$, you can see that $C = 0$. The third property can be proved simila by comparing the derivatives of $\ln(x^n)$ and $n \ln x$. Finally, using the second and th properties, you can prove the fourth property.

$$\ln\left(\frac{a}{b}\right) = \ln[a(b^{-1})] = \ln a + \ln(b^{-1}) = \ln a - \ln b$$

Example 1 shows how logarithmic properties can be used to expand logarithn expressions.

Example 1 Expanding Logarithmic Expressions

a. $\ln\frac{10}{9} = \ln 10 - \ln 9$ Property 4

b. $\ln\sqrt{3x+2} = \ln(3x+2)^{1/2}$ Rewrite with rational exponent.

$= \frac{1}{2}\ln(3x+2)$ Property 3

c. $\ln\frac{6x}{5} = \ln(6x) - \ln 5$ Property 4

$= \ln 6 + \ln x - \ln 5$ Property 2

d. $\ln\frac{(x^2+3)^2}{x\sqrt[3]{x^2+1}} = \ln(x^2+3)^2 - \ln\left(x\sqrt[3]{x^2+1}\right)$

$= 2\ln(x^2+3) - [\ln x + \ln(x^2+1)^{1/3}]$

$= 2\ln(x^2+3) - \ln x - \ln(x^2+1)^{1/3}$

$= 2\ln(x^2+3) - \ln x - \frac{1}{3}\ln(x^2+1)$

When using the properties of logarithms to rewrite logarithmic functions, you m check to see whether the domain of the rewritten function is the same as the domair the original. For instance, the domain of $f(x) = \ln x^2$ is all real numbers except $x =$ and the domain of $g(x) = 2\ln x$ is all positive real numbers. (See Figure 5.4.)

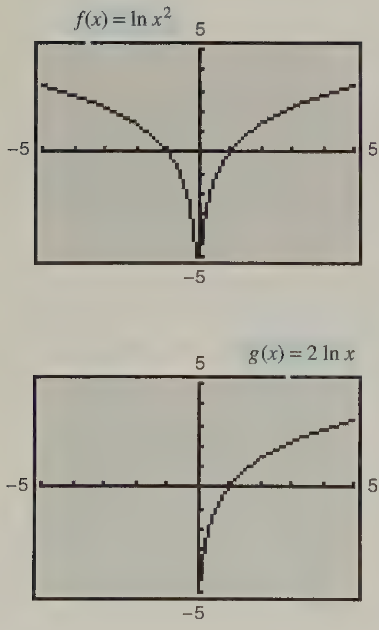
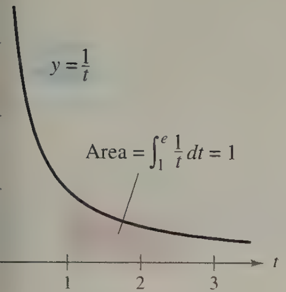


Figure 5.4

The Number *e*

It is likely that you have studied logarithms in an algebra course. There, without the benefit of calculus, logarithms would have been defined in terms of a **base** number. For example, common logarithms have a base of 10 because $\log_{10} 10 = 1$. (We will say more about this in Section 5.5.)

To define the **base for the natural logarithm,** we use the fact that the natural logarithmic function is continuous, is one-to-one, and has a range of $(-\infty, \infty)$. Hence, there must be a unique real number x such that $\ln x = 1$, as shown in Figure 5.5. This number is denoted by the letter e. It can be shown that e is irrational and has the following decimal approximation.



$e \approx 2.72$

he base for the natural logarithm
se $\ln e = 1$.
re 5.5

$$e \approx 2.71828182846$$

Definition of *e*

The letter e denotes the positive real number such that

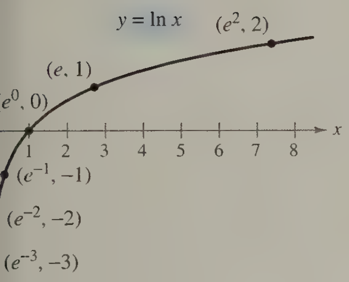
$$\ln e = \int_1^e \frac{1}{t}\,dt = 1.$$

FOR FURTHER INFORMATION To learn more about the number e, see the article "Unexpected Occurrences of the Number e" by Harris S. Shultz and Bill Leonard in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

Once you know that $\ln e = 1$, you can use logarithmic properties to evaluate the natural logarithms of several other numbers. For example, by using the property

$$\begin{aligned}\ln(e^n) &= n \ln e\\ &= n(1)\\ &= n\end{aligned}$$

you can evaluate $\ln(e^n)$ for various powers of n, as shown in the table and in Figure 5.6.



= e^n, then $\ln x = n$.
re 5.6

x	$\frac{1}{e^3} \approx 0.050$	$\frac{1}{e^2} \approx 0.135$	$\frac{1}{e} \approx 0.368$	$e^0 = 1$	$e \approx 2.718$	$e^2 \approx 7.389$
ln *x*	-3	-2	-1	0	1	2

The logarithms given in the table above are convenient because the x-values are integer powers of e. Most logarithmic expressions are, however, best evaluated with a calculator.

Example 2 Evaluating Natural Logarithmic Expressions

a. $\ln 2 \approx 0.693$

b. $\ln 32 \approx 3.466$

c. $\ln 0.1 \approx -2.303$

The Derivative of the Natural Logarithmic Function

The derivative of the natural logarithmic function is given in Theorem 5.3. The f part of the theorem follows from the definition of the natural logarithmic functior an antiderivative. The second part of the theorem is simply the Chain Rule versior the first part.

THEOREM 5.3 Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[\ln x] = \dfrac{1}{x},\ x > 0$ **2.** $\dfrac{d}{dx}[\ln u] = \dfrac{1}{u}\dfrac{du}{dx} = \dfrac{u'}{u},\ u > 0$

Example 3 Differentiation of Logarithmic Functions

a. $\dfrac{d}{dx}[\ln(2x)] = \dfrac{u'}{u} = \dfrac{2}{2x} = \dfrac{1}{x}$ $u = 2x$

b. $\dfrac{d}{dx}[\ln(x^2+1)] = \dfrac{u'}{u} = \dfrac{2x}{x^2+1}$ $u = x^2 + 1$

c. $\dfrac{d}{dx}[x \ln x] = x\left(\dfrac{d}{dx}[\ln x]\right) + (\ln x)\left(\dfrac{d}{dx}[x]\right)$ Product Rule

$$= x\left(\frac{1}{x}\right) + (\ln x)(1)$$

$$= 1 + \ln x$$

d. $\dfrac{d}{dx}[(\ln x)^3] = 3(\ln x)^2 \dfrac{d}{dx}[\ln x]$ Chain Rule

$$= 3(\ln x)^2 \frac{1}{x}$$

EXPLORATION

Use a graphing utility to graph

$$y_1 = \frac{1}{x}$$

and

$$y_2 = \frac{d}{dx}[\ln x]$$

in the same viewing window, in which $0.1 \le x \le 5$ and $-2 \le y \le 8$. Explain why the graphs appear to be identical.

Napier used logarithmic properties to simplify *calculations* involving produ quotients, and powers. Of course, given the availability of calculators, there is now li need for this particular application of logarithms. However, there is great value in us logarithmic properties to simplify *differentiation* involving products, quotients, powers.

Example 4 Logarithmic Properties as Aids to Differentiation

Differentiate $f(x) = \ln\sqrt{x+1}$.

Solution Because

$$f(x) = \ln\sqrt{x+1} = \ln(x+1)^{1/2} = \frac{1}{2}\ln(x+1)$$ Rewrite before differentiating.

you can write the following.

$$f'(x) = \frac{1}{2}\left(\frac{1}{x+1}\right) = \frac{1}{2(x+1)}$$ Differentiate.

---

iC *indicates that in the* Interactive 3.0 *CD-ROM and* Internet 3.0 *versions of this (available at* college.hmco.com) *you will find an Open Exploration, which further explores example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

Example 5 Logarithmic Properties as Aids to Differentiation

Differentiate $f(x) = \ln \dfrac{x(x^2+1)^2}{\sqrt{2x^3-1}}$.

Solution

$$\begin{aligned}
f(x) &= \ln \frac{x(x^2+1)^2}{\sqrt{2x^3-1}} && \text{Write original function.}\\
&= \ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1) && \text{Rewrite before differentiating.}\\
f'(x) &= \frac{1}{x} + 2\left(\frac{2x}{x^2+1}\right) - \frac{1}{2}\left(\frac{6x^2}{2x^3-1}\right) && \text{Differentiate.}\\
&= \frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x^2}{2x^3-1} && \text{Simplify.}
\end{aligned}$$

NOTE In Examples 4 and 5, be sure that you see the benefit of applying logarithmic properties *before* differentiating. Consider, for instance, the difficulty of direct differentiation of the function given in Example 5.

On occasion, it is convenient to use logarithms as aids in differentiating *nonlogarithmic* functions. This procedure is called **logarithmic differentiation.**

Example 6 Logarithmic Differentiation

Find the derivative of $y = \dfrac{(x-2)^2}{\sqrt{x^2+1}}$, $x \neq 2$.

Solution Note that $y > 0$ for all $x \neq 2$ and hence that $\ln y$ is defined. Begin by taking the natural logarithms of both sides of the equation. Then apply logarithmic properties and differentiate implicitly. Finally, solve for y'.

$$\begin{aligned}
y &= \frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2 && \text{Write original equation.}\\
\ln y &= \ln \frac{(x-2)^2}{\sqrt{x^2+1}} && \text{Take natural log of both sides.}\\
\ln y &= 2\ln(x-2) - \frac{1}{2}\ln(x^2+1) && \text{Logarithmic properties}\\
\frac{y'}{y} &= 2\left(\frac{1}{x-2}\right) - \frac{1}{2}\left(\frac{2x}{x^2+1}\right) && \text{Differentiate.}\\
&= \frac{2}{x-2} - \frac{x}{x^2+1} && \text{Simplify.}\\
y' &= y\left(\frac{2}{x-2} - \frac{x}{x^2+1}\right) && \text{Solve for } y'.\\
&= \frac{(x-2)^2}{\sqrt{x^2+1}}\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] && \text{Substitute for } y.\\
&= \frac{(x-2)(x^2+2x+2)}{(x^2+1)^{3/2}} && \text{Simplify.}
\end{aligned}$$

Because the natural logarithm is undefined for negative numbers, you will oft encounter expressions of the form $\ln|u|$. The following theorem states that you c differentiate functions of the form $y = \ln|u|$ as if the absolute value sign were r present.

THEOREM 5.4 Derivative Involving Absolute Value

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}.$$

Proof If $u > 0$, then $|u| = u$, and the result follows from Theorem 5.3. If $u <$ then $|u| = -u$, and you have

$$\begin{aligned}\frac{d}{dx}[\ln|u|] &= \frac{d}{dx}[\ln(-u)]\\ &= \frac{-u'}{-u}\\ &= \frac{u'}{u}.\end{aligned}$$

Example 7 Derivative Involving Absolute Value

Find the derivative of

$$f(x) = \ln|\cos x|.$$

Solution Using Theorem 5.4, let $u = \cos x$ and write

$$\begin{aligned}\frac{d}{dx}[\ln|\cos x|] &= \frac{u'}{u} && \frac{d}{dx}[\ln|u|] = \frac{u'}{u}\\ &= \frac{-\sin x}{\cos x} && u = \cos x\\ &= -\tan x. && \text{Simplify.}\end{aligned}$$

Example 8 Finding Relative Extrema

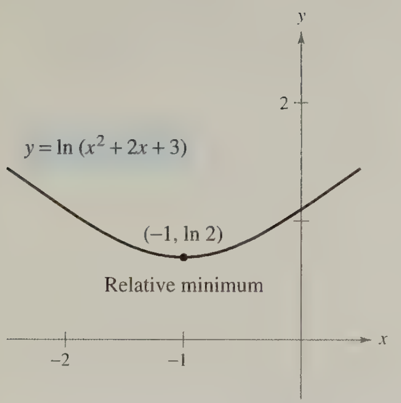
Locate the relative extrema of

$$y = \ln(x^2 + 2x + 3).$$

Solution Differentiating y, you obtain

$$\frac{dy}{dx} = \frac{2x + 2}{x^2 + 2x + 3}.$$

Because $dy/dx = 0$ when $x = -1$, you can apply the First Derivative Test a conclude that the point $(-1, \ln 2)$ is a relative minimum. Because there are no ot critical points, it follows that this is the only relative extremum (see Figure 5.7).



The derivative of y changes from negative to positive at $x = -1$.
Figure 5.7

XERCISES FOR SECTION 5.1

Complete the table below. Use a graphing utility and Simpson's Rule with $n = 10$ to approximate the integral

$$\int_1^x \frac{1}{t}\,dt.$$

x	0.5	1.5	2	2.5	3	3.5	4
$\int_1^x (1/t)\,dt$							

(a) Plot the points generated in Exercise 1 and connect them with a smooth curve. Compare the result with the graph of $y = \ln x$.

(b) Use a graphing utility to graph $y = \int_1^x (1/t)\,dt$ for $0.2 \le x \le 4$. Compare the result with the graph of $y = \ln x$.

Exercises 3–6, use a graphing utility to evaluate the logarithm (a) using the natural logarithm key, and (b) using the gration capabilities to evaluate the integral

$$\frac{1}{t}\,dt.$$

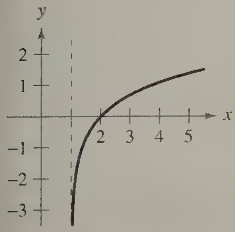
$\ln 45$

4. $\ln 8.3$

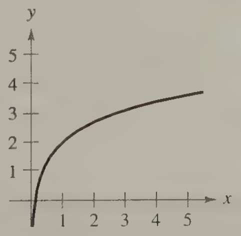
$\ln 0.8$

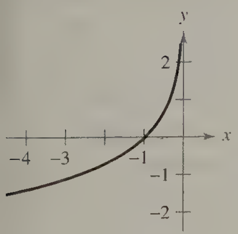
6. $\ln 0.6$

Exercises 7–10, match the function with its graph. [The phs are labeled (a), (b), (c), and (d).]

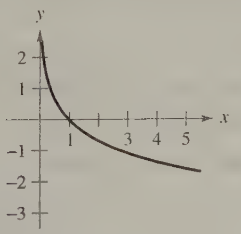


(b)





(d)



$f(x) = \ln x + 2$

8. $f(x) = -\ln x$

$f(x) = \ln(x - 1)$

10. $f(x) = -\ln(-x)$

Exercises 11–16, sketch the graph of the function and state its nain.

$f(x) = 3 \ln x$

12. $f(x) = -2 \ln x$

$f(x) = \ln 2x$

14. $f(x) = \ln|x|$

$f(x) = \ln(x - 1)$

16. $g(x) = 2 + \ln x$

In Exercises 17 and 18, use the properties of logarithms to approximate the indicated logarithms, given that $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$.

17. (a) $\ln 6$ (b) $\ln \frac{2}{3}$ (c) $\ln 81$ (d) $\ln \sqrt{3}$

18. (a) $\ln 0.25$ (b) $\ln 24$ (c) $\ln \sqrt[3]{12}$ (d) $\ln \frac{1}{72}$

In Exercises 19–28, use the properties of logarithms to expand the logarithmic expression.

19. $\ln \frac{2}{3}$

20. $\ln \sqrt{2^3}$

21. $\ln \dfrac{xy}{z}$

22. $\ln(xyz)$

23. $\ln \sqrt[3]{a^2 + 1}$

24. $\ln \sqrt{a - 1}$

25. $\ln\left(\dfrac{x^2 - 1}{x^3}\right)^3$

26. $\ln(3e^2)$

27. $\ln z(z - 1)^2$

28. $\ln \dfrac{1}{e}$

In Exercises 29–34, write the expression as a logarithm of a single quantity.

29. $\ln(x - 2) - \ln(x + 2)$

30. $3 \ln x + 2 \ln y - 4 \ln z$

31. $\frac{1}{3}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)]$

32. $2[\ln x - \ln(x + 1) - \ln(x - 1)]$

33. $2 \ln 3 - \frac{1}{2} \ln(x^2 + 1)$

34. $\frac{3}{2}[\ln(x^2 + 1) - \ln(x + 1) - \ln(x - 1)]$

In Exercises 35 and 36, show that $f = g$ by using a graphing utility to graph f and g in the same viewing window.

35. $f(x) = \ln \dfrac{x^2}{4},\ x > 0, \quad g(x) = 2 \ln x - \ln 4$

36. $f(x) = \ln \sqrt{x(x^2 + 1)}, \quad g(x) = \frac{1}{2}[\ln x + \ln(x^2 + 1)]$

In Exercises 37–40, find the limit.

37. $\lim\limits_{x \to 3^+} \ln(x - 3)$

38. $\lim\limits_{x \to 6^-} \ln(6 - x)$

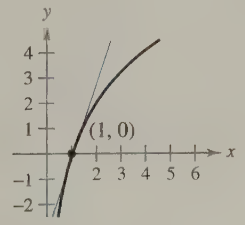
39. $\lim\limits_{x \to 2} \ln[x^2(3 - x)]$

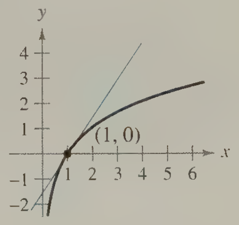
40. $\lim\limits_{x \to 5^+} \ln \dfrac{x}{\sqrt{x - 4}}$

In Exercises 41–44, find the slope of the tangent line to the graph of the logarithmic function at the point (1, 0).

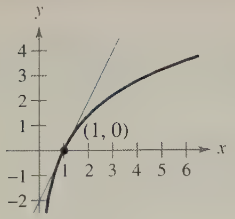
41. $y = \ln x^3$

42. $y = \ln x^{3/2}$

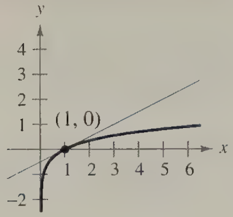




43. $y = \ln x^2$



44. $y = \ln x^{1/2}$



In Exercises 45–70, find the derivative of the function.

45. $g(x) = \ln x^2$

46. $h(x) = \ln(2x^2 + 1)$

47. $y = (\ln x)^4$

48. $y = x \ln x$

49. $y = \ln\left(x\sqrt{x^2 - 1}\right)$

50. $y = \ln\sqrt{x^2 - 4}$

51. $f(x) = \ln\left(\dfrac{x}{x^2 + 1}\right)$

52. $f(x) = \ln\left(\dfrac{2x}{x + 3}\right)$

53. $g(t) = \dfrac{\ln t}{t^2}$

54. $h(t) = \dfrac{\ln t}{t}$

55. $y = \ln(\ln x^2)$

56. $y = \ln(\ln x)$

57. $y = \ln\sqrt{\dfrac{x + 1}{x - 1}}$

58. $y = \ln\sqrt[3]{\dfrac{x - 1}{x + 1}}$

59. $f(x) = \ln\left(\dfrac{\sqrt{4 + x^2}}{x}\right)$

60. $f(x) = \ln\left(x + \sqrt{4 + x^2}\right)$

61. $y = \dfrac{-\sqrt{x^2 + 1}}{x} + \ln\left(x + \sqrt{x^2 + 1}\right)$

62. $y = \dfrac{-\sqrt{x^2 + 4}}{2x^2} - \dfrac{1}{4}\ln\left(\dfrac{2 + \sqrt{x^2 + 4}}{x}\right)$

63. $y = \ln|\sin x|$

64. $y = \ln|\csc x|$

65. $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

66. $y = \ln|\sec x + \tan x|$

67. $y = \ln\left|\dfrac{-1 + \sin x}{2 + \sin x}\right|$

68. $y = \ln\sqrt{1 + \sin^2 x}$

69. $f(x) = \sin 2x \ln x^2$

70. $g(x) = \displaystyle\int_1^{\ln x} (t^2 + 3)\, dt$

In Exercises 71 and 72, (a) find an equation of the tangent line to the graph of f at the indicated point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

	Function	*Point*
71.	$f(x) = 3x^2 - \ln x$	$(1, 3)$
72.	$f(x) = 4 - x^2 - \ln\left(\frac{1}{2}x + 1\right)$	$(0, 4)$

In Exercises 73 and 74, use implicit differentiation to find dy/dx.

73. $x^2 - 3\ln y + y^2 = 10$

74. $\ln xy + 5x = 30$

In Exercises 75 and 76, show that the function is a solution of the differential equation.

	Function	*Differential Equation*
75.	$y = 2\ln x + 3$	$xy'' + y' = 0$
76.	$y = x\ln x - 4x$	$x + y - xy' = 0$

In Exercises 77–82, locate any relative extrema and inflect points. Use a graphing utility to confirm your results.

77. $y = \dfrac{x^2}{2} - \ln x$

78. $y = x - \ln x$

79. $y = x \ln x$

80. $y = \dfrac{\ln x}{x}$

81. $y = \dfrac{x}{\ln x}$

82. $y = x^2 \ln \dfrac{x}{4}$

Linear and Quadratic Approximations **In Exercises 83 and use a graphing utility to graph the function. Then graph**

$$P_1(x) = f(1) + f'(1)(x - 1)$$

and

$$P_2(x) = f(1) + f'(1)(x - 1) + \tfrac{1}{2}f''(1)(x - 1)^2$$

in the same viewing window. Compare the values of f, P_1, P_2 and their first derivatives at $x = 1$.

83. $f(x) = \ln x$

84. $f(x) = x \ln x$

In Exercises 85 and 86, use Newton's Method to approximate three decimal places, the x-coordinate of the point of inter tion of the graphs of the two equations. Use a graphing utilit verify your result.

85. $y = \ln x$, $y = -x$

86. $y = \ln x$, $y = 3 - x$

In Exercises 87–92, find dy/dx using logarithmic differentiat

87. $y = x\sqrt{x^2 - 1}$

88. $y = \sqrt{(x - 1)(x - 2)(x -}$

89. $y = \dfrac{x^2\sqrt{3x - 2}}{(x - 1)^2}$

90. $y = \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$

91. $y = \dfrac{x(x - 1)^{3/2}}{\sqrt{x + 1}}$

92. $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

Getting at the Concept

93. In your own words, state the properties of the natur logarithmic function.

94. Define the base for the natural logarithmic function.

95. Explain why $\ln e^x = x$.

96. Let f be a function that is positive and differentiable on tl entire real line. Let $g(x) = \ln f(x)$.

(a) If g is increasing, must f be increasing? Explain.

(b) If the graph of f is concave upward, must the graph g be concave upward? Explain.

97. Consider the function $f(x) = x - 2\ln x$ on $[1, 3]$.

(a) Explain why Rolle's Theorem (Section 3.2) does n apply.

(b) Do you think the conclusion of Rolle's Theorem is tr for f? Explain.

. ***Home Mortgage*** The term t (in years) of a \$120,000 home mortgage at 10% interest can be approximated by

$$t = \frac{5.315}{-6.7968 + \ln x}, \quad x > 1000$$

where x is the monthly payment in dollars.

(a) Use a graphing utility to graph the model.

(b) Use the model to approximate the term of a home mortgage for which the monthly payment is \$1167.41. What is the total amount paid?

(c) Use the model to approximate the term of a home mortgage for which the monthly payment is \$1068.45. What is the total amount paid?

(d) Find the instantaneous rate of change of t with respect to x when $x = 1167.41$ and $x = 1068.45$.

(e) Write a short paragraph describing the benefit of the higher monthly payment.

. ***Sound Intensity*** The relationship between the number of decibels β and the intensity of a sound I in watts per centimeter squared is

$$\beta = 10 \log_{10}\left(\frac{I}{10^{-16}}\right).$$

Use the properties of logarithms to write the formula in simpler form, and determine the number of decibels of a sound with an intensity of 10^{-10} watts per square centimeter.

. ***Modeling Data*** The table shows the temperature T (°F) at which water boils at selected pressures p (pounds per square inch). *(Source: Standard Handbook of Mechanical Engineers)*

p	5	10	14.696 (1 atm)	20
T	162.24°	193.21°	212.00°	227.96°

p	30	40	60	80	100
T	250.33°	267.25°	292.71°	312.03°	327.81°

A model that approximates the data is

$$T = 87.97 + 34.96 \ln p + 7.91\sqrt{p}.$$

(a) Use a graphing utility to plot the data and graph the model.

(b) Find the rate of change of T with respect to p when $p = 10$ and $p = 70$.

(c) Use a graphing utility to graph T'. Find

$$\lim_{p\to\infty} T'(p)$$

and interpret the result in the context of the problem.

101. ***Modeling Data*** The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is one atmosphere (1.033227 kilograms per square centimeter). The table shows the pressure p (in atmospheres) at a given altitude h (in kilometers).

h	0	5	10	15	20	25
p	1	0.55	0.25	0.12	0.06	0.02

(a) Use a graphing utility to find a model of the form $p = a + b \ln h$ for the data. Explain why the result is an error message.

(b) Use a graphing utility to find the logarithmic model $h = a + b \ln p$ for the data.

(c) Use a graphing utility to plot the data and graph the logarithmic model.

(d) Use the model to estimate the altitude at which the pressure is 0.75 atmosphere.

(e) Use the model to estimate the pressure at an altitude of 13 kilometers.

(f) Find the rate of change of pressure when $h = 5$ and $h = 20$. Interpret the results in the context of the problem.

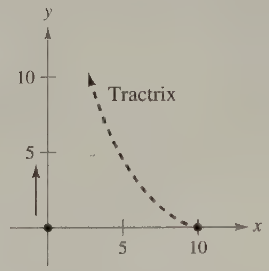
102. ***Tractrix*** A person walking along a dock drags a boat by a 10-meter rope. The boat travels along a path known as a *tractrix* (see figure). The equation of this path is

$$y = 10 \ln\left(\frac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2}.$$

(a) Use a graphing utility to graph the function.

(b) What is the slope of this path when $x = 5$ and $x = 9$?

(c) What does the slope of the path approach as $x \to 10$?



103. ***Conjecture*** Use a graphing utility to graph f and g in the same viewing window and determine which is increasing at the faster rate for "large" values of x. What can you conclude about the rate of growth of the natural logarithmic function?

(a) $f(x) = \ln x$, $\quad g(x) = \sqrt{x}$

(b) $f(x) = \ln x$, $\quad g(x) = \sqrt[4]{x}$

104. Prove that the natural logarithmic function is one-to-one.

True or False? **In Exercises 105 and 106, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

105. $\ln(x + 25) = \ln x + \ln 25$

106. If $y = \ln \pi$, then $y' = 1/\pi$.

Section 5.2 The Natural Logarithmic Function: Integration

- Use the Log Rule for Integration to integrate a rational function.
- Integrate trigonometric functions.

EXPLORATION

Integrating Rational Functions Early in Chapter 4, you learned rules that allowed you to integrate *any* polynomial function. The Log Rule presented in this section goes a long way toward enabling you to integrate rational functions. For instance, each of the following functions can be integrated with the Log Rule.

$\dfrac{2}{x}$	Example 1
$\dfrac{1}{4x-1}$	Example 2
$\dfrac{x}{x^2+1}$	Example 3
$\dfrac{3x^2+1}{x^3+x}$	Example 4a
$\dfrac{x+1}{x^2+2x}$	Example 4c
$\dfrac{1}{3x+2}$	Example 4d
$\dfrac{x^2+x+1}{x^2+1}$	Example 5
$\dfrac{2x}{(x+1)^2}$	Example 6

There are still some rational functions that cannot be integrated using the Log Rule. Give examples of these functions, and explain your reasoning.

Log Rule for Integration

The differentiation rules

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \qquad \text{and} \qquad \frac{d}{dx}[\ln|u|] = \frac{u'}{u}$$

that you studied in the preceding section produce the following integration rule.

THEOREM 5.5 Log Rule for Integration

Let u be a differentiable function of x.

1. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$ **2.** $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$

Because $du = u'\,dx$, the second formula can also be written as

$$\int \frac{u'}{u}\,dx = \ln|u| + C.$$

Alternative form of Log Rule

Example 1 Using the Log Rule for Integration

$$\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx$$

$$= 2\ln|x| + C \qquad \text{Log Rule for Integration}$$

$$= \ln(x^2) + C \qquad \text{Property of logarithms}$$

Because x^2 cannot be negative, the absolute value is unnecessary in the final form the antiderivative.

Example 2 Using the Log Rule with a Change of Variables

Find $\displaystyle\int \frac{1}{4x-1}\,dx$.

Solution If you let $u = 4x - 1$, then $du = 4\,dx$.

$$\int \frac{1}{4x-1}\,dx = \frac{1}{4}\int\left(\frac{1}{4x-1}\right)4\,dx \qquad \text{Multiply and divide by 4.}$$

$$= \frac{1}{4}\int \frac{1}{u}\,du \qquad \text{Substitute: } u = 4x-1.$$

$$= \frac{1}{4}\ln|u| + C \qquad \text{Apply Log Rule.}$$

$$= \frac{1}{4}\ln|4x-1| + C \qquad \text{Back-substitute.}$$

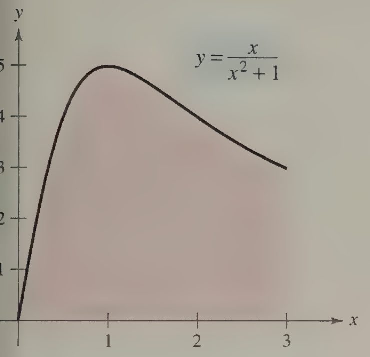
Example 3 uses the alternative form of the Log Rule. To apply this rule, look for quotients in which the numerator is the derivative of the denominator.

Example 3 Finding Area with the Log Rule

Find the area of the region bounded by the graph of

$$y = \frac{x}{x^2+1}$$

the x-axis, and the line $x = 3$.



$$\text{a} = \int_0^3 \frac{x}{x^2+1}\,dx$$

e area of the region bounded by the graph y, the x-axis, and $x = 3$ is $\frac{1}{2}\ln 10$.

ure 5.8

Solution From Figure 5.8, you can see that the area of the region is given by the definite integral

$$\int_0^3 \frac{x}{x^2+1}\,dx.$$

If you let $u = x^2 + 1$, then $u' = 2x$. To apply the Log Rule, multiply and divide by 2 as follows.

$$\begin{aligned}
\int_0^3 \frac{x}{x^2+1}\,dx &= \frac{1}{2}\int_0^3 \frac{2x}{x^2+1}\,dx && \text{Multiply and divide by 2.}\\
&= \frac{1}{2}\Big[\ln(x^2+1)\Big]_0^3 && \int \frac{u'}{u}\,dx = \ln|u| + C\\
&= \frac{1}{2}(\ln 10 - \ln 1)\\
&= \frac{1}{2}\ln 10 && \ln 1 = 0\\
&\approx 1.151
\end{aligned}$$

Example 4 Recognizing Quotient Forms of the Log Rule

a. $\displaystyle\int \frac{3x^2+1}{x^3+x}\,dx = \ln|x^3+x| + C$ $\qquad u = x^3 + x$

b. $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx = \ln|\tan x| + C$ $\qquad u = \tan x$

c. $\displaystyle\int \frac{x+1}{x^2+2x}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x}\,dx$ $\qquad u = x^2 + 2x$

$$= \frac{1}{2}\ln|x^2+2x| + C$$

d. $\displaystyle\int \frac{1}{3x+2}\,dx = \frac{1}{3}\int \frac{3}{3x+2}\,dx$ $\qquad u = 3x + 2$

$$= \frac{1}{3}\ln|3x+2| + C$$

With antiderivatives involving logarithms, it is easy to obtain forms that look quite different but are still equivalent. For instance, which of the following are equivalent to the antiderivative listed in Example 4d?

$$\ln|(3x+2)^{1/3}| + C, \quad \frac{1}{3}\ln\left|x + \tfrac{2}{3}\right| + C, \quad \ln|3x+2|^{1/3} + C$$

Integrals to which the Log Rule can be applied often appear in disguised fo For instance, if a rational function has a *numerator of degree greater than or equa that of the denominator,* division may reveal a form to which you can apply the L Rule. This is illustrated in Example 5.

Example 5 Using Long Division Before Integrating

Find $\int \frac{x^2+x+1}{x^2+1}\,dx.$

Solution Begin by using long division to rewrite the integrand.

$$\frac{x^2+x+1}{x^2+1} \quad \Rightarrow \quad \begin{array}{r} 1 \\ x^2+1\,\overline{)\,x^2+x+1} \\ \underline{x^2 \qquad\; +1} \\ x \end{array} \quad \Rightarrow \quad 1+\frac{x}{x^2+1}$$

Now, you can integrate to obtain

$$\int \frac{x^2+x+1}{x^2+1}\,dx = \int \left(1+\frac{x}{x^2+1}\right)dx \qquad \text{Rewrite using long division.}$$

$$= \int dx + \frac{1}{2}\int \frac{2x}{x^2+1}\,dx \qquad \text{Rewrite as two integrals.}$$

$$= x + \frac{1}{2}\ln(x^2+1) + C. \qquad \text{Integrate.}$$

Check this result by differentiating to obtain the original integrand.

The next example gives another instance in which the use of the Log Rule disguised. In this case, a change of variables helps you recognize the Log Rule.

Example 6 Change of Variables with the Log Rule

Find $\int \frac{2x}{(x+1)^2}\,dx.$

Solution If you let $u = x + 1$, then $du = dx$ and $x = u - 1$.

$$\int \frac{2x}{(x+1)^2}\,dx = \int \frac{2(u-1)}{u^2}\,du \qquad \text{Substitute.}$$

$$= 2\int \left(\frac{u}{u^2} - \frac{1}{u^2}\right)du \qquad \text{Rewrite as two fractions.}$$

$$= 2\int \frac{du}{u} - 2\int u^{-2}\,du \qquad \text{Rewrite as two integrals.}$$

$$= 2\ln|u| - 2\left(\frac{u^{-1}}{-1}\right) + C \qquad \text{Integrate.}$$

$$= 2\ln|u| + \frac{2}{u} + C \qquad \text{Simplify.}$$

$$= 2\ln|x+1| + \frac{2}{x+1} + C \qquad \text{Back-substitute.}$$

Check this result by differentiating to obtain the original integrand.

TECHNOLOGY If you have access to a computer algebra system, try using it to find the indefinite integrals in Examples 5 and 6. How does the form of the antiderivative that it gives you compare with that given in Examples 5 and 6?

As you study the methods shown in Examples 5 and 6, be aware that both methods involve rewriting a disguised integrand so that it fits one or more of the basic integration formulas. Throughout the remaining sections of Chapter 5 and in Chapter 7 we will devote much time to integration techniques. To master these techniques, you must recognize the "form-fitting" nature of integration. In this sense, integration is not nearly as straightforward as differentiation. Differentiation takes the form

> *"Here is the question; what is the answer?"*

Integration is more like

> *"Here is the answer; what is the question?"*

We suggest the following guidelines for integration.

UDY TIP Keep in mind that you check your answer to an integration blem by differentiating the answer. instance, in Example 7, the derivative $= \ln|\ln x| + C$ is $y' = 1/(x \ln x)$.

Guidelines for Integration

1. Learn a basic list of integration formulas. (Including those given in this section, you now have 12 formulas: the Power Rule, the Log Rule, and ten trigonometric rules. By the end of Section 5.9, this list will have expanded to 20 basic rules.)
2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of u that will make the integrand conform to the formula.
3. If you cannot find a u-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, or addition and subtraction of the same quantity. Be creative.
4. If you have access to computer software that will find antiderivatives symbolically, use it.

Example 7 u-Substitution and the Log Rule

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x \ln x}$.

Solution The solution can be written as an indefinite integral.

$$y = \int \frac{1}{x \ln x}\,dx$$

Because the integrand is a quotient whose denominator is raised to the first power, you should try the Log Rule. There are three basic choices for u. The choices $u = x$ and $u = x \ln x$ fail to fit the u'/u form of the Log Rule. However, the third choice does fit. Letting $u = \ln x$ produces $u' = 1/x$, and you obtain the following.

$$\int \frac{1}{x \ln x}\,dx = \int \frac{1/x}{\ln x}\,dx \qquad \text{Divide numerator and denominator by } x.$$

$$= \int \frac{u'}{u}\,dx \qquad \text{Substitute: } u = \ln x.$$

$$= \ln|u| + C \qquad \text{Apply Log Rule.}$$

$$= \ln|\ln x| + C \qquad \text{Back-substitute.}$$

So, the solution is $y = \ln|\ln x| + C$.

Integrals of Trigonometric Functions

In Section 4.1, you looked at six trigonometric integration rules—the six that co spond directly to differentiation rules. With the Log Rule, you can now complete set of basic trigonometric integration formulas.

Example 8 Using a Trigonometric Identity

Find $\int \tan x \, dx$.

Solution This integral does not seem to fit any formulas on our basic list. Howe by using a trigonometric identity, you obtain the following.

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

Knowing that $D_x[\cos x] = -\sin x$, you can let $u = \cos x$ and write

$$\begin{aligned} \int \tan x \, dx &= -\int \frac{-\sin x}{\cos x} \, dx && \text{Trigonometric identity} \\ &= -\int \frac{u'}{u} \, dx && \text{Substitute: } u = \cos x. \\ &= -\ln|u| + C && \text{Apply Log Rule.} \\ &= -\ln|\cos x| + C. && \text{Back-substitute.} \end{aligned}$$

Example 8 uses a trigonometric identity to derive an integration rule for tangent function. In the next example, we take a rather unusual step (multiplying dividing by the same quantity) to derive an integration rule for the secant function

Example 9 Derivation of the Secant Formula

Find $\int \sec x \, dx$.

Solution Consider the following procedure.

$$\begin{aligned} \int \sec x \, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \end{aligned}$$

Letting u be the denominator of this quotient produces

$$u = \sec x + \tan x \quad \Longrightarrow \quad u' = \sec x \tan x + \sec^2 x.$$

Therefore, you can conclude that

$$\begin{aligned} \int \sec x \, dx &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx && \text{Rewrite integrand.} \\ &= \int \frac{u'}{u} \, dx && \text{Substitute: } u = \sec x + \tan x. \\ &= \ln|u| + C && \text{Apply Log Rule.} \\ &= \ln|\sec x + \tan x| + C. && \text{Back-substitute.} \end{aligned}$$

With the results of Examples 8 and 9, you now have integration formulas for sin x, cos x, tan x, and sec x. All six trigonometric rules are summarized below.

Integrals of the Six Basic Trigonometric Functions

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

OTE Using trigonometric identities d properties of logarithms, you could vrite these six integration rules in ner forms. For instance, you could ite

$\int \csc u\,du = \ln|\csc u - \cot u| + C.$

ee Exercises 51–54.)

Example 10 Integrating Trigonometric Functions

Evaluate $\displaystyle\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\,dx.$

Solution Using $1 + \tan^2 x = \sec^2 x$, you can write

$$\begin{aligned}\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\,dx &= \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx \\ &= \int_0^{\pi/4} \sec x\,dx & \sec x \geq 0 \text{ for } 0 \leq x \leq \frac{\pi}{4}. \\ &= \ln|\sec x + \tan x|\Big]_0^{\pi/4} \\ &= \ln\left(\sqrt{2} + 1\right) - \ln 1 \\ &\approx 0.8814.\end{aligned}$$

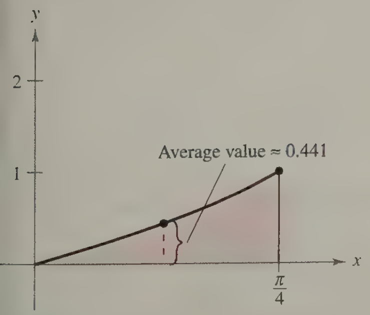
Example 11 Finding an Average Value

Find the average value of $f(x) = \tan x$ on the interval $[0, \pi/4]$.

Solution

$$\begin{aligned}\text{Average value} &= \frac{1}{(\pi/4) - 0}\int_0^{\pi/4} \tan x\,dx & \text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{4}{\pi}\int_0^{\pi/4} \tan x\,dx & \text{Simplify.} \\ &= \frac{4}{\pi}\Big[-\ln|\cos x|\Big]_0^{\pi/4} & \text{Integrate.} \\ &= -\frac{4}{\pi}\left[\ln\left(\frac{\sqrt{2}}{2}\right) - \ln(1)\right] \\ &= -\frac{4}{\pi}\ln\left(\frac{\sqrt{2}}{2}\right) \\ &\approx 0.441\end{aligned}$$

The average value is about 0.441, as indicated in Figure 5.9.



gure 5.9

EXERCISES FOR SECTION 5.2

In Exercises 1–24, find the indefinite integral.

1. $\int \frac{5}{x}\,dx$ **2.** $\int \frac{10}{x}\,dx$ **3.** $\int \frac{1}{x+1}\,dx$

4. $\int \frac{1}{x-5}\,dx$ **5.** $\int \frac{1}{3-2x}\,dx$ **6.** $\int \frac{1}{3x+2}\,dx$

7. $\int \frac{x}{x^2+1}\,dx$ **8.** $\int \frac{x^2}{3-x^3}\,dx$

9. $\int \frac{x^2-4}{x}\,dx$ **10.** $\int \frac{x}{\sqrt{9-x^2}}\,dx$

11. $\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx$ **12.** $\int \frac{x(x+2)}{x^3+3x^2-4}\,dx$

13. $\int \frac{x^2-3x+2}{x+1}\,dx$ **14.** $\int \frac{2x^2+7x-3}{x-2}\,dx$

15. $\int \frac{x^3-3x^2+5}{x-3}\,dx$ **16.** $\int \frac{x^3-6x-20}{x+5}\,dx$

17. $\int \frac{x^4+x-4}{x^2+2}\,dx$ **18.** $\int \frac{x^3-3x^2+4x-9}{x^2+3}\,dx$

19. $\int \frac{(\ln x)^2}{x}\,dx$ **20.** $\int \frac{1}{x\ln(x^3)}\,dx$

21. $\int \frac{1}{\sqrt{x}+1}\,dx$ **22.** $\int \frac{1}{x^{2/3}(1+x^{1/3})}\,dx$

23. $\int \frac{2x}{(x-1)^2}\,dx$ **24.** $\int \frac{x(x-2)}{(x-1)^3}\,dx$

In Exercises 25–28, find the indefinite integral by u-substitution. (*Hint:* Let u be the denominator of the integrand.)

25. $\int \frac{1}{1+\sqrt{2x}}\,dx$ **26.** $\int \frac{1}{1+\sqrt{3x}}\,dx$

27. $\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx$ **28.** $\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx$

In Exercises 29–36, find the indefinite integral.

29. $\int \frac{\cos\theta}{\sin\theta}\,d\theta$ **30.** $\int \tan 5\theta\,d\theta$

31. $\int \csc 2x\,dx$ **32.** $\int \sec\frac{x}{2}\,dx$

33. $\int \frac{\cos t}{1+\sin t}\,dt$ **34.** $\int \frac{\csc^2 t}{\cot t}\,dt$

35. $\int \frac{\sec x\tan x}{\sec x-1}\,dx$ **36.** $\int (\sec t+\tan t)\,dt$

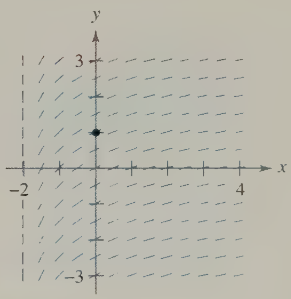
In Exercises 37–40, solve the differential equation. Use a graphing utility to graph three solutions, one of which passes through the indicated point.

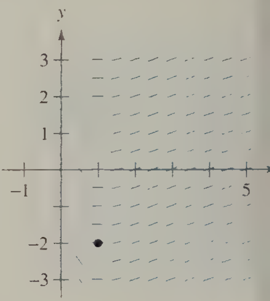
37. $\frac{dy}{dx} = \frac{3}{2-x}$, $(1, 0)$ **38.** $\frac{dy}{dx} = \frac{2x}{x^2-9}$, $(0, 4)$

39. $\frac{ds}{d\theta} = \tan 2\theta$, $(0, 2)$ **40.** $\frac{dr}{dt} = \frac{\sec^2 t}{\tan t+1}$, $(\pi, 4)$

Slope Fields **In Exercises 41 and 42, a differential equatio point, and a slope field are given. (a) Sketch two approxim solutions of the differential equation on the slope field, on which passes through the indicated point. (b) Use integratio find the particular solution of the differential equation and u graphing utility to graph the solution. Compare the result w the sketches in part (a). To print an enlarged copy of the gra go to the website *www.mathgraphs.com*.**

41. $\frac{dy}{dx} = \frac{1}{x+2}$, $(0, 1)$ **42.** $\frac{dy}{dx} = \frac{\ln x}{x}$, $(1, -2)$





In Exercises 43–50, evaluate the definite integral. Use a gra ing utility to verify your result.

43. $\int_0^4 \frac{5}{3x+1}\,dx$ **44.** $\int_{-1}^{1} \frac{1}{x+2}\,dx$

45. $\int_1^e \frac{(1+\ln x)^2}{x}\,dx$ **46.** $\int_e^{e^2} \frac{1}{x\ln x}\,dx$

47. $\int_0^2 \frac{x^2-2}{x+1}\,dx$ **48.** $\int_0^1 \frac{x-1}{x+1}\,dx$

49. $\int_1^2 \frac{1-\cos\theta}{\theta-\sin\theta}\,d\theta$ **50.** $\int_{0.1}^{0.2} (\csc 2\theta-\cot 2\theta)^2\,d\theta$

In Exercises 51–54, show that the two formulas are equivale

51. $\int \tan x\,dx = -\ln|\cos x| + C$

$\int \tan x\,dx = \ln|\sec x| + C$

52. $\int \cot x\,dx = \ln|\sin x| + C$

$\int \cot x\,dx = -\ln|\csc x| + C$

53. $\int \sec x\,dx = \ln|\sec x+\tan x| + C$

$\int \sec x\,dx = -\ln|\sec x-\tan x| + C$

54. $\int \csc x\,dx = -\ln|\csc x+\cot x| + C$

$\int \csc x\,dx = \ln|\csc x-\cot x| + C$

Exercises 55–60, use a computer algebra system to find or aluate the integral.

. $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$

56. $\displaystyle\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

. $\displaystyle\int \cos(1-x)\,dx$

58. $\displaystyle\int \frac{\tan^2 2x}{\sec 2x}\,dx$

. $\displaystyle\int_{\pi/4}^{\pi/2} (\csc x - \sin x)\,dx$

60. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx$

Exercises 61–64, find $F'(x)$.

. $F(x) = \displaystyle\int_1^x \frac{1}{t}\,dt$

62. $F(x) = \displaystyle\int_0^x \tan t\,dt$

. $F(x) = \displaystyle\int_x^{3x} \frac{1}{t}\,dt$

64. $F(x) = \displaystyle\int_1^{x^2} \frac{1}{t}\,dt$

proximation **In Exercises 65 and 66, determine which value st approximates the area of the region between the x-axis and e graph of the function over the given interval. (Make your lection on the basis of a sketch of the region and not by rforming any calculations.)**

. $f(x) = \sec x, \quad [0, 1]$

(a) 6 (b) -6 (c) $\frac{1}{2}$ (d) 1.25 (e) 3

. $f(x) = \dfrac{2x}{x^2+1}, \quad [0, 4]$

(a) 3 (b) 7 (c) -2 (d) 5 (e) 1

ea **In Exercises 67–70, find the area of the region bounded the graphs of the equations. Use a graphing utility to graph e region and verify your result.**

. $y = \dfrac{x^2+4}{x}, x = 1, x = 4, y = 0$

. $y = \dfrac{x+4}{x}, x = 1, x = 4, y = 0$

. $y = 2\sec\dfrac{\pi x}{6}, x = 0, x = 2, y = 0$

. $y = 2x - \tan(0.3x), x = 1, x = 4, y = 0$

Getting at the Concept

n Exercises 71–74, state the integration formula you would use to perform the integration. Do not integrate.

1. $\displaystyle\int \sqrt[3]{x}\,dx$

72. $\displaystyle\int \frac{x}{(x^2+4)^3}\,dx$

3. $\displaystyle\int \frac{x}{x^2+4}\,dx$

74. $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx$

5. What is the first step when integrating

$$\int \frac{x^2}{x+1}\,dx?$$

6. Make a list of the integration formulas studied so far in the text.

In Exercises 77–80, find the average value of the function over the interval.

Function	*Interval*	*Function*	*Interval*
77. $f(x) = \dfrac{8}{x^2}$	$[2, 4]$	**78.** $f(x) = \dfrac{4(x+1)}{x^2}$	$[2, 4]$
79. $f(x) = \dfrac{\ln x}{x}$	$[1, e]$	**80.** $f(x) = \sec\dfrac{\pi x}{6}$	$[0, 2]$

81. ***Population Growth*** A population of bacteria is changing at a rate of

$$\frac{dP}{dt} = \frac{3000}{1+0.25t}$$

where t is the time in days. The initial population (when $t = 0$) is 1000. Write an equation that gives the population at any time t, and find the population when $t = 3$ days.

82. ***Heat Transfer*** Find the time required for an object to cool from 300°F to 250°F by evaluating

$$t = \frac{10}{\ln 2}\int_{250}^{300} \frac{1}{T-100}\,dT$$

where t is time in minutes.

83. ***Average Price*** The demand equation for a product is

$$p = \frac{90{,}000}{400+3x}.$$

Find the *average* price p on the interval $40 \le x \le 50$.

84. ***Sales*** The rate of change in sales S is inversely proportional to time t $(t > 1)$ measured in weeks. Find S as a function of t if sales after 2 and 4 weeks are 200 units and 300 units.

85. ***Orthogonal Trajectory***

(a) Use a graphing utility to graph the equation $2x^2 - y^2 = 8$.

(b) Evaluate the integral to find y^2 in terms of x.

$y^2 = e^{-\int(1/x)\,dx}$

For a particular value of the constant of integration, graph the result on the same screen used in part (a).

(c) Verify that the tangents to the graphs of parts (a) and (b) are perpendicular at the points of intersection.

86. Graph the function

$$f_k(x) = \frac{x^k - 1}{k}$$

for $k = 1, 0.5$, and 0.1 on $[0, 10]$. Find $\displaystyle\lim_{k\to 0^+} f_k(x)$.

True or False? **In Exercises 87–90, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

87. $(\ln x)^{1/2} = \frac{1}{2}(\ln x)$

88. $\int \ln x\,dx = (1/x) + C$

89. $\displaystyle\int \frac{1}{x}\,dx = \ln|cx|, \quad c \neq 0$

90. $\displaystyle\int_{-1}^{2} \frac{1}{x}\,dx = \Big[\ln|x|\Big]_{-1}^{2} = \ln 2 - \ln 1 = \ln 2$

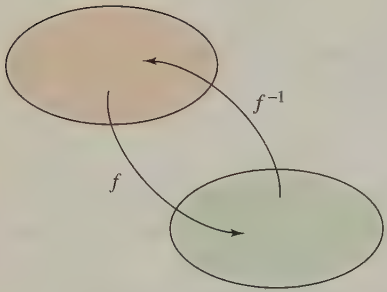
Section 5.3 Inverse Functions

- Verify that one function is the inverse function of another function.
- Determine whether a function has an inverse function.
- Find the derivative of an inverse function.

Inverse Functions

Recall from Section P.3 that a function can be represented by a set of ordered pa For instance, the function $f(x) = x + 3$ and $A = \{1, 2, 3, 4\}$ to $B = \{4, 5, 6, 7\}$ c be written as

$$f: \{(1, 4), (2, 5), (3, 6), (4, 7)\}.$$



Domain of f = range of f^{-1}
Domain of f^{-1} = range of f
Figure 5.10

By interchanging the first and second coordinates of each ordered pair, you can fo the **inverse function** of f. This function is denoted by f^{-1}. It is a function from B A, and can be written as

$$f^{-1}: \{(4, 1), (5, 2), (6, 3), (7, 4)\}.$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown Figure 5.10. The functions f and f^{-1} have the effect of "undoing" each other. That when you form the composition of f with f^{-1} or the composition of f^{-1} with f, y obtain the identity function.

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$

EXPLORATION

Finding Inverse Functions Explain how to "undo" each of the following functions. Then use your explanation to write the inverse function of f.

a. $f(x) = x - 5$
b. $f(x) = 6x$
c. $f(x) = \dfrac{x}{2}$
d. $f(x) = 3x + 2$
e. $f(x) = x^3$
f. $f(x) = 4(x - 2)$

Use a graphing utility to graph each function and its inverse function in the same "square" viewing window. What observation can you make about each pair of graphs?

Definition of Inverse Function

A function g is the **inverse function** of the function f if

$f(g(x)) = x$ for each x in the domain of g

and

$g(f(x)) = x$ for each x in the domain of f.

The function g is denoted by f^{-1} (read "f inverse").

NOTE Although the notation used to denote an inverse function resembles *exponen notation*, it is a different use of -1 as a superscript. That is, in general, $f^{-1}(x) \neq 1/f(x)$.

Here are some important observations about inverse functions.

1. If g is the inverse function of f, then f is the inverse function of g.
2. The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to domain of f.
3. A function need not have an inverse function, but if it does, the inverse functio unique (see Exercise 99).

You can think of f^{-1} as undoing what has been done by f. For example, subtr tion can be used to undo addition, and division can be used to undo multiplicati Use the definition of an inverse function to check the following.

$f(x) = x + c$ and $f^{-1}(x) = x - c$ are inverse functions of each oth

$f(x) = cx$ and $f^{-1}(x) = \dfrac{x}{c}$, $c \neq 0$, are inverse functions of each oth

Example 1 Verifying Inverse Functions

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \qquad \text{and} \qquad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

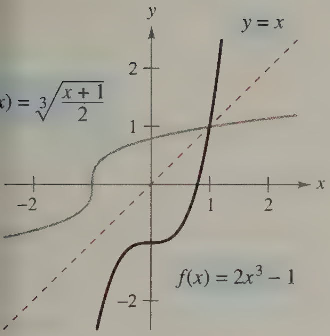
Solution Because the domains and ranges of both f and g consist of all real numbers, you can conclude that both composite functions exist for all x. The composite of f with g is given by

$$\begin{aligned} f(g(x)) &= 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 \\ &= 2\left(\frac{x+1}{2}\right) - 1 \\ &= x + 1 - 1 \\ &= x. \end{aligned}$$

The composite of g with f is given by

$$\begin{aligned} g(f(x)) &= \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}} \\ &= \sqrt[3]{\frac{2x^3}{2}} \\ &= \sqrt[3]{x^3} \\ &= x. \end{aligned}$$

Because $f(g(x)) = x$ and $g(f(x)) = x$, you can conclude that f and g are inverse functions of each other (see Figure 5.11).



nd *g* are inverse functions of each other.
ure 5.11

STUDY TIP In Example 1, try comparing the functions f and g verbally.

For f: First cube x, then multiply by 2, then subtract 1.

For g: First add 1, then divide by 2, then take the cube root.

Do you see the "undoing pattern"?

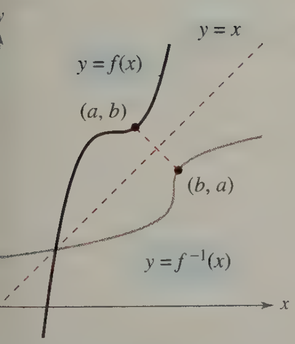
In Figure 5.11, the graphs of f and $g = f^{-1}$ appear to be mirror images of each other with respect to the line $y = x$. The graph of f^{-1} is a **reflection** of the graph of f in the line $y = x$. This idea is generalized in the following theorem.

> **THEOREM 5.6 Reflective Property of Inverse Functions**
>
> The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

Proof If (a, b) is on the graph of f, then $f(a) = b$ and you can write

$$f^{-1}(b) = f^{-1}(f(a)) = a.$$

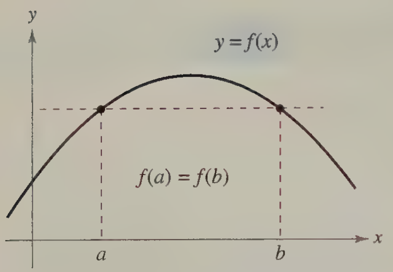
So, (b, a) is on the graph of f^{-1}, as shown in Figure 5.12. A similar argument will prove the theorem in the other direction.



graph of f^{-1} is a reflection of the graph
in the line $y = x$.
ure 5.12

Existence of an Inverse Function

Not every function has an inverse, and Theorem 5.6 suggests a graphical test for th that do—the **horizontal line test** for an inverse function. This test states that a funct f has an inverse function if and only if every horizontal line intersects the graph of most once (see Figure 5.13). The following theorem formally states why the horiz tal line test is valid. (Recall from Section 3.3 that a function is *strictly monotonic* i is either increasing on its entire domain or decreasing on its entire domain.)



If a horizontal line intersects the graph of f twice, then f is not one-to-one.
Figure 5.13

THEOREM 5.7 The Existence of an Inverse Function

1. A function has an inverse function if and only if it is one-to-one.
2. If f is strictly monotonic on its entire domain, then it is one-to-one anc therefore has an inverse function.

Proof To prove the second part of the theorem, recall from Section P.3 that f is o to-one if for x_1 and x_2 in its domain

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2.$$

The *contrapositive* of this implication is logically equivalent and states that

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

Now, choose x_1 and x_2 in the domain of f. If $x_1 \neq x_2$, then, because f is strictly mo tonic, it follows that either

$$f(x_1) < f(x_2) \qquad \text{or} \qquad f(x_1) > f(x_2).$$

In either case, $f(x_1) \neq f(x_2)$. So, f is one-to-one on the interval. The proof of the f part of the theorem is left as an exercise (see Exercise 100).

Example 2 The Existence of an Inverse Function

Which of the functions has an inverse function?

a. $f(x) = x^3 + x - 1$ **b.** $f(x) = x^3 - x + 1$

Solution

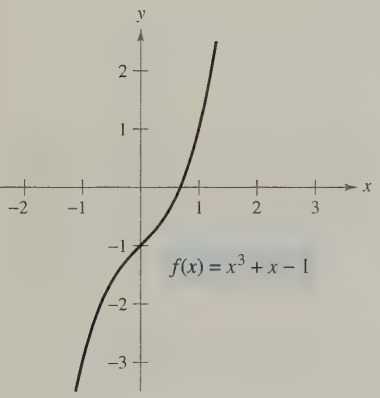
a. From the graph of f given in Figure 5.14a, it appears that f is increasing over entire domain. To verify this, note that the derivative, $f'(x) = 3x^2 + 1$, is posi for all real values of x. Therefore, f is strictly monotonic and it must have inverse function.

b. From the graph in Figure 5.14b, you can see that the function does not pass the h izontal line test. In other words, it is not one-to-one. For instance, f has the sa value when $x = -1$, 0, and 1.

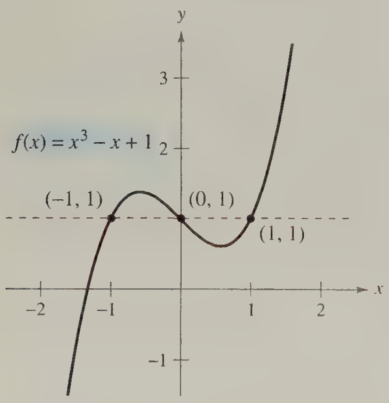
$$f(-1) = f(1) = f(0) = 1 \qquad \text{Not one-to-one}$$

Therefore, by Theorem 5.7, f does not have an inverse function.

NOTE Often it is easier to prove that a function *has* an inverse function than to find inverse function. For instance, it would be difficult algebraically to find the inverse functio the function in Example 2a.



(a) Because f is increasing over its entire domain, it has an inverse function.

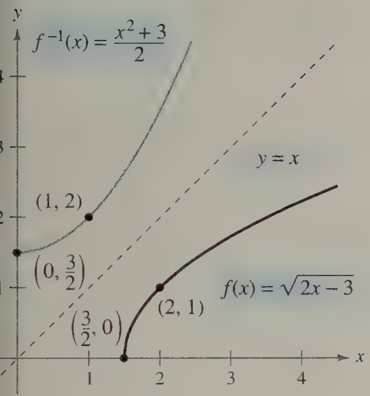


(b) Because f is not one-to-one, it does not have an inverse function.
Figure 5.14

The following guidelines suggest a procedure for finding an inverse function.

Guidelines for Finding an Inverse Function

1. Use Theorem 5.7 to determine whether the function given by $y = f(x)$ has an inverse function.
2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} to be the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.



ne domain of f^{-1}, $[0, \infty)$, is the range
f.
gure 5.15

Example 3 Finding an Inverse Function

Find the inverse function of $f(x) = \sqrt{2x - 3}$.

Solution The function has an inverse function because it is increasing on its entire domain (see Figure 5.15). To find an equation for the inverse function, let $y = f(x)$ and solve for x in terms of y.

$$\begin{aligned} \sqrt{2x-3} &= y && \text{Let } y = f(x). \\ 2x - 3 &= y^2 && \text{Square both sides.} \\ x &= \frac{y^2+3}{2} && \text{Solve for } x. \\ y &= \frac{x^2+3}{2} && \text{Interchange } x \text{ and } y. \\ f^{-1}(x) &= \frac{x^2+3}{2} && \text{Replace } y \text{ by } f^{-1}(x). \end{aligned}$$

The domain of f^{-1} is the range of f, which is $[0, \infty)$. You can verify this result as follows.

$$f(f^{-1}(x)) = \sqrt{2\left(\frac{x^2+3}{2}\right) - 3} = \sqrt{x^2} = x, \qquad x \geq 0$$

$$f^{-1}(f(x)) = \frac{\left(\sqrt{2x-3}\right)^2 + 3}{2} = \frac{2x - 3 + 3}{2} = x, \qquad x \geq \frac{3}{2}$$

NOTE Remember that any letter can be used to represent the independent variable. So,

$$f^{-1}(y) = \frac{y^2+3}{2}$$

$$f^{-1}(x) = \frac{x^2+3}{2}$$

$$f^{-1}(s) = \frac{s^2+3}{2}$$

all represent the same function.

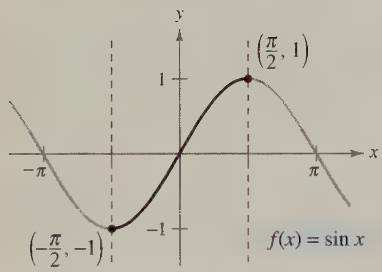
Theorem 5.7 is useful in the following type of problem. Suppose you are giver function that is *not* one-to-one on its domain. By restricting the domain to an interv on which the function is strictly monotonic, you can conclude that the new functi *is* one-to-one on the restricted domain.

Example 4 Testing Whether a Function Is One-to-One

Show that the sine function

$$f(x) = \sin x$$

is not one-to-one on the entire real line. Then show that $[-\pi/2, \pi/2]$ is the large interval, centered at the origin, for which f is strictly monotonic.



f is one-to-one on the interval $[-\pi/2, \pi/2]$.
Figure 5.16

Solution It is clear that f is not one-to-one, because many different x-values yie the same y-value. For instance,

$$\sin(0) = 0 = \sin(\pi).$$

Moreover, f is increasing on the open interval $(-\pi/2, \pi/2)$, because its derivative

$$f'(x) = \cos x$$

is positive there. Finally, because the left and right endpoints correspond to relati extrema of the sine function, you can conclude that f is increasing on the clos interval $[-\pi/2, \pi/2]$ *and* that in any larger interval the function is not strict monotonic (see Figure 5.16).

Derivative of an Inverse Function

The next two theorems discuss the derivative of an inverse function. The reasonableness of Theorem 5.8 follows from the reflective property of inverse functions shown in Figure 5.12. Proofs of the two theorems are given in Appendix B.

THEOREM 5.8 Continuity and Differentiability of Inverse Functions

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable at c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

THEOREM 5.9 The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \qquad f'(g(x)) \neq 0.$$

EXPLORATION

Graph the inverse functions

$$f(x) = x^3$$

and

$$g(x) = x^{1/3}.$$

Calculate the slope of f at (1, 1), (2, 8), and (3, 27), and the slope of g at (1, 1), (8, 2), and (27, 3). What do you observe? What happens at (0, 0)?

Example 5 Evaluating the Derivative of an Inverse Function

Let $f(x) = \frac{1}{4}x^3 + x - 1$.

a. What is the value of $f^{-1}(x)$ when $x = 3$?

b. What is the value of $(f^{-1})'(x)$ when $x = 3$?

Solution Notice that f is one-to-one and hence has an inverse function.

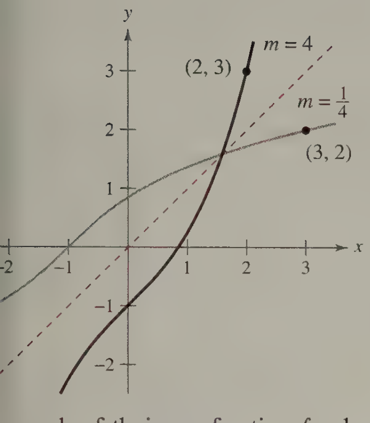
a. Because $f(x) = 3$ when $x = 2$, you know that $f^{-1}(3) = 2$.

b. Because the function f is differentiable and has an inverse function, you can apply Theorem 5.9 (with $g = f^{-1}$) to write

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)}.$$

Moreover, using $f'(x) = \frac{3}{4}x^2 + 1$, you can conclude that

$$(f^{-1})'(3) = \frac{1}{f'(2)} = \frac{1}{\frac{3}{4}(2^2) + 1} = \frac{1}{4}.$$



e graphs of the inverse functions f and ^{1} have reciprocal slopes at points (a, b) d (b, a).

gure 5.17

In Example 5, note that at the point (2, 3) the slope of the graph of f is 4 and at the point (3, 2) the slope of the graph of f^{-1} is $\frac{1}{4}$ (see Figure 5.17). This reciprocal relationship (which follows from Theorem 5.9) is sometimes written as

$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

Example 6 Graphs of Inverse Functions Have Reciprocal Slopes

Let $f(x) = x^2$ (for $x \geq 0$) and let $f^{-1}(x) = \sqrt{x}$. Show that the slopes of the graphs of f and f^{-1} are reciprocals at each of the following points.

a. (2, 4) and (4, 2)

b. (3, 9) and (9, 3)

Solution The derivatives of f and f^{-1} are given by

$$f'(x) = 2x \qquad \text{and} \qquad (f^{-1})'(x) = \frac{1}{2\sqrt{x}}.$$

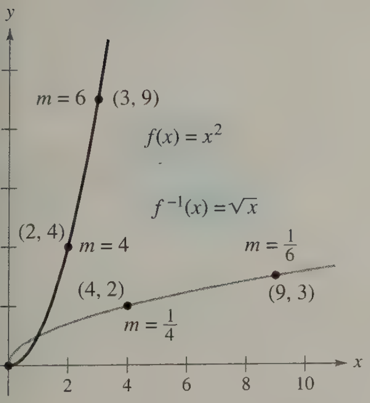
a. At (2, 4), the slope of the graph of f is $f'(2) = 2(2) = 4$. At (4, 2), the slope of the graph of f^{-1} is

$$(f^{-1})'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4}.$$

b. At (3, 9), the slope of the graph of f is $f'(3) = 2(3) = 6$. At (9, 3), the slope of the graph of f^{-1} is

$$(f^{-1})'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}.$$

So, in both cases, the slopes are reciprocals, as shown in Figure 5.18.



(0, 0), the derivative of f is 0, and the rivative of f^{-1} does not exist.

gure 5.18

EXERCISES FOR SECTION 5.3

In Exercises 1–8, show that f and g are inverse functions (a) analytically and (b) graphically.

1. $f(x) = 5x + 1,$ $g(x) = (x - 1)/5$

2. $f(x) = 3 - 4x,$ $g(x) = (3 - x)/4$

3. $f(x) = x^3,$ $g(x) = \sqrt[3]{x}$

4. $f(x) = 1 - x^3,$ $g(x) = \sqrt[3]{1 - x}$

5. $f(x) = \sqrt{x - 4},$ $g(x) = x^2 + 4, \quad x \geq 0$

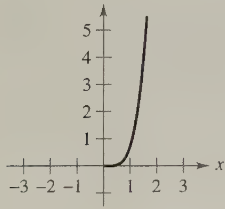
6. $f(x) = 16 - x^2, \quad x \geq 0,$ $g(x) = \sqrt{16 - x}$

7. $f(x) = 1/x,$ $g(x) = 1/x$

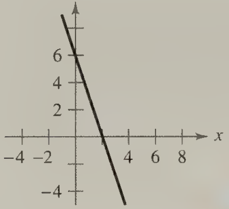
8. $f(x) = \dfrac{1}{1 + x}, \quad x \geq 0,$ $g(x) = \dfrac{1 - x}{x}, \quad 0 < x \leq 1$

In Exercises 9–12, match the graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]

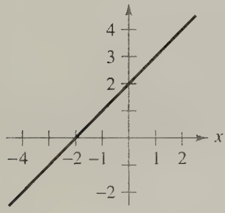
(a)



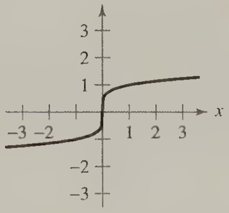
(b)



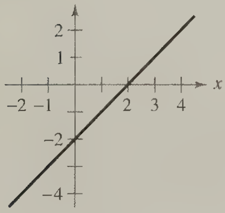
(c)



(d)



9.



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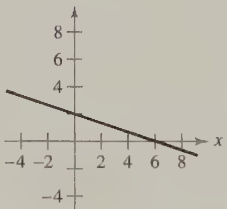

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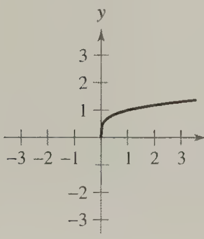

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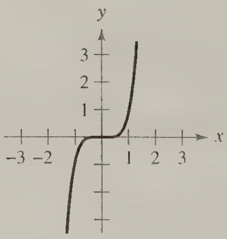
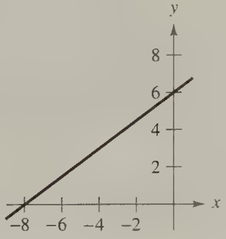
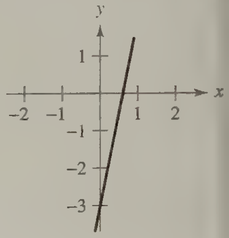


In Exercises 13–16, use the horizontal line test to determi whether the function is one-to-one on its entire domain a therefore has an inverse function. To print an enlarged copy the graph, go to the website *www.mathgraphs.com*.

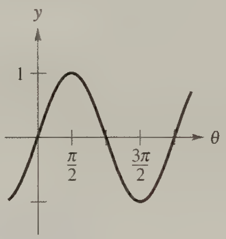
13. $f(x) = \frac{3}{4}x + 6$



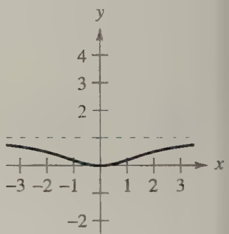
14. $f(x) = 5x - 3$



15. $f(\theta) = \sin \theta$



16. $f(x) = \dfrac{x^2}{x^2 + 4}$



In Exercises 17–22, use a graphing utility to graph the functi Determine whether the function is one-to-one on its ent domain.

17. $h(s) = \dfrac{1}{s - 2} - 3$

18. $g(t) = \dfrac{1}{\sqrt{t^2 + 1}}$

19. $f(x) = \ln x$

20. $f(x) = 5x\sqrt{x - 1}$

21. $g(x) = (x + 5)^3$

22. $h(x) = |x + 4| - |x -$

In Exercises 23–28, use the derivative to determine whether function is strictly monotonic on its entire domain and theref has an inverse function.

23. $f(x) = (x + a)^3 + b$

24. $f(x) = \cos \dfrac{3x}{2}$

25. $f(x) = \dfrac{x^4}{4} - 2x^2$

26. $f(x) = x^3 - 6x^2 + 12$

27. $f(x) = 2 - x - x^3$

28. $f(x) = \ln(x - 3)$

In Exercises 29–36, find the inverse function of f. Graph hand) f and f^{-1}. Describe the relationship between the grap

29. $f(x) = 2x - 3$

30. $f(x) = 3x$

31. $f(x) = x^5$

32. $f(x) = x^3 - 1$

33. $f(x) = \sqrt{x}$

34. $f(x) = x^2, \quad x \geq 0$

35. $f(x) = \sqrt{4 - x^2}, \quad x \geq 0$

36. $f(x) = \sqrt{x^2 - 4}, \quad x \geq 2$

Exercises 37–42, find the inverse function of f. Use a graph- g utility to graph f and f^{-1} in the same viewing window. escribe the relationship between the graphs.

. $f(x) = \sqrt[3]{x-1}$ **38.** $f(x) = 3\sqrt[5]{2x-1}$

. $f(x) = x^{2/3}, \quad x \geq 0$ **40.** $f(x) = x^{3/5}$

. $f(x) = \dfrac{x}{\sqrt{x^2+7}}$ **42.** $f(x) = \dfrac{x+2}{x}$

Exercises 43 and 44, use the graph of the function to complete the table and sketch the graph of f^{-1}. To int an enlarged copy of the graph, go to the website *vw.mathgraphs.com*.

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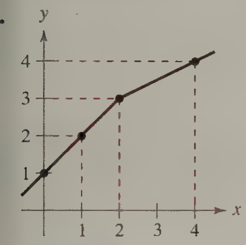
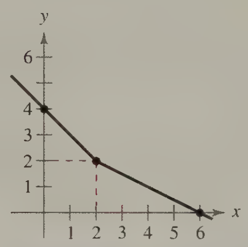


x	1	2	3	4
$f^{-1}(x)$				

44.



x	0	2	4
$f^{-1}(x)$			

. ***Cost*** Suppose you need 50 pounds of two commodities costing \$1.25 and \$1.60 per pound.

(a) Verify that the total cost is

$$y = 1.25x + 1.60(50 - x)$$

where x is the number of pounds of the less expensive commodity.

(b) Find the inverse function of the cost function. What does each variable represent in the inverse function?

(c) Use the context of the problem to determine the domain of the inverse function.

(d) Determine the number of pounds of the less expensive commodity purchased if the total cost is \$73.

. ***Think About It*** The function $f(x) = k(2 - x - x^3)$ is one-to-one and $f^{-1}(3) = -2$. Find k.

Exercises 47–52, show that f is strictly monotonic on the licated interval and therefore has an inverse function on that erval.

Function	*Interval*
. $f(x) = (x-4)^2$	$[4, \infty)$
. $f(x) = \lvert x+2 \rvert$	$[-2, \infty)$
. $f(x) = \dfrac{4}{x^2}$	$(0, \infty)$
. $f(x) = \cot x$	$(0, \pi)$
. $f(x) = \cos x$	$[0, \pi]$
$f(x) = \sec x$	$\left[0, \dfrac{\pi}{2}\right)$

In Exercises 53 and 54, find the inverse function of f over the indicated interval. Use a graphing utility to graph f and f^{-1} in the same viewing window. Describe the relationship between the graphs.

Function	*Interval*
53. $f(x) = \dfrac{x}{x^2-4}$	$-2 < x < 2$
54. $f(x) = 2 - \dfrac{3}{x^2}$	$0 < x < 10$

Graphical Reasoning **In Exercises 55–58, (a) use a graphing utility to graph the function, (b) use the *drawing* feature of a graphing utility to draw the inverse of the function, and (c) determine whether the graph of the inverse relation is an inverse function. Explain your reasoning.**

55. $f(x) = x^3 + x + 4$

56. $h(x) = x\sqrt{4 - x^2}$

57. $g(x) = \dfrac{3x^2}{x^2+1}$

58. $f(x) = \dfrac{4x}{\sqrt{x^2+15}}$

In Exercises 59–62, determine whether the function is one-to-one. If it is, find its inverse function.

59. $f(x) = \sqrt{x-2}$

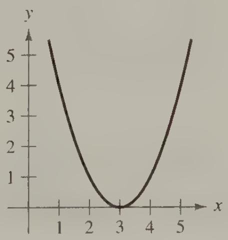
60. $f(x) = -3$

61. $f(x) = \lvert x-2 \rvert, \quad x \leq 2$

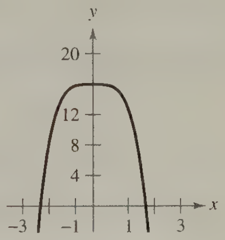
62. $f(x) = ax + b, \quad a \neq 0$

In Exercises 63–66, delete part of the domain so that the function that remains is one-to-one. Find the inverse function of the remaining function and give the domain of the inverse function. (*Note:* There is more than one correct answer.)

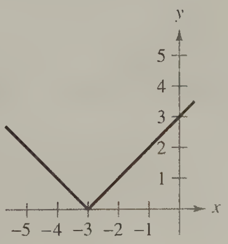
63. $f(x) = (x-3)^2$



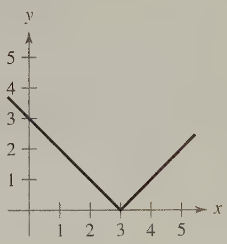
64. $f(x) = 16 - x^4$



65. $f(x) = \lvert x+3 \rvert$



66. $f(x) = \lvert x-3 \rvert$



Think About It **In Exercises 67–70, decide whether the function has an inverse function. If so, what is the inverse function?**

67. $g(t)$ is the volume of water that has passed through a water line t minutes after a control valve is opened.

68. $h(t)$ is the height of the tide t hours after midnight, where $0 \le t < 24$.

69. $C(t)$ is the cost of a long distance call lasting t minutes.

70. $A(r)$ is the area of a circle of radius r.

In Exercises 71–76, find $(f^{-1})'(a)$ for the function f and real number a.

Function	*Real Number*
71. $f(x) = x^3 + 2x - 1$	$a = 2$
72. $f(x) = \frac{1}{27}(x^5 + 2x^3)$,	$a = -11$
73. $f(x) = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$	$a = \frac{1}{2}$
74. $f(x) = \cos 2x, \quad 0 \le x \le \frac{\pi}{2}$	$a = 1$
75. $f(x) = x^3 - \frac{4}{x}$	$a = 6$
76. $f(x) = \sqrt{x - 4}$	$a = 2$

In Exercises 77–80, (a) find the domains of f and f^{-1}, (b) find the ranges of f and f^{-1}, (c) graph f and f^{-1}, and (d) show that the slopes of the graphs of f and f^{-1} are reciprocals at the indicated points.

Functions	*Point*
77. $f(x) = x^3$	$\left(\frac{1}{2}, \frac{1}{8}\right)$
$f^{-1}(x) = \sqrt[3]{x}$	$\left(\frac{1}{8}, \frac{1}{2}\right)$
78. $f(x) = 3 - 4x$	$(1, -1)$
$f^{-1}(x) = \dfrac{3 - x}{4}$	$(-1, 1)$
79. $f(x) = \sqrt{x - 4}$	$(5, 1)$
$f^{-1}(x) = x^2 + 4, \quad x \ge 0$	$(1, 5)$
80. $f(x) = \dfrac{4}{1 + x^2}, \quad x \ge 0$	$(1, 2)$
$f^{-1}(x) = \sqrt{\dfrac{4 - x}{x}}$	$(2, 1)$

In Exercises 81 and 82, find dy/dx at the indicated point for the equation.

81. $x = y^3 - 7y^2 + 2$ $(-4, 1)$

82. $x = 2 \ln(y^2 - 3)$ $(0, 4)$

In Exercises 83–86, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the indicated value.

83. $(f^{-1} \circ g^{-1})(1)$

84. $(g^{-1} \circ f^{-1})(-3)$

85. $(f^{-1} \circ f^{-1})(6)$

86. $(g^{-1} \circ g^{-1})(-4)$

In Exercises 87–90, use the functions $f(x) = x + 4$ a $g(x) = 2x - 5$ to find the indicated function.

87. $g^{-1} \circ f^{-1}$

88. $f^{-1} \circ g^{-1}$

89. $(f \circ g)^{-1}$

90. $(g \circ f)^{-1}$

Getting at the Concept

91. Describe how to find the inverse function of a one-to-one function given by an equation in x and y. Give an example

92. Describe the relationship between the graph of a function and the graph of its inverse function.

93. Give an example of a function that does *not* have an inverse function.

94. State the theorem that gives the method for finding the derivative of an inverse function.

In Exercises 95 and 96, the derivative of the function has the same sign for all x in its domain, but the function is not one-to-one. Explain.

95. $f(x) = \tan x$

96. $f(x) = \dfrac{x}{x^2 - 4}$

97. Prove that if f and g are one-to-one functions, th $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

98. Prove that if f has an inverse function, then $(f^{-1})^{-1} = f$.

99. Prove that if a function has an inverse function, then inverse function is unique.

100. Prove that a function has an inverse function if and only i is one-to-one.

True or False? **In Exercises 101–104, determine whether t statement is true or false. If it is false, explain why or give example that shows it is false.**

101. If f is an even function, then f^{-1} exists.

102. If the inverse function of f exists, then the y-intercept of f an x-intercept of f^{-1}.

103. If $f(x) = x^n$ where n is odd, then f^{-1} exists.

104. There exists no function f such that $f = f^{-1}$.

105. Is the converse of the second part of Theorem 5.7 true? T is, if a function is one-to-one (and hence has an inverse fu tion), then must the function be strictly monotonic? If prove it. If not, give a counterexample.

106. Let f be twice-differentiable and one-to-one on an open int val I. Show that its inverse function g satisfies

$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}.$$

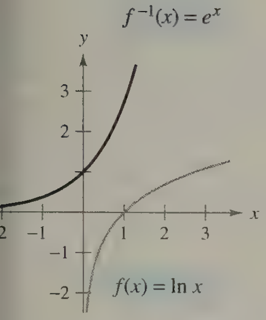
If f is increasing and concave downward, what is the conc ity of $f^{-1} = g$?

107. If $f(x) = \displaystyle\int_2^x \frac{dt}{\sqrt{1 + t^4}}$, find $(f^{-1})'(0)$.

Section 5.4 Exponential Functions: Differentiation and Integration

- Develop properties of the natural exponential function.
- Differentiate natural exponential functions.
- Integrate natural exponential functions.

The Natural Exponential Function



ne inverse function of the natural logarith-
ic function is the natural exponential
nction.

igure 5.19

The function $f(x) = \ln x$ is increasing on its entire domain, and hence it has an inverse function f^{-1}. The domain of f^{-1} is the set of all reals, and the range is the set of positive reals, as shown in Figure 5.19. So, for any real number x,

$$f(f^{-1}(x)) = \ln[f^{-1}(x)] = x. \qquad x \text{ is any real number.}$$

If x happens to be rational, then

$$\ln(e^x) = x \ln e = x(1) = x. \qquad x \text{ is a rational number.}$$

Because the natural logarithmic function is one-to-one, you can conclude that $f^{-1}(x)$ and e^x agree for *rational* values of x. The following definition extends the meaning of e^x to include *all* real values of x.

Definition of the Natural Exponential Function

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** and is denoted by

$$f^{-1}(x) = e^x.$$

That is,

$$y = e^x \qquad \text{if and only if} \qquad x = \ln y.$$

The inverse relationship between the natural logarithmic function and the natural exponential function can be summarized as follows.

$$\ln(e^x) = x \qquad \text{and} \qquad e^{\ln x} = x \qquad \text{Inverse relationship}$$

THE NUMBER e

The symbol e was first used by mathematician Leonhard Euler to represent the base of natural logarithms in a letter to another mathematician, Christian Goldbach, in 1731.

Example 1 **Solving Exponential Equations**

Solve $7 = e^{x+1}$.

Solution You can convert from exponential form to logarithmic form by *taking the natural log of both sides* of the equation.

$$7 = e^{x+1} \qquad \text{Write original equation.}$$
$$\ln 7 = \ln(e^{x+1}) \qquad \text{Take natural log of both sides.}$$
$$\ln 7 = x + 1 \qquad \text{Apply inverse property.}$$
$$-1 + \ln 7 = x \qquad \text{Solve for } x.$$
$$0.946 \approx x \qquad \text{Use a calculator.}$$

Check this solution in the original equation.

Example 2 Solving a Logarithmic Equation

Solve $\ln(2x - 3) = 5$.

Solution To convert from logarithmic form to exponential form, you can *exponen* *ate both sides* of the logarithmic equation.

$$\ln(2x - 3) = 5 \qquad \text{Write original equation.}$$
$$e^{\ln(2x-3)} = e^5 \qquad \text{Exponentiate both sides.}$$
$$2x - 3 = e^5 \qquad \text{Apply inverse property.}$$
$$x = \tfrac{1}{2}(e^5 + 3) \qquad \text{Solve for } x.$$
$$x \approx 75.707 \qquad \text{Use a calculator.}$$

The familiar rules for operating with rational exponents can be extended to t natural exponential function, as indicated in the following theorem.

THEOREM 5.10 Operations with Exponential Functions

Let a and b be any real numbers.

1. $e^a e^b = e^{a+b}$ **2.** $\dfrac{e^a}{e^b} = e^{a-b}$

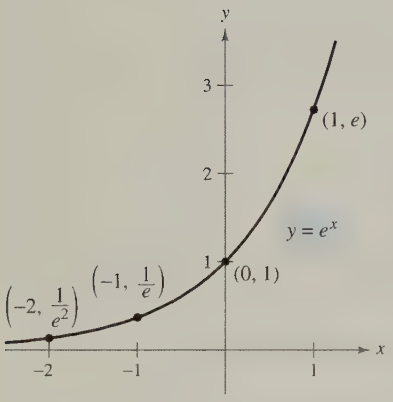
Proof To prove Property 1, you can write

$$\begin{aligned}\ln(e^a e^b) &= \ln(e^a) + \ln(e^b)\\ &= a + b\\ &= \ln(e^{a+b}).\end{aligned}$$

Because the natural log function is one-to-one, you can conclude that

$$e^a e^b = e^{a+b}.$$

The proof of the second property is left to you (see Exercise 130).



The natural exponential function is increasing, and its graph is concave upward.
Figure 5.20

In Section 5.3, you learned that an inverse function f^{-1} shares many properti with f. So, the natural exponential function inherits the following properties from t natural logarithmic function (see Figure 5.20).

Properties of the Natural Exponential Function

1. The domain of $f(x) = e^x$ is $(-\infty, \infty)$, and the range is $(0, \infty)$.
2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.
3. The graph of $f(x) = e^x$ is concave upward on its entire domain.
4. $\displaystyle\lim_{x\to-\infty} e^x = 0$ and $\displaystyle\lim_{x\to\infty} e^x = \infty$

Derivatives of Exponential Functions

One of the most intriguing (and useful) characteristics of the natural exponential function is that *it is its own derivative*. In other words, it is a solution to the differential equation $y' = y$. This result is stated in the next theorem.

R FURTHER INFORMATION To find
about derivatives of exponential
ctions of order 1/2, see the article
Child's Garden of Fractional
rivatives" by Marcia Kleinz and
omas J. Osler in *The College*
thematics Journal. To view this article,
to the website *www.matharticles.com*.

THEOREM 5.11 The Derivative of the Natural Exponential Function

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[e^x] = e^x$ **2.** $\dfrac{d}{dx}[e^u] = e^u\dfrac{du}{dx}$

Proof To prove Property 1, use the fact that $\ln e^x = x$, and differentiate both sides of the equation.

$$\ln e^x = x \qquad \text{Definition of exponential function}$$

$$\frac{d}{dx}[\ln e^x] = \frac{d}{dx}[x] \qquad \text{Differentiate both sides with respect to } x.$$

$$\frac{1}{e^x}\frac{d}{dx}[e^x] = 1$$

$$\frac{d}{dx}[e^x] = e^x$$

The derivative of e^u follows from the Chain Rule.

NOTE You can interpret this theorem geometrically by saying that the slope of the graph of $f(x) = e^x$ at any point (x, e^x) is equal to the y-coordinate of the point.

Example 3 Differentiating Exponential Functions

a. $\dfrac{d}{dx}[e^{2x-1}] = e^u\dfrac{du}{dx} = 2e^{2x-1}$ $\qquad u = 2x - 1$

b. $\dfrac{d}{dx}[e^{-3/x}] = e^u\dfrac{du}{dx} = \left(\dfrac{3}{x^2}\right)e^{-3/x} = \dfrac{3e^{-3/x}}{x^2}$ $\qquad u = -\dfrac{3}{x}$

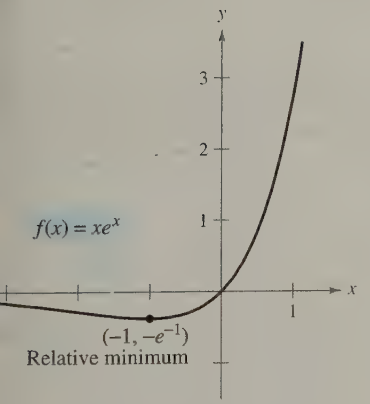
Example 4 Locating Relative Extrema

Find the relative extrema of $f(x) = xe^x$.

Solution The derivative of f is given by

$$f'(x) = x(e^x) + e^x(1) \qquad \text{Product Rule}$$
$$= e^x(x + 1).$$

Because e^x is never 0, the derivative is 0 only when $x = -1$. Moreover, by the First Derivative Test, you can determine that this corresponds to a relative minimum, as shown in Figure 5.21. Because the derivative $f'(x) = e^x(x + 1)$ is defined for all x, there are no other critical points.



e derivative of f changes from negative to
itive at $x = -1$.
ure 5.21

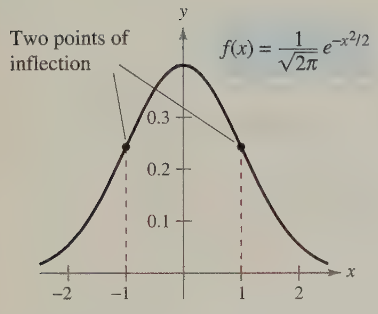
Example 5 The Normal Probability Density Function

Show that the *normal probability density function*

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

has points of inflection when $x = \pm 1$.

Solution To locate possible points of inflection, find the x-values for which second derivative is 0.



The bell-shaped curve given by a normal probability density function

Figure 5.22

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \qquad \text{Write original function.}$$

$$f'(x) = \frac{1}{\sqrt{2\pi}}(-x)e^{-x^2/2} \qquad \text{First derivative}$$

$$f''(x) = \frac{1}{\sqrt{2\pi}}[(-x)(-x)e^{-x^2/2} + (-1)e^{-x^2/2}] \qquad \text{Product Rule}$$

$$= \frac{1}{\sqrt{2\pi}}(e^{-x^2/2})(x^2 - 1) \qquad \text{Second derivative}$$

Therefore, $f''(x) = 0$ when $x = \pm 1$, and you can apply the techniques of Chapter 3 conclude that these values yield the two points of inflection shown in Figure 5.22.

NOTE The general form of a normal probability density function (whose mean is 0 given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}$$

where σ is the standard deviation (σ is the lowercase Greek letter sigma). This "bell-sha curve" has points of inflection when $x = \pm\sigma$.

Example 6 M.D.s in the United States

The number y of medical doctors (in thousands) in the United States from 19 through 1997, can be modeled by

$$y = 475{,}520e^{0.0271t}$$

where $t = 0$ represents 1980. At what rate was the number of M.D.s changing 1992? *(Source: American Medical Association)*

Solution The derivative of the given model is

$$y' = (0.0271)(475{,}520)e^{0.0271t}$$
$$\approx 12{,}887e^{0.0271t}.$$

By evaluating the derivative when $t = 12$, you can conclude that the rate of change 1992 was about

17,839 doctors per year.

The graph of this model is shown in Figure 5.23.

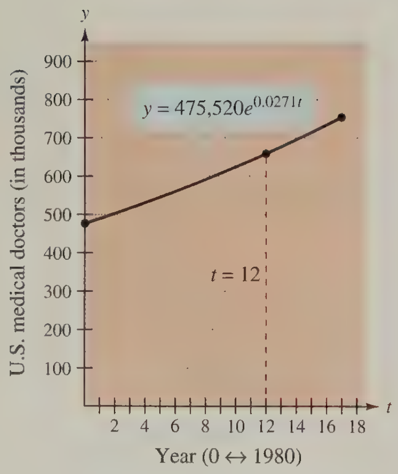


Figure 5.23

Integrals of Exponential Functions

Each differentiation formula in Theorem 5.11 has a corresponding integration formula.

THEOREM 5.12 Integration Rules for Exponential Functions

Let u be a differentiable function of x.

1. $\displaystyle\int e^x\,dx = e^x + C$ **2.** $\displaystyle\int e^u\,du = e^u + C$

Example 7 Integrating Exponential Functions

Find $\int e^{3x+1}\,dx$.

Solution If you let $u = 3x + 1$, then $du = 3\,dx$.

$$\begin{aligned}
\int e^{3x+1}\,dx &= \frac{1}{3}\int e^{3x+1}(3)\,dx && \text{Multiply and divide by 3.}\\
&= \frac{1}{3}\int e^u\,du && \text{Substitute: } u = 3x + 1.\\
&= \frac{1}{3}e^u + C && \text{Apply Exponential Rule.}\\
&= \frac{e^{3x+1}}{3} + C && \text{Back-substitute.}
\end{aligned}$$

NOTE In Example 7, the missing *constant* factor 3 was introduced to create $du = 3\,dx$. However, remember that you cannot introduce a missing *variable* factor in the integrand. For instance,

$$\int e^{-x^2}\,dx \neq \frac{1}{x}\int e^{-x^2}\,(x\,dx).$$

Example 8 Integrating Exponential Functions

Find $\int 5xe^{-x^2}\,dx$.

Solution If you let $u = -x^2$, then $du = -2x\,dx$ or $x\,dx = -du/2$.

$$\begin{aligned}
\int 5xe^{-x^2}\,dx &= \int 5e^{-x^2}(x\,dx) && \text{Regroup integrand.}\\
&= \int 5e^u\left(-\frac{du}{2}\right) && \text{Substitute: } u = -x^2.\\
&= -\frac{5}{2}\int e^u\,du && \text{Factor } -\tfrac{5}{2} \text{ out of integral.}\\
&= -\frac{5}{2}e^u + C && \text{Apply Exponential Rule.}\\
&= -\frac{5}{2}e^{-x^2} + C && \text{Back-substitute.}
\end{aligned}$$

Example 9 Integrating Exponential Functions

a. $$\int \frac{e^{1/x}}{x^2}\,dx = -\int \overbrace{e^{1/x}}^{e^u}\overbrace{\left(-\frac{1}{x^2}\right)dx}^{du} \qquad u = \frac{1}{x}$$

$$= -e^{1/x} + C$$

b. $$\int \sin x\, e^{\cos x}\,dx = -\int \overbrace{e^{\cos x}}^{e^u}\overbrace{(-\sin x\,dx)}^{du} \qquad u = \cos x$$

$$= -e^{\cos x} + C$$

Example 10 Finding Areas Bounded by Exponential Functions

Evaluate the definite integral.

a. $\displaystyle\int_0^1 e^{-x}\,dx$ **b.** $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\,dx$ **c.** $\displaystyle\int_{-1}^0 [e^x\cos(e^x)]\,dx$

Solution

a. $$\int_0^1 e^{-x}\,dx = -e^{-x}\Big]_0^1 \qquad \text{See Figure 5.24(a).}$$

$$= -e^{-1} - (-1)$$

$$= 1 - \frac{1}{e}$$

$$\approx 0.632$$

b. $$\int_0^1 \frac{e^x}{1+e^x}\,dx = \ln(1+e^x)\Big]_0^1 \qquad \text{See Figure 5.24(b).}$$

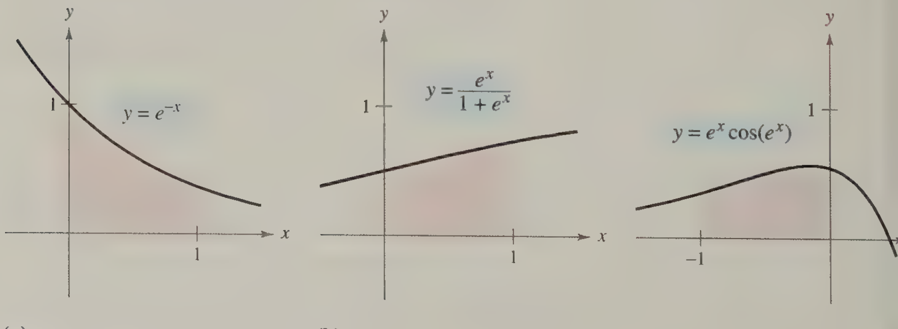
$$= \ln(1+e) - \ln 2$$

$$\approx 0.620$$

c. $$\int_{-1}^0 [e^x\cos(e^x)]\,dx = \sin(e^x)\Big]_{-1}^0 \qquad \text{See Figure 5.24(c).}$$

$$= \sin 1 - \sin(e^{-1})$$

$$\approx 0.482$$



(a) (b) (c)

Figure 5.24

XERCISES FOR SECTION 5.4

**Exercises 1–4, write the exponential equation as a logarith-
equation or vice versa.**

$e^0 = 1$ | **2.** $e^{-2} = 0.1353\ldots$

$\ln 2 = 0.6931\ldots$ | **4.** $\ln 0.5 = -0.6931\ldots$

Exercises 5–18, solve for x accurate to three decimal places.

$e^{\ln x} = 4$ | **6.** $e^{\ln 2x} = 12$

$e^x = 12$ | **8.** $4e^x = 83$

$9 - 2e^x = 7$ | **10.** $-6 + 3e^x = 8$

$50e^{-x} = 30$ | **12.** $200e^{-4x} = 15$

$\ln x = 2$ | **14.** $\ln x^2 = 10$

$\ln(x - 3) = 2$ | **16.** $\ln 4x = 1$

$\ln \sqrt{x + 2} = 1$ | **18.** $\ln(x - 2)^2 = 12$

Exercises 19–22, sketch the graph of the function.

$y = e^{-x}$ | **20.** $y = \frac{1}{2}e^x$

$y = e^{-x^2}$ | **22.** $y = e^{-x/2}$

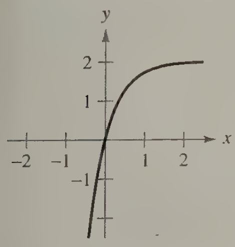
Use a graphing utility to graph $f(x) = e^x$ and the given function in the same viewing window. How are the two graphs related?

(a) $g(x) = e^{x-2}$ (b) $h(x) = -\frac{1}{2}e^x$ (c) $q(x) = e^{-x} + 3$

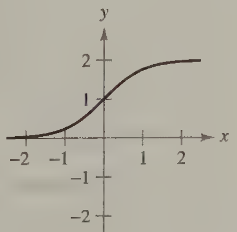
Use a graphing utility to graph the function. Use the graph to determine any asymptotes of the function.

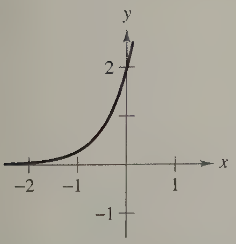
(a) $f(x) = \dfrac{8}{1 + e^{-0.5x}}$ (b) $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

Exercises 25–28, match the equation with the correct graph. ume that a and C are positive real numbers. [The graphs are eled (a), (b), (c), and (d).]

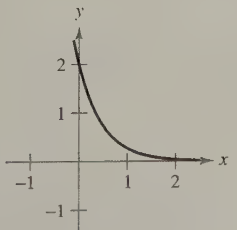


(b)





(d)



$y = Ce^{ax}$ | **26.** $y = Ce^{-ax}$

$y = C(1 - e^{-ax})$ | **28.** $y = \dfrac{C}{1 + e^{-ax}}$

In Exercises 29–32, illustrate that the functions are inverses of each other by graphing both functions on the same set of coordinate axes.

29. $f(x) = e^{2x}$
$g(x) = \ln \sqrt{x}$

30. $f(x) = e^{x/3}$
$g(x) = \ln x^3$

31. $f(x) = e^x - 1$
$g(x) = \ln(x + 1)$

32. $f(x) = e^{x-1}$
$g(x) = 1 + \ln x$

33. ***Graphical Analysis*** Use a graphing utility to graph

$$f(x) = \left(1 + \frac{0.5}{x}\right)^x \quad \text{and} \quad g(x) = e^{0.5}$$

in the same viewing window. What is the relationship between f and g as $x \to \infty$?

34. ***Conjecture*** Use the result of Exercise 33 to make a conjecture about the value of

$$\left(1 + \frac{r}{x}\right)^x$$

as $x \to \infty$.

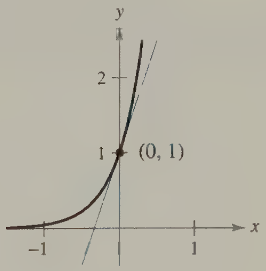
In Exercises 35 and 36, compare the given number with the number e. Is the number less than or greater than e?

35. $\left(1 + \dfrac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$ (See Exercise 34.)

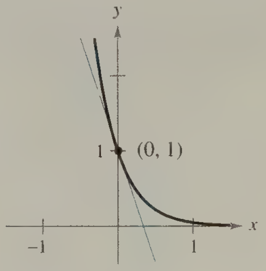
36. $1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{120} + \dfrac{1}{720} + \dfrac{1}{5040}$

In Exercises 37 and 38, find the slope of the tangent line to the graph of the function at the point (0, 1).

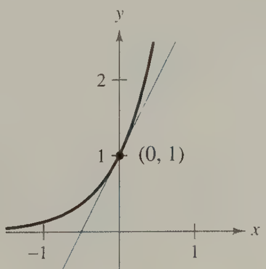
37. (a) $y = e^{3x}$



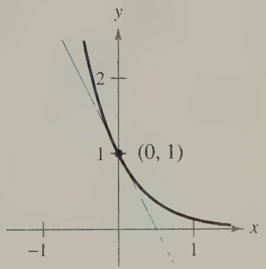
(b) $y = e^{-3x}$



38. (a) $y = e^{2x}$



(b) $y = e^{-2x}$



In Exercises 39–58, find the derivative of the function.

39. $f(x) = e^{2x}$

40. $f(x) = e^{1-x}$

41. $y = e^{-2x+x^2}$

42. $y = e^{-x^2}$

43. $y = e^{\sqrt{x}}$

44. $y = x^2e^{-x}$

45. $g(t) = (e^{-t} + e^t)^3$

46. $g(t) = e^{-3/t^2}$

47. $y = \ln(e^{x^2})$

48. $y = \ln\left(\dfrac{1 + e^x}{1 - e^x}\right)$

49. $y = \ln(1 + e^{2x})$

50. $y = \ln \dfrac{e^x + e^{-x}}{2}$

51. $y = \dfrac{2}{e^x + e^{-x}}$

52. $y = \dfrac{e^x - e^{-x}}{2}$

53. $y = x^2e^x - 2xe^x + 2e^x$

54. $y = xe^x - e^x$

55. $f(x) = e^{-x} \ln x$

56. $f(x) = e^3 \ln x$

57. $y = e^x(\sin x + \cos x)$

58. $y = \ln e^x$

In Exercises 59 and 60, use implicit differentiation to find dy/dx.

59. $xe^y - 10x + 3y = 0$

60. $e^{xy} + x^2 - y^2 = 10$

In Exercises 61 and 62, find the second derivative of the function.

61. $f(x) = (3 + 2x)e^{-3x}$

62. $g(x) = \sqrt{x} + e^x \ln x$

In Exercises 63 and 64, show that the function $y = f(x)$ is a solution of the differential equation.

63. $y = e^x(\cos \sqrt{2}x + \sin \sqrt{2}x)$

$y'' - 2y' + 3y = 0$

64. $y = e^x(3 \cos 2x - 4 \sin 2x)$

$y'' - 2y' + 5y = 0$

In Exercises 65–72, find the extrema and the points of inflection (if any exist) of the function. Use a graphing utility to graph the function and confirm your results.

65. $f(x) = \dfrac{e^x + e^{-x}}{2}$

66. $f(x) = \dfrac{e^x - e^{-x}}{2}$

67. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-2)^2/2}$

68. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-3)^2/2}$

69. $f(x) = x^2e^{-x}$

70. $f(x) = xe^{-x}$

71. $g(t) = 1 + (2 + t)e^{-t}$

72. $f(x) = -2 + e^{3x}(4 - 2x)$

73. ***Area*** Find the area of the largest rectangle that can be inscribed under the curve $y = e^{-x^2}$ in the first and second quadrants.

74. ***Area*** Perform the following steps to find the maximum of the rectangle shown in the figure.

(a) Solve for c in the equation $f(c) = f(c + x)$.

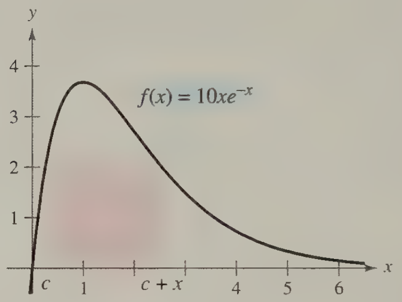
(b) Use the result in part (a) to write the area A as a functio x. [*Hint:* $A = xf(c)$]

(c) Use a graphing utility to graph the area function. Use graph to approximate the dimensions of the rectangl maximum area. Determine the maximum area.

(d) Use a graphing utility to graph the expression for c fo in part (a). Use the graph to approximate

$$\lim_{x \to 0^+} c \quad \text{and} \quad \lim_{x \to \infty} c.$$

Use this result to describe the changes in dimensions position of the rectangle for $0 < x < \infty$.



75. Verify that the function

$$y = \frac{L}{1 + ae^{-x/b}}, \qquad a > 0, b > 0, L > 0$$

increases at a maximum rate when $y = L/2$.

76. Find the point on the graph of $y = e^{-x}$ where the normal lin the curve passes through the origin. (Use Newton's Metho the root-finding capabilities of a graphing utility.)

77. Find, to three decimal places, the value of x such that

$e^{-x} = x$.

(Use Newton's Method or the root-finding capabilities graphing utility.)

78. ***Depreciation*** The value V of an item t years after purchased is

$V = 15{,}000e^{-0.6286t}, \qquad 0 \le t \le 10.$

(a) Use a graphing utility to graph the function.

(b) Find the rate of change of V with respect to t when t and $t = 5$.

(c) Use a graphing utility to graph the tangent line to the fu tion when $t = 1$ and $t = 5$.

79. ***Writing*** Consider the function

$$f(x) = \frac{2}{1 + e^{1/x}}.$$

(a) Use a graphing utility to graph f.

(b) Write a short paragraph explaining why the graph h horizontal asymptote at $y = 1$ and why the function h nonremovable discontinuity at $x = 0$.

Harmonic Motion The displacement from equilibrium of a mass oscillating on the end of a spring suspended from a ceiling is

$$y = 1.56e^{-0.22t}\cos 4.9t$$

where y is the displacement in feet and t is the time in seconds. Use a graphing utility to graph the displacement function on the interval $[0, 10]$. Find a value of t past which the displacement is less than 3 inches from equilibrium.

Modeling Data A meteorologist measures the atmospheric pressure P (in kilograms per square meter) at altitude h (in kilometers). The data are shown below.

h	0	5	10	15	20
P	10,332	5583	2376	1240	517

(a) Use a graphing utility to plot the points $(h, \ln P)$. Use the regression capabilities of the graphing utility to find a linear model for the revised data points.

(b) The line in part (a) has the form $\ln P = ah + b$. Write the equation in exponential form.

(c) Use a graphing utility to plot the original data and graph the exponential model in part (b).

(d) Find the rate of change of the pressure when $h = 5$ and $h = 18$.

Modeling Data A 1994 Chevrolet Camaro coupe with a 6-cylinder engine, 5-speed transmission, and air conditioning had a retail price of \$17,040. A local dealership had the following guide for the approximate value of the car for the years 1995 through 2000.

Year	1994	1995	1996	1997
Value	\$17,040	\$14,590	\$12,845	\$10,995

Year	1998	1999	2000
Value	\$9,220	\$8,095	\$6,835

In each of the following, let V represent the value of the automobile in the year t, with $t = 4$ corresponding to 1994.

(a) Use a computer algebra system to find linear and quadratic models for the data. Plot the data and graph the models.

(b) What does the slope represent in the linear model in part (a)?

(c) Use a computer algebra system to fit an exponential model to the data.

(d) Determine the horizontal asymptote of the exponential model found in part (c). Interpret its meaning in the context of the problem.

(e) Find the rate of decrease in the value of the car when $t = 5$ and $t = 9$ using the exponential model.

Linear and Quadratic Approximations **In Exercises 83 and 84, use a graphing utility to graph the function. Then graph**

$$P_1(x) = f(0) + f'(0)(x - 0)$$

and

$$P_2(x) = f(0) + f'(0)(x - 0) + \tfrac{1}{2}f''(0)(x - 0)^2$$

in the same viewing window. Compare the values of f, P_1, and P_2 and their first derivatives at $x = 0$.

83. $f(x) = e^{x/2}$ **84.** $f(x) = e^{-x^2/2}$

85. ***Finding a Pattern*** Use a graphing utility to compare the graph of the function $y = e^x$ with the graphs of each of the following functions.

(a) $y_1 = 1 + \dfrac{x}{1!}$

(b) $y_2 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!}$

(c) $y_3 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$

86. Identify the pattern of successive polynomials in Exercise 85. Extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of $y = e^x$. What do you think this pattern implies?

In Exercises 87–108, find or evaluate the integral.

87. $\displaystyle\int e^{5x}(5)\,dx$ **88.** $\displaystyle\int e^{-x^4}(-4x^3)\,dx$

89. $\displaystyle\int_0^1 e^{-2x}\,dx$ **90.** $\displaystyle\int_3^4 e^{3-x}\,dx$

91. $\displaystyle\int xe^{-x^2}\,dx$ **92.** $\displaystyle\int x^2e^{x^3/2}\,dx$

93. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ **94.** $\displaystyle\int \frac{e^{1/x^2}}{x^3}\,dx$

95. $\displaystyle\int \frac{e^{-x}}{1 + e^{-x}}\,dx$ **96.** $\displaystyle\int \frac{e^{2x}}{1 + e^{2x}}\,dx$

97. $\displaystyle\int_1^3 \frac{e^{3/x}}{x^2}\,dx$ **98.** $\displaystyle\int_0^{\sqrt{2}} xe^{-(x^2/2)}\,dx$

99. $\displaystyle\int e^x\sqrt{1 - e^x}\,dx$ **100.** $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

101. $\displaystyle\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$ **102.** $\displaystyle\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\,dx$

103. $\displaystyle\int \frac{5 - e^x}{e^{2x}}\,dx$ **104.** $\displaystyle\int \frac{e^{2x} + 2e^x + 1}{e^x}\,dx$

105. $\displaystyle\int e^{\sin \pi x}\cos \pi x\,dx$ **106.** $\displaystyle\int e^{\sec 2x}\sec 2x\tan 2x\,dx$

107. $\displaystyle\int e^{-x}\tan(e^{-x})\,dx$ **108.** $\displaystyle\int \ln(e^{2x-1})\,dx$

In Exercises 109 and 110, solve the differential equation.

109. $\dfrac{dy}{dx} = xe^{ax^2}$ **110.** $\dfrac{dy}{dx} = (e^x - e^{-x})^2$

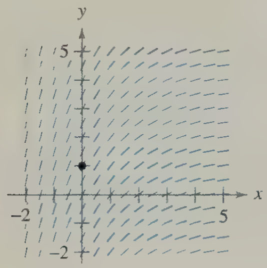
In Exercises 111 and 112, find the particular solution that satisfies the initial conditions.

111. $f''(x) = \frac{1}{2}(e^x + e^{-x})$,
$f(0) = 1, f'(0) = 0$

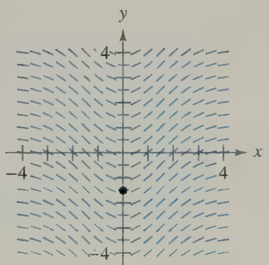
112. $f''(x) = \sin x + e^{2x}$,
$f(0) = \frac{1}{4}, f'(0) = \frac{1}{2}$

Slope Fields **In Exercises 113 and 114, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the indicated point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

113. $\dfrac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$



114. $\dfrac{dy}{dx} = xe^{-0.2x^2}$, $\left(0, -\frac{3}{2}\right)$



Area **In Exercises 115–118, find the area of the region bounded by the graphs of the equations. Use a graphing utility to graph the region and verify your result.**

115. $y = e^x, y = 0, x = 0, x = 5$

116. $y = e^{-x}, y = 0, x = a, x = b$

117. $y = xe^{-x^2/4}, y = 0, x = 0, x = \sqrt{6}$

118. $y = e^{-2x} + 2, y = 0, x = 0, x = 2$

119. Given the exponential function $f(x) = e^x$, show that

(a) $f(u - v) = \dfrac{f(u)}{f(v)}$. (b) $f(kx) = [f(x)]^k$.

120. Approximate each integral using the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule with $n = 12$. Then use the integration capabilities of a graphing utility to approximate the integrals and compare the results.

(a) $\displaystyle\int_0^4 \sqrt{x}\, e^x\, dx$ (b) $\displaystyle\int_0^2 2xe^{-x}\, dx$

121. ***Probability*** A car battery has an average lifetime of 48 months with a standard deviation of 6 months. The battery lives are normally distributed. The probability that a given battery will last between 48 months and 60 months is

$$0.0665 \int_{48}^{60} e^{-0.0139(t-48)^2}\, dt.$$

Use the integration capabilities of a graphing utility to approximate the integral. Interpret the resulting probability.

122. ***Probability*** The median waiting time (in minutes) for pec
waiting for service in a convenience store is given by the sc
tion of the equation

$$\int_0^x 0.3e^{-0.3t}\, dt = \frac{1}{2}.$$

Solve the equation.

123. Given $e^x \ge 1$ for $x \ge 0$, it follows that

$$\int_0^x e^t\, dt \ge \int_0^x 1\, dt.$$

Perform this integration to derive the inequality $e^x \ge 1$
for $x \ge 0$.

124. ***Modeling Data*** A valve on a storage tank is opened fc
hours to release a chemical in a manufacturing process.
flow rate R (in liters per hour) at time t (in hours) is giver
the table.

t	0	1	2	3	4
R	425	240	118	71	36

(a) Use the regression capabilities of a graphing utility to f
a linear model for the points $(t, \ln R)$. Write the result
equation of the form $\ln R = at + b$ in exponential for

(b) Use a graphing utility to plot the data and graph the ex
nential model.

(c) Use the definite integral to approximate the number
liters of chemical released during the 4 hours.

Getting at the Concept

125. In your own words, state the properties of the natur
exponential function.

126. Describe the relationship between the graph of $f(x) = \ln$
and $g(x) = e^x$.

127. Is there a function f such that $f(x) = f'(x)$? If so, identify i

128. Without integrating, state the integration formula you ca
use to integrate each of the following.

(a) $\displaystyle\int \frac{e^x}{e^x + 1}\, dx$ (b) $\displaystyle\int xe^{x^2}\, dx$

129. Explain why $\displaystyle\int_0^2 e^{-x}\, dx > 0$.

130. Prove that $\dfrac{e^a}{e^b} = e^{a-b}$.

131. Let $f(x) = \dfrac{\ln x}{x}$.

(a) Graph f on $(0, \infty)$ and show that f is strictly decreas
on (e, ∞).

(b) Show that if $e \le A < B$, then $A^B > B^A$.

(c) Use part (b) to show that $e^\pi > \pi^e$.

Section 5.5 Bases Other than e and Applications

- Define exponential functions that have bases other than e.
- Differentiate and integrate exponential functions that have bases other than e.
- Use exponential functions to model compound interest and exponential growth.

Bases Other than e

The **base** of the natural exponential function is e. This "natural" base can be used to assign a meaning to a general base a.

Definition of Exponential Function to Base a

If a is a positive real number ($a \neq 1$) and x is any real number, then the **exponential function to the base a** is denoted by a^x and is defined by

$$a^x = e^{(\ln a)x}.$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

These functions obey the usual laws of exponents. For instance, here are some familiar properties.

1. $a^0 = 1$

2. $a^x a^y = a^{x+y}$

3. $\dfrac{a^x}{a^y} = a^{x-y}$

4. $(a^x)^y = a^{xy}$

When modeling the half-life of a radioactive sample, it is convenient to use $\frac{1}{2}$ as the base of the exponential model.

Example 1 Radioactive Half-Life Model

The half-life of carbon-14 is about 5730 years. If 1 gram of carbon-14 is present in a sample, how much will be present in 10,000 years?

Solution Let $t = 0$ represent the present time and let y represent the amount (in grams) of carbon-14 in the sample. Using a base of $\frac{1}{2}$, you can model y by the equation

$$y = \left(\frac{1}{2}\right)^{t/5730}.$$

Notice that when $t = 5730$, the amount is reduced to half of the original amount.

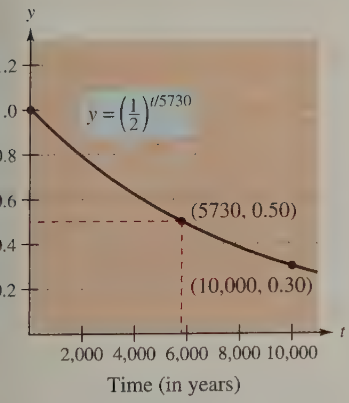
$$y = \left(\frac{1}{2}\right)^{5730/5730} = \frac{1}{2} \text{ gram}$$

When $t = 11{,}460$, the amount is reduced to a quarter of the original amount, and so on. To find the amount of carbon-14 after 10,000 years, substitute 10,000 for t.

$$y = \left(\frac{1}{2}\right)^{10{,}000/5730}$$

$$\approx 0.30 \text{ gram}$$

The graph of y is shown in Figure 5.25.



half-life of carbon-14 is about 5730
.

re 5.25

Logarithmic functions to bases other than e can be defined in much the same as exponential functions to other bases are defined.

NOTE In precalculus, you learned that $\log_a x$ is the value to which a should be raised to produce x. This agrees with the definition given here because

$$\begin{aligned} a^{\log_a x} &= a^{(1/\ln a)\ln x} \\ &= (e^{\ln a})^{(1/\ln a)\ln x} \\ &= e^{(\ln a/\ln a)\ln x} \\ &= e^{\ln x} \\ &= x. \end{aligned}$$

Definition of Logarithmic Function to Base *a*

If a is a positive real number ($a \neq 1$) and x is any positive real number, then the **logarithmic function to the base a** is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$

Logarithmic functions to the base a have properties similar to those of natural logarithmic function given in Theorem 5.2.

1. $\log_a 1 = 0$ Log of 1
2. $\log_a xy = \log_a x + \log_a y$ Log of a product
3. $\log_a x^n = n \log_a x$ Log of a power
4. $\log_a \dfrac{x}{y} = \log_a x - \log_a y$ Log of a quotient

From the definitions of the exponential and logarithmic functions to the base follows that $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions of each other.

Properties of Inverse Functions

1. $y = a^x$ if and only if $x = \log_a y$
2. $a^{\log_a x} = x$, for $x > 0$
3. $\log_a a^x = x$, for all x

The logarithmic function to the base 10 is called the **common logarith** **function.** Thus, for common logarithms, $y = 10^x$ if and only if $x = \log_{10} y$.

Example 2 Bases Other than *e*

Solve for x in each equation.

a. $3^x = \dfrac{1}{81}$ **b.** $\log_2 x = -4$

Solution

a. To solve this equation, you can apply the logarithmic function to the base 3 to both sides of the equation.

$$\begin{aligned} 3^x &= \frac{1}{81} \\ \log_3 3^x &= \log_3 \frac{1}{81} \\ x &= \log_3 3^{-4} \\ x &= -4 \end{aligned}$$

b. To solve this equation, you can a the exponential function to the ba to both sides of the equation.

$$\begin{aligned} \log_2 x &= -4 \\ 2^{\log_2 x} &= 2^{-4} \\ x &= \frac{1}{2^4} \\ x &= \frac{1}{16} \end{aligned}$$

Differentiation and Integration

To differentiate exponential and logarithmic functions to other bases, you have three options: (1) use the definitions of a^x and $\log_a x$ and differentiate using the rules for the natural exponential and logarithmic functions, (2) use logarithmic differentiation, or (3) use the following differentiation rules for bases other than e.

THEOREM 5.13 Derivatives for Bases Other than e

Let a be a positive real number $(a \neq 1)$ and let u be a differentiable function of x.

1. $\dfrac{d}{dx}[a^x] = (\ln a)a^x$

2. $\dfrac{d}{dx}[a^u] = (\ln a)a^u\dfrac{du}{dx}$

3. $\dfrac{d}{dx}[\log_a x] = \dfrac{1}{(\ln a)x}$

4. $\dfrac{d}{dx}[\log_a u] = \dfrac{1}{(\ln a)u}\dfrac{du}{dx}$

Proof By definition, $a^x = e^{(\ln a)x}$. Therefore, you can prove the first rule by letting $u = (\ln a)x$ and differentiating with base e to obtain

$$\frac{d}{dx}[a^x] = \frac{d}{dx}[e^{(\ln a)x}] = e^u\frac{du}{dx} = e^{(\ln a)x}(\ln a) = (\ln a)a^x.$$

To prove the third rule, you can write

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{1}{\ln a}\ln x\right] = \frac{1}{\ln a}\left(\frac{1}{x}\right) = \frac{1}{(\ln a)x}.$$

The second and fourth rules are simply the Chain Rule versions of the first and third rules. ◻

NOTE These differentiation rules are similar to those for the natural exponential function and natural logarithmic function. In fact, they differ only by the constant factors $\ln a$ and $1/\ln a$. This points out one reason why, for calculus, e is the most convenient base.

Example 3 Differentiating Functions to Other Bases

Find the derivative of each of the following.

a. $y = 2^x$

b. $y = 2^{3x}$

c. $y = \log_{10} \cos x$

Solution

a. $y' = \dfrac{d}{dx}[2^x] = (\ln 2)2^x$

b. $y' = \dfrac{d}{dx}[2^{3x}] = (\ln 2)2^{3x}(3) = (3 \ln 2)2^{3x}$

Try writing 2^{3x} as 8^x and differentiating to see that you obtain the same result.

c. $y' = \dfrac{d}{dx}[\log_{10} \cos x] = \dfrac{-\sin x}{(\ln 10)\cos x} = -\dfrac{1}{\ln 10}\tan x$

Occasionally, an integrand involves an exponential function to a base other t e. When this occurs, there are two options: (1) convert to base e using the form $a^x = e^{(\ln a)x}$ and then integrate, or (2) integrate directly, using the integration form (which follows from Theorem 5.13)

$$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C.$$

Example 4 Integrating an Exponential Function to Another Base

Find $\int 2^x\,dx$.

Solution

$$\int 2^x\,dx = \frac{1}{\ln 2}2^x + C$$

When we introduced the Power Rule, $D_x[x^n] = nx^{n-1}$, in Chapter 2, we requi the exponent n to be a rational number. We now extend the rule to cover any real va of n. Try to prove this theorem using logarithmic differentiation.

THEOREM 5.14 The Power Rule for Real Exponents

Let n be any real number and let u be a differentiable function of x.

1. $\dfrac{d}{dx}[x^n] = nx^{n-1}$ **2.** $\dfrac{d}{dx}[u^n] = nu^{n-1}\dfrac{du}{dx}$

The next example compares the derivatives of four types of functions. E function uses a different differentiation formula, depending on whether the base exponent are constants or variables.

Example 5 Comparing Variables and Constants

a. $\dfrac{d}{dx}[e^e] = 0$ Constant Rule

b. $\dfrac{d}{dx}[e^x] = e^x$ Exponential Rule

c. $\dfrac{d}{dx}[x^e] = ex^{e-1}$ Power Rule

d. $y = x^x$ Logarithmic differentiation

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{y'}{y} = x\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x$$

$$y' = y(1 + \ln x) = x^x(1 + \ln x)$$

NOTE Be sure you see that there is no simple differentiation rule for calculating the derivative of $y = x^x$. In general, if $y = u(x)^{v(x)}$, you need to use logarithmic differentiation.

Applications of Exponential Functions

n	A
1	\$1080.00
2	\$1081.60
4	\$1082.43
12	\$1083.00
365	\$1083.28

Suppose P dollars is deposited in an account at an annual interest rate r (in decimal form). If interest accumulates in the account, what is the balance in the account at the end of 1 year? The answer depends on the number of times n the interest is compounded according to the formula

$$A = P\left(1 + \frac{r}{n}\right)^n.$$

For instance, the result for a deposit of \$1000 at 8% interest compounded n times a year is shown in the upper table at the left.

As n increases, the balance A approaches a limit. To develop this limit, we use the following theorem. To test the reasonableness of this theorem, try evaluating $[(x + 1)/x]^x$ for several values of x, as shown in the lower table at the left. (A proof of this theorem is given in Appendix B.)

x	$\left(\frac{x+1}{x}\right)^x$
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

THEOREM 5.15 A Limit Involving e

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e$$

Now, let's take another look at the formula for the balance A in an account in which the interest is compounded n times per year. By taking the limit as n approaches infinity, you obtain the following.

$$\begin{aligned}
A &= \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^n && \text{Take limit as } n\to\infty.\\
&= P \lim_{n\to\infty}\left[\left(1 + \frac{1}{n/r}\right)^{n/r}\right]^r && \text{Rewrite.}\\
&= P\left[\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x\right]^r && \text{Let } x = n/r. \text{ Then } x\to\infty \text{ as } n\to\infty.\\
&= Pe^r && \text{Apply Theorem 5.15.}
\end{aligned}$$

This limit produces the balance after 1 year of **continuous compounding.** So, for a deposit of \$1000 at 8% interest compounded continuously, the balance at the end of 1 year would be

$$A = 1000e^{0.08} \approx \$1083.29.$$

These results are summarized as follows.

Summary of Compound Interest Formulas

Let P = amount of deposit, t = number of years, A = balance after t years, r = annual interest rate (decimal form), and n = number of compoundings per year.

1. Compounded n times per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

2. Compounded continuously: $A = Pe^{rt}$

Example 6 Comparing Continuous and Quarterly Compounding

A deposit of \$2500 is made in an account that pays an annual interest rate of
Find the balance in the account at the end of 5 years if the interest is compounded
quarterly, (b) monthly, and (c) continuously.

Solution

$$\textbf{a. } A = P\left(1+\frac{r}{n}\right)^{nt} = 2500\left(1+\frac{0.05}{4}\right)^{4(5)}$$

Compounded quarterly

$$= 2500(1.0125)^{20}$$

$$\approx \$3205.09$$

$$\textbf{b. } A = P\left(1+\frac{r}{n}\right)^{nt} = 2500\left(1+\frac{0.05}{12}\right)^{12(5)}$$

Compounded monthly

$$\approx 2500(1.0041667)^{60}$$

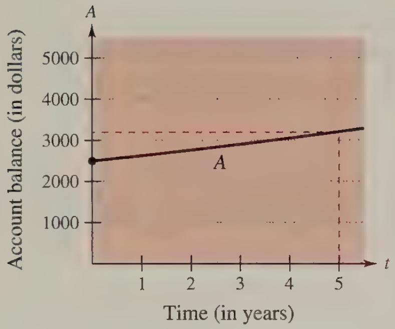
$$\approx \$3208.40$$

$$\textbf{c. } A = Pe^{rt} = 2500[e^{0.05(5)}]$$

Compounded continuously

$$= 2500e^{0.25}$$

$$\approx \$3210.06$$



The balance in a savings account grows exponentially.
Figure 5.26

Figure 5.26 shows how the balance increases over the 5-year period. Notice that
scale used in the figure does not graphically distinguish among the three type
exponential growth in (a), (b), and (c).

Example 7 Bacterial Culture Growth

A bacterial culture is growing according to the *logistics growth function*

$$y = \frac{1.25}{1+0.25e^{-0.4t}}, \quad t \geq 0$$

where y is the weight of the culture in grams and t is the time in hours. Find the we
of the culture after (a) 0 hours, (b) 1 hour, and (c) 10 hours. (d) What is the limit
approaches infinity?

Solution

$$\textbf{a. } \text{When } t = 0, \quad y = \frac{1.25}{1+0.25e^{-0.4(0)}}$$

$$= 1 \text{ gram.}$$

$$\textbf{b. } \text{When } t = 1, \quad y = \frac{1.25}{1+0.25e^{-0.4(1)}}$$

$$\approx 1.071 \text{ grams.}$$

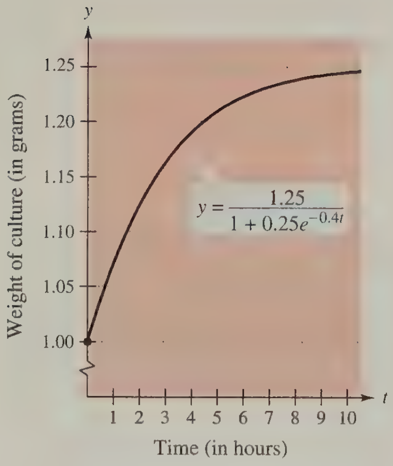
$$\textbf{c. } \text{When } t = 10, \quad y = \frac{1.25}{1+0.25e^{-0.4(10)}}$$

$$\approx 1.244 \text{ grams.}$$

d. Finally, taking the limit as t approaches infinity, you obtain

$$\lim_{t\to\infty} \frac{1.25}{1+0.25e^{-0.4t}} = \frac{1.25}{1+0} = 1.25 \text{ grams.}$$

The graph of the function is shown in Figure 5.27.



The limit of the weight of the culture as $t \to \infty$ is 1.25 grams.
Figure 5.27

XERCISES FOR SECTION 5.5

reciation **In Exercises 1–4, the time in which a machine reciates to one-half its purchase price is given. Find a model yields the fraction of the purchase price as a function of and determine that fraction at time t_0.**

Depreciation time	t_0
3 years	6 years
8 years	16 years
7 years	10 years
5 years	2 years

Exercises 5–8, evaluate the expression without using a ulator.

$\log_2 \frac{1}{8}$ **6.** $\log_{27} 9$

$\log_7 1$ **8.** $\log_a \dfrac{1}{a}$

xercises 9–12, write the exponential equation as a logarith- equation or vice versa.

(a) $2^3 = 8$ **10.** (a) $27^{2/3} = 9$

(b) $3^{-1} = \frac{1}{3}$ (b) $16^{3/4} = 8$

(a) $\log_{10} 0.01 = -2$ **12.** (a) $\log_3 \frac{1}{9} = -2$

(b) $\log_{0.5} 8 = -3$ (b) $49^{1/2} = 7$

xercises 13–18, sketch the graph of the function by hand.

$y = 3^x$ **14.** $y = 3^{x-1}$

$y = \left(\frac{1}{3}\right)^x$ **16.** $y = 2^{x^2}$

$h(x) = 5^{x-2}$ **18.** $y = 3^{-|x|}$

xercises 19–24, solve for *x* or *b*.

(a) $\log_{10} 1000 = x$ **20.** (a) $\log_3 \frac{1}{81} = x$

(b) $\log_{10} 0.1 = x$ (b) $\log_6 36 = x$

(a) $\log_3 x = -1$ **22.** (a) $\log_b 27 = 3$

(b) $\log_2 x = -4$ (b) $\log_b 125 = 3$

(a) $x^2 - x = \log_5 25$

(b) $3x + 5 = \log_2 64$

(a) $\log_3 x + \log_3(x - 2) = 1$

(b) $\log_{10}(x + 3) - \log_{10} x = 1$

xercises 25–34, solve the equation accurate to three decimal es.

$3^{2x} = 75$ **26.** $5^{6x} = 8320$

$2^{3-z} = 625$ **28.** $3(5^{x-1}) = 86$

$\left(1 + \dfrac{0.09}{12}\right)^{12t} = 3$ **30.** $\left(1 + \dfrac{0.10}{365}\right)^{365t} = 2$

$\log_2(x - 1) = 5$ **32.** $\log_{10}(t - 3) = 2.6$

$\log_3 x^2 = 4.5$ **34.** $\log_5 \sqrt{x - 4} = 3.2$

In Exercises 35–38, use a graphing utility to graph the function and approximate its zero(s) accurate to three decimal places.

35. $g(x) = 6(2^{1-x}) - 25$

36. $f(t) = 300(1.0075^{12t}) - 735.41$

37. $h(s) = 32 \log_{10}(s - 2) + 15$

38. $g(x) = 1 - 2 \log_{10}[x(x - 3)]$

In Exercises 39 and 40, illustrate that the functions are inverse functions of each other by sketching their graphs on the same set of coordinate axes.

39. $f(x) = 4^x$
$g(x) = \log_4 x$

40. $f(x) = 3^x$
$g(x) = \log_3 x$

In Exercises 41–56, find the derivative of the function.

41. $f(x) = 4^x$ **42.** $g(x) = 2^{-x}$

43. $y = 5^{x-2}$ **44.** $y = x(6^{-2x})$

45. $g(t) = t^2 2^t$ **46.** $f(t) = \dfrac{3^{2t}}{t}$

47. $h(\theta) = 2^{-\theta} \cos \pi\theta$ **48.** $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$

49. $y = \log_3 x$ **50.** $y = \log_{10} 2x$

51. $f(x) = \log_2 \dfrac{x^2}{x - 1}$ **52.** $h(x) = \log_3 \dfrac{x\sqrt{x - 1}}{2}$

53. $y = \log_5 \sqrt{x^2 - 1}$ **54.** $y = \log_{10} \dfrac{x^2 - 1}{x}$

55. $g(t) = \dfrac{10 \log_4 t}{t}$ **56.** $f(t) = t^{3/2} \log_2 \sqrt{t + 1}$

In Exercises 57–60, use logarithmic differentiation to find *dy/dx*.

57. $y = x^{2/x}$

58. $y = x^{x-1}$

59. $y = (x - 2)^{x+1}$

60. $y = (1 + x)^{1/x}$

In Exercises 61–68, find or evaluate the integral.

61. $\displaystyle\int 3^x \, dx$ **62.** $\displaystyle\int 5^{-x} \, dx$

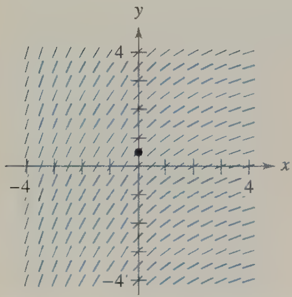
63. $\displaystyle\int_{-1}^{2} 2^x \, dx$ **64.** $\displaystyle\int_{-2}^{0} (3^3 - 5^2) \, dx$

65. $\displaystyle\int x(5^{-x^2}) \, dx$ **66.** $\displaystyle\int (3 - x)7^{(3-x)^2} \, dx$

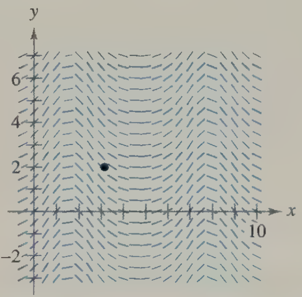
67. $\displaystyle\int \frac{3^{2x}}{1 + 3^{2x}} \, dx$ **68.** $\displaystyle\int 2^{\sin x} \cos x \, dx$

Slope Fields **In Exercises 69 and 70, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the indicated point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

69. $\dfrac{dy}{dx} = 0.4^{x/3}, \quad \left(0, \frac{1}{2}\right)$



70. $\dfrac{dy}{dx} = e^{\sin x} \cos x, \quad (\pi, 2)$



Getting at the Concept

71. List some applications of the exponential functions $f(x) = a^x$ and $g(x) = a^{-x}$.

72. Describe how to use a calculator to find the logarithm of a number if the base is not 10 or e.

73. The table of values below was obtained by evaluating a function. Determine which of the statements may be true and which must be false, and explain why.

(a) y is an exponential function of x.

(b) y is a logarithmic function of x.

(c) x is an exponential function of y.

(d) y is a linear function of x.

x	1	2	8
y	0	1	3

74. Consider the function $f(x) = \log_{10} x$.

(a) What is the domain of f?

(b) Find f^{-1}.

(c) If x is a real number between 1000 and 10,000, determine the interval in which $f(x)$ will be found.

(d) Determine the interval in which x will be found if $f(x)$ is negative.

(e) If $f(x)$ is increased by one unit, x must have been increased by what factor?

(f) Find the ratio of x_1 to x_2 given that $f(x_1) = 3n$ and $f(x_2) = n$.

75. ***Ordering Functions*** Order the functions

$$f(x) = \log_2 x, \quad g(x) = x^x, \quad h(x) = x^2, \quad \text{and} \quad k(x) = 2^x$$

from the one with the greatest rate of growth to the one wit smallest rate of growth for "large" values of x.

76. Given the exponential function $f(x) = a^x$, show that

(a) $f(u + v) = f(u) \cdot f(v)$.

(b) $f(2x) = [f(x)]^2$.

77. ***Inflation*** If the annual rate of inflation averages 5% ove next 10 years, the approximate cost C of goods or serv during any year in that decade is

$$C(t) = P(1.05)^t$$

where t is the time in years and P is the present cost.

(a) If the price of an oil change for your car is presently $24 estimate the price 10 years from now.

(b) Find the rate of change of C with respect to t when t and $t = 8$.

(c) Verify that the rate of change of C is proportional t What is the constant of proportionality?

78. ***Depreciation*** After t years, the value of a car purchase $20,000 is

$$V(t) = 20{,}000\left(\tfrac{3}{4}\right)^t.$$

(a) Use a graphing utility to graph the function and deter the value of the car 2 years after it was purchased.

(b) Find the rate of change of V with respect to t when t and $t = 4$.

(c) Use a graphing utility to graph $V'(t)$ and determine horizontal asymptote of $V'(t)$. Interpret its meaning i context of the problem.

Compound Interest **In Exercises 79–82, complete the tab determine the balance A for P dollars invested at rate r years and compounded n times per year.**

n	1	2	4	12	365	Continuous compounding
A						

79. $P = \$1000$

$r = 3\frac{1}{2}\%$

$t = 10$ years

80. $P = \$2500$

$r = 6\%$

$t = 20$ years

81. $P = \$1000$

$r = 5\%$

$t = 30$ years

82. $P = \$5000$

$r = 7\%$

$t = 25$ years

npound Interest **In Exercises 83–86, complete the table to ermine the amount of money P (present value) that should nvested at rate r to produce a balance of \$100,000 in t years.**

1	10	20	30	40	50

$r = 5\%$ **84.** $r = 6\%$

Compounded continuously Compounded continuously

$r = 5\%$ **86.** $r = 7\%$

Compounded monthly Compounded daily

Compound Interest Assume that you can earn 6% on an investment, compounded daily. Which of the following options would yield the greatest balance after 8 years?

(a) \$20,000 now

(b) \$30,000 after 8 years

(c) \$8000 now and \$20,000 after 4 years

(d) \$9000 now, \$9000 after 4 years, and \$9000 after 8 years

Compound Interest Consider a deposit of \$100 placed in an account for 20 years at $r\%$ compounded continuously. Use a graphing utility to graph the exponential functions giving the growth of the investment over the 20 years for each of the following interest rates. Compare the ending balances for each of the rates.

(a) $r = 3\%$ (b) $r = 5\%$ (c) $r = 6\%$

Timber Yield The yield V (in millions of cubic feet per acre) for a stand of timber at age t is

$$V = 6.7e^{(-48.1)/t}$$

where t is measured in years.

(a) Find the limiting volume of wood per acre as t approaches infinity.

(b) Find the rate at which the yield is changing when $t = 20$ years and $t = 60$ years.

Learning Theory In a group project in learning theory, a mathematical model for the proportion P of correct responses after n trials was found to be

$$P = \frac{0.86}{1 + e^{-0.25n}}.$$

(a) Find the limiting proportion of correct responses as n approaches infinity.

(b) Find the rate at which P is changing after $n = 3$ trials and $n = 10$ trials.

Forest Defoliation To estimate the amount of defoliation caused by the gypsy moth during a year, a forester counts the number of egg masses on $\frac{1}{40}$ of an acre the preceding fall. The percent of defoliation y is approximated by

$$y = \frac{300}{3 + 17e^{-0.0625x}}$$

where x is the number of egg masses in thousands. *(Source: USDA Forest Service)*

(a) Use a graphing utility to graph the function.

(b) Estimate the percent of defoliation if 2000 egg masses are counted.

(c) Estimate the number of egg masses that existed if you observe that approximately $\frac{2}{3}$ of a forest is defoliated.

(d) Use calculus to estimate the value of x for which y is increasing most rapidly.

92. *Population Growth* A lake is stocked with 500 fish, and their population increases according to the logistics curve

$$p(t) = \frac{10,000}{1 + 19e^{-t/5}}$$

where t is measured in months.

(a) Use a graphing utility to graph the function.

(b) What is the limiting size of the fish population?

(c) At what rates is the fish population changing at the end of 1 month and at the end of 10 months?

(d) After how many months is the population increasing most rapidly?

93. *Modeling Data* The breaking strength B (in tons) of a steel cable of diameter d (in inches) is given in the table.

d	0.50	0.75	1.00	1.25	1.50	1.75
B	9.85	21.8	38.3	59.2	84.4	114.0

(a) Use the regression capabilities of a graphing utility to fit an exponential model to the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Find the rate of growth of the model when $d = 0.8$ and $d = 1.5$.

94. *Comparing Models* The amount y (in billions of dollars) given to philanthropy (from individuals, foundations, corporations, and charitable bequests) in the United States for the years 1991 through 1997 is given in the table, with $x = 1$ corresponding to 1991. *(Source: AAFRC Trust for Philanthropy)*

x	1	2	3	4	5	6	7
y	105.0	110.4	116.5	119.2	124.3	133.5	143.5

(a) Use the regression capabilities of a graphing utility to find the following models for the data.

$y_1 = ax + b$ $y_2 = a + b \ln x$

$y_3 = ab^x$ $y_4 = ax^b$

(b) Use a graphing utility to plot the data and graph each of the models. Which model do you think best fits the data?

(c) Interpret the slope of the linear model in the context of the problem.

(d) Find the rate of change of each of the models for the year 1996. Which model is increasing at the greatest rate in 1996?

95. ***Conjecture***

(a) Use a graphing utility to approximate the integrals of the functions

$$f(t) = 4\left(\frac{3}{8}\right)^{2t/3}, \quad g(t) = 4\left(\frac{\sqrt[3]{9}}{4}\right)^{t}, \text{ and } h(t) = 4e^{-0.653886t}$$

on the interval $[0, 4]$.

(b) Use a graphing utility to graph the three functions.

(c) Use the results in parts (a) and (b) to make a conjecture about the three functions. Could you make the conjecture using only part (a)? Explain. Prove your conjecture analytically.

96. ***Area*** Find the area of the region bounded by the graphs of $y = 3^x$, $y = 0$, $x = 0$, and $x = 3$.

97. ***Continuous Cash Flow*** The present value P of a continuous cash flow of \$2000 per year earning 6% interest compounded continuously over 10 years is

$$P = \int_0^{10} 2000e^{-0.06t}\,dt.$$

Find P.

98. Complete the table to demonstrate that e can also be defined as $\lim_{x \to 0^+} (1 + x)^{1/x}$.

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1 + x)^{1/x}$					

In Exercises 99 and 100, find an exponential function that the experimental data collected over time t.

99.

t	0	1	2	3	4
y	1200.00	720.00	432.00	259.20	155.52

100.

t	0	1	2	3	4
y	600.00	630.00	661.50	694.58	729.30

True or False? **In Exercises 101–106, determine whether statement is true or false. If it is false, explain why or give example that shows it is false.**

101. $e = 271{,}801/99{,}990$.

102. If $f(x) = \ln x$, then $f(e^{n+1}) - f(e^n) = 1$ for any value of

103. The functions $f(x) = 2 + e^x$ and $g(x) = \ln(x - 2)$ are inve functions of each other.

104. The exponential function $y = Ce^x$ is a solution of the dif ential equation $d^n y/dx^n = y$, $n = 1, 2, 3, \ldots$.

105. The graphs of $f(x) = e^x$ and $g(x) = e^{-x}$ meet at right angl

106. If $f(x) = g(x)e^x$, then the only zeros of f are the zeros of

107. Solve the logistics differential equation

$$\frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right), \quad y(0) = 1$$

and obtain the logistics growth function of Example 7.

$$\left[\textit{Hint: } \frac{1}{y\left(\frac{5}{4} - y\right)} = \frac{4}{5}\left(\frac{1}{y} + \frac{1}{\frac{5}{4} - y}\right)\right]$$

108. Find an equation of the tangent line to $y = x^{\sin x}$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

SECTION PROJECT USING GRAPHING UTILITIES TO ESTIMATE SLOP

Let $f(x) = \begin{cases} |x|^x, & x \neq 0 \\ 1, & x = 0. \end{cases}$

(a) Use a graphing utility to graph f in the viewing window $-3 \le x \le 3$, $-2 \le y \le 2$. What is the domain of f?

(b) Use the *zoom* and *trace* features of a graphing utility to estimate

$$\lim_{x \to 0} f(x).$$

(c) Write a short paragraph explaining why the function f is continuous for all real numbers.

(d) Visually estimate the slope of f at the point $(0, 1)$.

(e) Explain why the derivative of a function can be approximated by the formula

$$\frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

for small values of Δx. Use this formula to approximate t slope of f at the point $(0, 1)$.

$$f'(0) \approx \frac{f(0 + \Delta x) - f(0 - \Delta x)}{2\Delta x} = \frac{f(\Delta x) - f(-\Delta x)}{2\Delta x}$$

What do you think the slope of the graph of f is at $(0, 1)$?

(f) Find a formula for the derivative of f and determine $f'($ Write a short paragraph explaining how a graphing utility mig lead you to approximate the slope of a graph incorrectly.

(g) Use your formula for the derivative of f to find the relati extrema of f. Verify your answer with a graphing utility.

FOR FURTHER INFORMATION For more information using graphing utilities to estimate slope, see the artic "Computer-Aided Delusions" by Richard L. Hall in *The Colle Mathematics Journal.* To view this article, go to the webs *www.matharticles.com.*

Section 5.6 Differential Equations: Growth and Decay

- Use separation of variables to solve a simple differential equation.
- Use exponential functions to model growth and decay in applied problems.

Differential Equations

Up to now in the text, you have learned to solve only two types of differential equations—those of the forms

$$y' = f(x) \quad \text{and} \quad y'' = f(x).$$

In this section, you will learn how to solve a more general type of differential equation. The strategy is to rewrite the equation so that each variable occurs on only one side of the equation. This strategy is called *separation of variables*. (You will study this strategy in detail in Section 5.7.)

Example 1 Solving a Differential Equation

Solve the differential equation $y' = 2x/y$.

Solution

$$\begin{aligned} y' &= \frac{2x}{y} && \text{Write original equation.} \\ yy' &= 2x && \text{Multiply both sides by } y. \\ \int yy'\,dx &= \int 2x\,dx && \text{Integrate with respect to } x. \\ \int y\,dy &= \int 2x\,dx && dy = y'\,dx \\ \frac{1}{2}y^2 &= x^2 + C_1 && \text{Apply Power Rule.} \\ y^2 - 2x^2 &= C && \text{Rewrite, letting } C = 2C_1. \end{aligned}$$

So, the general solution is given by

$$y^2 - 2x^2 = C.$$

You can use implicit differentiation to check this result.

TE When you integrate both sides
e equation in Example 1, you don't
l to add a constant of integration to
sides of the equation. If you did,
would obtain the same result as in
mple 1.

$$\begin{aligned} \int y\,dy &= \int 2x\,dx \\ \tfrac{1}{2}y^2 + C_2 &= x^2 + C_3 \\ \tfrac{1}{2}y^2 &= x^2 + (C_3 - C_2) \\ \tfrac{1}{2}y^2 &= x^2 + C_1 \end{aligned}$$

In practice, most people prefer to use Leibniz notation and differentials when applying separation of variables. Using this notation, the solution of Example 1 is as follows.

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x}{y} \\ y\,dy &= 2x\,dx \\ \int y\,dy &= \int 2x\,dx \\ \frac{1}{2}y^2 &= x^2 + C_1 \\ y^2 - 2x^2 &= C \end{aligned}$$

EXPLORATION

Example 1, the general solution of
differential equation is

$$y^2 - 2x^2 = C.$$

e a graphing utility to sketch
eral particular solutions—those
en by $C = \pm 2$, $C = \pm 1$, and
$= 0$. Describe the solutions
phically. Is the following
tement true of each solution?

The slope of the graph at the point (x, y) is equal to twice the ratio of x and y.

plain your reasoning. Are all
ves for which this statement is
e represented by the general
ution?

Growth and Decay Models

In many applications, the rate of change of a variable y is proportional to the valu
y. If y is a function of time t, the proportion can be written as follows.

$$\overset{\text{Rate of change of } y}{\frac{dy}{dt}} \overset{\text{is}}{=} \overset{\text{proportional to } y.}{ky}$$

The general solution of this differential equation is given in the following theoren

THEOREM 5.16 Exponential Growth and Decay Model

If y is a differentiable function of t such that $y > 0$ and $y' = ky$, for some constant k, then

$$y = Ce^{kt}.$$

C is the **initial value** of y, and k is the **proportionality constant. Exponential growth** occurs when $k > 0$, and **exponential decay** occurs when $k < 0$.

NOTE Differentiate the function $y = Ce^{kt}$ with respect to t, and verify that $y' = ky$.

Proof

$$y' = ky \qquad \text{Write original equation.}$$

$$\frac{y'}{y} = k \qquad \text{Separate variables.}$$

$$\int \frac{y'}{y}\,dt = \int k\,dt \qquad \text{Integrate with respect to } t.$$

$$\int \frac{1}{y}\,dy = \int k\,dt \qquad dy = y'\,dt$$

$$\ln y = kt + C_1 \qquad \text{Find antiderivative of each side.}$$

$$y = e^{kt}e^{C_1} \qquad \text{Solve for } y.$$

$$y = Ce^{kt} \qquad \text{Let } C = e^{C_1}.$$

So, all solutions of $y' = ky$ are of the form $y = Ce^{kt}$.

Example 2 Using an Exponential Growth Model

The rate of change of y is proportional to y. When $t = 0$, $y = 2$. When $t = 2$, y =
What is the value of y when $t = 3$?

Solution Because $y' = ky$, you know that y and t are related by the equa
$y = Ce^{kt}$. You can find the values of the constants C and k by applying the in
conditions.

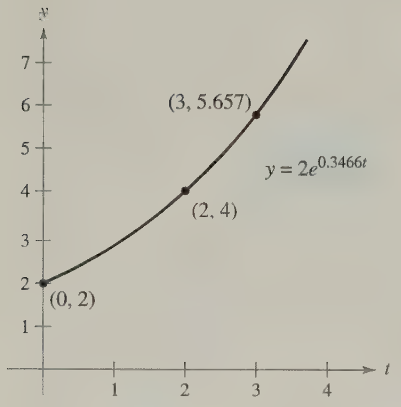
$$2 = Ce^0 \quad \Rightarrow \quad C = 2 \qquad \text{When } t = 0,\ y = 2.$$

$$4 = 2e^{2k} \quad \Rightarrow \quad k = \frac{1}{2}\ln 2 \approx 0.3466 \qquad \text{When } t = 2,\ y = 4.$$

Therefore, the model is

$$y \approx 2e^{0.3466t}.$$

When $t = 3$, the value of y is $2e^{0.3466(3)} \approx 5.657$ (see Figure 5.28).



If the rate of change of y is proportional to y, then y follows an exponential model.
Figure 5.28

Radioactive decay is measured in terms of *half-life*—the number of years required for half of the atoms in a sample of radioactive material to decay. The half-lives of some common radioactive isotopes are as follows.

Uranium (^{238}U)	4,510,000,000 years
Plutonium (^{239}Pu)	24,360 years
Carbon (^{14}C)	5730 years
Radium (^{226}Ra)	1620 years
Einsteinium (^{254}Es)	270 days
Nobelium (^{257}No)	23 seconds

e worst nuclear accident in history
ppened in 1986 at the Chernobyl
clear plant near Kiev in the Ukraine.
explosion destroyed one of the plant's
r reactors, releasing large amounts of
ioactive isotopes into the atmosphere.

Example 3 Radioactive Decay

Suppose that 10 grams of the plutonium isotope Pu-239 was released in the Chernobyl nuclear accident. How long will it take for the 10 grams to decay to 1 gram?

Solution Let y represent the mass (in grams) of the plutonium. Because the rate of decay is proportional to y, you know that

$$y = Ce^{kt}$$

where t is the time in years. To find the values of the constants C and k, apply the initial conditions. Using the fact that $y = 10$ when $t = 0$, you can write

$$10 = Ce^{k(0)} = Ce^0$$

which implies that $C = 10$. Next, using the fact that $y = 5$ when $t = 24{,}360$, you can write

$$\begin{aligned} 5 &= 10e^{k(24{,}360)} \\ \frac{1}{2} &= e^{24{,}360k} \\ \frac{1}{24{,}360}\ln\frac{1}{2} &= k \\ -2.8454 \times 10^{-5} &\approx k. \end{aligned}$$

Therefore, the model is

$$y = 10e^{-0.000028454t}. \qquad \text{Half-life model}$$

To find the time it would take for 10 grams to decay to 1 gram, you can solve for t in the equation

$$1 = 10e^{-0.000028454t}.$$

The solution is approximately 80,923 years.

NOTE The exponential decay model in Example 3 could also be written as $y = 10\left(\frac{1}{2}\right)^{t/24{,}360}$. This model is much easier to derive, but for some applications, it is not as convenient to use.

From Example 3, notice that in an exponential growth or decay problem, it is easy to solve for C when you are given the value of y at $t = 0$. The next example demonstrates a procedure for solving for C and k when you do not know the value of y at $t = 0$.

Example 4 Population Growth

Suppose an experimental population of fruit flies increases according to the law exponential growth. There were 100 flies after the second day of the experiment a 300 flies after the fourth day. Approximately how many flies were in the origir population?

Solution Let $y = Ce^{kt}$ be the number of flies at time t, where t is measured in day Because $y = 100$ when $t = 2$ and $y = 300$ when $t = 4$, you can write

$$100 = Ce^{2k} \quad \text{and} \quad 300 = Ce^{4k}.$$

From the first equation, you know that $C = 100e^{-2k}$. Substituting this value into t second equation produces the following.

$$\begin{aligned} 300 &= 100e^{-2k}e^{4k} \\ 300 &= 100e^{2k} \\ \ln 3 &= 2k \\ \frac{1}{2}\ln 3 &= k \\ 0.5493 &\approx k \end{aligned}$$

Therefore, the exponential growth model is

$$y = Ce^{0.5493t}.$$

To solve for C, reapply the condition $y = 100$ when $t = 2$ and obtain

$$\begin{aligned} 100 &= Ce^{0.5493(2)} \\ C &= 100e^{-1.0986} \approx 33. \end{aligned}$$

So, the original population (when $t = 0$) consisted of approximately $y = C =$ flies, as indicated in Figure 5.29.

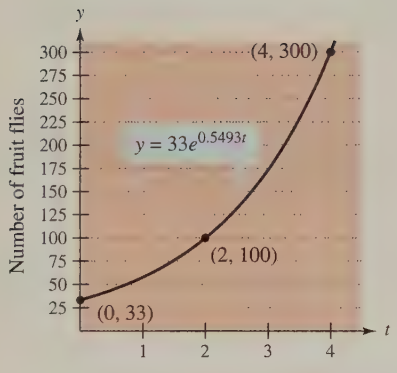


Figure 5.29

Example 5 Declining Sales

Four months after it stops advertising, a manufacturing company notices that its sal have dropped from 100,000 units per month to 80,000 units per month. If the sal follow an exponential pattern of decline, what will they be after another 2 months?

Solution Use the exponential decay model $y = Ce^{kt}$, where t is measured in montl as shown in Figure 5.30. From the initial condition ($t = 0$), you know that C 100,000. Moreover, because $y = 80{,}000$ when $t = 4$, you have

$$\begin{aligned} 80{,}000 &= 100{,}000e^{4k} \\ 0.8 &= e^{4k} \\ \ln(0.8) &= 4k \\ -0.0558 &\approx k. \end{aligned}$$

So, after 2 more months ($t = 6$), you can expect the monthly sales rate to be

$$\begin{aligned} y &\approx 100{,}000e^{-0.0558(6)} \\ &\approx 71{,}500 \text{ units.} \end{aligned}$$

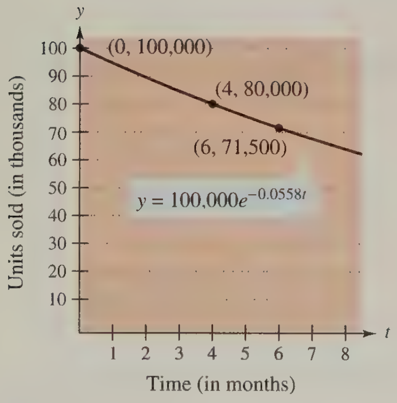


Figure 5.30

In Examples 2 through 5, you did not actually have to solve the differential equation

$$y' = ky.$$

(This was done once in the proof of Theorem 5.16.) The next example demonstrates a problem whose solution involves the separation of variables technique. The example concerns **Newton's Law of Cooling,** which states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium.

ECHNOLOGY If you didn't read he text at the beginning of the chapter n page 312, turn back and read it ow. There you can see how data ollected using a *Texas Instruments Calculator-Based Laboratory (CBL) ystem* can be used to experimentally erive a model for Newton's Law of ooling.

Example 6 Newton's Law of Cooling

Let y represent the temperature (in °F) of an object in a room whose temperature is kept at a constant 60°. If the object cools from 100° to 90° in 10 minutes, how much longer will it take for its temperature to decrease to 80°?

Solution From Newton's Law of Cooling, you know that the rate of change in y is proportional to the difference between y and 60. This can be written as

$$y' = k(y - 60), \qquad 80 \le y \le 100.$$

To solve this differential equation, you can use separation of variables, as follows.

$$\frac{dy}{dt} = k(y - 60) \qquad \text{Differential equation}$$

$$\left(\frac{1}{y - 60}\right) dy = k\,dt \qquad \text{Separate variables.}$$

$$\int \frac{1}{y - 60}\,dy = \int k\,dt \qquad \text{Integrate both sides.}$$

$$\ln|y - 60| = kt + C_1 \qquad \text{Find antiderivative of each side.}$$

Because $y > 60$, $|y - 60| = y - 60$, and you can omit the absolute value signs. Using exponential notation, you have

$$y - 60 = e^{kt + C_1} \qquad y = 60 + Ce^{kt}. \qquad C = e^{C_1}$$

Using $y = 100$ when $t = 0$, you obtain $100 = 60 + Ce^{k(0)} = 60 + C$, which implies that $C = 40$. Because $y = 90$ when $t = 10$, it follows that

$$90 = 60 + 40e^{k(10)}$$

$$30 = 40e^{10k}$$

$$k = \tfrac{1}{10}\ln\tfrac{3}{4} \approx -0.02877.$$

So, the model is

$$y = 60 + 40e^{-0.02877t} \qquad \text{Cooling model}$$

and finally, when $y = 80$, you obtain

$$80 = 60 + 40e^{-0.02877t}$$

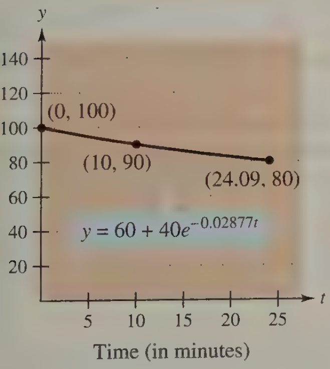
$$20 = 40e^{-0.02877t}$$

$$\tfrac{1}{2} = e^{-0.02877t}$$

$$\ln\tfrac{1}{2} = -0.02877t$$

$$t \approx 24.09 \text{ minutes.}$$

Therefore, it will require about 14.09 *more* minutes for the object to cool to a temperature of 80° (see Figure 5.31).



ure 5.31

EXERCISES FOR SECTION 5.6

In Exercises 1–10, solve the differential equation.

1. $\dfrac{dy}{dx} = x + 2$

2. $\dfrac{dy}{dx} = 4 - x$

3. $\dfrac{dy}{dx} = y + 2$

4. $\dfrac{dy}{dx} = 4 - y$

5. $y' = \dfrac{5x}{y}$

6. $y' = \dfrac{\sqrt{x}}{3y}$

7. $y' = \sqrt{x}\,y$

8. $y' = x(1 + y)$

9. $(1 + x^2)y' - 2xy = 0$

10. $xy + y' = 100x$

In Exercises 11–14, write and solve the differential equation that models the verbal statement.

11. The rate of change of Q with respect to t is inversely proportional to the square of t.

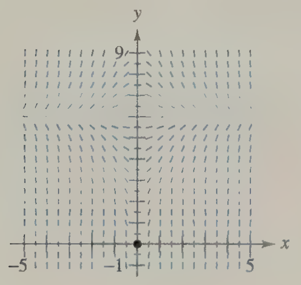
12. The rate of change of P with respect to t is proportional to $10 - t$.

13. The rate of change of N with respect to s is proportional to $250 - s$.

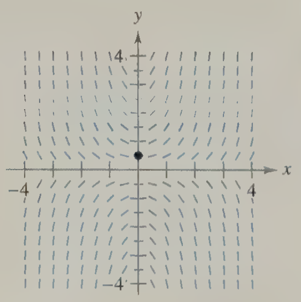
14. The rate of change of y with respect to x varies jointly as x and $L - y$.

Slope Fields **In Exercises 15 and 16, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the indicated point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

15. $\dfrac{dy}{dx} = x(6 - y)$, $(0, 0)$



16. $\dfrac{dy}{dx} = xy$, $\left(0, \frac{1}{2}\right)$



In Exercises 17–20, find the function $y = f(t)$ passing through the point $(0, 10)$ with the given first derivative. Use a graphing utility to graph the solution.

17. $\dfrac{dy}{dt} = \dfrac{1}{2}t$

18. $\dfrac{dy}{dt} = -\dfrac{3}{4}\sqrt{t}$

19. $\dfrac{dy}{dt} = -\dfrac{1}{2}y$

20. $\dfrac{dy}{dt} = \dfrac{3}{4}y$

In Exercises 21–24, write and solve the differential equati that models the verbal statement. Evaluate the solution at t specified value of the independent variable.

21. The rate of change of y is proportional to y. When $x = 0$, $y =$ and when $x = 3$, $y = 10$. What is the value of y when $x =$

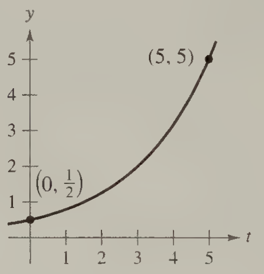
22. The rate of change of N is proportional to N. When $t =$ $N = 250$ and when $t = 1$, $N = 400$. What is the value of when $t = 4$?

23. The rate of change of V is proportional to V. When $t =$ $V = 20{,}000$ and when $t = 4$, $V = 12{,}500$. What is the value V when $t = 6$?

24. The rate of change of P is proportional to P. When $t =$ $P = 5000$ and when $t = 1$, $P = 4750$. What is the value o when $t = 5$?

In Exercises 25–28, find the exponential function $y = Ce^{kt}$ th passes through the two given points.

25.



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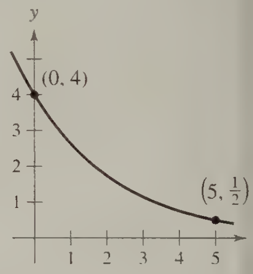

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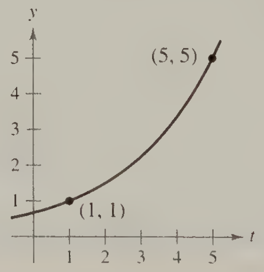

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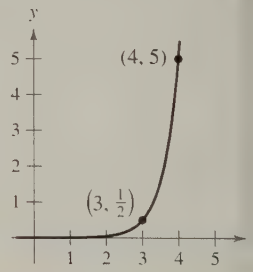


Getting at the Concept

29. In your own words, describe what is meant by a differential equation. Give an example.

30. Give the differential equation that models exponential growth and decay.

In Exercises 31 and 32, determine the quadrants in which the solution of the differential equation is an increasing function. Explain. (Do not solve the differential equation.)

31. $\dfrac{dy}{dx} = \dfrac{1}{2}xy$

32. $\dfrac{dy}{dx} = \dfrac{1}{2}x^2y$

lioactive Decay **In Exercises 33–40, complete the table for radioactive isotope.**

Isotope	*Half-Life (in years)*	*Initial Quantity*	*Amount After 1,000 Years*	*Amount After 10,000 Years*
^{226}Ra	1620	10g		
^{226}Ra	1620		1.5g	
^{226}Ra	1620			0.5g
^{14}C	5730			2g
^{14}C	5730	5g		
^{14}C	5730		3.2g	
^{239}Pu	24,360		2.1g	
^{239}Pu	24,360			0.4g

Radioactive Decay Radioactive radium has a half-life of approximately 1620 years. What percent of a given amount remains after 100 years?

Carbon Dating Carbon-14 dating assumes that the carbon dioxide on earth today has the same radioactive content as it did centuries ago. If this is true, the amount of ^{14}C absorbed by a tree that grew several centuries ago should be the same as the amount of ^{14}C absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much of the radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal? (The half-life of ^{14}C is 5730 years.)

npound Interest **In Exercises 43–48, complete the table for vings account in which interest is compounded continuously.**

Initial Investment	*Annual Rate*	*Time to Double*	*Amount After 10 Years*
\$1000	6%		
\$20,000	$5\frac{1}{2}\%$		
\$750		$7\frac{3}{4}$ yr	
\$10,000		5 yr	
\$500			\$1292.85
\$2000			\$5436.56

npound Interest **In Exercises 49–52, find the principal P t must be invested at rate r, compounded monthly, so that 0,000 will be available for retirement in t years.**

$r = 7\frac{1}{2}\%, \quad t = 20$

50. $r = 6\%, \quad t = 40$

$r = 8\%, \quad t = 35$

52. $r = 9\%, \quad t = 25$

npound Interest **In Exercises 53–56, find the time necessary \$1000 to double if it is invested at a rate of r compounded (a) ually, (b) monthly, (c) daily, and (d) continuously.**

$r = 7\%$

54. $r = 6\%$

$r = 8.5\%$

56. $r = 5.5\%$

Population **In Exercises 57–60, the population (in millions) of a country in 1999 and the continuous annual rate of change k of the population for the years 1990 through 2000 are given. Find the exponential growth model $P = Ce^{kt}$ for the population by letting $t = 0$ correspond to 2000. Use the model to predict the population of the city in 2010.** ***(Source: U.S. Census Bureau, "International Data Base")***

	Country	*1999 Population*	*k*
57.	Bulgaria	8.2	−0.009
58.	Cambodia	11.6	0.031
59.	Jordan	4.6	0.036
60.	Lithuania	3.6	−0.004

61. ***Writing*** Use the results of Exercises 57–60 and the exponential model $P = Ce^{kt}$ to discuss the relationship between the sign of k and the change in population for a given country.

62. ***Modeling Data*** One hundred bacteria are started in a culture and the number N of bacteria is counted each hour for 5 hours. The results are shown in the table, where t is the time in hours.

t	0	1	2	3	4	5
N	100	126	151	198	243	297

(a) Use the regression capabilities of a graphing utility to find an exponential model for the data.

(b) Use the model to estimate the time required for the population to quadruple in size.

63. ***Atmospheric Pressure*** Atmospheric pressure P (measured in millimeters of mercury) decreases exponentially with increasing altitude x (measured in meters). The pressure is 760 millimeters of mercury at sea level ($x = 0$) and 672.71 millimeters of mercury at an altitude of 1000 meters. Find the pressure at an altitude of 3000 meters.

64. ***Revenue*** Because of a slump in the economy, a company finds that its annual revenue has dropped from \$742,000 in 1998 to \$632,000 in 2000. If the revenue is following an exponential pattern of decline, what is the expected revenue for 2002? (Let $t = 0$ represent 1998.)

65. ***Learning Curve*** The management at a certain factory has found that a worker can produce at most 30 units in a day. The learning curve for the number of units N produced per day after a new employee has worked t days is

$N = 30(1 - e^{kt})$.

After 20 days on the job, a particular worker produces 19 units.

(a) Find the learning curve for this worker.

(b) How many days should pass before this worker is producing 25 units per day?

66. ***Learning Curve*** If in Exercise 65 management requires a new employee to produce at least 20 units per day after 30 days on the job, find (a) the learning curve that describes this minimum requirement and (b) the number of days before a minimal achiever is producing 25 units per day.

67. ***Sales*** The sales S (in thousands of units) of a new product after it has been on the market for t years is

$S = Ce^{k/t}$.

(a) Find S as a function of t if 5000 units have been sold after 1 year and the saturation point for the market is 30,000 units (that is, $\lim_{t\to\infty} S = 30$).

(b) How many units will have been sold after 5 years?

(c) Use a graphing utility to graph this sales function.

68. ***Sales*** The sales S (in thousands of units) of a new product after it has been on the market for t years is

$S = 25(1 - e^{kt})$.

(a) Find S as a function of t if 4000 units have been sold after 1 year.

(b) How many units will saturate this market?

(c) How many units will have been sold after 5 years?

(d) Use a graphing utility to graph this sales function.

69. ***Forestry*** The value of a tract of timber is

$V(t) = 100{,}000e^{0.8\sqrt{t}}$

where t is the time in years, with $t = 0$ corresponding to 1998. If money earns interest continuously at 10%, the present value of the timber at any time t is

$A(t) = V(t)e^{-0.10t}$.

Find the year in which the timber should be harvested to maximize the present value function.

70. ***Modeling Data*** The table shows the net receipts and the amounts required to service the national debt of the United States from 1990 through 1999. The monetary amounts are given in billions of dollars. *(Source: U.S. Office of Management and Budget)*

Year	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999
Receipts	1032.0	1055.0	1091.3	1154.4	1258.6	1351.8	1453.1	1579.3	1721.8	1806.3
Interest	264.7	285.5	292.3	292.5	296.3	332.4	344.0	355.8	363.8	353.4

(a) Use the regression capabilities of a graphing utility to find an exponential model R for the receipts and a quartic model I for the amount required to service the debt. Let t represent the time in years, with $t = 0$ corresponding to 1990.

(b) Use a graphing utility to plot the points corresponding to the receipts, and graph the corresponding model. Based on the model, what is the continuous rate of growth of the receipts?

(c) Use a graphing utility to plot the points corresponding to
amount required to service the debt, and graph the quar
model.

(d) Find a function $P(t)$ that approximates the percent of
receipts that is required to service the national debt. Us
graphing utility to graph this function.

71. ***Sound Intensity*** The level of sound β, in decibels, with
intensity of I is

$\beta(I) = 10\log_{10}\frac{I}{I_0}$

where I_0 is an intensity of 10^{-16} watts per square centimet
corresponding roughly to the faintest sound that can be hea
Determine $\beta(I)$ for the following.

(a) $I = 10^{-14}$ watts per square centimeter (whisper)

(b) $I = 10^{-9}$ watts per square centimeter (busy street corne

(c) $I = 10^{-6.5}$ watts per square centimeter (air hammer)

(d) $I = 10^{-4}$ watts per square centimeter (threshold of pain

72. ***Noise Level*** With the installation of noise suppression ma
rials, the noise level in an auditorium was reduced from 93
80 decibels. Use the function in Exercise 71 to find the perc
decrease in the intensity level of the noise as a result of
installation of these materials.

73. ***Earthquake Intensity*** On the Richter scale, the magnitude
of an earthquake of intensity I is

$R = \frac{\ln I - \ln I_0}{\ln 10}$

where I_0 is the minimum intensity used for comparis
Assume that $I_0 = 1$.

(a) Find the intensity of the 1906 San Francisco earthqua
($R = 8.3$).

(b) Find the factor by which the intensity is increased if
Richter scale measurement is doubled.

(c) Find dR/dI.

74. ***Newton's Law of Cooling*** When an object is removed fro
furnace and placed in an environment with a constant tempe
ture of 80°F, its core temperature is 1500°F. One hour after i
removed, the core temperature is 1120°F. Find the core temp
ature 5 hours after the object is removed from the furnace.

True or False? **In Exercises 75–78, determine whether**
statement is true or false. If it is false, explain why or give
example that shows it is false.

75. In exponential growth the rate of growth is constant.

76. In linear growth the rate of growth is constant.

77. If prices are rising at a rate of 0.5% per month, then they
rising at a rate of 6% per year.

78. The differential equation modeling exponential growth
$dy/dx = ky$ where k is a constant.

Section 5.7 Differential Equations: Separation of Variables

- Use initial conditions to find particular solutions of differential equations.
- Recognize and solve differential equations that can be solved by separation of variables.
- Recognize and solve homogeneous differential equations.
- Use a differential equation to model and solve an applied problem.

General and Particular Solutions

Several times in the text, we have identified physical phenomena that can be described by differential equations. For example, in Section 5.6, you saw that problems involving radioactive decay, population growth, and Newton's Law of Cooling can be formulated in terms of differential equations.

A function $y = f(x)$ is called a **solution** of a differential equation if the equation is satisfied when y and its derivatives are replaced by $f(x)$ and its derivatives. For example, differentiation and substitution would show that $y = e^{-2x}$ is a solution of the differential equation $y' + 2y = 0$. It can be shown that every solution of this differential equation is of the form

$$y = Ce^{-2x} \qquad \text{General solution of } y' + 2y = 0$$

TE First order linear differential ations are discussed in Appendix A.

where C is any real number. This solution is called the **general solution.** Some differential equations have **singular solutions** that cannot be written as special cases of the general solution. However, we will not consider such solutions in this text. The **order** of a differential equation is determined by the highest-order derivative in the equation. For instance, $y' = 4y$ is a first-order differential equation.

In Section 4.1, Example 8, you saw that the second-order differential equation $s''(t) = -32$ has the general solution

$$s(t) = -16t^2 + C_1t + C_2 \qquad \text{General solution of } s''(t) = -32$$

which contains two arbitrary constants. It can be shown that a differential equation of order n has a general solution with n arbitrary constants.

Example 1 Verifying Solutions

Determine whether the given function is a solution of the differential equation $y'' - y = 0$.

a. $y = \sin x$ **b.** $y = 4e^{-x}$ **c.** $y = Ce^x$

Solution

a. Because $y = \sin x$, $y' = \cos x$, and $y'' = -\sin x$, it follows that

$$y'' - y = -\sin x - \sin x = -2\sin x \neq 0.$$

Hence, $y = \sin x$ is *not* a solution.

b. Because $y = 4e^{-x}$, $y' = -4e^{-x}$, and $y'' = 4e^{-x}$, it follows that

$$y'' - y = 4e^{-x} - 4e^{-x} = 0.$$

Hence, $y = 4e^{-x}$ *is* a solution.

c. Because $y = Ce^x$, $y' = Ce^x$, and $y'' = Ce^x$, it follows that

$$y'' - y = Ce^x - Ce^x = 0.$$

Hence, $y = Ce^x$ *is* a solution for any value of C.

Geometrically, the general solution of a first-order differential equation represe a family of curves known as **solution curves,** one for each value assigned to t arbitrary constant. For instance, you can verify that every function of the form

$$y = \frac{C}{x}$$ General solution of $xy' + y = 0$

is a solution of the differential equation $xy' + y = 0$. Figure 5.32 shows some of t solution curves corresponding to different values of C.

As discussed in Section 4.1, **particular solutions** of a differential equation a obtained from **initial conditions** that give the value of the dependent variable or o of its derivatives for a particular value of the independent variable. The term "init condition" stems from the fact that, often in problems involving time, the value the dependent variable or one of its derivatives is known at the *initial* time $t =$ For instance, the second-order differential equation $s''(t) = -32$ having the gene solution

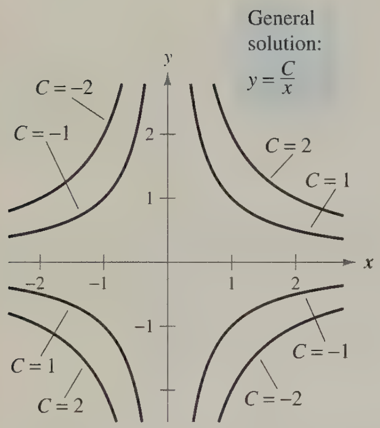
$$s(t) = -16t^2 + C_1t + C_2$$ General solution of $s''(t) = -32$

might have the following initial conditions.

$$s(0) = 80, \qquad s'(0) = 64$$ Initial conditions

In this case, the initial conditions yield the particular solution

$$s(t) = -16t^2 + 64t + 80.$$ Particular solution



Solution curves for $xy' + y = 0$
Figure 5.32

Example 2 Finding a Particular Solution

For the differential equation

$$xy' - 3y = 0$$

verify that $y = Cx^3$ is a solution, and find the particular solution determined by t initial condition $y = 2$ when $x = -3$.

Solution You know that $y = Cx^3$ is a solution because $y' = 3Cx^2$ and

$$\begin{aligned} xy' - 3y &= x(3Cx^2) - 3(Cx^3) \\ &= 0. \end{aligned}$$

Furthermore, the initial condition $y = 2$ when $x = -3$ yields

$$y = Cx^3$$ General solution

$$2 = C(-3)^3$$ Substitute initial condition.

$$-\frac{2}{27} = C$$ Solve for C.

and you can conclude that the particular solution is

$$y = -\frac{2x^3}{27}.$$ Particular solution

Try checking this solution by substituting for y and y' in the original different equation.

NOTE To determine a particular solution, the number of initial conditions must match number of constants in the general solution.

Separation of Variables

Consider a differential equation that can be written in the form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

where M is a continuous function of x alone and N is a continuous function of y alone. As you saw in the preceding section, for this type of equation, all x terms can be collected with dx and all y terms with dy, and a solution can be obtained by integration. Such equations are said to be **separable,** and the solution procedure is called *separation of variables*. Here are some examples of differential equations that are separable.

R FURTHER INFORMATION For an mple from engineering of a differen- equation that is separable, see the cle "Designing a Rose Cutter" by . Hartzler in *The College Mathematics rnal.* To view this article, go to the bsite *www.matharticles.com.*

Original Differential Equation	*Rewritten with Variables Separated*
$x^2 + 3y\dfrac{dy}{dx} = 0$	$3y\,dy = -x^2\,dx$
$\sin xy' = \cos x$	$dy = \cot x\,dx$
$\dfrac{xy'}{e^y + 1} = 2$	$\dfrac{1}{e^y + 1}\,dy = \dfrac{2}{x}\,dx$

Example 3 Separation of Variables

Find the general solution of $(x^2 + 4)\dfrac{dy}{dx} = xy$.

Solution To begin, note that $y = 0$ is a solution. To find other solutions, assume that $y \neq 0$ and separate variables as follows.

$$(x^2 + 4)\,dy = xy\,dx$$

Differential form

$$\frac{dy}{y} = \frac{x}{x^2 + 4}\,dx$$

Separate variables.

Now, integrate to obtain

$$\int \frac{dy}{y} = \int \frac{x}{x^2 + 4}\,dx$$

Integrate.

$$\begin{aligned}
\ln|y| &= \frac{1}{2}\ln(x^2 + 4) + C_1 \\
\ln|y| &= \ln\sqrt{x^2 + 4} + C_1 \\
|y| &= e^{C_1}\sqrt{x^2 + 4} \\
y &= \pm e^{C_1}\sqrt{x^2 + 4}.
\end{aligned}$$

Because $y = 0$ is also a solution, you can write the general solution as

$$y = C\sqrt{x^2 + 4}.$$

General solution

NOTE We encourage you to check your solutions throughout this chapter. In Example 3 you can check the solution $y = C\sqrt{x^2 + 4}$ by differentiating and substituting into the original equation.

$$(x^2 + 4)\frac{dy}{dx} = xy$$

Original equation

$$(x^2 + 4)\frac{Cx}{\sqrt{x^2 + 4}} \overset{?}{=} x\left(C\sqrt{x^2 + 4}\right)$$

Substitute.

$$Cx\sqrt{x^2 + 4} = Cx\sqrt{x^2 + 4}$$

Solution checks.

In some cases it is not feasible to write the general solution in the explicit fo $y = f(x)$. The next example illustrates such a solution. Implicit differentiation can used to verify this solution.

Example 4 Finding a Particular Solution

Given the initial condition $y(0) = 1$, find the particular solution of the equation

$$xy\,dx + e^{-x^2}(y^2 - 1)\,dy = 0.$$

Solution Note that $y = 0$ is a solution of the differential equation—but this soluti does not satisfy the initial condition. Hence, you can assume that $y \neq 0$. To separ variables, you must rid the first term of y and the second term of e^{-x^2}. So, you shou multiply by e^{x^2}/y and obtain the following.

$$\begin{aligned} xy\,dx + e^{-x^2}(y^2 - 1)\,dy &= 0 \\ e^{-x^2}(y^2 - 1)\,dy &= -xy\,dx \\ \int\left(y - \frac{1}{y}\right)dy &= \int -xe^{x^2}\,dx \\ \frac{y^2}{2} - \ln y &= -\frac{1}{2}e^{x^2} + C \end{aligned}$$

From the initial condition $y(0) = 1$, you have $\frac{1}{2} - 0 = -\frac{1}{2} + C$, which implies t $C = 1$. So, the particular solution has the implicit form

$$\begin{aligned} \frac{y^2}{2} - \ln y &= -\frac{1}{2}e^{x^2} + 1 \\ y^2 - \ln y^2 + e^{x^2} &= 2. \end{aligned}$$

You can check this by differentiating and rewriting to get the original equation.

Example 5 Finding a Particular Solution Curve

Find the equation of the curve that passes through the point $(1, 3)$ and has a slope y/x^2 at the point (x, y), as shown in Figure 5.33.

Solution Because the slope of the curve is given by y/x^2, you have

$$\frac{dy}{dx} = \frac{y}{x^2}$$

with the initial condition $y(1) = 3$. Separating variables and integrating produces

$$\begin{aligned} \int\frac{dy}{y} &= \int\frac{dx}{x^2}, \qquad y \neq 0 \\ \ln|y| &= -\frac{1}{x} + C_1 \\ y &= e^{-(1/x)+C_1} = Ce^{-1/x}. \end{aligned}$$

Because $y = 3$ when $x = 1$, it follows that $3 = Ce^{-1}$ and $C = 3e$. Therefore, equation of the specified curve is

$$y = (3e)e^{-1/x} = 3e^{(x-1)/x}, \qquad x > 0.$$

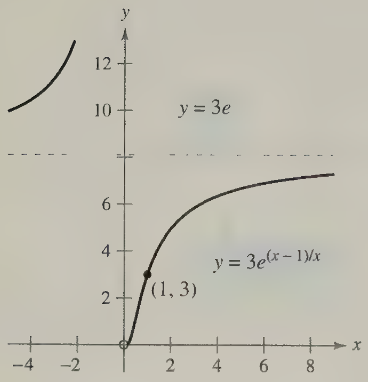


Figure 5.33

Homogeneous Differential Equations

Some differential equations that are not separable in x and y can be made separable by a change of variables. This is true for differential equations of the form $y' = f(x, y)$, where f is a **homogeneous function.** The function given by $f(x, y)$ is **homogeneous of degree n** if

$$f(tx, ty) = t^n f(x, y) \qquad \text{Homogeneous function of degree } n$$

where n is a real number.

TE The notation $f(x, y)$ is used to
note a function of two variables in
ch the same way as $f(x)$ denotes a
ction of one variable. You will study
ctions of two variables in detail in
apter 12.

Example 6 Verifying Homogeneous Functions

a. $f(x, y) = x^2y - 4x^3 + 3xy^2$ is a homogeneous function of degree 3 because

$$\begin{aligned} f(tx, ty) &= (tx)^2(ty) - 4(tx)^3 + 3(tx)(ty)^2 \\ &= t^3(x^2y) - t^3(4x^3) + t^3(3xy^2) \\ &= t^3(x^2y - 4x^3 + 3xy^2) \\ &= t^3f(x, y). \end{aligned}$$

b. $f(x, y) = xe^{x/y} + y\sin(y/x)$ is a homogeneous function of degree 1 because

$$\begin{aligned} f(tx, ty) &= txe^{tx/ty} + ty\sin\frac{ty}{tx} \\ &= t\left(xe^{x/y} + y\sin\frac{y}{x}\right) \\ &= tf(x, y). \end{aligned}$$

c. $f(x, y) = x + y^2$ is not a homogeneous function because

$$f(tx, ty) = tx + t^2y^2 = t(x + ty^2) \neq t^n(x + y^2).$$

d. $f(x, y) = x/y$ is a homogeneous function of degree 0 because

$$f(tx, ty) = \frac{tx}{ty} = t^0\frac{x}{y}.$$

Definition of Homogeneous Differential Equation

A **homogeneous differential equation** is an equation of the form

$$M(x, y)\,dx + N(x, y)dy = 0$$

where M and N are homogeneous functions of the same degree.

Example 7 Testing for Homogeneous Differential Equations

a. $(x^2 + xy)\,dx + y^2\,dy = 0$ is homogeneous of degree 2.

b. $x^3\,dx = y^3\,dy$ is homogeneous of degree 3.

c. $(x^2 + 1)\,dx + y^2\,dy = 0$ is *not* a homogeneous differential equation.

To solve a homogeneous differential equation by the method of separation variables, we use the following change of variables theorem.

THEOREM 5.17 Change of Variables for Homogeneous Equations

If $M(x, y)\,dx + N(x, y)\,dy = 0$ is homogeneous, then it can be transformed into a differential equation whose variables are separable by the substitution

$$y = vx$$

where v is a differentiable function of x.

Example 8 Solving a Homogeneous Differential Equation

Find the general solution of

$$(x^2 - y^2)\,dx + 3xy\,dy = 0.$$

STUDY TIP The substitution $y = vx$ will yield a differential equation that is separable with respect to the variables x and v. You must express your final solution, however, in terms of x and y.

Solution Because $(x^2 - y^2)$ and $3xy$ are both homogeneous of degree 2, let $y =$ to obtain $dy = x\,dv + v\,dx$. Then, by substitution, you have

$$\begin{aligned}(x^2 - v^2x^2)\,dx + 3x(vx)(\overbrace{x\,dv + v\,dx}^{dy}) &= 0\\ (x^2 + 2v^2x^2)\,dx + 3x^3v\,dv &= 0\\ x^2(1 + 2v^2)\,dx + x^2(3vx)\,dv &= 0.\end{aligned}$$

Dividing by x^2 and separating variables produces

$$\begin{aligned}(1 + 2v^2)\,dx &= -3vx\,dv\\ \int\frac{dx}{x} &= \int\frac{-3v}{1 + 2v^2}\,dv\\ \ln|x| &= -\frac{3}{4}\ln(1 + 2v^2) + C_1\\ 4\ln|x| &= -3\ln(1 + 2v^2) + \ln|C|\\ \ln x^4 &= \ln|C(1 + 2v^2)^{-3}|\\ x^4 &= C(1 + 2v^2)^{-3}.\end{aligned}$$

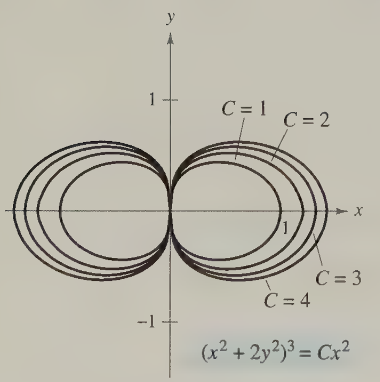
Substituting for v produces the following general solution.

$$x^4 = C\left[1 + 2\left(\frac{y}{x}\right)^2\right]^{-3}$$

$$\left(1 + \frac{2y^2}{x^2}\right)^3 x^4 = C$$

$$(x^2 + 2y^2)^3 = Cx^2 \qquad \text{General solution}$$

You can check this by differentiating and rewriting to get the original equation.



General solutions of
$(x^2 - y^2)\,dx + 3xy\,dy = 0$
Figure 5.34

TECHNOLOGY If you have access to a graphing utility, try using it to graph seve of the solutions in Example 8. For instance, Figure 5.34 shows the graphs of

$$(x^2 + 2y^2)^3 = Cx^2$$

for $C = 1, 2, 3,$ and 4.

Applications

Example 9 Wildlife Population

The rate of change of the number of coyotes $N(t)$ in a population is directly proportional to $650 - N(t)$, where t is the time in years. When $t = 0$, the population is 300, and when $t = 2$, the population has increased to 500. Find the population when $t = 3$.

Solution Because the rate of change of the population is proportional to $650 - N(t)$, you can write the following differential equation.

$$\frac{dN}{dt} = k(650 - N)$$

You can solve this equation using separation of variables.

$$dN = k(650 - N)\,dt \qquad \text{Differential form}$$

$$\frac{dN}{650 - N} = k\,dt \qquad \text{Separate variables.}$$

$$-\ln|650 - N| = kt + C_1 \qquad \text{Integrate.}$$

$$\ln|650 - N| = -kt - C_1$$

$$650 - N = e^{-kt - C_1} \qquad \text{Assume } N < 650.$$

$$N = 650 - Ce^{-kt} \qquad \text{General solution}$$

Using $N = 300$ when $t = 0$, you can conclude that $C = 350$, which produces

$$N = 650 - 350e^{-kt}.$$

Then, using $N = 500$ when $t = 2$, it follows that

$$500 = 650 - 350e^{-2k} \quad \Rightarrow \quad e^{-2k} = \tfrac{3}{7} \quad \Rightarrow \quad k \approx 0.4236.$$

So, the model for the coyote population is

$$N = 650 - 350e^{-0.4236t}. \qquad \text{Model for population}$$

When $t = 3$, you can approximate the population to be

$$N = 650 - 350e^{-0.4236(3)} \approx 552 \text{ coyotes.}$$

The model for the population is shown in Figure 5.35.

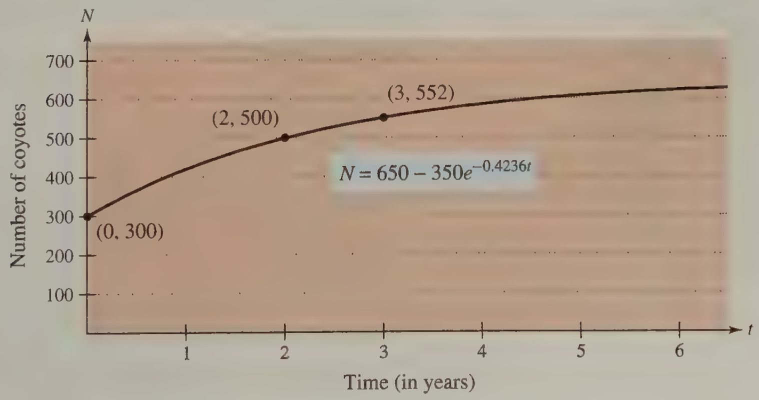
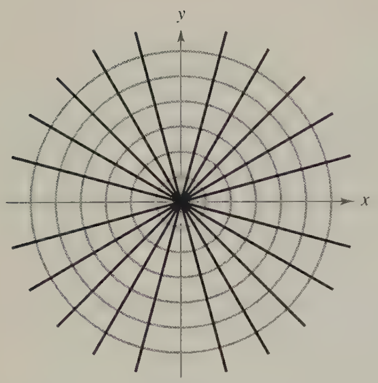


Figure 5.35



Each line $y = Kx$ is an orthogonal trajectory to the family of circles.
Figure 5.36

A common problem in electrostatics, thermodynamics, and hydrodynami involves finding a family of curves, each of which is orthogonal to all members of given family of curves. For example, Figure 5.36 shows a family of circles

$$x^2 + y^2 = C \qquad \text{Family of circles}$$

each of which intersects the lines in the family

$$y = Kx \qquad \text{Family of lines}$$

at right angles. Two such families of curves are said to be **mutually orthogonal,** ar each curve in one of the families is called an **orthogonal trajectory** of the oth family. In electrostatics, lines of force are orthogonal to the *equipotential curves.* thermodynamics, the flow of heat across a plane surface is orthogonal to the *isothe mal curves.* In hydrodynamics, the flow (stream) lines are orthogonal trajectories the *velocity potential curves.*

Example 10 Finding Orthogonal Trajectories

Describe the orthogonal trajectories for the family of curves given by

$$y = \frac{C}{x}$$

for $C \neq 0$. Sketch several members of each family.

Solution First, solve the given equation for C and write $xy = C$. Then, by differe tiating implicitly with respect to x, you obtain the differential equation

$$xy' + y = 0 \qquad \text{Differential equation}$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}. \qquad \text{Slope of given family}$$

Because y' represents the slope of the given family of curves at (x, y), it follows th the orthogonal family has the negative reciprocal slope x/y, and we write

$$\frac{dy}{dx} = \frac{x}{y}. \qquad \text{Slope of orthogonal family}$$

Now you can find the orthogonal family by separating variables and integrating.

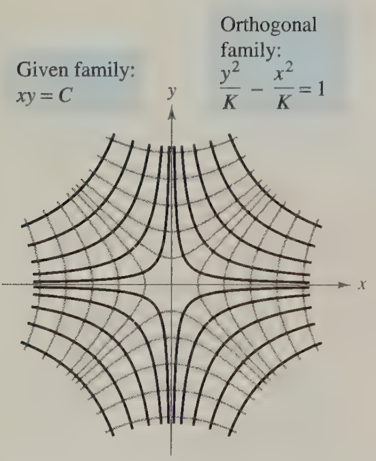
$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1$$

Therefore, each orthogonal trajectory is a hyperbola given by

$$\frac{y^2}{K} - \frac{x^2}{K} = 1. \qquad 2C_1 = K \neq 0$$

The centers are at the origin, and the transverse axes are vertical for $K > 0$ a horizontal for $K < 0$. Several trajectories are shown in Figure 5.37.



Orthogonal trajectories
Figure 5.37

Lab Series LAB 7

EXERCISES FOR SECTION 5.7

Exercises 1–6, verify the solution of the differential equation.

Solution	*Differential Equation*
1. $y = Ce^{4x}$	$y' = 4y$
2. $x^2 + y^2 = Cy$	$y' = 2xy/(x^2 - y^2)$
3. $y = C_1 \cos x + C_2 \sin x$	$y'' + y = 0$
4. $y = C_1e^{-x} \cos x + C_2e^{-x} \sin x$	$y'' + 2y' + 2y = 0$
5. $y = -\cos x \ln\lvert\sec x + \tan x\rvert$	$y'' + y = \tan x$
6. $y = \frac{2}{3}(e^{-2x} + e^x)$	$y'' + 2y' = 2e^x$

Exercises 7–12, determine whether the function is a solution f the differential equation $y^{(4)} - 16y = 0$.

7. $y = 3 \cos x$
8. $y = 3 \cos 2x$
9. $y = e^{-2x}$
10. $y = 5 \ln x$
1. $y = C_1e^{2x} + C_2e^{-2x} + C_3 \sin 2x + C_4 \cos 2x$
2. $y = 3e^{2x} - 4 \sin 2x$

Exercises 13–18, determine whether the function is a solution f the differential equation $xy' - 2y = x^3e^x$.

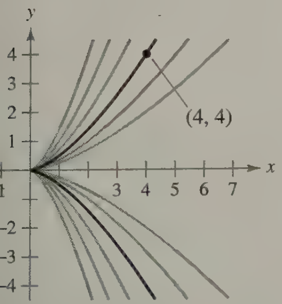
3. $y = x^2$
14. $y = x^2e^x$
5. $y = x^2(2 + e^x)$
16. $y = \sin x$
7. $y = \ln x$
18. $y = x^2e^x - 5x^2$

9. ***Think About It*** It is known that $y = Ce^{kx}$ is a solution of the differential equation $y' = 0.07y$. Is it possible to determine C or k from the information given? If so, find its value.

0. ***Think About It*** It is known that $y = A \sin \omega t$ is a solution of the differential equation $y'' + 16y = 0$. Find the value of ω.

Exercises 21 and 22, some of the curves corresponding to fferent values of C in the general solution of the differential quation are given. Find the particular solution that passes rough the point indicated on the graph.

Solution	*Differential Equation*
1. $y^2 = Cx^3$	$2xy' - 3y = 0$
2. $2x^2 - y^2 = C$	$yy' - 2x = 0$



gure for 21

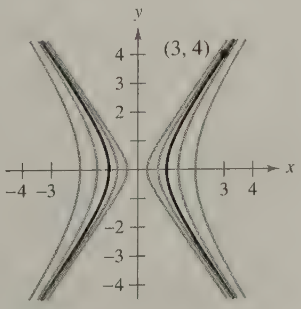


Figure for 22

In Exercises 23 and 24, the general solution of the differential equation is given. Use a graphing utility to graph the particular solutions for the given values of C.

23. $4yy' - x = 0$
$4y^2 - x^2 = C$
$C = 0, C = \pm 1, C = \pm 4$

24. $yy' + x = 0$
$x^2 + y^2 = C$
$C = 0, C = 1, C = 4$

In Exercises 25–30, verify that the general solutions satisfy the differential equation. Then find the particular solution that satisfies the initial condition.

25. $y = Ce^{-2x}$
$y' + 2y = 0$
$y = 3$ when $x = 0$

26. $3x^2 + 2y^2 = C$
$3x + 2yy' = 0$
$y = 3$ when $x = 1$

27. $y = C_1 \sin 3x + C_2 \cos 3x$
$y'' + 9y = 0$
$y = 2$ when $x = \pi/6$
$y' = 1$ when $x = \pi/6$

28. $y = C_1 + C_2 \ln x$
$xy'' + y' = 0$
$y = 0$ when $x = 2$
$y' = \frac{1}{2}$ when $x = 2$

29. $y = C_1x + C_2x^3$
$x^2y'' - 3xy' + 3y = 0$
$y = 0$ when $x = 2$
$y' = 4$ when $x = 2$

30. $y = e^{2x/3}(C_1 + C_2x)$
$9y'' - 12y' + 4y = 0$
$y = 4$ when $x = 0$
$y = 0$ when $x = 3$

In Exercises 31–42, use integration to find a general solution of the differential equation.

31. $\dfrac{dy}{dx} = 3x^2$
32. $\dfrac{dy}{dx} = x^3 - 4x$
33. $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2}$
34. $\dfrac{dy}{dx} = \dfrac{e^x}{1 + e^x}$
35. $\dfrac{dy}{dx} = \dfrac{x - 2}{x}$
36. $\dfrac{dy}{dx} = x \cos x^2$
37. $\dfrac{dy}{dx} = \sin 2x$
38. $\dfrac{dy}{dx} = \tan^2 x$
39. $\dfrac{dy}{dx} = x\sqrt{x - 3}$
40. $\dfrac{dy}{dx} = x\sqrt{5 - x}$
41. $\dfrac{dy}{dx} = xe^{x^2}$
42. $\dfrac{dy}{dx} = 5e^{-x/2}$

In Exercises 43–54, find the general solution of the differential equation.

43. $\dfrac{dy}{dx} = \dfrac{x}{y}$
44. $\dfrac{dy}{dx} = \dfrac{x^2 + 2}{3y^2}$
45. $\dfrac{dr}{ds} = 0.05r$
46. $\dfrac{dr}{ds} = 0.05s$
47. $(2 + x)y' = 3y$
48. $xy' = y$
49. $yy' = \sin x$
50. $yy' = 6 \cos(\pi x)$
51. $\sqrt{1 - 4x^2}\, y' = x$
52. $\sqrt{x^2 - 9}\, y' = 5x$
53. $y \ln x - xy' = 0$
54. $4yy' - 3e^x = 0$

In Exercises 55–64, find the particular solution that satisfies the initial condition.

Differential Equation	*Initial Condition*
55. $yy' - e^x = 0$	$y(0) = 4$
56. $\sqrt{x} + \sqrt{y}\,y' = 0$	$y(1) = 4$
57. $y(x + 1) + y' = 0$	$y(-2) = 1$
58. $2xy' - \ln x^2 = 0$	$y(1) = 2$
59. $y(1 + x^2)y' - x(1 + y^2) = 0$	$y(0) = \sqrt{3}$
60. $y\sqrt{1 - x^2}\,y' - x\sqrt{1 - y^2} = 0$	$y(0) = 1$
61. $\dfrac{du}{dv} = uv \sin v^2$	$u(0) = 1$
62. $\dfrac{dr}{ds} = e^{r-2s}$	$r(0) = 0$
63. $dP - kP\,dt = 0$	$P(0) = P_0$
64. $dT + k(T - 70)\,dt = 0$	$T(0) = 140$

In Exercises 65 and 66, find an equation of the graph that passes through the point and has the indicated slope.

Point	*Slope*
65. $(1, 1)$	$y' = -\dfrac{9x}{16y}$
66. $(8, 2)$	$y' = \dfrac{2y}{3x}$

In Exercises 67 and 68, find all functions *f* having the indicated property.

67. The tangent to the graph of f at the point (x, y) intersects the x-axis at $(x + 2, 0)$.

68. All tangents to the graph of f pass through the origin.

In Exercises 69–76, determine whether the function is homogeneous, and if it is, determine its degree.

69. $f(x, y) = x^3 - 4xy^2 + y^3$

70. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

71. $f(x, y) = \dfrac{x^2y^2}{\sqrt{x^2 + y^2}}$

72. $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

73. $f(x, y) = 2 \ln xy$

74. $f(x, y) = \tan(x + y)$

75. $f(x, y) = 2 \ln \dfrac{x}{y}$

76. $f(x, y) = \tan \dfrac{y}{x}$

In Exercises 77–82, solve the homogeneous differential equation.

77. $y' = \dfrac{x + y}{2x}$

78. $y' = \dfrac{x^3 + y^3}{xy^2}$

79. $y' = \dfrac{x - y}{x + y}$

80. $y' = \dfrac{x^2 + y^2}{2xy}$

81. $y' = \dfrac{xy}{x^2 - y^2}$

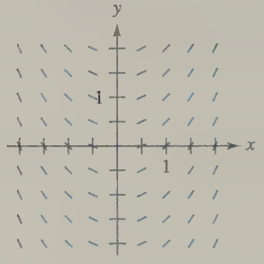
82. $y' = \dfrac{2x + 3y}{x}$

In Exercises 83–86, find the particular solution that satisfies th initial condition.

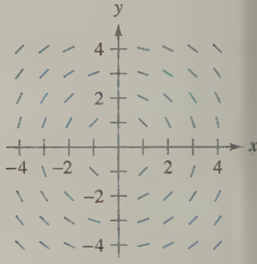
Differential Equation	*Initial Condition*
83. $x\,dy - (2xe^{-y/x} + y)\,dx = 0$	$y(1) = 0$
84. $-y^2\,dx + x(x + y)\,dy = 0$	$y(1) = 1$
85. $\left(x \sec \dfrac{y}{x} + y\right) dx - x\,dy = 0$	$y(1) = 0$
86. $(2x^2 + y^2)\,dx + xy\,dy = 0$	$y(1) = 0$

Slope Fields **In Exercises 87–90, sketch a few solutions of th differential equation on the slope field and then find the gener solution analytically. To print an enlarged copy of the graph, g to the website *www.mathgraphs.com*.**

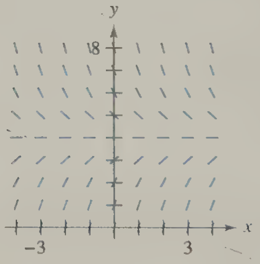
87. $\dfrac{dy}{dx} = x$



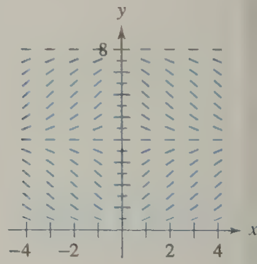
88. $\dfrac{dy}{dx} = -\dfrac{x}{y}$



89. $\dfrac{dy}{dx} = 4 - y$



90. $\dfrac{dy}{dx} = 0.25x(4 - y)$



In Exercises 91–94, use a computer algebra system to sketch th slope field for the differential equation, and graph the solutio satisfying the specified initial condition.

91. $\dfrac{dy}{dx} = 0.5y, \; y(0) = 6$

92. $\dfrac{dy}{dx} = 2 - y, \; y(0) = 4$

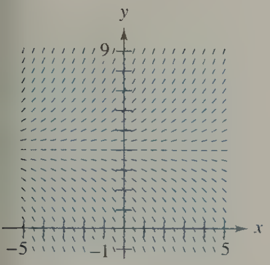
93. $\dfrac{dy}{dx} = 0.02y(10 - y), \; y(0) = 2$

94. $\dfrac{dy}{dx} = 0.2x(2 - y), \; y(0) = 9$

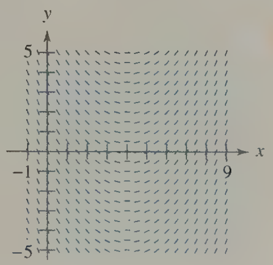
95. ***Radioactive Decay*** The rate of decomposition of radioacti radium is proportional to the amount present at any time. T half-life of radioactive radium is 1620 years. What percent o present amount will remain after 25 years?

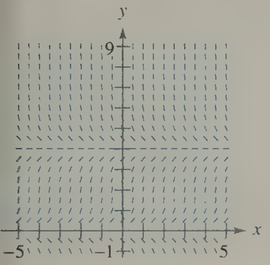
96. ***Chemical Reaction*** In a chemical reaction, a certain co pound changes into another compound at a rate proportional the unchanged amount. If initially there is 20 grams of t original compound, and there is 16 grams after 1 hour, wh will 75 percent of the compound be changed?

pe Fields **In Exercises 97–100, (a) write a differential ation for the statement, (b) match the differential equation h a possible slope field, and (c) verify your result by using a phing utility to graph a slope field for the differential ation. [The slope fields are labeled (a), (b), (c), and (d).] To nt an enlarged copy of the graph, go to the website *w.mathgraphs.com.***

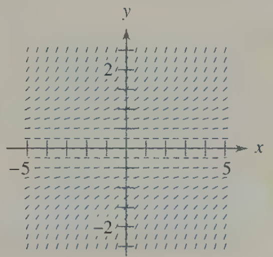


(b)





(d)



. The rate of change of y with respect to x is proportional to the difference between y and 4.

. The rate of change of y with respect to x is proportional to the difference between x and 4.

. The rate of change of y with respect to x is proportional to the product of y and the difference between y and 4.

. The rate of change of y with respect to x is proportional to y^2.

. ***Weight Gain*** A calf that weighs 60 pounds at birth gains weight at the rate

$$\frac{dw}{dt} = k(1200 - w)$$

where w is weight in pounds and t is time in years. Solve the differential equation.

(a) Use a computer algebra system to solve the differential equation for $k = 0.8$, $k = 0.9$, and $k = 1$. Graph the three solutions.

(b) If the animal is sold when its weight reaches 800 pounds, find the time of sale for each of the models in part (a).

(c) What is the maximum weight of the animal for each of the models?

. ***Weight Gain*** A calf that weighs w_0 pounds at birth gains weight at the rate

$$\frac{dw}{dt} = 1200 - w$$

where w is weight in pounds and t is time in years. Solve the differential equation.

103. ***Sailing*** Ignoring resistance, a sailboat starting from rest accelerates (dv/dt) at a rate proportional to the difference between the velocities of the wind and the boat.

(a) Write the velocity as a function of time if the wind is blowing at 20 knots, and after 1 minute the boat is moving at 5 knots.

(b) Use the result in part (a) to write the distance traveled by the boat as a function of time.

104. ***Radio Reception*** In hilly areas, radio reception may be poor. Consider a situation where an FM transmitter is located at the point $(-1, 1)$ behind a hill modeled by the graph of $y = x - x^2$, and a radio receiver is on the opposite side of the hill. (Assume that the x-axis represents ground level at the base of the hill.)

(a) What is the closest the radio can be to the hill so that reception is unobstructed?

(b) Write the closest position x of the radio as a function of h if the transmitter is located at $(-1, h)$.

(c) Use a graphing utility to graph the function in part (b). Determine the vertical asymptote of the function and interpret the result.

In Exercises 105–110, find the orthogonal trajectories of the family. Use a graphing utility to graph several members of each family.

105. $x^2 + y^2 = C$

106. $x^2 - 2y^2 = C$

107. $x^2 = Cy$

108. $y^2 = 2Cx$

109. $y^2 = Cx^3$

110. $y = Ce^x$

Getting at the Concept

111. In your own words, describe the difference between a general solution of a differential equation and a particular solution.

112. When determining a particular solution, how do you determine how many initial conditions are required?

113. State the test for determining if a differential equation is homogeneous. Give an example.

114. In your own words, describe the relationship between two families of curves that are mutually orthogonal.

True or False? **In Exercises 115–118, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

115. If $y = f(x)$ is a solution of a first-order differential equation, then $y = f(x) + C$ is also a solution.

116. The differential equation $y' = xy - 2y + x - 2$ can be written in separated variables form.

117. The function $f(x, y) = x^2 + xy + 2$ is homogeneous.

118. The families $x^2 + y^2 = 2Cy$ and $x^2 + y^2 = 2Kx$ are mutually orthogonal.

Section 5.8 Inverse Trigonometric Functions: Differentiation

- Develop properties of the six inverse trigonometric functions.
- Differentiate an inverse trigonometric function.
- Review the basic differentiation formulas for elementary functions.

Inverse Trigonometric Functions

This section begins with a rather surprising statement: *None of the six basic trigon metric functions has an inverse function.* This statement is true because all trigonometric functions are periodic and hence not one-to-one. In this section y will examine these six functions to see whether their domains can be redefined such a way that they will have inverse functions on the *restricted domains.*

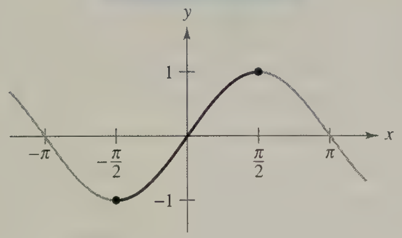
In Example 4 of Section 5.3, you saw that the sine function is increasing (a therefore is one-to-one) on the interval $[-\pi/2, \pi/2]$ (see Figure 5.38). On t interval you can define the inverse of the *restricted* sine function to be

$$y = \arcsin x \qquad \text{if and only if} \qquad \sin y = x$$

where $-1 \le x \le 1$ and $-\pi/2 \le \arcsin x \le \pi/2$.

Under suitable restrictions, each of the six trigonometric functions is one-to-o and so has an inverse function, as indicated in the following definition.

$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



The sine function is one-to-one on $[-\pi/2, \pi/2]$.
Figure 5.38

NOTE The term "iff" is used to represent the phrase "if and only if."

Definition of Inverse Trigonometric Functions

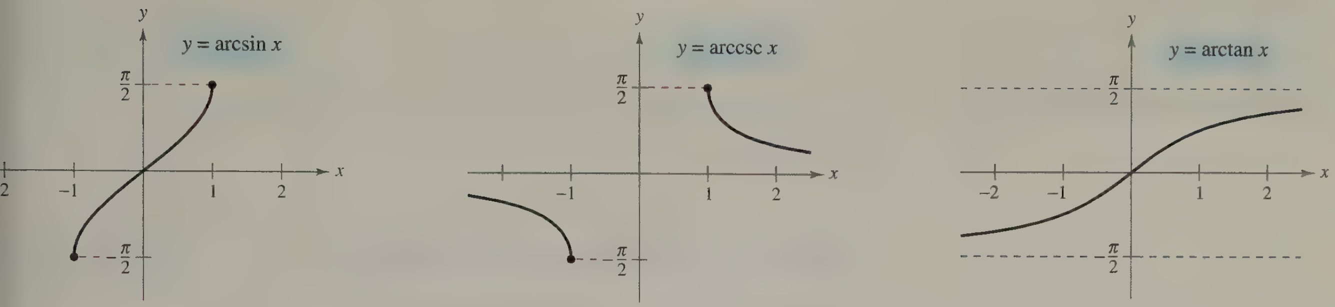
Function	*Domain*	*Range*
$y = \arcsin x$ iff $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \operatorname{arccot} x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \operatorname{arcsec} x$ iff $\sec y = x$	$\lvert x\rvert \ge 1$	$0 \le y \le \pi,\quad y \ne \frac{\pi}{2}$
$y = \operatorname{arccsc} x$ iff $\csc y = x$	$\lvert x\rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\quad y \ne 0$

NOTE The term "arcsin x" is read as "the arcsine of x" or sometimes "the angle whose s is x." An alternative notation for the inverse sine function is "$\sin^{-1} x$."

EXPLORATION

The Inverse Secant Function In the definition above, the inverse secant func tion is defined by restricting the domain of the secant function to the intervals $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. Most other texts and reference books agree with this, but some disagree. What other domains might make sense? Explain your reasoning graphi cally. Most calculators do not have a key for the inverse secant function. How ca you use a calculator to evaluate the inverse secant function?

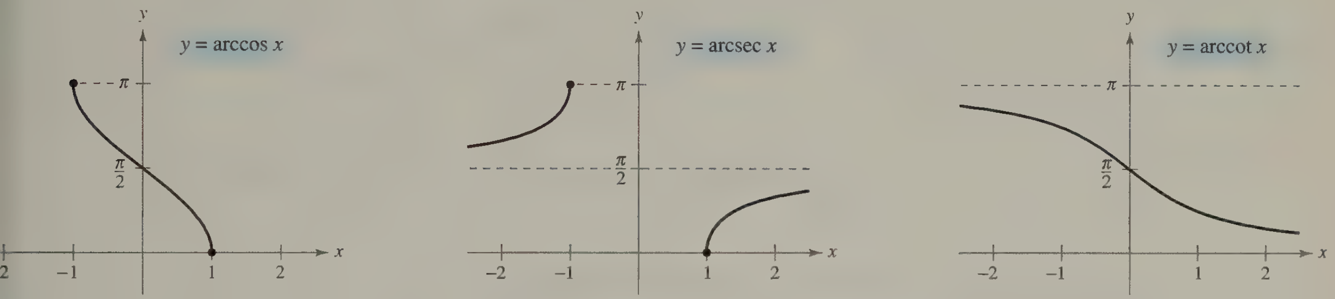
The graphs of the six inverse trigonometric functions are shown in Figure 5.39.



main: $[-1, 1]$
ge: $[-\pi/2, \pi/2]$

Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$

Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$



main: $[-1, 1]$
ge: $[0, \pi]$
ure 5.39

Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[0, \pi/2) \cup (\pi/2, \pi]$

Domain: $(-\infty, \infty)$
Range: $(0, \pi)$

Example 1 Evaluating Inverse Trigonometric Functions

Evaluate each of the following.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\arccos 0$ **c.** $\arctan \sqrt{3}$ **d.** $\arcsin(0.3)$

TE When evaluating inverse onometric functions, remember that denote *angles in radian measure.*

Solution

a. By definition, $y = \arcsin\left(-\frac{1}{2}\right)$ implies that $\sin y = -\frac{1}{2}$. In the interval $[-\pi/2, \pi/2]$, the correct value of y is $-\pi/6$.

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

b. By definition, $y = \arccos 0$ implies that $\cos y = 0$. In the interval $[0, \pi]$, you have $y = \pi/2$.

$$\arccos 0 = \frac{\pi}{2}$$

c. By definition, $y = \arctan \sqrt{3}$ implies that $\tan y = \sqrt{3}$. In the interval $(-\pi/2, \pi/2)$, you have $y = \pi/3$.

$$\arctan \sqrt{3} = \frac{\pi}{3}$$

d. Using a calculator set in *radian mode* produces

$$\arcsin(0.3) \approx 0.3047.$$

EXPLORATION

Graph $y = \arccos(\cos x)$ for $-4\pi \le x \le 4\pi$. Why isn't the graph the same as the graph of $y = x$?

Inverse functions have the properties

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

When applying these properties to inverse trigonometric functions, remember that t trigonometric functions have inverse functions only in restricted domains. For x-valu outside these domains, these two properties do not hold. For example, $\arcsin(\sin \pi)$ equal to 0, not π.

Properties of Inverse Trigonometric Functions

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

If $|x| \ge 1$ and $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$, then

$$\sec(\operatorname{arcsec} x) = x \quad \text{and} \quad \operatorname{arcsec}(\sec y) = y.$$

Similar properties hold for the other inverse trigonometric functions.

Example 2 Solving an Equation

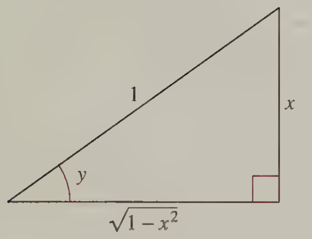
$$\arctan(2x - 3) = \frac{\pi}{4} \qquad \text{Original equation}$$

$$\tan[\arctan(2x - 3)] = \tan\frac{\pi}{4} \qquad \text{Take tangent of both sides.}$$

$$2x - 3 = 1 \qquad \tan(\arctan x) = x$$

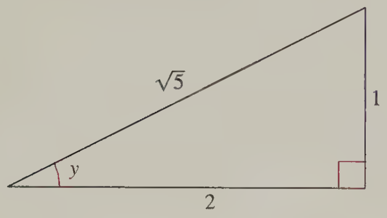
$$x = 2 \qquad \text{Solve for } x.$$

Some problems in calculus require that you evaluate expressions such $\cos(\arcsin x)$, as illustrated in Example 3.



$y = \arcsin x$

Figure 5.40



$y = \operatorname{arcsec}\dfrac{\sqrt{5}}{2}$

Figure 5.41

Example 3 Using Right Triangles

a. Given $y = \arcsin x$, where $0 < y < \pi/2$, find $\cos y$.

b. Given $y = \operatorname{arcsec}(\sqrt{5}/2)$, find $\tan y$.

Solution

a. Because $y = \arcsin x$, you know that $\sin y = x$. This relationship between x an can be represented by a right triangle, as shown in Figure 5.40.

$$\cos y = \cos(\arcsin x) = \frac{\text{adj.}}{\text{hyp.}} = \sqrt{1 - x^2}$$

(This result is also valid for $-\pi/2 < y < 0$.)

b. Use the right triangle shown in Figure 5.41.

$$\tan y = \tan\left[\operatorname{arcsec}\left(\frac{\sqrt{5}}{2}\right)\right] = \frac{\text{opp.}}{\text{adj.}} = \frac{1}{2}$$

Derivatives of Inverse Trigonometric Functions

In Section 5.1 you saw that the derivative of the *transcendental* function $f(x) = \ln x$ is the *algebraic* function $f'(x) = 1/x$. You will now see that the derivatives of the inverse trigonometric functions also are algebraic (even though the inverse trigonometric functions are themselves transcendental).

The following theorem lists the derivatives of the six inverse trigonometric functions. Note that the derivatives of arccos u, arccot u, and arccsc u are the *negatives* of the derivatives of arcsin u, arctan u, and arcsec u, respectively.

TE There is no common agreement the definition of arcsec x (or arccsc x) negative values of x. When we ined the range of the arcsecant, we se to preserve the reciprocal identity

$$\text{arcsec } x = \arccos \frac{1}{x}.$$

example, to evaluate arcsec(-2), can write

$$\text{arcsec}(-2) = \arccos(-0.5) \approx 2.09.$$

e of the consequences of the definition he inverse secant function given in text is that its graph has a positive pe at every x-value in its domain. e Figure 5.39.) This accounts for the olute value sign in the formula for the ivative of arcsec x.

THEOREM 5.18 Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\text{arccot } u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\text{arcsec } u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\text{arccsc } u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

To derive these formulas, you can use implicit differentiation. For instance, if $y = \arcsin x$, then $\sin y = x$ and $(\cos y)y' = 1$. (See Exercise 76.)

TECHNOLOGY If your graphing tility does not have the arcsecant unction, you can obtain its graph sing

$$f(x) = \text{arcsec } x = \arccos \frac{1}{x}.$$

Example 4 Differentiating Inverse Trigonometric Functions

a. $\dfrac{d}{dx}[\arcsin(2x)] = \dfrac{2}{\sqrt{1-(2x)^2}} = \dfrac{2}{\sqrt{1-4x^2}}$

b. $\dfrac{d}{dx}[\arctan(3x)] = \dfrac{3}{1+(3x)^2} = \dfrac{3}{1+9x^2}$

c. $\dfrac{d}{dx}[\arcsin \sqrt{x}] = \dfrac{(1/2)\,x^{-1/2}}{\sqrt{1-x}} = \dfrac{1}{2\sqrt{x}\sqrt{1-x}} = \dfrac{1}{2\sqrt{x-x^2}}$

d. $\dfrac{d}{dx}[\text{arcsec } e^{2x}] = \dfrac{2e^{2x}}{e^{2x}\sqrt{(e^{2x})^2-1}} = \dfrac{2e^{2x}}{e^{2x}\sqrt{e^{4x}-1}} = \dfrac{2}{\sqrt{e^{4x}-1}}$

The absolute value sign is not necessary because $e^{2x} > 0$.

Example 5 A Derivative That Can Be Simplified

Differentiate $y = \arcsin x + x\sqrt{1-x^2}$.

Solution

$$\begin{aligned} y' &= \frac{1}{\sqrt{1-x^2}} + x\left(\frac{1}{2}\right)(-2x)(1-x^2)^{-1/2} + \sqrt{1-x^2} \\ &= \frac{1}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} \\ &= \sqrt{1-x^2} + \sqrt{1-x^2} \\ &= 2\sqrt{1-x^2} \end{aligned}$$

TE From Example 5, you can see e of the benefits of inverse trigonotric functions—they can be used to egrate common algebraic functions. instance, from the result shown in example, it follows that

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right).$$

Example 6 Analyzing an Inverse Trigonometric Graph

Analyze the graph of $y = (\arctan x)^2$.

Solution From the derivative

$$y' = 2(\arctan x)\left(\frac{1}{1+x^2}\right)$$
$$= \frac{2\arctan x}{1+x^2}$$

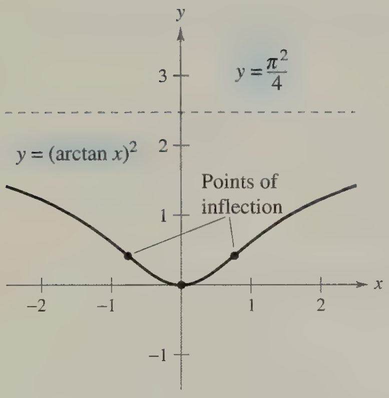
you can see that the only critical number is $x = 0$. By the First Derivative Test, th value corresponds to a relative minimum. From the second derivative

$$y'' = \frac{(1+x^2)\left(\frac{2}{1+x^2}\right) - (2\arctan x)(2x)}{(1+x^2)^2}$$
$$= \frac{2(1-2x\arctan x)}{(1+x^2)^2}$$

it follows that points of inflection occur when $2x \arctan x = 1$. Using Newton Method, these points occur when $x \approx \pm 0.765$. Finally, because

$$\lim_{x\to\pm\infty} (\arctan x)^2 = \frac{\pi^2}{4}$$

it follows that the graph has a horizontal asymptote at $y = \pi^2/4$. The graph is show in Figure 5.42.



The graph of $y = (\arctan x)^2$ has a horizontal asymptote at $y = \pi^2/4$.
Figure 5.42

Example 7 Maximizing an Angle

A photographer is taking a picture of a 4-foot painting hung in an art gallery. The ca era lens is 1 foot below the lower edge of the painting, as shown in Figure 5.43. Ho far should the camera be from the painting to maximize the angle subtended by t camera lens?

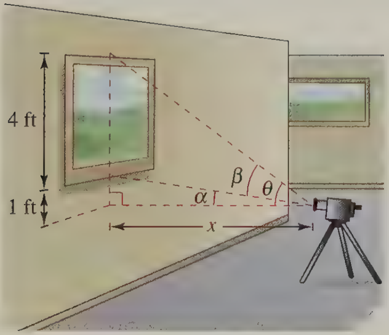
Solution In Figure 5.43, let β be the angle to be maximized.

$$\beta = \theta - \alpha$$
$$= \operatorname{arccot}\frac{x}{5} - \operatorname{arccot} x$$

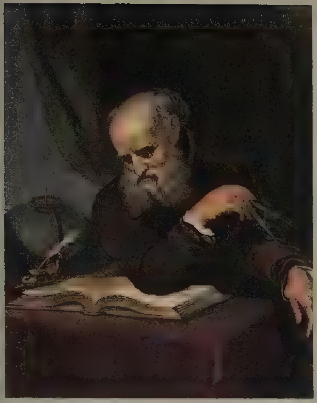
Differentiating produces

$$\frac{d\beta}{dx} = \frac{-1/5}{1+(x^2/25)} - \frac{-1}{1+x^2}$$
$$= \frac{-5}{25+x^2} + \frac{1}{1+x^2}$$
$$= \frac{4(5-x^2)}{(25+x^2)(1+x^2)}.$$

Because $d\beta/dx = 0$ when $x = \sqrt{5}$, you can conclude from the First Derivative T that this distance yields a maximum value of β. So, the distance is $x \approx 2.236$ feet a the angle is $\beta \approx 0.7297$ radians $\approx 41.81°$.



The camera should be 2.236 feet from the painting to maximize the angle β.
Figure 5.43



The Granger Collection

GALILEO GALILEI (1564–1642)

Galileo's approach to science departed from the accepted Aristotelian view that nature had describable *qualities*, such as "fluidity" and "potentiality." He chose to describe the physical world in terms of measurable *quantities*, such as time, distance, force, and mass.

Review of Basic Differentiation Rules

In the 1600s, Europe was ushered into the scientific age by such great thinkers as Descartes, Galileo, Huygens, Newton, and Kepler. These men believed that nature is governed by basic laws—laws that can, for the most part, be written in terms of mathematical equations. One of the most influential publications of this period—*Dialogue on the Great World Systems*, by Galileo Galilei—has become a classic description of modern scientific thought.

As mathematics has developed during the past few hundred years, a small number of elementary functions has proven sufficient for modeling most* phenomena in physics, chemistry, biology, engineering, economics, and a variety of other fields. An **elementary function** is a function from the following list or one that can be formed as the sum, product, quotient, or composition of functions in the list.

Algebraic Functions	*Transcendental Functions*
Polynomial functions	Logarithmic functions
Rational functions	Exponential functions
Functions involving radicals	Trigonometric functions
	Inverse trigonometric functions

With the differentiation rules introduced so far in the text, you can differentiate *any* elementary function. For convenience, we summarize these differentiation rules here.

Basic Differentiation Rules for Elementary Functions

1. $\frac{d}{dx}[cu] = cu'$
2. $\frac{d}{dx}[u \pm v] = u' \pm v'$
3. $\frac{d}{dx}[uv] = uv' + vu'$
4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}[c] = 0$
6. $\frac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\frac{d}{dx}[x] = 1$
8. $\frac{d}{dx}[|u|] = \frac{u}{|u|}(u'), \quad u \neq 0$
9. $\frac{d}{dx}[\ln u] = \frac{u'}{u}$
10. $\frac{d}{dx}[e^u] = e^u u'$
11. $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$
12. $\frac{d}{dx}[a^u] = (\ln a)a^u u'$
13. $\frac{d}{dx}[\sin u] = (\cos u)u'$
14. $\frac{d}{dx}[\cos u] = -(\sin u)u'$
15. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$
17. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$
18. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1 - u^2}}$
20. $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1 - u^2}}$
21. $\frac{d}{dx}[\arctan u] = \frac{u'}{1 + u^2}$
22. $\frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1 + u^2}$
23. $\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2 - 1}}$
24. $\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2 - 1}}$

---

[*] *Some important functions used in engineering and science (such as Bessel functions and gamma functions) are not elementary functions.*

EXERCISES FOR SECTION 5.8

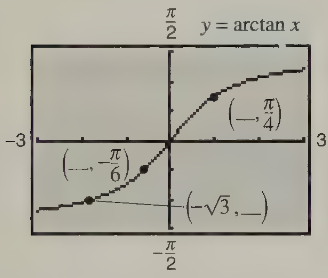
Numerical and Graphical Analysis **In Exercises 1 and 2, (a) use a graphing utility to complete the table, (b) plot the points in the table and graph the function by hand, (c) use a graphing utility to graph the function and compare the result with your hand-drawn graph in part (b), and (d) determine any intercepts and symmetry of the graph.**

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y											

1. $y = \arcsin x$ **2.** $y = \arccos x$

3. ***True or False?*** Decide whether the following statement is true or false, and explain: Because $\cos(-\pi/3) = \frac{1}{2}$, it follows that $\arccos \frac{1}{2} = -\pi/3$.

4. Determine the missing coordinates of the points on the graph of the function.



In Exercises 5–12, evaluate the expression without using a calculator.

5. $\arcsin \frac{1}{2}$ **6.** $\arcsin 0$

7. $\arccos \frac{1}{2}$ **8.** $\arccos 0$

9. $\arctan \frac{\sqrt{3}}{3}$ **10.** $\operatorname{arccot}(-\sqrt{3})$

11. $\operatorname{arccsc}(-\sqrt{2})$ **12.** $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

In Exercises 13–16, use a calculator to approximate the value. Round your answer to two decimal places.

13. $\arccos(-0.8)$ **14.** $\arcsin(-0.39)$

15. $\operatorname{arcsec} 1.269$ **16.** $\arctan(-3)$

In Exercises 17–20, evaluate the expression without using a calculator. (*Hint:* See Example 3.)

17. (a) $\sin\left(\arctan \frac{3}{4}\right)$ **18.** (a) $\tan\left(\arccos \frac{\sqrt{2}}{2}\right)$

(b) $\sec\left(\arcsin \frac{4}{5}\right)$ (b) $\cos\left(\arcsin \frac{5}{13}\right)$

19. (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right]$ **20.** (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right]$

(b) $\csc\left[\arctan\left(-\frac{5}{12}\right)\right]$ (b) $\tan\left[\arcsin\left(-\frac{5}{6}\right)\right]$

In Exercises 21–28, write the expression in algebraic form.

21. $\cos(\arcsin 2x)$ **22.** $\sec(\arctan 4x)$

23. $\sin(\operatorname{arcsec} x)$ **24.** $\cos(\operatorname{arccot} x)$

25. $\tan\left(\operatorname{arcsec} \frac{x}{3}\right)$ **26.** $\sec[\arcsin(x - 1)]$

27. $\csc\left(\arctan \frac{x}{\sqrt{2}}\right)$ **28.** $\cos\left(\arcsin \frac{x - h}{r}\right)$

In Exercises 29 and 30, use a graphing utility to graph f and g the same viewing window to verify that they are equal. Expla why they are equal. Identify any asymptotes of the graphs.

29. $f(x) = \sin(\arctan 2x)$, $g(x) = \dfrac{2x}{\sqrt{1 + 4x^2}}$

30. $f(x) = \tan\left(\arccos \frac{x}{2}\right)$, $g(x) = \dfrac{\sqrt{4 - x^2}}{x}$

In Exercises 31–34, solve the equation for x.

31. $\arcsin(3x - \pi) = \frac{1}{2}$ **32.** $\arctan(2x - 5) = -1$

33. $\arcsin\sqrt{2x} = \arccos\sqrt{x}$ **34.** $\arccos x = \operatorname{arcsec} x$

In Exercises 35 and 36, verify each identity.

35. (a) $\operatorname{arccsc} x = \arcsin \frac{1}{x}$, $x \geq 1$

(b) $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$, $x > 0$

36. (a) $\arcsin(-x) = -\arcsin x$, $|x| \leq 1$

(b) $\arccos(-x) = \pi - \arccos x$, $|x| \leq 1$

In Exercises 37–40, sketch the graph of the function. Use graphing utility to verify your graph.

37. $f(x) = \arcsin(x - 1)$ **38.** $f(x) = \arctan x + \frac{\pi}{2}$

39. $f(x) = \operatorname{arcsec} 2x$ **40.** $f(x) = \arccos \frac{x}{4}$

In Exercises 41–60, find the derivative of the function.

41. $f(x) = 2\arcsin(x - 1)$ **42.** $f(t) = \arcsin t^2$

43. $g(x) = 3\arccos \frac{x}{2}$ **44.** $f(x) = \operatorname{arcsec} 2x$

45. $f(x) = \arctan \frac{x}{a}$ **46.** $f(x) = \arctan\sqrt{x}$

47. $g(x) = \dfrac{\arcsin 3x}{x}$ **48.** $h(x) = x^2 \arctan x$

49. $h(t) = \sin(\arccos t)$ **50.** $f(x) = \arcsin x + \operatorname{arcco}$

51. $y = x\arccos x - \sqrt{1 - x^2}$ **52.** $y = \ln(t^2 + 4) - \frac{1}{2}\arctan$

53. $y = \frac{1}{2}\left(\frac{1}{2}\ln\frac{x + 1}{x - 1} + \arctan x\right)$

. $y = \frac{1}{2}\left[x\sqrt{4-x^2} + 4\arcsin\left(\frac{x}{2}\right)\right]$

. $y = x\arcsin x + \sqrt{1-x^2}$

. $y = x\arctan 2x - \frac{1}{4}\ln(1+4x^2)$

. $y = 8\arcsin\frac{x}{4} - \frac{x\sqrt{16-x^2}}{2}$

. $y = 25\arcsin\frac{x}{5} - x\sqrt{25-x^2}$

. $y = \arctan x + \frac{x}{1+x^2}$ **60.** $y = \arctan\frac{x}{2} - \frac{1}{2(x^2+4)}$

inear and Quadratic Approximations **In Exercises 61 and 62, se a computer algebra system to find the linear approximation**

$$(x) = f(a) + f'(a)(x-a)$$

nd the quadratic approximation

$$(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2$$

the function f at $x = a$. Sketch the graph of the function and s linear and quadratic approximations.

. $f(x) = \arcsin x$ **62.** $f(x) = \arctan x$

$a = \frac{1}{2}$ $a = 1$

Exercises 63–66, find any relative extrema of the function.

. $f(x) = \operatorname{arcsec} x - x$ **64.** $f(x) = \arcsin x - 2x$

. $f(x) = \arctan x - \arctan(x-4)$

. $h(x) = \arcsin x - 2\arctan x$

Getting at the Concept

67. Explain why the domains of the trigonometric functions are restricted when finding the inverse trigonometric functions.

68. Explain why $\tan\pi = 0$ does not imply that $\arctan 0 = \pi$.

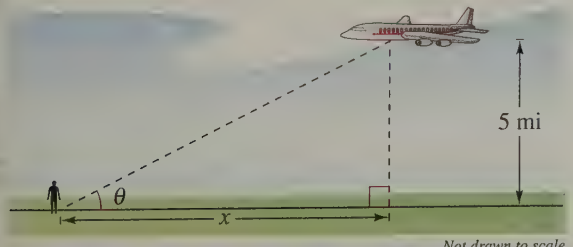
69. Explain how to graph $y = \operatorname{arccot} x$ on a graphing utility that does not have the arccotangent function.

70. Are the derivatives of the inverse trigonometric functions algebraic or transcendental functions? List the derivatives of the inverse trigonometric functions.

. ***Angular Rate of Change*** An airplane flies at an altitude of 5 miles toward a point directly over an observer. Consider θ and x as shown in the figure.

(a) Write θ as a function of x.

(b) If the speed of the plane is 400 miles per hour, find $d\theta/dt$ when $x = 10$ miles and $x = 3$ miles.



72. ***Writing*** Repeat Exercise 71 if the altitude of the plane is 3 miles and describe how the altitude affects the rate of change of θ.

73. ***Angular Rate of Change*** In a free-fall experiment, an object is dropped from a height of 256 feet. A camera on the ground 500 feet from the point of impact records the fall of the object.

(a) Find the position function giving the height of the object at time t assuming the object is released at time $t = 0$. At what time will the object reach ground level?

(b) Find the rate of change of the angle of elevation of the camera when $t = 1$ and $t = 2$.

74. ***Angular Rate of Change*** A television camera at ground level is filming the lift-off of a space shuttle at a point 750 meters from the launch pad. Let θ be the angle of elevation of the shuttle and let s be the distance between the camera and the shuttle. Write θ as a function of s for the period of time when the shuttle is moving vertically. Differentiate the result to find $d\theta/dt$ in terms of s and ds/dt.

75. Prove that

$$\arctan x + \arctan y = \arctan\frac{x+y}{1-xy}, \qquad xy \neq 1.$$

Use this formula to show that

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}.$$

76. Verify each differentiation formula.

(a) $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$ (b) $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$

(c) $\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}}$

(d) $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$

(e) $\frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$ (f) $\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$

77. ***Existence of an Inverse*** Determine the values of k such that the function $f(x) = kx + \sin x$ has an inverse function.

78. ***Think About It*** Use a graphing utility to graph

$f(x) = \sin x$ and $g(x) = \arcsin(\sin x)$.

(a) Why isn't the graph of g the line $y = x$?

(b) Determine the extrema of g.

True or False? **In Exercises 79–82, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

79. The slope of the graph of the inverse tangent function is positive for all x.

80. The range of $y = \arcsin x$ is $[0, \pi]$.

81. $\frac{d}{dx}[\arctan(\tan x)] = 1$ for all x in the domain.

82. $\arcsin^2 x + \arccos^2 x = 1$

Section 5.9 Inverse Trigonometric Functions: Integration

- Integrate functions whose antiderivatives involve inverse trigonometric functions.
- Use the method of completing the square to integrate a function.
- Review the basic integration formulas involving elementary functions.

Integrals Involving Inverse Trigonometric Functions

The derivatives of the six inverse trigonometric functions fall into three pairs. In eac pair, the derivative of one function is the negative of the other. For example,

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

and

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}.$$

When listing the *antiderivative* that corresponds to each of the inverse trigonometr functions, you need to use only one member from each pair. It is conventional to u $\arcsin x$ as the antiderivative of $1/\sqrt{1-x^2}$, rather than $-\arccos x$. The next theore gives one antiderivative formula for each of the three pairs. The proofs of these int gration rules are left to you (see Exercise 61).

NOTE For a proof of part 2 of Theorem 5.19, see the article "A Direct Proof of the Integral Formula for Arctangent" by Arnold J. Insel in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

THEOREM 5.19 Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x, and let $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$

2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

3. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

Example 1 Integration with Inverse Trigonometric Functions

a. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \arcsin\frac{x}{2} + C$

b. $\displaystyle\int \frac{dx}{2+9x^2} = \frac{1}{3}\int \frac{3\,dx}{(\sqrt{2})^2+(3x)^2}$ $\quad u = 3x,\ a = \sqrt{2}$

$$= \frac{1}{3\sqrt{2}}\arctan\frac{3x}{\sqrt{2}} + C$$

c. $\displaystyle\int \frac{dx}{x\sqrt{4x^2-9}} = \int \frac{2\,dx}{2x\sqrt{(2x)^2-3^2}}$ $\quad u = 2x,\ a = 3$

$$= \frac{1}{3}\operatorname{arcsec}\frac{|2x|}{3} + C$$

The integrals in Example 1 are fairly straightforward applications of integrati formulas. Unfortunately, this is not typical. The integration formulas for inver trigonometric functions can be disguised in many ways.

TECHNOLOGY PITFALL Computer software that can perform symbolic integration is useful for integrating functions such as the one in Example 2. When using such software, however, you must remember that it can fail to find an antiderivative for two reasons. First, some elementary functions simply do not have antiderivatives that are elementary functions. Second, every symbolic integration utility has limitations—you might have entered a function that the software was not programmed to handle. You should also remember that antiderivatives involving trigonometric functions or logarithmic functions can be written in many different forms. For instance, when we used a symbolic integration utility to find the integral in Example 2, we obtained

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \arctan\sqrt{e^{2x}-1} + C.$$

Try showing that this antiderivative is equivalent to that obtained in Example 2.

Example 2 Integration by Substitution

Find $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$.

Solution As it stands, this integral doesn't fit any of the three inverse trigonometric formulas. Using the substitution $u = e^x$, however, produces the following.

$$u = e^x \quad \Rightarrow \quad du = e^x\,dx \quad \Rightarrow \quad dx = \frac{du}{e^x} = \frac{du}{u}$$

With this substitution, you can integrate as follows.

$$\begin{aligned}
\int \frac{dx}{\sqrt{e^{2x}-1}} &= \int \frac{dx}{\sqrt{(e^x)^2-1}} && \text{Write } e^{2x} \text{ as } (e^x)^2.\\
&= \int \frac{du/u}{\sqrt{u^2-1}} && \text{Substitute.}\\
&= \int \frac{du}{u\sqrt{u^2-1}} && \text{Rewrite to fit Arcsecant Rule.}\\
&= \operatorname{arcsec}\frac{|u|}{1} + C && \text{Apply Arcsecant Rule.}\\
&= \operatorname{arcsec} e^x + C && \text{Back-substitute.}
\end{aligned}$$

Example 3 Rewriting as the Sum of Two Quotients

Find $\displaystyle\int \frac{x+2}{\sqrt{4-x^2}}\,dx$.

Solution This integral does not appear to fit any of the basic integration formulas. By splitting the integrand into two parts, however, you can see that the first part can be found with the Power Rule and the second part yields an inverse sine function.

$$\begin{aligned}
\int \frac{x+2}{\sqrt{4-x^2}}\,dx &= \int \frac{x}{\sqrt{4-x^2}}\,dx + \int \frac{2}{\sqrt{4-x^2}}\,dx\\
&= -\frac{1}{2}\int (4-x^2)^{-1/2}(-2x)\,dx + 2\int \frac{1}{\sqrt{4-x^2}}\,dx\\
&= -\frac{1}{2}\left[\frac{(4-x^2)^{1/2}}{1/2}\right] + 2\arcsin\frac{x}{2} + C\\
&= -\sqrt{4-x^2} + 2\arcsin\frac{x}{2} + C
\end{aligned}$$

Completing the Square

Completing the square helps when quadratic functions are involved in the integrand. For example, the quadratic $x^2 + bx + c$ can be written as the difference of two squares by adding and subtracting $(b/2)^2$.

$$\begin{aligned}
x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c\\
&= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c
\end{aligned}$$

Example 4 Completing the Square

Find $\int \frac{dx}{x^2 - 4x + 7}$.

Solution You can write the denominator as the sum of two squares as follows.

$$x^2 - 4x + 7 = (x^2 - 4x + 4) - 4 + 7 = (x - 2)^2 + 3 = u^2 + a^2$$

Now, in this completed square form, let $u = x - 2$ and $a = \sqrt{3}$.

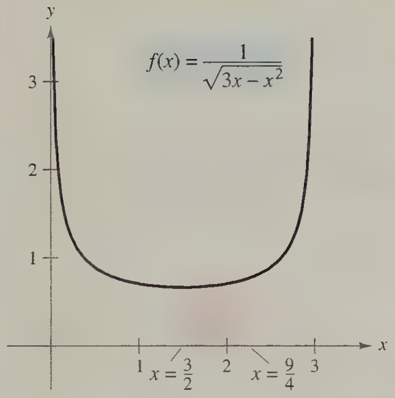
$$\int \frac{dx}{x^2 - 4x + 7} = \int \frac{dx}{(x - 2)^2 + 3} = \frac{1}{\sqrt{3}} \arctan \frac{x - 2}{\sqrt{3}} + C$$

If the leading coefficient is not 1, it helps to factor before completing the squa
For instance, you can complete the square of $2x^2 - 8x + 10$ as follows.

$$\begin{aligned} 2x^2 - 8x + 10 &= 2(x^2 - 4x + 5) \\ &= 2(x^2 - 4x + 4 - 4 + 5) \\ &= 2[(x - 2)^2 + 1] \end{aligned}$$

To complete the square when the coefficient of x^2 is negative, use the same "factori
process" illustrated above. For instance, you can complete the square for $3x - x^2$
follows.

$$\begin{aligned} 3x - x^2 &= -(x^2 - 3x) \\ &= -\left[x^2 - 3x + \left(\tfrac{3}{2}\right)^2 - \left(\tfrac{3}{2}\right)^2\right] \\ &= \left(\tfrac{3}{2}\right)^2 - \left(x - \tfrac{3}{2}\right)^2 \end{aligned}$$



The area of the region bounded by the graph of f, the x-axis, $x = \frac{3}{2}$, and $x = \frac{9}{4}$ is $\pi/6$.
Figure 5.44

Example 5 Completing the Square (Negative Leading Coefficient)

Find the area of the region bounded by the graph of

$$f(x) = \frac{1}{\sqrt{3x - x^2}}$$

the x-axis, and the lines $x = \frac{3}{2}$ and $x = \frac{9}{4}$.

Solution From Figure 5.44, you can see that the area is given by

$$\text{Area} = \int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}}\, dx.$$

Using the completed square form derived above, you can integrate as follows.

$$\begin{aligned} \int_{3/2}^{9/4} \frac{dx}{\sqrt{3x - x^2}} &= \int_{3/2}^{9/4} \frac{dx}{\sqrt{(3/2)^2 - [x - (3/2)]^2}} \\ &= \arcsin \frac{x - (3/2)}{3/2} \bigg]_{3/2}^{9/4} \\ &= \arcsin \frac{1}{2} - \arcsin 0 \\ &= \frac{\pi}{6} \\ &\approx 0.524 \end{aligned}$$

TECHNOLOGY With definite integrals such as the one given in Example 5, remember that you can resort to a numerical solution. For instance, applying Simpson's Rule (with $n = 6$) to the integral in the example, you obtain

$$\int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}}\, dx \approx 0.523599.$$

This differs from the exact value of the integral ($\pi/6 \approx 0.5235988$) by less than one millionth.

Review of Basic Integration Rules

You have now completed the introduction of the **basic integration rules.** To be efficient at applying these rules, you should have practiced enough so that each rule is committed to memory.

Basic Integration Rules ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int e^u\,du = e^u + C$
7. $\int a^u\,du = \left(\dfrac{1}{\ln a}\right)a^u + C$
8. $\int \sin u\,du = -\cos u + C$
9. $\int \cos u\,du = \sin u + C$
10. $\int \tan u\,du = -\ln|\cos u| + C$
11. $\int \cot u\,du = \ln|\sin u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u\,du = \tan u + C$
15. $\int \csc^2 u\,du = -\cot u + C$
16. $\int \sec u \tan u\,du = \sec u + C$
17. $\int \csc u \cot u\,du = -\csc u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin \dfrac{u}{a} + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a} \arctan \dfrac{u}{a} + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a} \operatorname{arcsec} \dfrac{|u|}{a} + C$

You can learn a lot about the nature of integration by comparing this list with the summary of differentiation rules given in the preceding section. For differentiation, you now have rules that allow you to differentiate *any* elementary function. For integration, this is far from true.

The integration rules listed above are primarily those that we happened on when developing differentiation rules. We do not find integration rules for the antiderivative of a general product or quotient, the natural logarithmic function, or the inverse trigonometric functions. More importantly, you cannot apply any of the rules in this list unless you can create the proper du corresponding to the u in the formula. The point is that we need to work more on integration techniques, which we will do in Chapter 7. The next two examples should give you a better feeling for the integration problems that you *can* and *cannot* do with the techniques and rules you now know.

Example 6 Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniqu you have studied so far in the text.

a. $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$ **b.** $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-1}}$ **c.** $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$

Solution

a. You *can* find this integral (it fits the Arcsecant Rule).

$$\int \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arcsec}|x| + C$$

b. You *can* find this integral (it fits the Power Rule).

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{x^2-1}} &= \frac{1}{2}\int (x^2-1)^{-1/2}(2x)\,dx\\ &= \frac{1}{2}\left[\frac{(x^2-1)^{1/2}}{1/2}\right] + C\\ &= \sqrt{x^2-1} + C\end{aligned}$$

c. You *cannot* find this integral using present techniques. (You should scan the list basic integration rules to verify this conclusion.)

Example 7 Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniqu you have studied so far in the text.

a. $\displaystyle\int \frac{dx}{x\ln x}$ **b.** $\displaystyle\int \frac{\ln x\,dx}{x}$ **c.** $\displaystyle\int \ln x\,dx$

Solution

a. You *can* find this integral (it fits the Log Rule).

$$\begin{aligned}\int \frac{dx}{x\ln x} &= \int \frac{1/x}{\ln x}\,dx\\ &= \ln|\ln x| + C\end{aligned}$$

b. You *can* find this integral (it fits the Power Rule).

$$\begin{aligned}\int \frac{\ln x\,dx}{x} &= \int \left(\frac{1}{x}\right)(\ln x)^1\,dx\\ &= \frac{(\ln x)^2}{2} + C\end{aligned}$$

c. You *cannot* find this integral using present techniques.

NOTE Note in Examples 6 and 7 that the *simplest* functions are the ones that you cannot integrate.

EXERCISES FOR SECTION 5.9

Exercises 1–30, find or evaluate the integral.

$\int \frac{5}{\sqrt{9-x^2}}\,dx$ **2.** $\int \frac{3}{\sqrt{1-4x^2}}\,dx$

$\int_0^{1/6} \frac{1}{\sqrt{1-9x^2}}\,dx$ **4.** $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

$\int \frac{7}{16+x^2}\,dx$ **6.** $\int \frac{4}{1+9x^2}\,dx$

$\int_0^{\sqrt{3}/2} \frac{1}{1+4x^2}\,dx$ **8.** $\int_{\sqrt{3}}^3 \frac{1}{9+x^2}\,dx$

$\int \frac{1}{x\sqrt{4x^2-1}}\,dx$ **10.** $\int \frac{1}{4+(x-1)^2}\,dx$

$\int \frac{x^3}{x^2+1}\,dx$ **12.** $\int \frac{x^4-1}{x^2+1}\,dx$

$\int \frac{1}{\sqrt{1-(x+1)^2}}\,dx$ **14.** $\int \frac{t}{t^4+16}\,dt$

$\int \frac{t}{\sqrt{1-t^4}}\,dt$ **16.** $\int \frac{1}{x\sqrt{x^4-4}}\,dx$

$\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$ **18.** $\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}}\,dx$

$\int_{-1/2}^0 \frac{x}{\sqrt{1-x^2}}\,dx$ **20.** $\int_{-\sqrt{3}}^0 \frac{x}{1+x^2}\,dx$

$\int \frac{e^{2x}}{4+e^{4x}}\,dx$ **22.** $\int_1^2 \frac{1}{3+(x-2)^2}\,dx$

$\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$ **24.** $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx$

$\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx$ **26.** $\int \frac{3}{2\sqrt{x}(1+x)}\,dx$

$\int \frac{x-3}{x^2+1}\,dx$ **28.** $\int \frac{4x+3}{\sqrt{1-x^2}}\,dx$

$\int \frac{x+5}{\sqrt{9-(x-3)^2}}\,dx$ **30.** $\int \frac{x-2}{(x+1)^2+4}\,dx$

Exercises 31–42, find or evaluate the integral. (Complete the are, if necessary.)

$\int_0^2 \frac{dx}{x^2-2x+2}$ **32.** $\int_{-2}^2 \frac{dx}{x^2+4x+13}$

$\int \frac{2x}{x^2+6x+13}\,dx$ **34.** $\int \frac{2x-5}{x^2+2x+2}\,dx$

$\int \frac{1}{\sqrt{-x^2-4x}}\,dx$ **36.** $\int \frac{2}{\sqrt{-x^2+4x}}\,dx$

$\int \frac{x+2}{\sqrt{-x^2-4x}}\,dx$ **38.** $\int \frac{x-1}{\sqrt{x^2-2x}}\,dx$

$\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}\,dx$ **40.** $\int \frac{1}{(x-1)\sqrt{x^2-2x}}\,dx$

$\int \frac{x}{x^4+2x^2+2}\,dx$ **42.** $\int \frac{x}{\sqrt{9+8x^2-x^4}}\,dx$

In Exercises 43 and 44, use the specified substitution to find the integral.

43. $\int \sqrt{e^t-3}\,dt$

$u = \sqrt{e^t-3}$

44. $\int \frac{\sqrt{x-2}}{x+1}\,dx$

$u = \sqrt{x-2}$

Getting at the Concept

45. What is a perfect square trinomial?

46. What term must be added to $x^2 + 3x$ to complete the square? Explain how you found the term.

In Exercises 47–50, determine which of the integrals can be found using the basic integration formulas you have studied so far in the text.

47. (a) $\int \frac{1}{\sqrt{1-x^2}}\,dx$ (b) $\int \frac{x}{\sqrt{1-x^2}}\,dx$ (c) $\int \frac{1}{x\sqrt{1-x^2}}\,dx$

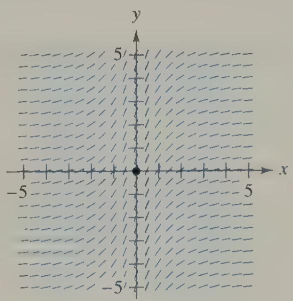
48. (a) $\int e^{x^2}\,dx$ (b) $\int xe^{x^2}\,dx$ (c) $\int \frac{1}{x^2}e^{1/x}\,dx$

49. (a) $\int \sqrt{x-1}\,dx$ (b) $\int x\sqrt{x-1}\,dx$ (c) $\int \frac{x}{\sqrt{x-1}}\,dx$

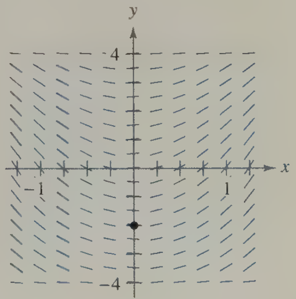
50. (a) $\int \frac{1}{1+x^4}\,dx$ (b) $\int \frac{x}{1+x^4}\,dx$ (c) $\int \frac{x^3}{1+x^4}\,dx$

Slope Fields **In Exercises 51 and 52, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the indicated point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

51. $\frac{dy}{dx} = \frac{3}{1+x^2}$, $(0, 0)$



52. $\frac{dy}{dx} = x\sqrt{16-y^2}$, $(0, -2)$



In Exercises 53 and 54, use a computer algebra system to graph the slope field for the differential equation and graph the solution satisfying the specified initial condition.

53. $\frac{dy}{dx} = \frac{10}{x\sqrt{x^2-1}}$

$y(3) = 0$

54. $\frac{dy}{dx} = \frac{2y}{\sqrt{16-x^2}}$

$y(0) = 2$

In Exercises 55 and 56, find the area of the region bounded by the graphs of the equations.

55. $y = \dfrac{1}{x^2 - 2x + 5}$, $y = 0$, $x = 1$, $x = 3$

56. $y = \dfrac{1}{\sqrt{4 - x^2}}$, $y = 0$, $x = 0$, $x = 1$

57. ***Approximation*** Determine which value best approximates the area of the region between the x-axis and the function

$$f(x) = \frac{1}{\sqrt{1 - x^2}}$$

over the interval $[-0.5, 0.5]$. (Make your selection on the basis of a sketch of the region and *not* by performing any calculations.)

(a) 4 (b) -3 (c) 1 (d) 2 (e) 3

58. ***Approximation*** Sketch the region whose area is represented by the integral

$$\int_0^1 \arcsin x \, dx$$

and use the integration capabilities of a graphing utility to approximate the area.

59. (a) Show that

$$\int_0^1 \frac{4}{1 + x^2} \, dx = \pi.$$

(b) Approximate the number π using Simpson's Rule (with $n = 6$) and the integral in part (a).

(c) Approximate the number π by using the integration capabilities of a graphing utility.

60. ***Investigation*** Consider the function

$$F(x) = \frac{1}{2} \int_x^{x+2} \frac{2}{t^2 + 1} \, dt.$$

(a) Write a short paragraph giving a geometric interpretation of the function $F(x)$ relative to the function

$$f(x) = \frac{2}{x^2 + 1}.$$

Use what you have written to guess the value of x that will make F maximum.

(b) Perform the specified integration to find an alternative form of $F(x)$. Use calculus to locate the value of x that will make F maximum and compare the result with your guess in part (a).

61. Verify each rule by differentiating $(a > 0)$.

(a) $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$

(b) $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

(c) $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

62. Consider the integral

$$\int \frac{1}{\sqrt{6x - x^2}} \, dx.$$

(a) Find the integral by completing the square of the radica

(b) Find the integral by making the substitution $u = \sqrt{x}$.

(c) The antiderivatives in parts (a) and (b) appear significan different. Use a graphing utility to graph each in the sa viewing window and determine the relationship betw the two antiderivatives. Find the domain of each.

63. ***Vertical Motion*** An object is projected upward from grou level with an initial velocity of 500 feet per second. In this ex cise, the goal is to analyze the motion of the object during upward flight.

(a) If air resistance is neglected, find the velocity of the obj as a function of time. Use a graphing utility to graph function.

(b) Use the result in part (a) to find the position function determine the maximum height attained by the object.

(c) If the air resistance is proportional to the square of velocity, you obtain the equation

$$\frac{dv}{dt} = -(32 + kv^2)$$

where -32 feet per second per second is the accelerat due to gravity and k is a constant. Find the velocity a function of time by solving the equation

$$\int \frac{dv}{32 + kv^2} = -\int dt.$$

(d) Use a graphing utility to graph the velocity function $v(t$ part (c) if $k = 0.001$. Use the graph to approximate the ti t_0 at which the object reaches its maximum height.

(e) Use the integration capabilities of a graphing utility approximate the integral

$$\int_0^{t_0} v(t) \, dt$$

where $v(t)$ and t_0 are those found in part (d). This is approximation of the maximum height of the object.

(f) Explain the difference between the results in parts (b) (e).

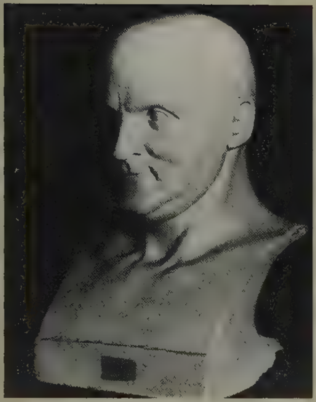
FOR FURTHER INFORMATION For more information on t topic, see "What Goes Up Must Come Down; Will Air Resistance Make It Return Sooner, or Later?" by John Lek in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

64. Graph $y_1 = \dfrac{x}{1 + x^2}$, $y_2 = \arctan x$, and $y_3 = x$ on $[0, 10]$.

Prove that $\dfrac{x}{1 + x^2} < \arctan x < x$ for $x > 0$.

Section 5.10 Hyperbolic Functions

- Develop properties of hyperbolic functions.
- Differentiate and integrate hyperbolic functions.
- Develop properties of inverse hyperbolic functions.
- Differentiate and integrate functions involving inverse hyperbolic functions.



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JOHANN HEINRICH LAMBERT (1728–1777)

The first person to publish a comprehensive study on hyperbolic functions was Johann Heinrich Lambert, a Swiss-German mathematician and colleague of Euler.

Hyperbolic Functions

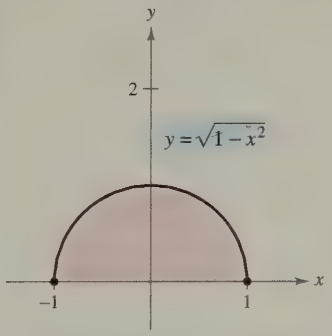
In this section you will look briefly at a special class of exponential functions called **hyperbolic functions.** The name *hyperbolic function* arose from comparison of the area of a semicircular region, as shown in Figure 5.45, with the area of a region under a hyperbola, as shown in Figure 5.46. The integral for the semicircular region involves an inverse trigonometric (circular) function:

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{1}{2}\Big[x\sqrt{1-x^2} + \arcsin x\Big]_{-1}^{1} = \frac{\pi}{2} \approx 1.571.$$

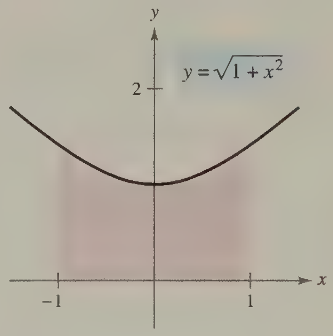
The integral for the hyperbolic region involves an inverse hyperbolic function:

$$\int_{-1}^{1} \sqrt{1+x^2}\,dx = \frac{1}{2}\Big[x\sqrt{1+x^2} + \sinh^{-1} x\Big]_{-1}^{1} \approx 2.296.$$

This is only one of many ways in which the hyperbolic functions are similar to the trigonometric functions.



Circle: $x^2 + y^2 = 1$
Figure 5.45



Hyperbola: $-x^2 + y^2 = 1$
Figure 5.46

OR FURTHER INFORMATION For nore information on the development of yperbolic functions, see the article "An ntroduction to Hyperbolic Functions in lementary Calculus" by Jerome osenthal in *Mathematics Teacher*. To iew this article, go to the website *ww.matharticles.com.*

Definition of the Hyperbolic Functions

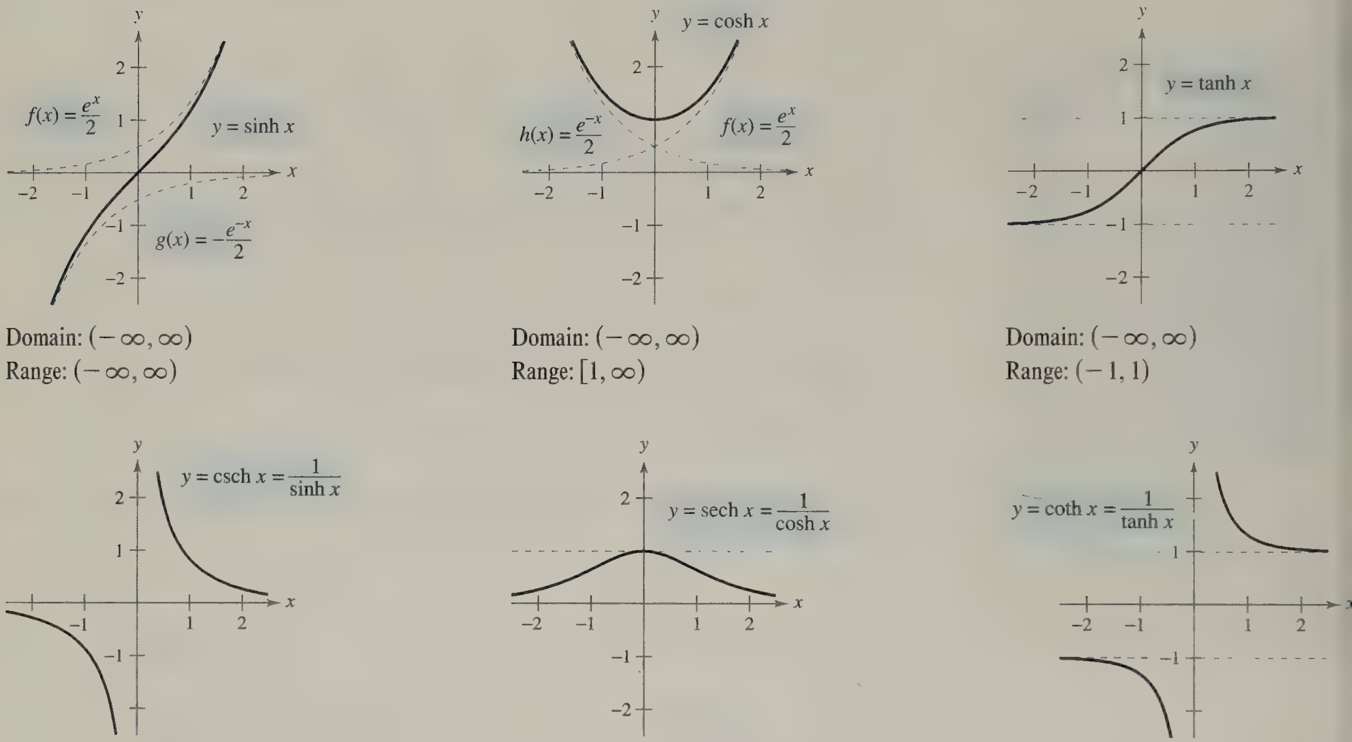
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}, \quad x \neq 0$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}, \quad x \neq 0$$

NOTE sinh x is read as "the hyperbolic sine of x," cosh x as "the hyperbolic cosine of x," and so on.

The graphs of the six hyperbolic functions and their domains and ranges are show in Figure 5.47. Note that the graph of $\sinh x$ can be obtained by *addition of ordinat* using the exponential functions $f(x) = \frac{1}{2}e^x$ and $g(x) = -\frac{1}{2}e^{-x}$. Likewise, the graph cosh x can be obtained by *addition of ordinates* using the exponential functio $f(x) = \frac{1}{2}e^x$ and $h(x) = \frac{1}{2}e^{-x}$.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Domain: $(-\infty, \infty)$
Range: $[1, \infty)$

Domain: $(-\infty, \infty)$
Range: $(-1, 1)$

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Domain: $(-\infty, \infty)$
Range: $(0, 1]$

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, -1) \cup (1, \infty)$

Figure 5.47

Many of the trigonometric identities have corresponding *hyperbolic identitie* For instance,

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}\\ &= \frac{4}{4}\\ &= 1\end{aligned}$$

and

$$\begin{aligned}2\sinh x \cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)\\ &= \frac{e^{2x} - e^{-2x}}{2}\\ &= \sinh 2x.\end{aligned}$$

FOR FURTHER INFORMATION To understand geometrically the relationship between the hyperbolic and exponential functions, see the article "A Short Proof Linking the Hyperbolic and Exponential Functions" by Michael J. Seery in *The AMATYC Review*. To view this article, go to the website *www.matharticles.com.*

Hyperbolic Identities

$$\cosh^2 x - \sinh^2 x = 1 \qquad \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1 \qquad \sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$$

$$\coth^2 x - \operatorname{csch}^2 x = 1 \qquad \cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$$

$$\sinh^2 x = \frac{-1 + \cosh 2x}{2} \qquad \cosh^2 x = \frac{1 + \cosh 2x}{2}$$

$$\sinh 2x = 2 \sinh x \cosh x \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$

Differentiation and Integration of Hyperbolic Functions

Because the hyperbolic functions are written in terms of e^x and e^{-x}, you can easily derive rules for their derivatives. The following theorem lists these derivatives with the corresponding integration rules.

THEOREM 5.20 Derivatives and Integrals of Hyperbolic Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh u] = (\cosh u)u' \qquad \int \cosh u\,du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = (\sinh u)u' \qquad \int \sinh u\,du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = (\operatorname{sech}^2 u)u' \qquad \int \operatorname{sech}^2 u\,du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -(\operatorname{csch}^2 u)u' \qquad \int \operatorname{csch}^2 u\,du = -\coth u + C$$

$$\frac{d}{dx}[\operatorname{sech} u] = -(\operatorname{sech} u \tanh u)u' \qquad \int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}[\operatorname{csch} u] = -(\operatorname{csch} u \coth u)u' \qquad \int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$$

Proof

$$\frac{d}{dx}[\sinh x] = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}[\tanh x] = \frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right] = \frac{\cosh x(\cosh x) - \sinh x(\sinh x)}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

In Exercises 87 and 91, you are asked to prove some of the other differentiation rules.

Example 1 Differentiation of Hyperbolic Functions

a. $\frac{d}{dx}[\sinh(x^2 - 3)] = 2x \cosh(x^2 - 3)$

b. $\frac{d}{dx}[\ln(\cosh x)] = \frac{\sinh x}{\cosh x} = \tanh x$

c. $\frac{d}{dx}[x \sinh x - \cosh x] = x \cosh x + \sinh x - \sinh x = x \cosh x$

Example 2 Finding Relative Extrema

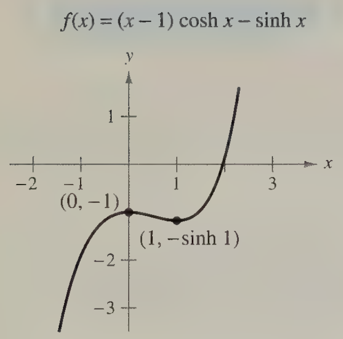
Find the relative extrema of $f(x) = (x - 1) \cosh x - \sinh x$.

Solution Begin by setting the first derivative of f equal to 0.

$$f'(x) = (x - 1) \sinh x + \cosh x - \cosh x = 0$$
$$(x - 1) \sinh x = 0$$

So, the critical numbers are $x = 1$ and $x = 0$. Using the Second Derivative Test, y can verify that the point $(0, -1)$ yields a relative maximum and the point $(1, -\sinh$ yields a relative minimum, as shown in Figure 5.48. Try using a graphing utility confirm this result. If your graphing utility does not have hyperbolic functions, y can use exponential functions as follows.

$$\begin{aligned} f(x) &= (x - 1)\left(\tfrac{1}{2}\right)(e^x + e^{-x}) - \tfrac{1}{2}(e^x - e^{-x}) \\ &= \tfrac{1}{2}(xe^x + xe^{-x} - e^x - e^{-x} - e^x + e^{-x}) \\ &= \tfrac{1}{2}(xe^x + xe^{-x} - 2e^x) \end{aligned}$$



$f''(0) < 0$, so $(0, -1)$ is a relative maximum. $f''(1) > 0$, so $(1, -\sinh 1)$ is a relative minimum.
Figure 5.48

When a uniform flexible cable, such as a telephone wire, is suspended from t points, it takes the shape of a **catenary,** as discussed in Example 3.

Example 3 Hanging Power Cables

Power cables are suspended between two towers, forming the catenary shown Figure 5.49. The equation for this catenary is

$$y = a \cosh \frac{x}{a}.$$

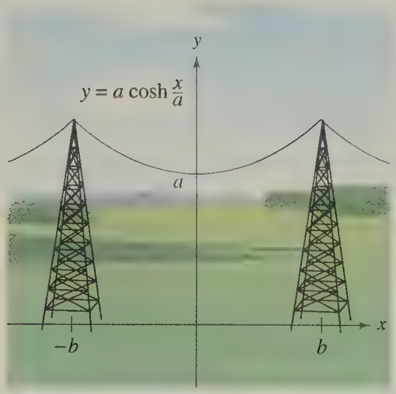
The distance between the two towers is $2b$. Find the slope of the catenary at the po where the cable meets the right-hand tower.

Solution Differentiating produces

$$y' = a\left(\frac{1}{a}\right) \sinh \frac{x}{a} = \sinh \frac{x}{a}.$$

At the point $(b, a \cosh(b/a))$, the slope (from the left) is given by

$$m = \sinh \frac{b}{a}.$$



Catenary
Figure 5.49

FOR FURTHER INFORMATION In Example 3, the cable is a catenary between t supports at the same height. To learn about the shape of a cable hanging betw supports of different heights, see the article "Reexamining the Catenary" by P Cella in *The College Mathematics Journal.* To view this article, go to the web *www.matharticles.com.*

Example 4 Integrating a Hyperbolic Function

Find $\int \cosh 2x \sinh^2 2x \, dx$.

Solution

$$\begin{aligned}\int \cosh 2x \sinh^2 2x \, dx &= \frac{1}{2}\int (\sinh 2x)^2(2\cosh 2x)\, dx \qquad u = \sinh 2x\\ &= \frac{1}{2}\left[\frac{(\sinh 2x)^3}{3}\right] + C\\ &= \frac{\sinh^3 2x}{6} + C\end{aligned}$$

Inverse Hyperbolic Functions

Unlike trigonometric functions, hyperbolic functions are not periodic. In fact, by looking back at Figure 5.47, you can see that four of the six hyperbolic functions are actually one-to-one (the hyperbolic sine, tangent, cosecant, and cotangent). So, you can apply Theorem 5.7 to conclude that these four functions have inverse functions. The other two (the hyperbolic cosine and secant) are one-to-one if their domains are restricted to the positive real numbers, and for this restricted domain they also have inverse functions. Because the hyperbolic functions are defined in terms of exponential functions, it is not surprising to find that the inverse hyperbolic functions can be written in terms of logarithmic functions, as shown in Theorem 5.21.

THEOREM 5.21 Inverse Hyperbolic Functions

Function	*Domain*
$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$	$(-\infty, \infty)$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$	$[1, \infty)$
$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$	$(-1, 1)$
$\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1}$	$(-\infty, -1) \cup (1, \infty)$
$\operatorname{sech}^{-1} x = \ln\frac{1+\sqrt{1-x^2}}{x}$	$(0, 1]$
$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	$(-\infty, 0) \cup (0, \infty)$

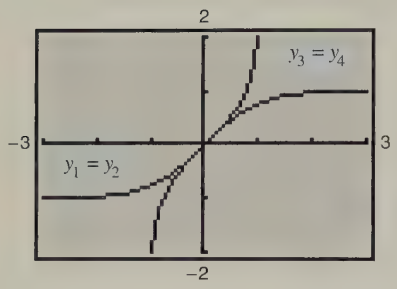
Proof The proof of this theorem is a straightforward application of the properties of the exponential and logarithmic functions. For example, if

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$

and

$$g(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

you can show that $f(g(x)) = x$ and $g(f(x)) = x$, which implies that g is the inverse function of f.



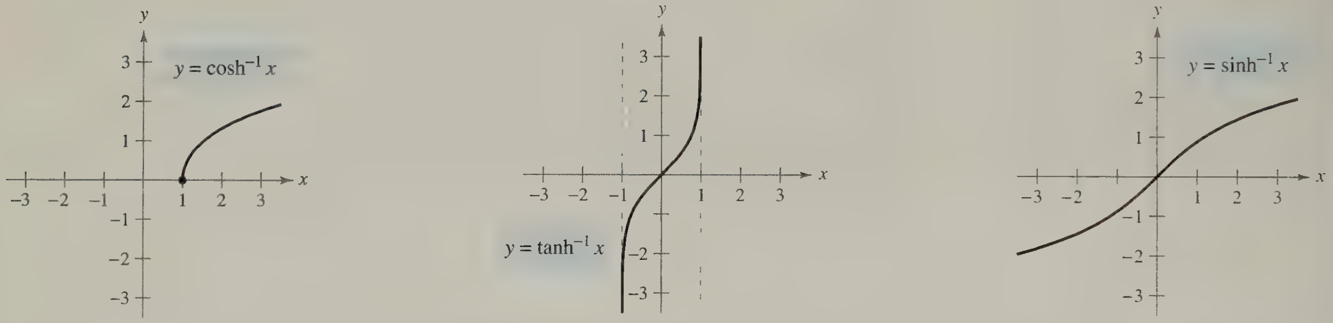
Graphs of the hyperbolic tangent function and the inverse hyperbolic tangent function
Figure 5.50

TECHNOLOGY You can use a graphing utility to confirm graphically th results of Theorem 5.21. For instance, try sketching the graphs of the followir functions.

$y_1 = \tanh x$	Hyperbolic tangent
$y_2 = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	Definition of hyperbolic tangent
$y_3 = \tanh^{-1} x$	Inverse hyperbolic tangent
$y_4 = \dfrac{1}{2} \ln \dfrac{1 + x}{1 - x}$	Definition of inverse hyperbolic tangent

The resulting display is shown in Figure 5.50. As you watch the graphs being traced out, notice that $y_1 = y_2$ and $y_3 = y_4$. Also notice that the graph of y_1 is the reflection of the graph of y_3 in the line $y = x$.

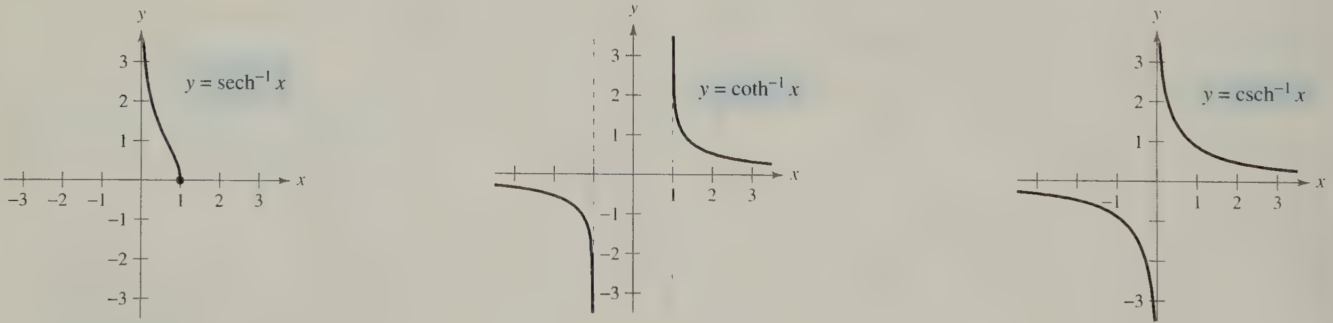
The graphs of the inverse hyperbolic functions are shown in Figure 5.51.



Domain: $[1, \infty)$
Range: $[0, \infty)$

Domain: $(-1, 1)$
Range: $(-\infty, \infty)$

Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



Domain: $(0, 1]$
Range: $[0, \infty)$

Domain: $(-\infty, -1) \cup (1, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Figure 5.51

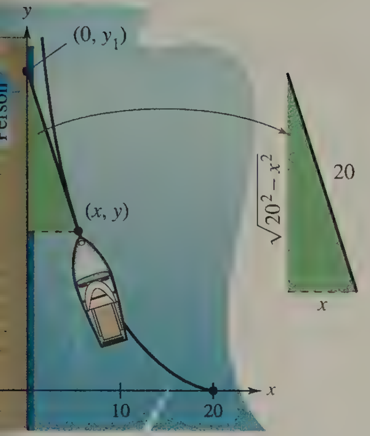
The inverse hyperbolic secant can be used to define a curve called a *tractrix* *pursuit curve*, as discussed in Example 5.

Example 5 A Tractrix

A person is holding a rope that is tied to a boat, as shown in Figure 5.52. As the person walks along the dock, the boat travels along a **tractrix,** given by the equation

$$y = a\ \text{sech}^{-1}\frac{x}{a} - \sqrt{a^2 - x^2}$$

where a is the length of the rope. If $a = 20$ feet, find the distance the person must walk to bring the boat 5 feet from the dock.



$= 20\ \text{sech}^{-1}\frac{x}{20} - \sqrt{20^2 - x^2}$

erson must walk 41.27 feet to bring the
at 5 feet from the dock.
gure 5.52

Solution In Figure 5.52, notice that the distance the person has walked is given by

$$\begin{aligned} y_1 = y + \sqrt{20^2 - x^2} &= \left(20\ \text{sech}^{-1}\frac{x}{20} - \sqrt{20^2 - x^2}\right) + \sqrt{20^2 - x^2} \\ &= 20\ \text{sech}^{-1}\frac{x}{20}. \end{aligned}$$

When $x = 5$, this distance is

$$\begin{aligned} y_1 = 20\ \text{sech}^{-1}\frac{5}{20} &= 20 \ln \frac{1 + \sqrt{1 - (1/4)^2}}{1/4} \\ &= 20 \ln\left(4 + \sqrt{15}\right) \\ &\approx 41.27 \text{ feet.} \end{aligned}$$

Differentiation and Integration of Inverse Hyperbolic Functions

The derivatives of the inverse hyperbolic functions, which resemble the derivatives of the inverse trigonometric functions, are listed in Theorem 5.22 with the corresponding integration formulas (in logarithmic form). You can verify each of these formulas by applying the logarithmic definitions of the inverse hyperbolic functions. (See Exercises 88–90.)

THEOREM 5.22 Differentiation and Integration Involving Inverse Hyperbolic Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh^{-1} u] = \frac{u'}{\sqrt{u^2 + 1}} \qquad \frac{d}{dx}[\cosh^{-1} u] = \frac{u'}{\sqrt{u^2 - 1}}$$

$$\frac{d}{dx}[\tanh^{-1} u] = \frac{u'}{1 - u^2} \qquad \frac{d}{dx}[\coth^{-1} u] = \frac{u'}{1 - u^2}$$

$$\frac{d}{dx}[\text{sech}^{-1} u] = \frac{-u'}{u\sqrt{1 - u^2}} \qquad \frac{d}{dx}[\text{csch}^{-1} u] = \frac{-u'}{|u|\sqrt{1 + u^2}}$$

$$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$$

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln\left|\frac{a + u}{a - u}\right| + C$$

$$\int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a} \ln \frac{a + \sqrt{a^2 \pm u^2}}{|u|} + C$$

Example 6 More About a Tractrix

For the tractrix given in Example 5, show that the boat is always pointing toward

person.

Solution For a point (x, y) on a tractrix, the slope of the graph gives the direction
the boat, as shown in Figure 5.52.

$$\begin{aligned} y' &= \frac{d}{dx}\left[20 \operatorname{sech}^{-1}\frac{x}{20} - \sqrt{20^2 - x^2}\right] \\ &= -20\left(\frac{1}{20}\right)\left[\frac{1}{(x/20)\sqrt{1 - (x/20)^2}}\right] - \left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{20^2 - x^2}}\right) \\ &= \frac{-20^2}{x\sqrt{20^2 - x^2}} + \frac{x}{\sqrt{20^2 - x^2}} \\ &= -\frac{\sqrt{20^2 - x^2}}{x} \end{aligned}$$

However, from Figure 5.52, you can see that the slope of the line segment connecti
the point $(0, y_1)$ with the point (x, y) is also $m = \left(-\sqrt{20^2 - x^2}\right)/x$. Thus, the boat
always pointing toward the person. (It is because of this property that a tractrix
called a *pursuit curve*.)

Example 7 Integration Using Inverse Hyperbolic Functions

Find $\displaystyle\int \frac{dx}{x\sqrt{4 - 9x^2}}$.

Solution Let $a = 2$ and $u = 3x$.

$$\begin{aligned} \int \frac{dx}{x\sqrt{4 - 9x^2}} &= \int \frac{3\,dx}{(3x)\sqrt{4 - 9x^2}} & \int \frac{du}{u\sqrt{a^2 - u^2}} \\ &= -\frac{1}{2}\ln\frac{2 + \sqrt{4 - 9x^2}}{|3x|} + C & -\frac{1}{a}\ln\frac{a + \sqrt{a^2 - u^2}}{|u|} + C \end{aligned}$$

Example 8 Integration Using Inverse Hyperbolic Functions

Find $\displaystyle\int \frac{dx}{5 - 4x^2}$.

Solution Let $a = \sqrt{5}$ and $u = 2x$.

$$\begin{aligned} \int \frac{dx}{5 - 4x^2} &= \frac{1}{2}\int \frac{2\,dx}{(\sqrt{5})^2 - (2x)^2} & \int \frac{du}{a^2 - u^2} \\ &= \frac{1}{2}\left(\frac{1}{2\sqrt{5}}\ln\left|\frac{\sqrt{5} + 2x}{\sqrt{5} - 2x}\right|\right) + C & \frac{1}{2a}\ln\left|\frac{a + u}{a - u}\right| + C \\ &= \frac{1}{4\sqrt{5}}\ln\left|\frac{\sqrt{5} + 2x}{\sqrt{5} - 2x}\right| + C \end{aligned}$$

EXERCISES FOR SECTION 5.10

Exercises 1–6, evaluate the function. If the value is not a tional number, give the answer to three-decimal-place curacy.

. (a) $\sinh 3$
(b) $\tanh(-2)$

2. (a) $\cosh 0$
(b) $\operatorname{sech} 1$

. (a) $\operatorname{csch}(\ln 2)$
(b) $\coth(\ln 5)$

4. (a) $\sinh^{-1} 0$
(b) $\tanh^{-1} 0$

. (a) $\cosh^{-1} 2$
(b) $\operatorname{sech}^{-1} \frac{2}{3}$

6. (a) $\operatorname{csch}^{-1} 2$
(b) $\coth^{-1} 3$

Exercises 7–12, verify the identity.

. $\tanh^2 x + \operatorname{sech}^2 x = 1$

. $\cosh^2 x = \dfrac{1 + \cosh 2x}{2}$

. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

. $\sinh 2x = 2 \sinh x \cosh x$

. $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$

. $\cosh x + \cosh y = 2 \cosh \dfrac{x + y}{2} \cosh \dfrac{x - y}{2}$

Exercises 13 and 14, use the value of the given hyperbolic nction to find the other hyperbolic functions.

. $\sinh x = \frac{3}{2}$, $\cosh x =$ ▨, $\tanh x =$ ▨,
$\operatorname{csch} x =$ ▨, $\operatorname{sech} x =$ ▨, $\coth x =$ ▨

. $\sinh x =$ ▨, $\cosh x =$ ▨, $\tanh x = \frac{1}{2}$,
$\operatorname{csch} x =$ ▨, $\operatorname{sech} x =$ ▨, $\coth x =$ ▨

Exercises 15–28, find the derivative of the function.

. $y = \sinh(1 - x^2)$

16. $y = \coth 3x$

. $f(x) = \ln(\sinh x)$

18. $g(x) = \ln(\cosh x)$

. $y = \ln\left(\tanh \dfrac{x}{2}\right)$

20. $y = x \cosh x - \sinh x$

$h(x) = \dfrac{1}{4} \sinh 2x - \dfrac{x}{2}$

22. $h(t) = t - \coth t$

$f(t) = \arctan(\sinh t)$

24. $g(x) = \operatorname{sech}^2 3x$

$g(x) = x^{\cosh x}$

26. $f(x) = e^{\sinh x}$

$y = (\cosh x - \sinh x)^2$

28. $y = \operatorname{sech}(x + 1)$

Exercises 29 and 30, find any relative extrema of the function d use a graphing utility to confirm your result.

$f(x) = \sin x \sinh x - \cos x \cosh x, \quad -4 \le x \le 4$

$f(x) = x \sinh(x - 1) - \cosh(x - 1)$

Exercises 31 and 32, use a graphing utility to graph the func- n and approximate any relative extrema of the function.

$g(x) = x \operatorname{sech} x$

32. $h(x) = 2 \tanh x - x$

In Exercises 33 and 34, show that the function satisfies the differential equation.

	Function	*Differential Equation*
33.	$y = a \sinh x$	$y''' - y' = 0$
34.	$y = a \cosh x$	$y'' - y = 0$

Linear and Quadratic Approximations **In Exercises 35 and 36, use a computer algebra system to find the linear approximation**

$$P_1(x) = f(a) + f'(a)(x - a)$$

and the quadratic approximation

$$P_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2} f''(a)(x - a)^2$$

to the function f at $x = a$. Use a graphing utility to graph the function and its linear and quadratic approximations.

35. $f(x) = \tanh x$
$a = 1$

36. $f(x) = \cosh x$
$a = 0$

Catenary **In Exercises 37 and 38, a model for power cables suspended between two towers is given. (a) Graph the model, (b) find the height of the cable at the towers and at the midpoint between the towers, and (c) find the slope of the model at the point where the cable meets the right-hand tower.**

37. $y = 10 + 15 \cosh \dfrac{x}{15}, \quad -15 \le x \le 15$

38. $y = 18 + 25 \cosh \dfrac{x}{25}, \quad -25 \le x \le 25$

In Exercises 39–54, find or evaluate the integral.

39. $\displaystyle\int \sinh(1 - 2x)\, dx$

40. $\displaystyle\int \frac{\cosh \sqrt{x}}{\sqrt{x}}\, dx$

41. $\displaystyle\int \cosh^2(x - 1) \sinh(x - 1)\, dx$

42. $\displaystyle\int \frac{\sinh x}{1 + \sinh^2 x}\, dx$

43. $\displaystyle\int \frac{\cosh x}{\sinh x}\, dx$

44. $\displaystyle\int \operatorname{sech}^2(2x - 1)\, dx$

45. $\displaystyle\int x \operatorname{csch}^2 \frac{x^2}{2}\, dx$

46. $\displaystyle\int \operatorname{sech}^3 x \tanh x\, dx$

47. $\displaystyle\int \frac{\operatorname{csch}(1/x) \coth(1/x)}{x^2}\, dx$

48. $\displaystyle\int \cosh^2 x\, dx$

49. $\displaystyle\int_0^4 \frac{1}{25 - x^2}\, dx$

50. $\displaystyle\int_0^4 \frac{1}{\sqrt{25 - x^2}}\, dx$

51. $\displaystyle\int_0^{\sqrt{2}/4} \frac{2}{\sqrt{1 - 4x^2}}\, dx$

52. $\displaystyle\int \frac{2}{x\sqrt{1 + 4x^2}}\, dx$

53. $\displaystyle\int \frac{x}{x^4 + 1}\, dx$

54. $\displaystyle\int \frac{\cosh x}{\sqrt{9 - \sinh^2 x}}\, dx$

In Exercises 55–62, find the derivative of the function.

55. $y = \cosh^{-1}(3x)$

56. $y = \tanh^{-1}\frac{x}{2}$

57. $y = \sinh^{-1}(\tan x)$

58. $y = \operatorname{sech}^{-1}(\cos 2x), \quad 0 < x < \pi/4$

59. $y = \coth^{-1}(\sin 2x)$

60. $y = (\operatorname{csch}^{-1} x)^2$

61. $y = 2x \sinh^{-1}(2x) - \sqrt{1 + 4x^2}$

62. $y = x \tanh^{-1} x + \ln\sqrt{1 - x^2}$

Getting at the Concept

63. Define the hyperbolic functions.

64. List the rules for differentiating inverse hyperbolic functions. List the corresponding integration formulas.

Tractrix **In Exercises 65 and 66, use the equation of the tractrix** $y = a \operatorname{sech}^{-1}\frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0.$

65. Find dy/dx.

66. Let L be the tangent line to the tractrix at the point P. If L intersects the y-axis at the point Q, show that the distance between P and Q is a.

In Exercises 67–74, find the indefinite integral using the formulas of Theorem 5.22.

67. $\int \frac{1}{\sqrt{1 + e^{2x}}}\,dx$

68. $\int \frac{x}{9 - x^4}\,dx$

69. $\int \frac{1}{\sqrt{x}\sqrt{1 + x}}\,dx$

70. $\int \frac{\sqrt{x}}{\sqrt{1 + x^3}}\,dx$

71. $\int \frac{-1}{4x - x^2}\,dx$

72. $\int \frac{dx}{(x + 2)\sqrt{x^2 + 4x + 8}}$

73. $\int \frac{1}{1 - 4x - 2x^2}\,dx$

74. $\int \frac{dx}{(x + 1)\sqrt{2x^2 + 4x + 8}}$

In Exercises 75–78, solve the differential equation.

75. $\frac{dy}{dx} = \frac{1}{\sqrt{80 + 8x - 16x^2}}$

76. $\frac{dy}{dx} = \frac{1}{(x - 1)\sqrt{-4x^2 + 8x - 1}}$

77. $\frac{dy}{dx} = \frac{x^3 - 21x}{5 + 4x - x^2}$

78. $\frac{dy}{dx} = \frac{1 - 2x}{4x - x^2}$

In Exercises 79–82, find the area of the region bounded by the graphs of the equations.

79. $y = \operatorname{sech}\frac{x}{2}, \quad y = 0, \quad x = -4, \quad x = 4$

80. $y = \tanh 2x, \quad y = 0, \quad x = 2$

81. $y = \frac{5x}{\sqrt{x^4 + 1}}, \quad y = 0, \quad x = 2$

82. $y = \frac{6}{\sqrt{x^2 - 4}}, \quad y = 0, \quad x = 3, \quad x = 5$

83. ***Chemical Reactions*** Suppose that chemicals A and B co
bine in a 3-to-1 ratio to form a compound. The amount
compound x being produced at any time t is proportional to
unchanged amounts of A and B remaining in the solution.
if 3 kilograms of A is mixed with 2 kilograms of B, you hav

$$\frac{dx}{dt} = k\left(3 - \frac{3x}{4}\right)\left(2 - \frac{x}{4}\right) = \frac{3k}{16}(x^2 - 12x + 32).$$

If 1 kilogram of the compound is formed after 10 minutes, f
the amount formed after 20 minutes by solving the integ
equation

$$\int \frac{3k}{16}\,dt = \int \frac{dx}{x^2 - 12x + 32}.$$

84. ***Vertical Motion*** An object is dropped from a height of
feet.

(a) Find the velocity of the object as a function of time (negl
air resistance on the object).

(b) Use the result in part (a) to find the position function.

(c) If the air resistance is proportional to the square of
velocity, then

$$\frac{dv}{dt} = -32 + kv^2$$

where -32 feet per second per second is the accelerat
due to gravity and k is a constant. Show that the velocit
as a function of time is

$$v(t) = -\sqrt{\frac{32}{k}}\tanh\left(\sqrt{32k}\,t\right)$$

by performing the following integration and simplify
the result.

$$\int \frac{dv}{32 - kv^2} = -\int dt$$

(d) Use the result in part (c) to find $\lim_{t\to\infty} v(t)$ and give
interpretation.

(e) Integrate the velocity function in part (c) and find
position s of the object as a function of t. Use a graph
utility to graph the position function when $k = 0.01$ and
position function in part (b) in the same viewing wind
Estimate the additional time required for the object to re
ground level when air resistance is not neglected.

85. ***Writing*** Give a written description of what you believe wo
happen if k were increased in Exercise 84. Then test your as
tion with a particular value of k.

86. Show that $\arctan(\sinh x) = \arcsin(\tanh x)$.

In Exercises 87–91, verify the differentiation formula.

87. $\frac{d}{dx}[\cosh x] = \sinh x$

88. $\frac{d}{dx}[\operatorname{sech}^{-1} x] = \frac{-1}{x\sqrt{1 -}}$

89. $\frac{d}{dx}[\cosh^{-1} x] = \frac{1}{\sqrt{x^2 - 1}}$

90. $\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{x^2 +}}$

91. $\frac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x$

SECTION PROJECT ST. LOUIS ARCH

Superstock, Inc.

The Gateway Arch in St. Louis, Missouri was constructed using the hyperbolic cosine function. The equation used to construct the arch was

$$y = 693.8597 - 68.7672 \cosh 0.0100333x,$$

$$-299.2239 \le x \le 299.2239$$

where x and y are measured in feet. Cross sections of the arch are equilateral triangles, and (x, y) traces the path of the centers of mass of the cross-sectional triangles. For each value of x, the area of the cross-sectional triangle is

$$A = 125.1406 \cosh 0.0100333x.$$

(*Source:* Owner's Manual for the Gateway Arch, *Saint Louis, MO, by William Thayer*)

(a) How high above the ground is the center of the highest triangle? (At ground level, $y = 0$.)

(b) What is the height of the arch? (*Hint:* For an equilateral triangle, $A = \sqrt{3}c^2$, where c is one-half the base of the triangle, and the center of mass of the triangle is located at two-thirds the height of the triangle.)

(c) How wide is the arch at ground level?

REVIEW EXERCISES FOR CHAPTER 5

5.1 **In Exercises 1 and 2, sketch the graph of the function y hand. Identify any asymptotes of the graph.**

$f(x) = \ln x + 3$

2. $f(x) = \ln(x - 3)$

Exercises 3 and 4, use the properties of logarithms to xpand the logarithmic function.

$\ln \sqrt[5]{\dfrac{4x^2 - 1}{4x^2 + 1}}$

4. $\ln[(x^2 + 1)(x - 1)]$

Exercises 5 and 6, write the expression as the logarithm of a ngle quantity.

$\ln 3 + \frac{1}{3}\ln(4 - x^2) - \ln x$

$3[\ln x - 2\ln(x^2 + 1)] + 2\ln 5$

Exercises 7 and 8, solve the equation for x.

$\ln \sqrt{x + 1} = 2$

8. $\ln x + \ln(x - 3) = 0$

Exercises 9–16, find the derivative of the function.

$g(x) = \ln \sqrt{x}$

10. $h(x) = \ln \dfrac{x(x - 1)}{x - 2}$

11. $f(x) = x\sqrt{\ln x}$

12. $f(x) = \ln[x(x^2 - 2)^{2/3}]$

13. $y = \dfrac{1}{b^2}\left[\ln(a + bx) + \dfrac{a}{a + bx}\right]$

14. $y = \dfrac{1}{b^2}[a + bx - a\ln(a + bx)]$

15. $y = -\dfrac{1}{a}\ln\dfrac{a + bx}{x}$

16. $y = -\dfrac{1}{ax} + \dfrac{b}{a^2}\ln\dfrac{a + bx}{x}$

5.2 **In Exercises 17–24, find or evaluate the integral.**

17. $\displaystyle\int \frac{1}{7x - 2}\,dx$

18. $\displaystyle\int \frac{x}{x^2 - 1}\,dx$

19. $\displaystyle\int \frac{\sin x}{1 + \cos x}\,dx$

20. $\displaystyle\int \frac{\ln \sqrt{x}}{x}\,dx$

21. $\displaystyle\int_1^4 \frac{x + 1}{x}\,dx$

22. $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

23. $\displaystyle\int_0^{\pi/3} \sec\theta\,d\theta$

24. $\displaystyle\int_0^{\pi/4} \tan\left(\frac{\pi}{4} - x\right)dx$

5.3 **In Exercises 25–30, (a) find the inverse of the function, (b) use a graphing utility to graph f and f^{-1} in the same viewing window, and (c) verify that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.**

25. $f(x) = \frac{1}{2}x - 3$ **26.** $f(x) = 5x - 7$

27. $f(x) = \sqrt{x + 1}$ **28.** $f(x) = x^3 + 2$

29. $f(x) = \sqrt[3]{x + 1}$ **30.** $f(x) = x^2 - 5, \quad x \geq 0$

In Exercise 31–34, find $(f^{-1})'(a)$ for the function f and real number a.

	Function	*Real number*
31.	$f(x) = x^3 + 2$	$a = -1$
32.	$f(x) = x\sqrt{x - 3}$	$a = 4$
33.	$f(x) = \tan x, \quad -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$	$a = \frac{\sqrt{3}}{3}$
34.	$f(x) = \ln x$	$a = 0$

5.4 **In Exercises 35 and 36, (a) find the inverse function of f, (b) using a graphing utility to graph f and f^{-1} in the same viewing window, and (c) verify that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.**

35. $f(x) = \ln \sqrt{x}$ **36.** $f(x) = e^{1-x}$

In Exercises 37 and 38, graph the function without the aid of a graphing utility.

37. $y = e^{-x/2}$ **38.** $y = 4e^{-x^2}$

In Exercises 39–46, find the derivative of the function.

39. $f(x) = \ln(e^{-x^2})$ **40.** $g(x) = \ln \dfrac{e^x}{1 + e^x}$

41. $g(t) = t^2e^t$ **42.** $h(z) = e^{-z^2/2}$

43. $y = \sqrt{e^{2x} + e^{-2x}}$ **44.** $y = 3e^{-3/t}$

45. $g(x) = \dfrac{x^2}{e^x}$ **46.** $f(\theta) = \frac{1}{2}e^{\sin 2\theta}$

In Exercises 47 and 48, use implicit differentiation to find dy/dx.

47. $y \ln x + y^2 = 0$ **48.** $\cos x^2 = xe^y$

In Exercises 49–56, find the indefinite integral.

49. $\int xe^{-3x^2}\,dx$ **50.** $\int \frac{e^{1/x}}{x^2}\,dx$

51. $\int \frac{e^{4x} - e^{2x} + 1}{e^x}\,dx$ **52.** $\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx$

53. $\int xe^{1-x^2}\,dx$ **54.** $\int x^2e^{x^3+1}\,dx$

55. $\int \frac{e^x}{e^x - 1}\,dx$ **56.** $\int \frac{e^{2x}}{e^{2x} + 1}\,dx$

57. Show that $y = e^x(a \cos 3x + b \sin 3x)$ satisfies the differential equation $y'' - 2y' + 10y = 0$.

58. ***Depreciation*** The value V of an item t years after it purchased is

$$V = 8000e^{-0.6t}, \quad 0 \leq t \leq 5.$$

(a) Use a graphing utility to graph the function.

(b) Find the rate of change of V with respect to t when $t =$ and $t = 4$.

(c) Use a graphing utility to sketch the tangent line to the fun tion when $t = 1$ and $t = 4$.

In Exercises 59 and 60, find the area of the region bounded the graphs of the equations.

59. $y = xe^{-x^2}, \quad y = 0, \quad x = 0, \quad x = 4$

60. $y = 2e^{-x}, \quad y = 0, \quad x = 0, \quad x = 2$

5.5 **In Exercises 61–64, sketch the graph of the function hand.**

61. $y = 3^{x/2}$ **62.** $y = 6(2^{-x^2})$

63. $y = \log_2(x - 1)$ **64.** $y = \log_4 x^2$

In Exercises 65–70, find the derivative of the function.

65. $f(x) = 3^{x-1}$ **66.** $f(x) = (4e)^x$

67. $y = x^{2x+1}$ **68.** $y = x(4^{-x})$

69. $g(x) = \log_3 \sqrt{1 - x}$ **70.** $h(x) = \log_5 \dfrac{x}{x - 1}$

In Exercises 71 and 72, find the indefinite integral.

71. $\int (x + 1)5^{(x+1)^2}\,dx$ **72.** $\int \frac{2^{1/t}}{t^2}\,dt$

73. ***Think About It*** Find the derivative of each function, giv that a is constant.

(a) $y = x^a$ (b) $y = a^x$ (c) $y = x^x$ (d) $y = a^a$

74. ***Climb Rate*** The time t (in minutes) for a small plane to cli to an altitude of h feet is

$$t = 50 \log_{10} \frac{18{,}000}{18{,}000 - h}$$

where 18,000 feet is the plane's absolute ceiling.

(a) Determine the domain of the function appropriate for t context of the problem.

(b) Use a graphing utility to graph the time function a identify any asymptotes.

(c) Find the time when the altitude is increasing at the great rate.

75. ***Compound Interest*** How large a deposit, at 7 percent inter compounded continuously, must be made to obtain a balance $10,000 in 15 years?

76. ***Compound Interest*** A deposit earns interest at a rate o percent compounded continuously and doubles in value in years. Find r.

.6

Air Pressure Under ideal conditions, air pressure decreases continuously with height above sea level at a rate proportional to the pressure at that height. If the barometer reads 30 inches at sea level and 15 inches at 18,000 feet, find the barometric pressure at 35,000 feet.

Radioactive Decay Radioactive radium has a half-life of approximately 1620 years. If the initial quantity is 5 grams, how much remains after 600 years?

Population Growth A population grows continuously at the rate of 1.5%. How long will it take the population to double?

Fuel Economy A certain automobile gets 28 miles per gallon of gasoline for speeds up to 50 miles per hour. Over 50 miles per hour, the number of miles per gallon drops at the rate of 12 percent for each 10 miles per hour.

(a) If s is the speed and y is the number of miles per gallon, find y as a function of s by solving the differential equation

$$\frac{dy}{ds} = -0.012y, \quad s > 50.$$

(b) Use the function in part (a) to complete the table.

Speed	50	55	60	65	70
Miles per gallon					

.7 **In Exercises 81–86, solve the differential equation.**

$\dfrac{dy}{dx} = \dfrac{x^2+3}{x}$

82. $\dfrac{dy}{dx} = \dfrac{e^{-2x}}{1+e^{-2x}}$

$y' - 2xy = 0$

84. $y' - e^y \sin x = 0$

$\dfrac{dy}{dx} = \dfrac{x^2+y^2}{2xy}$

86. $\dfrac{dy}{dx} = \dfrac{3(x+y)}{x}$

Verify that the general solution $y = C_1x + C_2x^3$ satisfies the differential equation $x^2y'' - 3xy' + 3y = 0$. Then find the particular solution that satisfies the initial condition $y = 0$ and $y' = 4$ when $x = 2$.

Vertical Motion A falling object encounters air resistance that is proportional to its velocity. If the acceleration due to gravity is -9.8 meters per second per second, the net change in velocity is

$$\frac{dv}{dt} = kv - 9.8.$$

(a) Find the velocity of the object as a function of time if the initial velocity is v_0.

(b) Use the result in part (a) to find the limit of the velocity as t approaches infinity.

(c) Integrate the velocity function found in part (a) to find the position function s.

5.8 **In Exercises 89 and 90, sketch the graph of the function by hand.**

89. $f(x) = 2\arctan(x+3)$

90. $h(x) = -3\arcsin 2x$

In Exercises 91 and 92, evaluate the expression without using a calculator. (*Hint:* Make a sketch of a right triangle.)

91. (a) $\sin\left(\arcsin \frac{1}{2}\right)$
(b) $\cos\left(\arcsin \frac{1}{2}\right)$

92. (a) $\tan(\operatorname{arccot} 2)$
(b) $\cos\left(\operatorname{arcsec} \sqrt{5}\right)$

In Exercises 93–98, find the derivative of the function.

93. $y = \tan(\arcsin x)$

94. $y = \arctan(x^2 - 1)$

95. $y = x \operatorname{arcsec} x$

96. $y = \frac{1}{2}\arctan e^{2x}$

97. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2}\arcsin x$

98. $y = \sqrt{x^2-4} - 2\operatorname{arcsec}\dfrac{x}{2}, \quad 2 < x < 4$

5.9 **In Exercises 99–106, find the indefinite integral.**

99. $\displaystyle\int \frac{1}{e^{2x}+e^{-2x}}\,dx$

100. $\displaystyle\int \frac{1}{3+25x^2}\,dx$

101. $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$

102. $\displaystyle\int \frac{1}{16+x^2}\,dx$

103. $\displaystyle\int \frac{x}{16+x^2}\,dx$

104. $\displaystyle\int \frac{4-x}{\sqrt{4-x^2}}\,dx$

105. $\displaystyle\int \frac{\arctan(x/2)}{4+x^2}\,dx$

106. $\displaystyle\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

107. ***Harmonic Motion*** A weight of mass m is attached to a spring and oscillates with simple harmonic motion. By Hooke's Law, you can determine that

$$\int \frac{dy}{\sqrt{A^2-y^2}} = \int \sqrt{\frac{k}{m}}\,dt$$

where A is the maximum displacement, t is the time, and k is a constant. Find y as a function of t, given that $y = 0$ when $t = 0$.

108. ***Think About It*** Sketch the region whose area is given by $\int_0^1 \arcsin x\,dx$. Then find the area of the region. Explain how you arrived at your answer.

5.10 **In Exercises 109 and 110, find the derivative of the function.**

109. $y = 2x - \cosh\sqrt{x}$

110. $y = x\tanh^{-1} 2x$

In Exercises 111 and 112, find the indefinite integral.

111. $\displaystyle\int \frac{x}{\sqrt{x^4-1}}\,dx$

112. $\displaystyle\int x^2 \operatorname{sech}^2 x^3\,dx$

P.S. Problem Solving

1. Find the value of a that maximizes the angle θ indicated in the figure. What is the approximate measure of this angle?

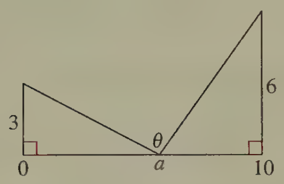


Figure for 1

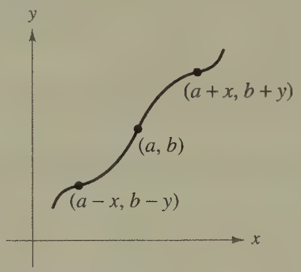


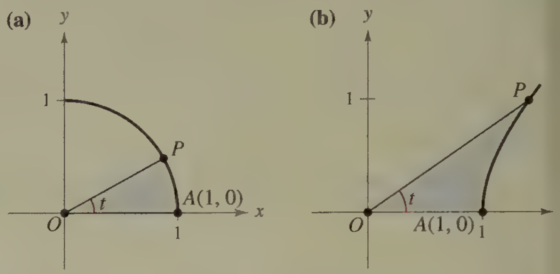
Figure for 2

2. Recall that the graph of a function $y = f(x)$ is symmetric with respect to the origin if whenever (x, y) is a point on the graph, $(-x, -y)$ is also a point on the graph. We say that the graph of the function $y = f(x)$ is **symmetric with respect to the point (a, b)** if whenever $(a - x, b - y)$ is a point on the graph, $(a + x, b + y)$ is also a point on the graph, as indicated in the figure.
   - (a) Sketch the graph of $y = \sin x$ on the interval $[0, 2\pi]$. Write a short paragraph explaining how the symmetry of the graph with respect to the point $(0, \pi)$ allows you to conclude that
   $$\int_0^{2\pi} \sin x \, dx = 0.$$
   - (b) Sketch the graph of $y = \sin x + 2$ on the interval $[0, 2\pi]$. Use the symmetry of the graph with respect to the point $(\pi, 2)$ to evaluate the integral
   $$\int_0^{2\pi} (\sin x + 2) \, dx.$$
   - (c) Sketch the graph of $y = \arccos x$ on the interval $[-1, 1]$. Use the symmetry of the graph to evaluate the integral
   $$\int_{-1}^{1} \arccos x \, dx.$$
   - (d) Evaluate the integral $\displaystyle\int_0^{\pi/2} \frac{1}{1 + (\tan x)^{\sqrt{2}}} \, dx.$

3. Let $f(x) = \sin(\ln x)$.
   - (a) Determine the domain of the function f.
   - (b) Find two values of x satisfying $f(x) = 1$.
   - (c) Find two values of x satisfying $f(x) = -1$.
   - (d) What is the range of the function f?
   - (e) Calculate $f'(x)$ and use calculus to find the maximum value of f on the interval $[1, 10]$.
   - (f) Use a graphing utility to graph f in the viewing window $[0, 5] \times [-2, 2]$ and estimate $\lim_{x \to 0^+} f(x)$, if it exists.
   - (g) Determine $\lim_{x \to 0^+} f(x)$ analytically, if it exists.

4. Graph the exponential function $y = a^x$ for $a = 0.5, 1.2$, and Which of these curves intersects the line $y = x$? Determine positive numbers a for which the curve $y = a^x$ intersects the l $y = x$.

5. (a) Let $P(\cos t, \sin t)$ be a point on the unit circle $x^2 + y^2 =$ in the first quadrant. Show that t is equal to twice the area the shaded circular sector AOP.



   (b) Let $P(\cosh t, \sinh t)$ be a point on the unit hyperb $x^2 - y^2 = 1$ in the first quadrant. Show that t is equal twice the area of the shaded region AOP. Begin by show that the area of the shaded region AOP is given by formula
   $$A(t) = \frac{1}{2} \cosh t \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1} \, dx.$$

6. Consider the three regions A, B, and C determined by the gra of $f(x) = \arcsin x$, as indicated in the figure.
   - (a) Calculate the areas of regions A and B.
   - (b) Use your answer in part (a) to evaluate the integral
   $$\int_{1/2}^{\sqrt{2}/2} \arcsin x \, dx.$$
   - (c) Use your answer in part (a) to evaluate the integral
   $$\int_1^3 \ln x \, dx.$$
   - (d) Use your answer in part (a) to evaluate the integral
   $$\int_1^{\sqrt{3}} \arctan x \, dx.$$

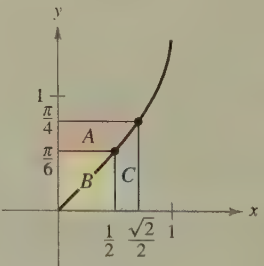


Figure for 6

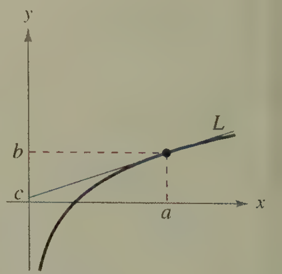
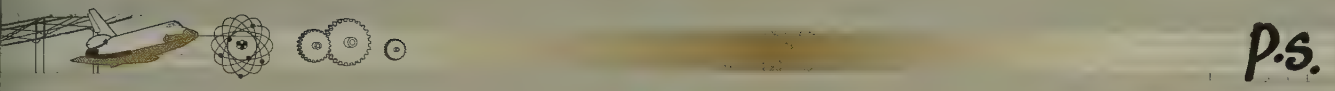
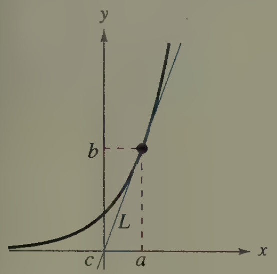


Figure for 7

7. Let L be the tangent line to the graph of the function $y = \ln$ the point (a, b). Show that the distance between b and c is alw equal to 1.



8. Let L be the tangent line to the graph of the function $y = e^x$ at the point (a, b). Show that the distance between a and c is always equal to 1.



9. Use integration by substitution to find the area under the curve

$$y = \frac{1}{\sqrt{x} + x}$$

between $x = 1$ and $x = 4$.

0. Use integration by substitution to find the area under the curve

$$y = \frac{1}{\sin^2 x + 4\cos^2 x}$$

between $x = 0$ and $x = \pi/4$.

1. The differential equation $dy/dt = ky^{1+\varepsilon}$, where k and ε are positive constants, is called the **doomsday equation.**

(a) Solve the doomsday equation

$$\frac{dy}{dt} = y^{1.01}$$

given that $y(0) = 1$.

Find the time T at which $\lim\limits_{t \to T^-} y(t) = \infty$.

(b) Solve the doomsday equation $dy/dt = ky^{1+\varepsilon}$ given that $y(0) = y_0$.

Explain why this equation is called the doomsday equation.

2. The differential equation $dy/dt = ky(L - y)$, where k and L are positive constants, is called the **logistics equation.**

(a) Solve the logistics equation

$$\frac{dy}{dt} = y(1 - y)$$

given that $y(0) = \frac{1}{4}$.

$$\left(\textit{Hint: } \frac{1}{y(1 - y)} = \frac{1}{y} + \frac{1}{1 - y}.\right)$$

(b) Graph the solution on the interval $-6 \le t \le 6$. Show that the rate of growth of the solution is maximum at the point of inflection.

(c) Solve the logistics equation $dy/dt = y(1 - y)$ given that $y(0) = 2$. How does this solution differ from that in part (a)?

3. A thermometer is taken from a room at 72°F to the outdoors, where the temperature is 20°F. Determine the reading on the thermometer after 5 minutes if the reading drops to 48°F after 1 minute.

14. A \$120,000 home mortgage for 35 years at $9\frac{1}{2}\%$ has a monthly payment of \$985.93. Part of the monthly payment goes for the interest charge on the unpaid balance and the remainder of the payment is used to reduce the principal. The amount that goes for interest is

$$u = M - \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}$$

and the amount that goes toward reduction of the principal is

$$v = \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}.$$

In these formulas, P is the size of the mortgage, r is the interest rate, M is the monthly payment, and t is the time in years.

(a) Use a graphing utility to graph each function in the same viewing window. (The viewing window should show all 35 years of mortgage payments.)

(b) In the early years of the mortgage, the larger part of the monthly payment goes for what purpose? Approximate the time when the monthly payment is evenly divided between interest and principal reduction.

(c) Use the graphs in part (a) to make a conjecture about the relationship between the slopes of the tangent lines to the two curves for a specified value of t. Give an analytical argument to verify your conjecture. Find $u'(15)$ and $v'(15)$.

(d) Repeat parts (a) and (b) for a repayment period of 20 years ($M = \$1118.56$). What can you conclude?

15. Let S represent sales of a new product (in thousands of units), let L represent the maximum level of sales (in thousands of units), and let t represent time (in months). The rate of change of S with respect to t varies jointly as the product of S and $L - S$.

(a) Write the differential equation for the sales model if $L = 100$, $S = 10$ when $t = 0$, and $S = 20$ when $t = 1$. Verify that

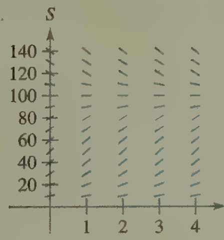
$$S = \frac{L}{1 + Ce^{-kt}}.$$

(b) At what time is the growth in sales increasing most rapidly?

(c) Use a graphing utility to graph the sales function.

(d) Sketch the solution in part (a) on the slope field shown in the figure below. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*

(e) If the estimated maximum level of sales is correct, use the slope field to describe the shape of the solution curves for sales if, at some period of time, sales exceed L.



Constructing an Arch Dam

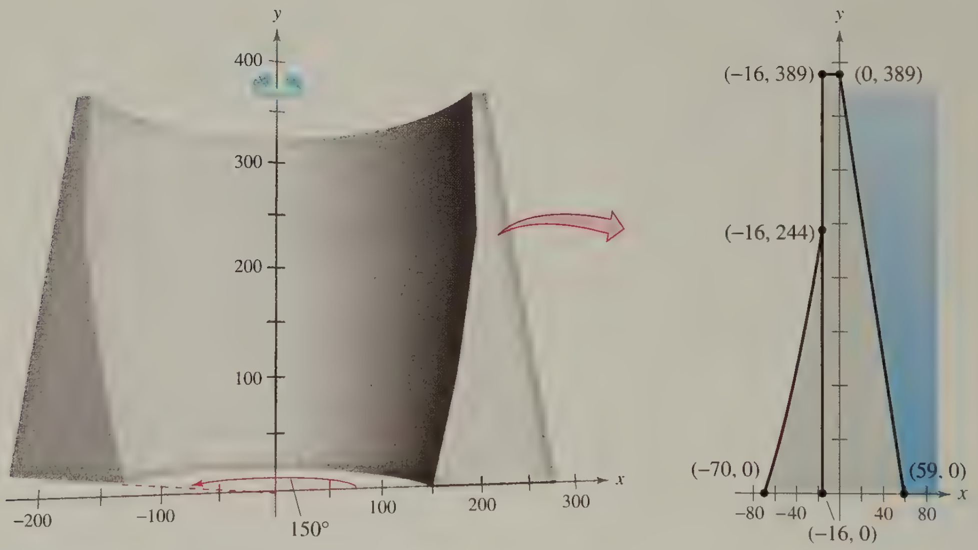
Dams were originally built to ensure water supplies during dry seasons. As technical knowledge has increased, they have begun serving other functions. Today, dams may be built to create recreational lakes, to power generators, and to prevent flooding. Every new dam creates concerns. A dam may upset an area's ecology and force the relocation of people and wildlife. Also, a poorly constructed dam endangers the entire surrounding region, creating the possibility of a massive disaster.

There are several designs used in dam construction, one of which is the arch dam. This design curves toward the water it contains, and is usually built in narrow canyons. The force of the water presses the edges of the dam against the walls of the canyon, so that the natural rock helps support the structure. This added support means that the arch dam can be built with less construction materials than its gravity-supported counterpart.

A cross section of a typical arch dam can be modeled as shown in the figure below. The model for this cross section is as follows.

$$f(x) = \begin{cases} 0.03x^2 + 7.1x + 350, & -70 \le x \le -16 \\ 389, & -16 < x < 0 \\ -6.593x + 389, & 0 \le x \le 59 \end{cases}$$

To form the arch dam, this cross section is swung through an arc, rotating it about the y-axis. The number of degrees through which it is rotated and the length of the axis of rotation vary, depending primarily on how much the water level varies. A possible configuration shows a rotation of 150° and an axis of rotation of 150 feet.



QUESTIONS

1. Find the area of a cross section of the dam.
2. Describe a strategy for estimating the volume of concrete that would be needed to build this dam.
3. Use the strategy to estimate the volume of concrete needed to build the dam described on this page.

The concepts presented here will be explored further in this chapter. For an extension of this application, see Lab 9 of the lab series that accompanies this text at college.hmco.com.

6

Applications of Integration

Henryk Kaiser/Leo de Wys

Hoover Dam, one of the highest concrete dams in the world, uses a gravity-arch construction. It relies on both the walls of the Black Canyon and its own mass to hold back the waters of the Colorado River.

Frank Crowe calculated the winning bid of $48,890,955 for Six Companies, the private contracting firm that built Hoover Dam. Under his leadership, the dam was completed two years early.

R FURTHER INFORMATION To learn more about the calculus of dam design, see *culus, Understanding Change*, a three-part, half-hour video production by MAP and funded by the National Science Foundation.

Section 6.1 Area of a Region Between Two Curves

- Find the area of a region between two curves using integration.
- Find the area of a region between intersecting curves using integration.
- Describe integration as an accumulation process.

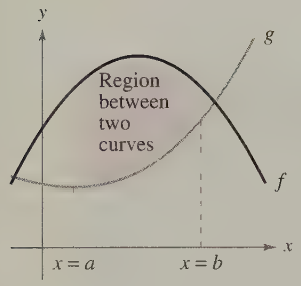
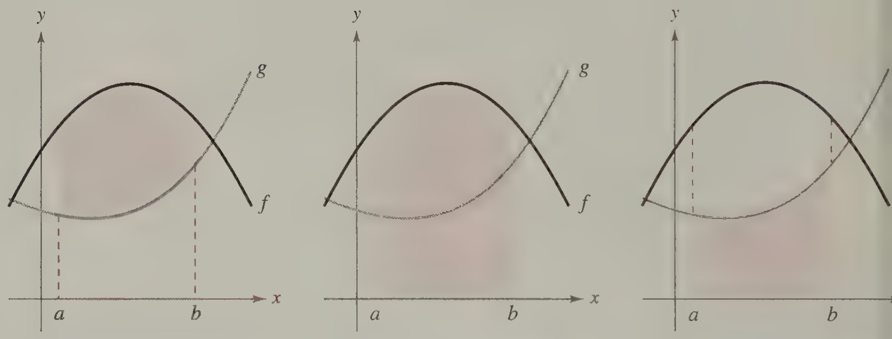


Figure 6.1

Area of a Region Between Two Curves

With a few modifications you can extend the application of definite integrals from area of a region *under* a curve to the area of a region *between* two curves. Consi two functions f and g that are continuous on the interval $[a, b]$. If, as in Figure 6 the graphs of both f and g lie above the x-axis, and the graph of g lies below the gra of f, you can geometrically interpret the area of the region between the graphs as area of the region under the graph of g subtracted from the area of the region un the graph of f, as shown in Figure 6.2.



Area of region between f and g	=	Area of region under f	−	Area of region under g

$$\int_a^b [f(x) - g(x)]\,dx \quad = \quad \int_a^b f(x)\,dx \quad - \quad \int_a^b g(x)\,dx$$

Figure 6.2

To verify the reasonableness of the result shown in Figure 6.2, you can partit the interval $[a, b]$ into n subintervals, each of width Δx. Then, as shown in Figure 6 sketch a **representative rectangle** of width Δx and height $f(x_i) - g(x_i)$, where x in the ith interval. The area of this representative rectangle is

$$\Delta A_i = (\text{height})(\text{width}) = [f(x_i) - g(x_i)]\Delta x.$$

By adding the areas of the n rectangles and taking the limit as $\|\Delta\| \to 0$ $(n \to \infty)$, y obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x.$$

Because f and g are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$ and limit exists. Therefore, the area of the given region is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x$$
$$= \int_a^b [f(x) - g(x)]\,dx.$$

Representative rectangle
Height: $f(x_i) - g(x_i)$
Width: Δx

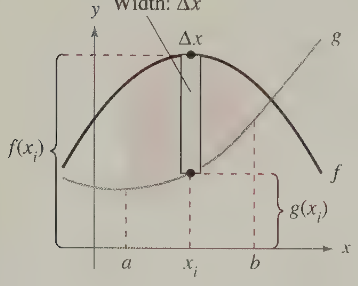


Figure 6.3

Area of a Region Between Two Curves

If f and g are continuous on $[a, b]$ and $g(x) \le f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

In Figure 6.1, the graphs of f and g are shown above the x-axis. This, however, is not necessary. The same integrand $[f(x) - g(x)]$ can be used as long as f and g are continuous and $g(x) \le f(x)$ for all x in the interval $[a, b]$. This result is summarized graphically in Figure 6.4.

OTE The height of a representative ectangle is $f(x) - g(x)$ regardless of the elative position of the x-axis, as shown Figure 6.4.

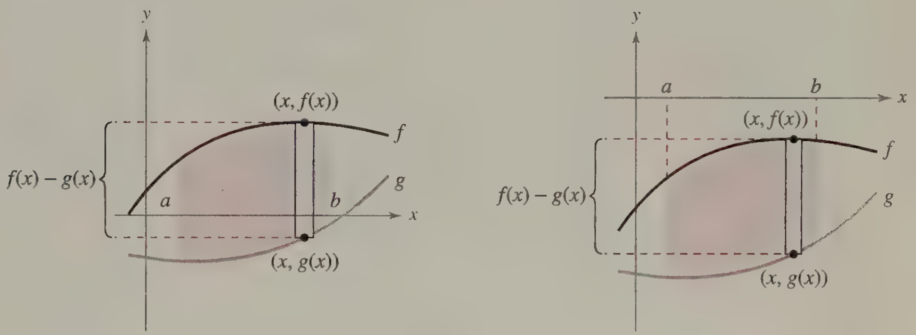
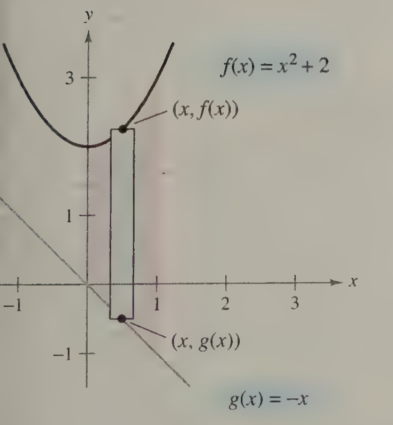


Figure 6.4

Representative rectangles are used throughout this chapter in various applications of integration. A vertical rectangle (of width Δx) implies integration with respect to x, whereas a horizontal rectangle (of width Δy) implies integration with respect to y.

Example 1 Finding the Area of a Region Between Two Curves

Find the area of the region bounded by the graphs of $y = x^2 + 2$, $y = -x$, $x = 0$, and $x = 1$.



egion bounded by the graph of f, the graph g, $x = 0$, and $x = 1$
gure 6.5

Solution Let $g(x) = -x$ and $f(x) = x^2 + 2$. Then $g(x) \le f(x)$ for all x in $[0, 1]$, as shown in Figure 6.5. Thus, the area of the representative rectangle is

$$\Delta A = [f(x) - g(x)]\,\Delta x = [(x^2 + 2) - (-x)]\,\Delta x$$

and the area of the region is

$$\begin{aligned} A = \int_a^b [f(x) - g(x)]\,dx &= \int_0^1 [(x^2 + 2) - (-x)]\,dx \\ &= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_0^1 \\ &= \frac{1}{3} + \frac{1}{2} + 2 \\ &= \frac{17}{6}. \end{aligned}$$

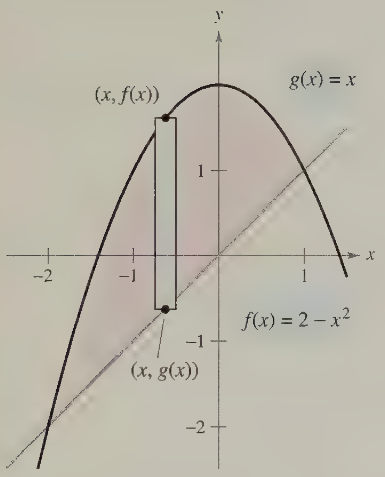
Area of a Region Between Intersecting Curves

In Example 1, the graphs of $f(x) = x^2 + 2$ and $g(x) = -x$ do not intersect, and th values of a and b are given explicitly. A more common problem involves the area a region bounded by two *intersecting* graphs, where the values of a and b must b calculated.

Example 2 A Region Lying Between Two Intersecting Graphs

Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

Solution In Figure 6.6, notice that the graphs of f and g have two points of inte section. To find the x-coordinates of these points, set $f(x)$ and $g(x)$ equal to each oth and solve for x.



Region bounded by the graph of f and the graph of g
Figure 6.6

$$2 - x^2 = x \qquad \text{Set } f(x) \text{ equal to } g(x).$$

$$-x^2 - x + 2 = 0 \qquad \text{Write in general form.}$$

$$-(x + 2)(x - 1) = 0 \qquad \text{Factor.}$$

$$x = -2 \text{ or } 1 \qquad \text{Solve for } x.$$

Thus, $a = -2$ and $b = 1$. Because $g(x) \le f(x)$ for all x in the interval $[-2, 1]$, th representative rectangle has an area of

$$\Delta A = [f(x) - g(x)]\,\Delta x$$
$$= [(2 - x^2) - x]\,\Delta x$$

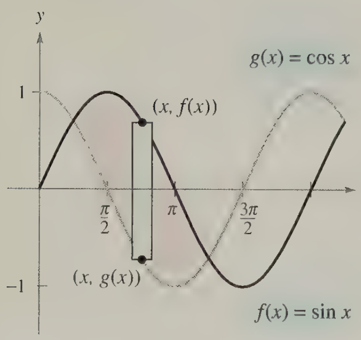
and the area of the region is

$$A = \int_{-2}^{1} [(2 - x^2) - x]\,dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1}$$
$$= \frac{9}{2}.$$

Example 3 A Region Lying Between Two Intersecting Graphs

The sine and cosine curves intersect infinitely many times, bounding regions of equ areas, as shown in Figure 6.7. Find the area of one of these regions.

Solution



One of the regions bounded by the graphs of the sine and cosine functions
Figure 6.7

$$\sin x = \cos x \qquad \text{Set } f(x) \text{ equal to } g(x).$$

$$\frac{\sin x}{\cos x} = 1 \qquad \text{Divide both sides by } \cos x.$$

$$\tan x = 1 \qquad \text{Trigonometric identity}$$

$$x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}, \quad 0 \le x \le 2\pi \qquad \text{Solve for } x.$$

So, $a = \pi/4$ and $b = 5\pi/4$. Because $\sin x \ge \cos x$ for all x in the interv $[\pi/4, 5\pi/4]$, the area of the region is

$$A = \int_{\pi/4}^{5\pi/4} [\sin x - \cos x]\,dx = \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4}$$
$$= 2\sqrt{2}.$$

If two curves intersect at *more* than two points, then to find the area of the region between the curves, you must find all points of intersection and check to see which curve is above the other in each interval determined by these points.

Example 4 Curves That Intersect at More Than Two Points

Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

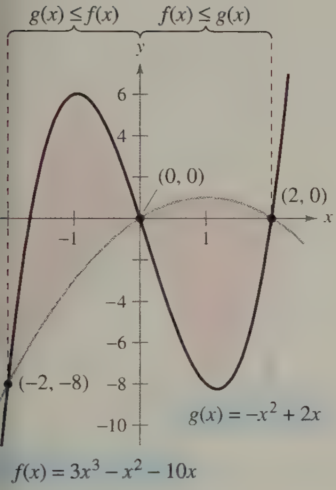
Solution Begin by setting $f(x)$ and $g(x)$ equal to each other and solving for x. This yields the x-values at each point of intersection of the two graphs.

$$3x^3 - x^2 - 10x = -x^2 + 2x \qquad \text{Set } f(x) \text{ equal to } g(x).$$

$$3x^3 - 12x = 0 \qquad \text{Write in general form.}$$

$$3x(x^2 - 4) = 0 \qquad \text{Factor.}$$

$$x = -2, 0, 2 \qquad \text{Solve for } x.$$



[−2, 0], $g(x) \le f(x)$ and in [0, 2],
) ≤ $g(x)$.
gure 6.8

So, the two graphs intersect when $x = -2, 0,$ and 2. In Figure 6.8, notice that $g(x) \le f(x)$ on the interval $[-2, 0]$. However, the two graphs switch at the origin, and $f(x) \le g(x)$ on the interval $[0, 2]$. Hence, you need two integrals—one for the interval $[-2, 0]$ and one for the interval $[0, 2]$.

$$\begin{aligned}
A &= \int_{-2}^{0} [f(x) - g(x)]\,dx + \int_{0}^{2} [g(x) - f(x)]\,dx \\
&= \int_{-2}^{0} (3x^3 - 12x)\,dx + \int_{0}^{2} (-3x^3 + 12x)\,dx \\
&= \left[\frac{3x^4}{4} - 6x^2\right]_{-2}^{0} + \left[\frac{-3x^4}{4} + 6x^2\right]_{0}^{2} \\
&= -(12 - 24) + (-12 + 24) \\
&= 24
\end{aligned}$$

NOTE In Example 4, notice that you get an incorrect result if you integrate from −2 to 2. Such integration produces

$$\int_{-2}^{2} [f(x) - g(x)]\,dx = \int_{-2}^{2} (3x^3 - 12x)\,dx = 0.$$

If the graph of a function of y is a boundary of a region, it is often convenient to use representative rectangles that are *horizontal* and find the area by integrating with respect to y. In general, to determine the area between two curves, you can use

$$A = \int_{x_1}^{x_2} \underbrace{[(\text{top curve}) - (\text{bottom curve})]\,dx}_{\text{in variable } x} \qquad \text{Vertical rectangles}$$

$$A = \int_{y_1}^{y_2} \underbrace{[(\text{right curve}) - (\text{left curve})]\,dy}_{\text{in variable } y} \qquad \text{Horizontal rectangles}$$

where (x_1, y_1) and (x_2, y_2) are either adjacent points of intersection of the two curves involved or points on the specified boundary lines.

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indicates that in the Interactive 3.0 *CD-ROM and* Internet 3.0 *versions of this text (available at* college.hmco.com) *you will find an Open Exploration, which further explores this example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

Example 5 Horizontal Representative Rectangles

Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.

Solution Consider

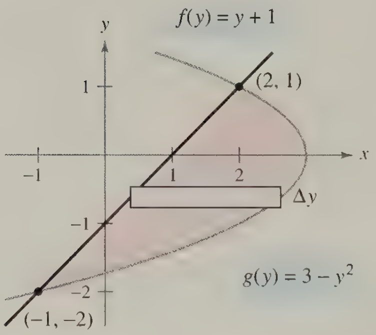
$$g(y) = 3 - y^2 \quad \text{and} \quad f(y) = y + 1.$$

These two curves intersect when $y = -2$ and $y = 1$, as shown in Figure 6.9. Becau $f(y) \le g(y)$ on this interval, you have

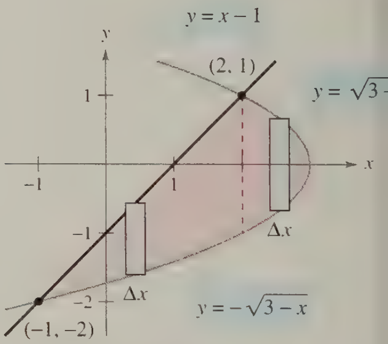
$$\begin{aligned}\Delta A &= [g(y) - f(y)]\,\Delta y\\ &= [(3 - y^2) - (y + 1)]\,\Delta y.\end{aligned}$$

Hence, the area is

$$\begin{aligned}A &= \int_{-2}^{1} [(3 - y^2) - (y + 1)]\,dy\\ &= \int_{-2}^{1} (-y^2 - y + 2)\,dy\\ &= \left[\frac{-y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^{1}\\ &= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}.\end{aligned}$$



Horizontal rectangles (integration with respect to *y*)
Figure 6.9



Vertical rectangles (integration with respec to *x*)
Figure 6.10

In Example 5, notice that by integrating with respect to *y* you need only c integral. If you had integrated with respect to *x*, you would have needed two integr because the upper boundary changes at $x = 2$, as shown in Figure 6.10.

$$\begin{aligned}A &= \int_{-1}^{2} \left[(x - 1) + \sqrt{3 - x}\right] dx + \int_{2}^{3} \left(\sqrt{3 - x} + \sqrt{3 - x}\right) dx\\ &= \int_{-1}^{2} [x - 1 + (3 - x)^{1/2}]\,dx + 2\int_{2}^{3} (3 - x)^{1/2}\,dx\\ &= \left[\frac{x^2}{2} - x - \frac{(3 - x)^{3/2}}{3/2}\right]_{-1}^{2} - 2\left[\frac{(3 - x)^{3/2}}{3/2}\right]_{2}^{3}\\ &= \left(2 - 2 - \frac{2}{3}\right) - \left(\frac{1}{2} + 1 - \frac{16}{3}\right) - 2(0) + 2\left(\frac{2}{3}\right) = \frac{9}{2}\end{aligned}$$

Integration as an Accumulation Process

In this section, we developed the integration formula for the area between two curves by using a rectangle as the *representative element*. For each new application in the remaining sections of this chapter, we will construct an appropriate representative element using precalculus formulas you already know. Each integration formula then will be obtained by summing or accumulating these representative elements.

Known precalculus formula ⇨ Representative element ⇨ New integration formula

For example, in this section we developed the area formula as follows.

$$A = (\text{height})(\text{width}) \quad ⇨ \quad \Delta A = [f(x) - g(x)]\,\Delta x \quad ⇨ \quad A = \int_a^b [f(x) - g(x)]\,dx$$

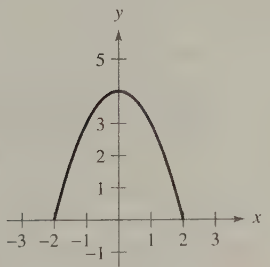
Example 6 Describing Integration as an Accumulation Process

Find the area of the region bounded by the graphs of $y = 4 - x^2$ and the x-axis. Describe the integration as an accumulation process.

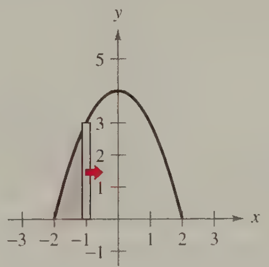
Solution The area of the region is given by

$$A = \int_{-2}^{2} (4 - x^2)\,dx.$$

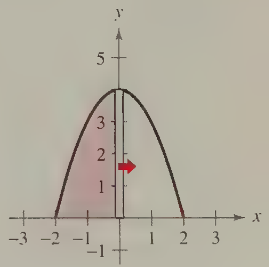
You can think of the integration as an accumulation of the areas of the rectangles formed as the representative rectangle slides from $x = -2$ to $x = 2$.



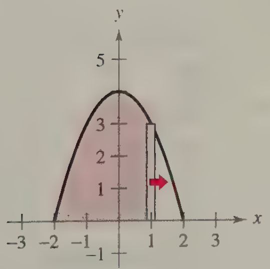
$$A = \int_{-2}^{-2} (4 - x^2)\,dx = 0$$



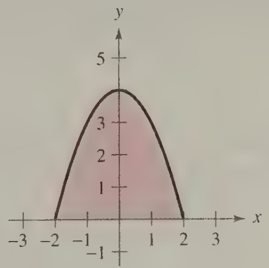
$$A = \int_{-2}^{-1} (4 - x^2)\,dx = \frac{5}{3}$$



$$A = \int_{-2}^{0} (4 - x^2)\,dx = \frac{16}{3}$$



$$A = \int_{-2}^{1} (4 - x^2)\,dx = 9$$

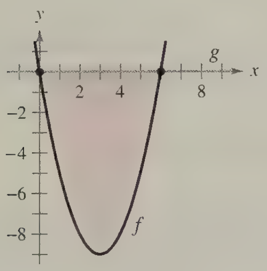


$$A = \int_{-2}^{2} (4 - x^2)\,dx = \frac{32}{3}$$

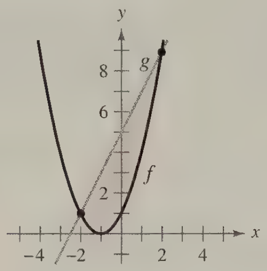
EXERCISES FOR SECTION 6.1

In Exercises 1–6, set up the definite integral that gives the area of the region.

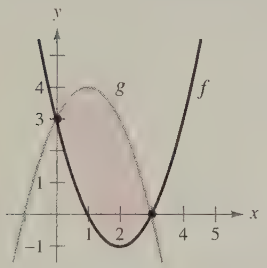
1. $f(x) = x^2 - 6x$
$g(x) = 0$



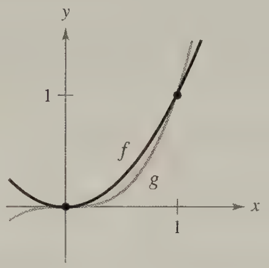
2. $f(x) = x^2 + 2x + 1$
$g(x) = 2x + 5$



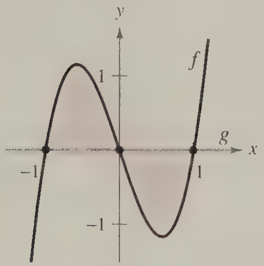
3. $f(x) = x^2 - 4x + 3$
$g(x) = -x^2 + 2x + 3$



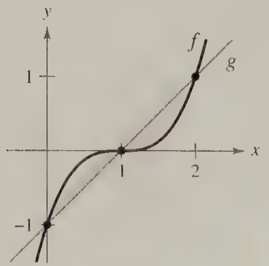
4. $f(x) = x^2$
$g(x) = x^3$



5. $f(x) = 3(x^3 - x)$
$g(x) = 0$



6. $f(x) = (x - 1)^3$
$g(x) = x - 1$



In Exercises 7–12, the integrand of the definite integral is a difference of two functions. Sketch the graph of each function and shade the region whose area is represented by the integral.

7. $\displaystyle\int_0^4 \left[(x + 1) - \frac{x}{2}\right] dx$

8. $\displaystyle\int_{-1}^{1} [(1 - x^2) - (x^2 - 1)]\, dx$

9. $\displaystyle\int_0^6 \left[4(2^{-x/3}) - \frac{x}{6}\right] dx$ **10.** $\displaystyle\int_2^3 \left[\left(\frac{x^3}{3} - x\right) - \frac{x}{3}\right] dx$

11. $\displaystyle\int_{-\pi/3}^{\pi/3} (2 - \sec x)\, dx$ **12.** $\displaystyle\int_{-\pi/4}^{\pi/4} (\sec^2 x - \cos x)\, dx$

Think About It **In Exercises 13 and 14, determine which valu best approximates the area of the region bounded by the graph of f and g. (Make your selection on the basis of a sketch of th region and *not* by performing any calculations.)**

13. $f(x) = x + 1, \quad g(x) = (x - 1)^2$
(a) -2 (b) 2 (c) 10 (d) 4 (e) 8

14. $f(x) = 2 - \frac{1}{2}x, \quad g(x) = 2 - \sqrt{x}$
(a) 1 (b) 6 (c) -3 (d) 3 (e) 4

In Exercises 15–30, sketch the region bounded by the graphs the algebraic functions and find the area of the region.

15. $y = \frac{1}{2}x^3 + 2, \ y = x + 1, \ x = 0, \ x = 2$

16. $y = -\frac{3}{8}x(x - 8), \ y = 10 - \frac{1}{2}x, \ x = 2, \ x = 8$

17. $f(x) = x^2 - 4x, \ g(x) = 0$

18. $f(x) = -x^2 + 4x + 1, \ g(x) = x + 1$

19. $f(x) = x^2 + 2x + 1, \ g(x) = 3x + 3$

20. $f(x) = -x^2 + 4x + 2, \ g(x) = x + 2$

21. $y = x, \ y = 2 - x, \ y = 0$

22. $y = \dfrac{1}{x^2}, \ y = 0, \ x = 1, \ x = 5$

23. $f(x) = \sqrt{3x} + 1, \ g(x) = x + 1$

24. $f(x) = \sqrt[3]{x - 1}, \ g(x) = x - 1$

25. $f(y) = y^2, \ g(y) = y + 2$

26. $f(y) = y(2 - y), \ g(y) = -y$

27. $f(y) = y^2 + 1, \ g(y) = 0, \ y = -1, \ y = 2$

28. $f(y) = \dfrac{y}{\sqrt{16 - y^2}}, \ g(y) = 0, \ y = 3$

29. $f(x) = \dfrac{10}{x}, \ x = 0, \ y = 2, \ y = 10$

30. $g(x) = \dfrac{4}{2 - x}, \ y = 4, \ x = 0$

In Exercises 31–40, use a graphing utility to graph the regio bounded by the graphs of the functions, and use the integratio capabilities of the graphing utility to find the area of the regio

31. $f(x) = x(x^2 - 3x + 3), \ g(x) = x^2$

32. $f(x) = x^3 - 2x + 1, \ g(x) = -2x, \ x = 1$

33. $y = x^2 - 4x + 3, \ y = 3 + 4x - x^2$

34. $y = x^4 - 2x^2, \ y = 2x^2$

35. $f(x) = x^4 - 4x^2, \ g(x) = x^2 - 4$

36. $f(x) = x^4 - 4x^2, \ g(x) = x^3 - 4x$

37. $f(x) = 1/(1 + x^2), \ g(x) = \frac{1}{2}x^2$

38. $f(x) = 6x/(x^2 + 1), \ y = 0, \ 0 \le x \le 3$

39. $y = \sqrt{1 + x^3}, \ y = \frac{1}{2}x + 2, \ x = 0$

40. $y = x\sqrt{\dfrac{4 - x}{4 + x}}, \ y = 0, \ x = 4$

Exercises 41–46, sketch the region bounded by the graphs of e transcendental functions, and find the area of the region.

1. $f(x) = 2\sin x,\ \ g(x) = \tan x,\ \ -\frac{\pi}{3} \le x \le \frac{\pi}{3}$

2. $f(x) = \sin x,\ \ g(x) = \cos 2x,\ \ -\frac{\pi}{2} \le x \le \frac{\pi}{6}$

3. $f(x) = \cos x,\ \ g(x) = 2 - \cos x,\ \ 0 \le x \le 2\pi$

4. $f(x) = \sec\frac{\pi x}{4}\tan\frac{\pi x}{4},\ \ g(x) = (\sqrt{2} - 4)x + 4,\ \ x = 0$

5. $f(x) = xe^{-x^2},\ \ y = 0,\ \ 0 \le x \le 1$

6. $f(x) = 3^x,\ \ g(x) = 2x + 1$

Exercises 47–50, use a graphing utility to graph the region ounded by the graphs of the functions, and use the integration pabilities of the graphing utility to find the area of the region.

7. $f(x) = 2\sin x + \sin 2x,\ \ y = 0,\ \ 0 \le x \le \pi$

3. $f(x) = 2\sin x + \cos 2x,\ \ y = 0,\ \ 0 < x \le \pi$

. $f(x) = \frac{1}{x^2}e^{1/x},\ \ y = 0,\ \ 1 \le x \le 3$

. $g(x) = \frac{4\ln x}{x},\ \ y = 0,\ \ x = 5$

Exercises 51 and 52, (a) use a graphing utility to graph the gion bounded by the graphs of the equations. (b) Set up the tegral giving the area of the region. Can you evaluate the inte-ral by hand? (c) Use the integration capabilities of a graphing ility to approximate the area.

. $y = \sqrt{\frac{x^3}{4 - x}},\ \ y = 0,\ \ x = 3$

. $y = \sqrt{x}\,e^x,\ \ y = 0,\ \ x = 0,\ \ x = 1$

Exercises 53–56, find the accumulation function F. Then aluate F at each specified value of the independent variable d graphically show the area given by each value of F.

. $F(x) = \int_0^x \left(\frac{1}{2}t + 1\right) dt$ (a) $F(0)$ (b) $F(2)$ (c) $F(6)$

. $F(x) = \int_0^x \left(\frac{1}{2}t^2 + 2\right) dt$ (a) $F(0)$ (b) $F(4)$ (c) $F(6)$

. $F(\alpha) = \int_{-1}^{\alpha} \cos\frac{\pi\theta}{2}\, d\theta$ (a) $F(-1)$ (b) $F(0)$ (c) $F\left(\frac{1}{2}\right)$

. $F(y) = \int_{-1}^{y} 4e^{x/2}\, dx$ (a) $F(-1)$ (b) $F(0)$ (c) $F(4)$

Exercises 57 and 58, use integration to find the area of the tri-gle having the given vertices.

. $(0, 0), (a, 0), (b, c)$ **58.** $(2, -3), (4, 6), (6, 1)$

Exercises 59 and 60, set up and evaluate the definite integral at gives the area of the region bounded by the graph of the nction and the tangent line to the graph at the indicated point.

. $f(x) = x^3, (1, 1)$ **60.** $f(x) = \frac{1}{x^2 + 1}, \left(1, \frac{1}{2}\right)$

Getting at the Concept

61. Suppose horizontal representative rectangles are used when finding the area of the region between two curves. Identify the variable of integration.

62. In your own words, describe how to proceed from a precalculus formula to a new integration formula when using integration to solve applied problems.

63. The graphs of $y = x^4 - 2x^2 + 1$ and $y = 1 - x^2$ intersect at three points. However, the area between the curves *can* be found by a single integral. Explain why this is so, and write an integral for this area.

64. The area of the region bounded by the graphs of $y = x^3$ and $y = x$ *cannot* be found by the single integral

$$\int_{-1}^{1} (x^3 - x)\, dx.$$

Explain why this is so. Use symmetry to write a single integral that does represent the area.

65. A college graduate has two job offers. The starting salary for each is \$32,000, and after eight years of service each will pay \$54,000. The salary increase for each offer is shown in the figure. From a strictly monetary viewpoint, which is the better offer? Explain.

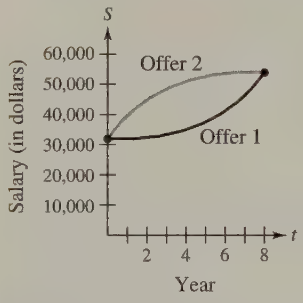


Figure for 65

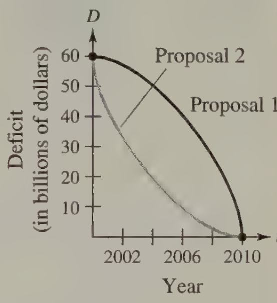


Figure for 66

66. A state legislature is debating two proposals for eliminating the annual budget deficits by the year 2010. The rate of decrease of the deficits for each proposal is shown in the figure. From the viewpoint of minimizing the cumulative state deficit, which is the better proposal? Explain.

In Exercises 67 and 68, find b such that the line $y = b$ divides the region bounded by the graphs of the two equations into two regions of equal area.

67. $y = 9 - x^2,\ \ y = 0$ **68.** $y = 9 - |x|,\ \ y = 0$

In Exercises 69 and 70, evaluate the limit and sketch the graph of the region whose area is represented by the limit.

69. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (x_i - x_i^2)\,\Delta x$

where $x_i = i/n$ and $\Delta x = 1/n$

70. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4 - x_i^2)\,\Delta x$

where $x_i = -2 + (4i/n)$ and $\Delta x = 4/n$

Revenue **In Exercises 71 and 72, two models R_1 and R_2 are given for revenue (in billions of dollars per year) for a large corporation. The model R_1 gives projected annual revenues from 2000 to 2005, with $t = 0$ corresponding to 2000, and R_2 gives projected revenues if there is a decrease in the rate of growth of corporate sales over the period. Approximate the total reduction in revenue if corporate sales are actually closer to the model R_2.**

71. $R_1 = 7.21 + 0.58t$

$R_2 = 7.21 + 0.45t$

72. $R_1 = 7.21 + 0.26t + 0.02t^2$

$R_2 = 7.21 + 0.1t + 0.01t^2$

73. ***Modeling Data*** The table shows the total receipts R and total expenditures E for the Old-Age and Survivors Insurance Trust Fund (Social Security Trust Fund) in billions of dollars. The time t is given in years, with $t = 1$ corresponding to 1991. *(Source: Social Security Administration)*

t	1	2	3	4	5
R	299.3	311.2	323.3	328.3	342.8
E	245.6	259.9	273.1	284.1	297.8

t	6	7	8	9
R	363.7	397.2	424.8	457.0
E	308.2	322.1	332.3	339.9

(a) Use a graphing utility to fit an exponential model to the data for receipts. Plot the data and graph the model.

(b) Use a graphing utility to fit an exponential model to the data for expenditures. Plot the data and graph the model.

(c) If the models are assumed true for the years 2000 through 2005, use integration to approximate the surplus revenue generated during those years.

(d) Will the models found in parts (a) and (b) intersect? Explain. Based on your answer and news reports about the fund, will these models be accurate for long-term analysis?

74. ***Profit*** The chief financial officer of a company reports that profits for the past fiscal year were \$893,000. The officer predicts that profits for the next 5 years will grow at a continuous annual rate somewhere between $3\frac{1}{2}\%$ and 5%. Estimate the cumulative difference in total profit over the 5 years based on the predicted range of growth rates.

75. ***Area*** The shaded region in the figure consists of all points whose distances to the center of the square are less than the distances to the edges of the square. Find the area of the region.

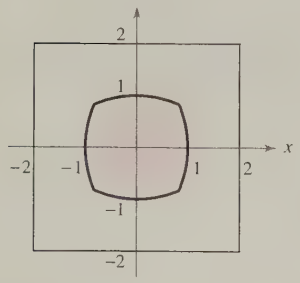


Figure for 75

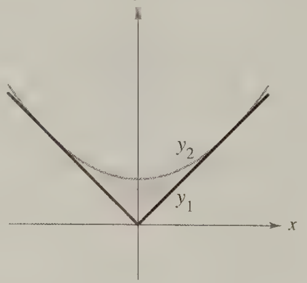


Figure for 76

76. ***Mechanical Design*** The surface of a machine part is t region between the graphs of $y_1 = |x|$ and $y_2 = 0.08x^2 +$ (see figure).

(a) Find k if the parabola is tangent to the graph of y_1.

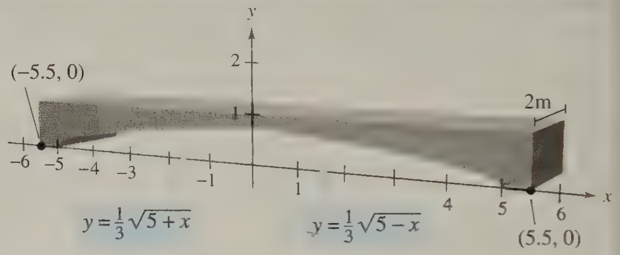
(b) Find the surface area of the machine part.

77. ***Building Design*** Concrete sections for a new building ha the dimensions (in meters) and shape shown in the figure.

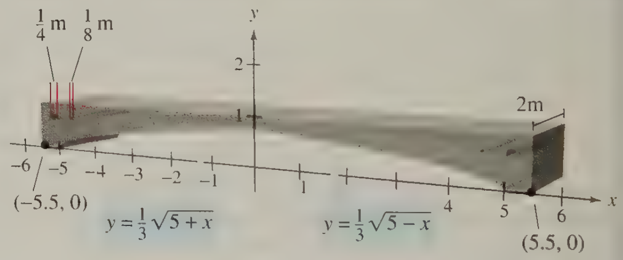
(a) Find the area of the face of the section superimposed on t rectangular coordinate system.

(b) Find the volume of concrete in one of the sections by mu tiplying the area in part (a) by 2 meters.

(c) One cubic meter of concrete weighs 5000 pounds. Find t weight of the section.



78. ***Building Design*** To decrease the weight and to aid in t hardening process, the concrete sections in Exercise 77 oft are not solid. Rework Exercise 77 to allow for cylindrical ope ings such as those shown in the figure.



True or False? **In Exercises 79–81, determine whether t statement is true or false. If it is false, explain why or give example that shows it is false.**

79. If the area of the region bounded by the graphs of f and is 1, then the area of the region bounded by the graphs $h(x) = f(x) + C$ and $k(x) = g(x) + C$ is also 1.

80. If $\int_a^b [f(x) - g(x)]\,dx = A$, then $\int_a^b [g(x) - f(x)]\,dx = -A$

81. If the graphs of f and g intersect midway between $x = a$ a $x = b$, then

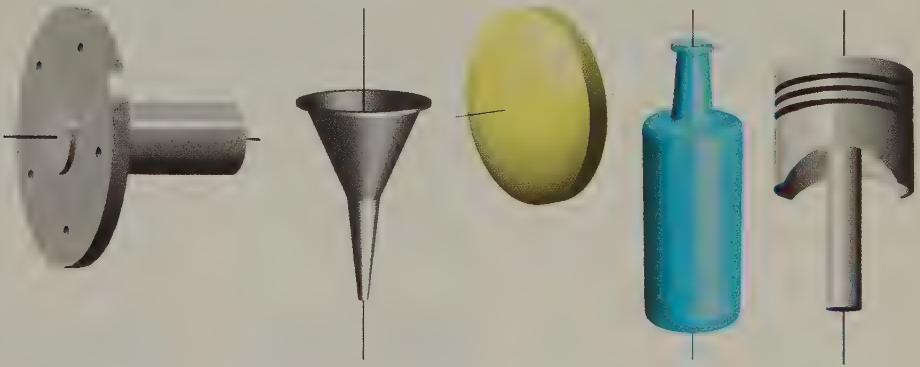
$$\int_a^b [f(x) - g(x)]\,dx = 0.$$

Section 6.2 Volume: The Disk Method

- Find the volume of a solid of revolution using the disk method.
- Find the volume of a solid of revolution using the washer method.
- Find the volume of a solid with known cross sections.

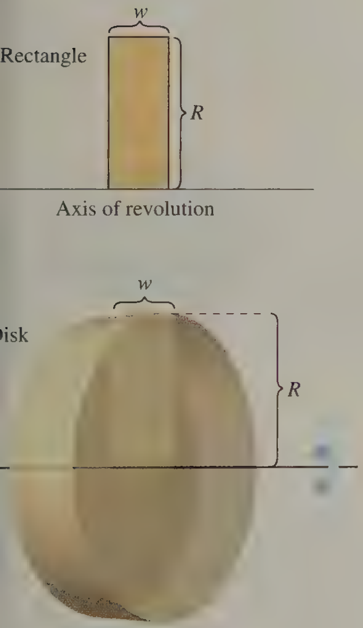
The Disk Method

In Chapter 4, we mentioned that area is only *one* of the many applications of the definite integral. Another important application is its use in finding the volume of a three-dimensional solid. In this section you will study a particular type of three-dimensional solid—one whose cross sections are similar. We begin with solids of revolution. Such solids are used commonly in engineering and manufacturing. Some examples are axles, funnels, pills, bottles, and pistons, as indicated in Figure 6.11.



Solids of revolution
Figure 6.11

If a region in the plane is revolved about a line, the resulting solid is a **solid of revolution,** and the line is called the **axis of revolution.** The simplest such solid is a right circular cylinder or **disk,** which is formed by revolving a rectangle about an axis adjacent to one side of the rectangle, as shown in Figure 6.12. The volume of such a disk is



ume of a disk: $\pi R^2 w$
ure 6.12

$$\begin{aligned}\text{Volume of disk} &= (\text{area of disk})(\text{width of disk})\\ &= \pi R^2 w\end{aligned}$$

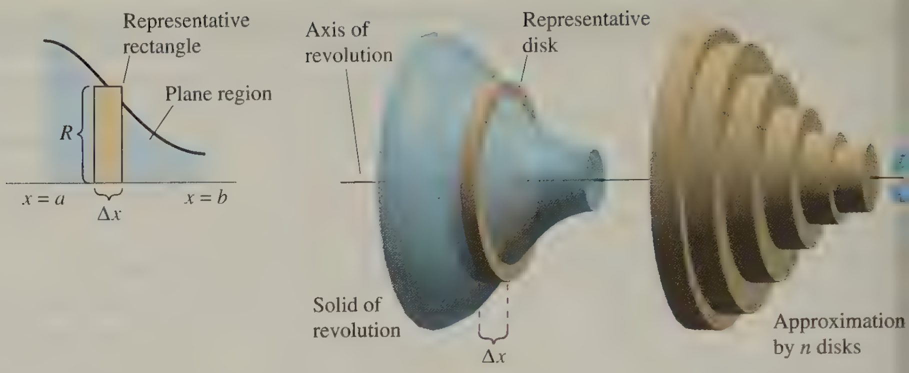
where R is the radius of the disk and w is the width.

To see how to use the volume of a disk to find the volume of a general solid of revolution, consider a solid of revolution formed by revolving the plane region in Figure 6.13 about the indicated axis. To determine the volume of this solid, consider a representative rectangle in the plane region. When this rectangle is revolved about the axis of revolution, it generates a representative disk whose volume is

$$\Delta V = \pi R^2 \Delta x.$$

Approximating the volume of the solid by n such disks of width Δx and radius $R(x_i)$ produces

$$\begin{aligned}\text{Volume of solid} &\approx \sum_{i=1}^{n} \pi[R(x_i)]^2\,\Delta x\\ &= \pi\sum_{i=1}^{n} [R(x_i)]^2\,\Delta x.\end{aligned}$$



Disk method
Figure 6.13

This approximation appears to become better and better as $\|\Delta\| \to 0$ $(n \to \infty)$. There-fore, you can define the volume of the solid as

$$\text{Volume of solid} = \lim_{\|\Delta\|\to 0} \pi \sum_{i=1}^{n} [R(x_i)]^2 \, \Delta x = \pi \int_a^b [R(x)]^2 \, dx.$$

Schematically, the disk method looks like this.

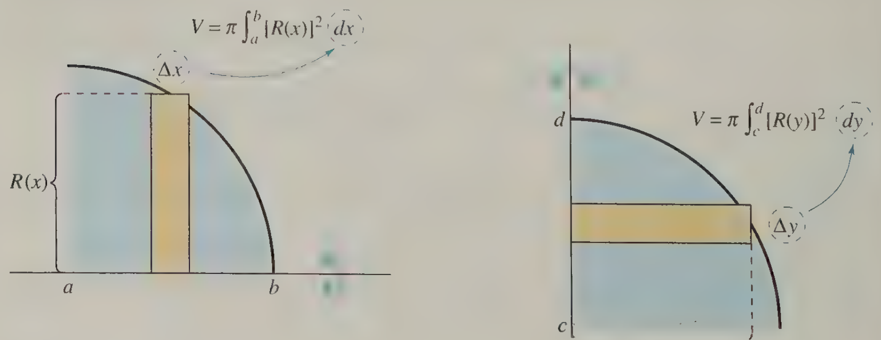
Known Precalculus Formula	*Representative Element*	*New Integration Formula*
Volume of disk $V = \pi R^2 w$	$\Delta V = \pi[R(x_i)]^2 \, \Delta x$	Solid of revolution $V = \pi \int_a^b [R(x)]^2 \, dx$

A similar formula can be derived if the axis of revolution is vertical.

The Disk Method

To find the volume of a solid of revolution with the **disk method,** use one of the following, as indicated in Figure 6.14.

Horizontal Axis of Revolution	*Vertical Axis of Revolution*
$\text{Volume} = V = \pi \int_a^b [R(x)]^2 \, dx$	$\text{Volume} = V = \pi \int_c^d [R(y)]^2 \, dy$


$V = \pi \int_a^b [R(x)]^2 \, dx$
Δx
R(x)
a
b
d
$V = \pi \int_c^d [R(y)]^2 \, dy$
Δy
c

Horizontal axis of revolution
Figure 6.14

Vertical axis of revolution

NOTE In Figure 6.14, note that you can determine the variable of integration by placing a representative rectangle in the *plane* region "perpendicular" to the axis of revolution. If the width of the rectangle is Δx, integrate with respect to x, and if the width of the rectangle is Δy, integrate with respect to y.

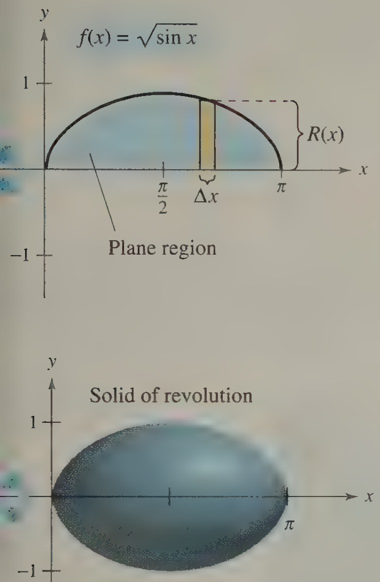
The simplest application of the disk method involves a plane region bounded by the graph of f and the x-axis. If the axis of revolution is the x-axis, the radius $R(x)$ is simply $f(x)$.

Example 1 Using the Disk Method

Find the volume of the solid formed by revolving the region bounded by the graph of

$$f(x) = \sqrt{\sin x}$$

and the x-axis $(0 \le x \le \pi)$ about the x-axis.



igure 6.15

Solution From the representative rectangle in the upper graph in Figure 6.15, you can see that the radius of this solid is

$$\begin{aligned} R(x) &= f(x) \\ &= \sqrt{\sin x}. \end{aligned}$$

So, the volume of the solid of revolution is

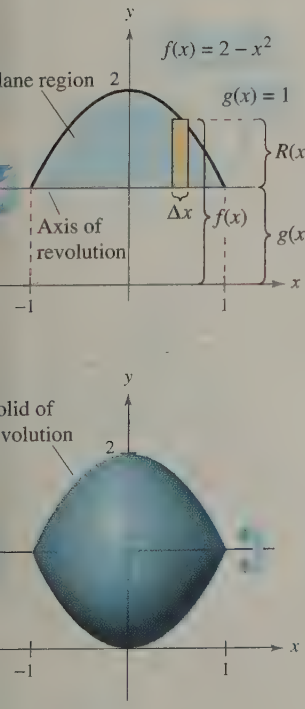
$$\begin{aligned} V = \pi\int_a^b [R(x)]^2\,dx &= \pi\int_0^{\pi} \left(\sqrt{\sin x}\right)^2 dx && \text{Apply disk method.} \\ &= \pi\int_0^{\pi} \sin x\,dx && \text{Simplify.} \\ &= \pi\Big[-\cos x\Big]_0^{\pi} && \text{Integrate.} \\ &= \pi(1+1) \\ &= 2\pi. \end{aligned}$$

Example 2 Revolving About a Line That Is Not a Coordinate Axis

Find the volume of the solid formed by revolving the region bounded by

$$f(x) = 2 - x^2$$

and $g(x) = 1$ about the line $y = 1$, as shown in Figure 6.16.



igure 6.16

Solution By equating $f(x)$ and $g(x)$, you can determine that the two graphs intersect when $x = \pm 1$. To find the radius, subtract $g(x)$ from $f(x)$.

$$\begin{aligned} R(x) &= f(x) - g(x) \\ &= (2 - x^2) - 1 \\ &= 1 - x^2 \end{aligned}$$

Finally, integrate between -1 and 1 to find the volume.

$$\begin{aligned} V = \pi\int_a^b [R(x)]^2\,dx &= \pi\int_{-1}^{1} (1 - x^2)^2\,dx && \text{Apply disk method.} \\ &= \pi\int_{-1}^{1} (1 - 2x^2 + x^4)\,dx && \text{Simplify.} \\ &= \pi\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1} && \text{Integrate.} \\ &= \frac{16\pi}{15} \end{aligned}$$

The Washer Method

The disk method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative **washer.** The washer is formed b revolving a rectangle about an axis, as shown in Figure 6.17. If r and R are the inne and outer radii of the washer and w is the width of the washer, the volume is given b

$$\text{Volume of washer} = \pi(R^2 - r^2)w.$$

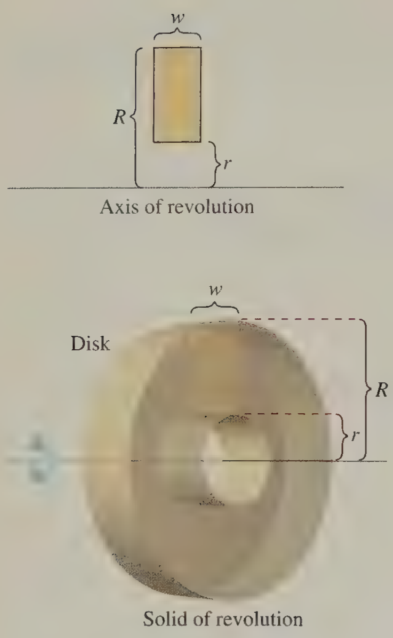


Figure 6.17

To see how this concept can be used to find the volume of a solid of revolution consider a region bounded by an **outer radius** $R(x)$ and an **inner radius** $r(x)$, a shown in Figure 6.18. If the region is revolved about its axis of revolution, the volum of the resulting solid is given by

$$V = \pi\int_a^b ([R(x)]^2 - [r(x)]^2)\,dx.$$ Washer method

Note that the integral involving the inner radius represents the volume of the hole and is *subtracted* from the integral involving the outer radius.

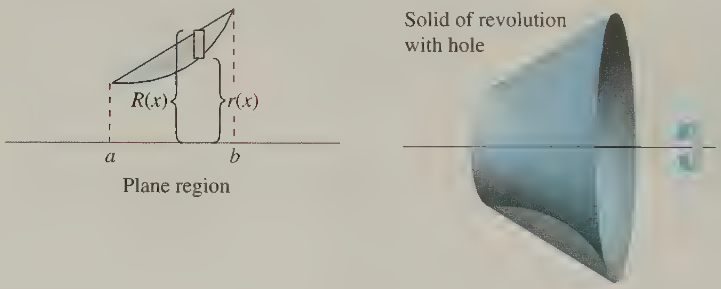
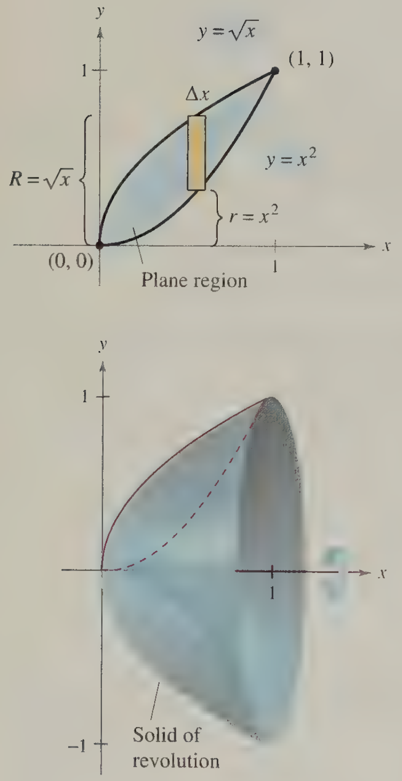


Figure 6.18

Example 3 Using the Washer Method

Find the volume of the solid formed by revolving the region bounded by the graphs o $y = \sqrt{x}$ and $y = x^2$ about the x-axis, as shown in Figure 6.19.



Solid of revolution

Figure 6.19

Solution In Figure 6.19, you can see that the outer and inner radii are as follows.

$R(x) = \sqrt{x}$ Outer radius

$r(x) = x^2$ Inner radius

Integrating between 0 and 1 produces

$$V = \pi\int_a^b ([R(x)]^2 - [r(x)]^2)\,dx$$ Apply washer method.

$$= \pi\int_0^1 \left[\left(\sqrt{x}\right)^2 - (x^2)^2\right] dx$$

$$= \pi\int_0^1 (x - x^4)\,dx$$ Simplify.

$$= \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$ Integrate.

$$= \frac{3\pi}{10}.$$

In each example so far, the axis of revolution has been *horizontal* and we have integrated with respect to x. In the next example, the axis of revolution is *vertical* and you must integrate with respect to y. In this example, you need two separate integrals to compute the volume.

Example 4 Integrating with Respect to *y*, Two-Integral Case

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis, as shown in Figure 6.20.

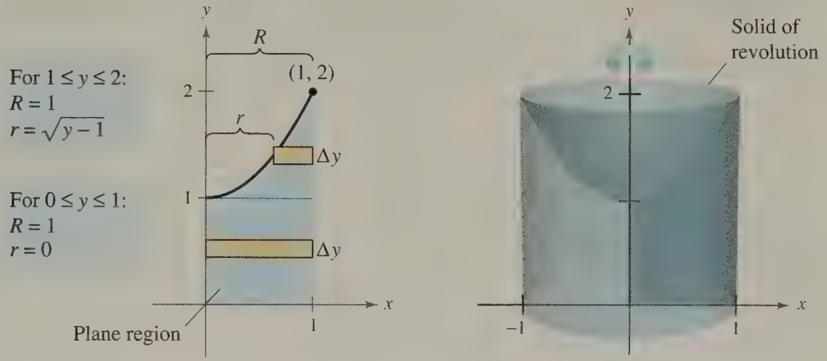


Figure 6.20

Solution For the region shown in Figure 6.20, the outer radius is simply $R = 1$. There is, however, no convenient formula that represents the inner radius. When $0 \le y \le 1$, $r = 0$, but when $1 \le y \le 2$, r is determined by the equation $y = x^2 + 1$, which implies that $r = \sqrt{y-1}$.

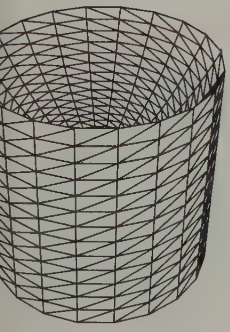
$$r(y) = \begin{cases} 0, & 0 \le y \le 1 \\ \sqrt{y-1}, & 1 \le y \le 2 \end{cases}$$

Using this definition of the inner radius, you can use two integrals to find the volume.

$$\begin{aligned} V &= \pi\int_0^1 (1^2 - 0^2)\,dy + \pi\int_1^2 \left[1^2 - \left(\sqrt{y-1}\right)^2\right]dy && \text{Apply washer method.} \\ &= \pi\int_0^1 1\,dy + \pi\int_1^2 (2-y)\,dy && \text{Simplify.} \\ &= \pi\Big[y\Big]_0^1 + \pi\left[2y - \frac{y^2}{2}\right]_1^2 && \text{Integrate.} \\ &= \pi + \pi\left(4 - 2 - 2 + \frac{1}{2}\right) = \frac{3\pi}{2} \end{aligned}$$

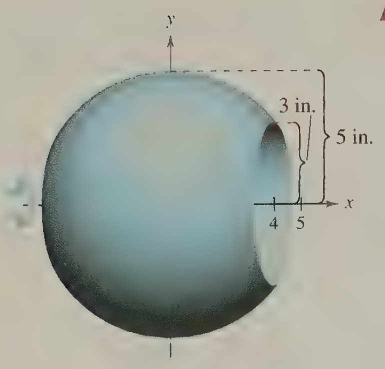
Note that the first integral $\pi\int_0^1 1\,dy$ represents the volume of a right circular cylinder of radius 1 and height 1. This portion of the volume could have been determined without using calculus.

TECHNOLOGY Some graphing utilities have the capability to generate (or have built-in software capable of generating) a solid of revolution. If you have access to such a utility, try using it to sketch some of the solids of revolution described in this section. For instance, the solid in Example 4 might appear like that shown in Figure 6.21.



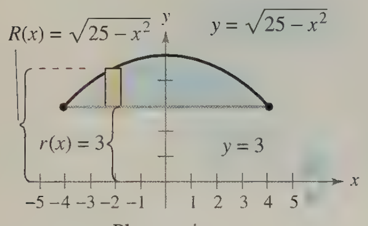
Generated by Mathematica

ure 6.21



Solid of revolution

(a)



Plane region

(b)

Figure 6.22

Example 5 Manufacturing

A manufacturer drills a hole through the center of a metal sphere of radius 5 inche as shown in Figure 6.22(a). The hole has a radius of 3 inches. What is the volume the resulting metal ring?

Solution You can imagine the ring to be generated by a segment of the circle who equation is $x^2 + y^2 = 25$, as shown in Figure 6.22(b). Because the radius of the ho is 3 inches, you can let $y = 3$ and solve the equation $x^2 + y^2 = 25$ to determine th the limits of integration are $x = \pm 4$. So, the inner and outer radii are $r(x) = 3$ a $R(x) = \sqrt{25 - x^2}$ and the volume is given by

$$
\begin{aligned}
V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2)\,dx &= \pi \int_{-4}^{4} \left[\left(\sqrt{25 - x^2}\right)^2 - (3)^2\,dx\right. \\
&= \pi \int_{-4}^{4} (16 - x^2)\,dx \\
&= \pi \left[16x - \frac{x^3}{3}\right]_{-4}^{4} \\
&= \frac{256\pi}{3} \text{ cubic inches.}
\end{aligned}
$$

Solids with Known Cross Sections

With the disk method, you can find the volume of a solid having a circular cro section whose area is $A = \pi R^2$. This method can be generalized to solids of a shape, as long as you know a formula for the area of an arbitrary cross section. Son common cross sections are squares, rectangles, triangles, semicircles, and trapezoic

Volumes of Solids with Known Cross Sections

1. For cross sections of area $A(x)$ taken perpendicular to the x-axis,

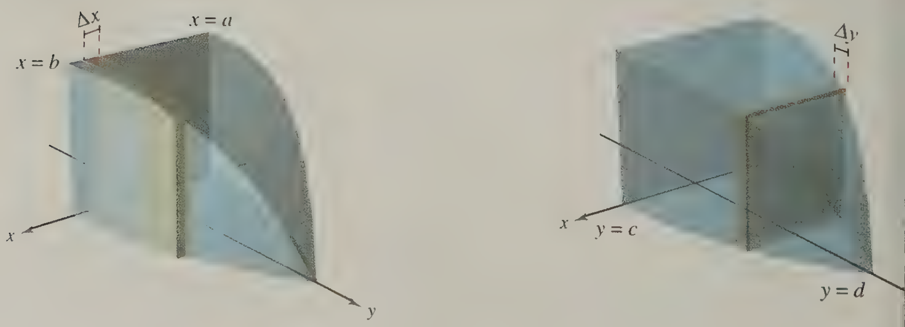
$$\text{Volume} = \int_a^b A(x)\,dx.$$

See Figure 6.23(a).

2. For cross sections of area $A(y)$ taken perpendicular to the y-axis,

$$\text{Volume} = \int_c^d A(y)\,dy.$$

See Figure 6.23(b).



(a) Cross sections perpendicular to x-axis

(b) Cross sections perpendicular to y-a

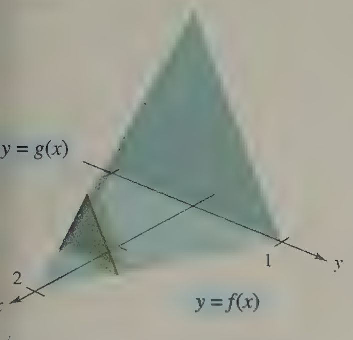
Figure 6.23

Example 6 Triangular Cross Sections

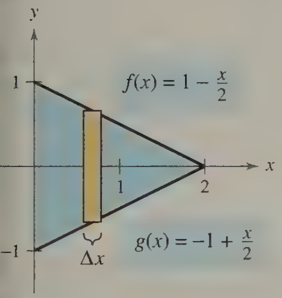
Find the volume of the solid shown in Figure 6.24. The base of the solid is the region bounded by the lines

$$f(x) = 1 - \frac{x}{2}, \qquad g(x) = -1 + \frac{x}{2}, \qquad \text{and} \qquad x = 0.$$

The cross sections perpendicular to the x-axis are equilateral triangles.



ross sections are equilateral triangles.



riangular base in xy-plane
igure 6.24

Solution The base and area of each triangular cross section are as follows.

$$\text{Base} = \left(1 - \frac{x}{2}\right) - \left(-1 + \frac{x}{2}\right) = 2 - x \qquad \text{Length of base}$$

$$\text{Area} = \frac{\sqrt{3}}{4}(\text{base})^2 \qquad \text{Area of equilateral triangle}$$

$$A(x) = \frac{\sqrt{3}}{4}(2 - x)^2 \qquad \text{Area of cross section}$$

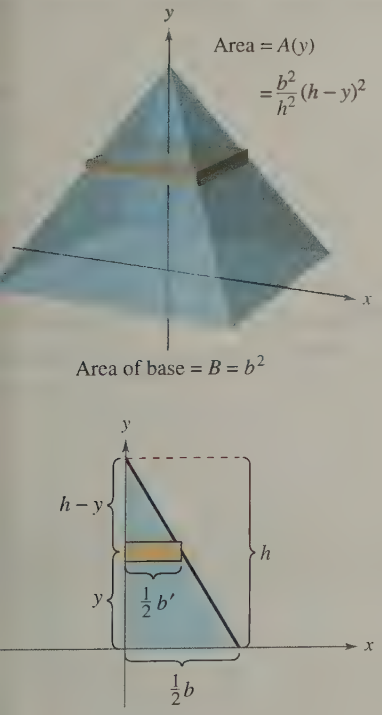
Because x ranges from 0 to 2, the volume of the solid is

$$V = \int_a^b A(x)\,dx = \int_0^2 \frac{\sqrt{3}}{4}(2 - x)^2\,dx$$

$$= -\frac{\sqrt{3}}{4}\left[\frac{(2 - x)^3}{3}\right]_0^2 = \frac{2\sqrt{3}}{3}.$$

Example 7 An Application to Geometry

Prove that the volume of a pyramid with a square base is $V = \frac{1}{3}hB$, where h is the height of the pyramid and B is the area of the base.



igure 6.25

Solution As shown in Figure 6.25, you can intersect the pyramid with a plane parallel to the base at height y to form a square cross section whose sides are of length b'. Using similar triangles, you can show that

$$\frac{b'}{b} = \frac{h - y}{h} \qquad \text{or} \qquad b' = \frac{b}{h}(h - y)$$

where b is the length of the sides of the base of the pyramid. So,

$$A(y) = (b')^2 = \frac{b^2}{h^2}(h - y)^2.$$

Integrating between 0 and h produces

$$V = \int_0^h A(y)\,dy = \int_0^h \frac{b^2}{h^2}(h - y)^2\,dy$$

$$= \frac{b^2}{h^2}\int_0^h (h - y)^2\,dy$$

$$= -\left(\frac{b^2}{h^2}\right)\left[\frac{(h - y)^3}{3}\right]_0^h$$

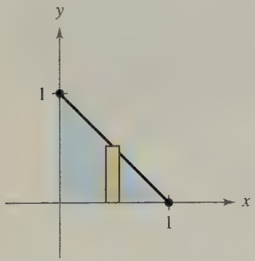
$$= \frac{b^2}{h^2}\left(\frac{h^3}{3}\right)$$

$$= \frac{1}{3}hB. \qquad B = b^2$$

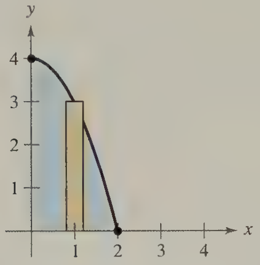
EXERCISES FOR SECTION 6.2

In Exercises 1–6, set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the x-axis.

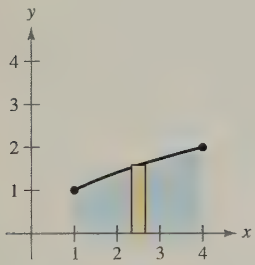
1. $y = -x + 1$



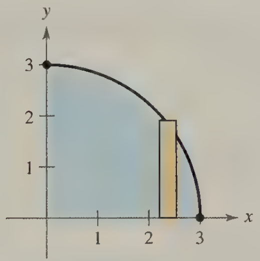
2. $y = 4 - x^2$



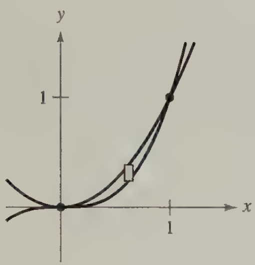
3. $y = \sqrt{x}$



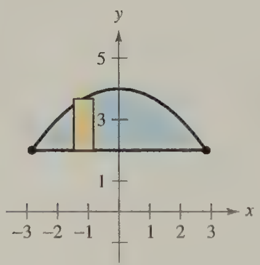
4. $y = \sqrt{9 - x^2}$



5. $y = x^2$, $y = x^3$

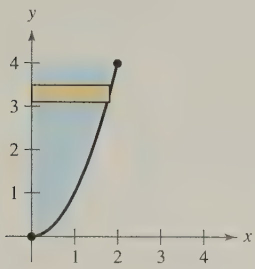


6. $y = 2$, $y = 4 - \dfrac{x^2}{4}$

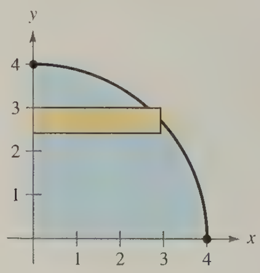


In Exercises 7–10, set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the y-axis.

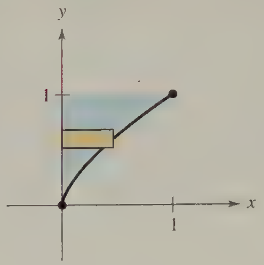
7. $y = x^2$



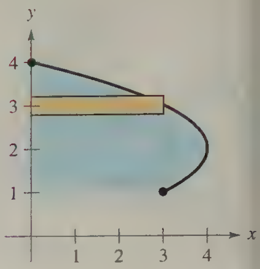
8. $y = \sqrt{16 - x^2}$



9. $y = x^{2/3}$



10. $x = -y^2 + 4y$



In Exercises 11–14, find the volume of the solid generated b revolving the region bounded by the graphs of the equation about the indicated lines.

11. $y = \sqrt{x}$, $y = 0$, $x = 4$

(a) the x-axis (b) the y-axis

(c) the line $x = 4$ (d) the line $x = 6$

12. $y = 2x^2$, $y = 0$, $x = 2$

(a) the y-axis (b) the x-axis

(c) the line $y = 8$ (d) the line $x = 2$

13. $y = x^2$, $y = 4x - x^2$

(a) the x-axis (b) the line $y = 6$

14. $y = 6 - 2x - x^2$, $y = x + 6$

(a) the x-axis (b) the line $y = 3$

In Exercises 15–18, find the volume of the solid generated b revolving the region bounded by the graphs of the equation about the line $y = 4$.

15. $y = x$, $y = 3$, $x = 0$

16. $y = \frac{1}{2}x^3$, $y = 4$, $x = 0$

17. $y = \dfrac{1}{1 + x}$, $y = 0$, $x = 0$, $x = 3$

18. $y = \sec x$, $y = 0$, $0 \le x \le \dfrac{\pi}{3}$

In Exercises 19–22, find the volume of the solid generated b revolving the region bounded by the graphs of the equation about the line $x = 6$.

19. $y = x$, $y = 0$, $y = 4$, $x = 6$

20. $y = 6 - x$, $y = 0$, $y = 4$, $x = 0$

21. $x = y^2$, $x = 4$

22. $xy = 6$, $y = 2$, $y = 6$, $x = 6$

n Exercises 23–30, find the volume of the solid generated by evolving the region bounded by the graphs of the equations bout the x-axis.

3. $y = \frac{1}{\sqrt{x+1}}$, $y = 0$, $x = 0$, $x = 3$

4. $y = x\sqrt{4 - x^2}$, $y = 0$

5. $y = \frac{1}{x}$, $y = 0$, $x = 1$, $x = 4$

6. $y = \frac{3}{x+1}$, $y = 0$, $x = 0$, $x = 8$

7. $y = e^{-x}$, $y = 0$, $x = 0$, $x = 1$

8. $y = e^{x/2}$, $y = 0$, $x = 0$, $x = 4$

9. $y = x^2 + 1$, $y = -x^2 + 2x + 5$, $x = 0$, $x = 3$

0. $y = \sqrt{x}$, $y = -\frac{1}{2}x + 4$, $x = 0$, $x = 8$

n Exercises 31 and 32, find the volume of the solid generated by evolving the region bounded by the graphs of the equations bout the y-axis.

1. $y = 3(2 - x)$, $y = 0$, $x = 0$

2. $y = 9 - x^2$, $y = 0$, $x = 2$, $x = 3$

n Exercises 33–38, use the integration capabilities of a graph- ng utility to approximate the volume of the solid generated by evolving the region bounded by the graphs of the equations bout the x-axis.

3. $y = \sin x$, $y = 0$, $x = 0$, $x = \pi$

4. $y = \cos x$, $y = 0$, $x = 0$, $x = \frac{\pi}{2}$

5. $y = e^{-x^2}$, $y = 0$, $x = 0$, $x = 2$

6. $y = \ln x$, $y = 0$, $x = 1$, $x = 3$

7. $y = e^{x/2} + e^{-x/2}$, $y = 0$, $x = -1$, $x = 2$

8. $y = 2\arctan(0.2x)$, $y = 0$, $x = 0$, $x = 5$

hink About It **In Exercises 39 and 40, determine which value est approximates the volume of the solid generated by revolv- ng the region bounded by the graphs of the equations about the -axis. (Make your selection on the basis of a sketch of the solid nd *not* by performing any calculations.)**

9. $y = e^{-x^2/2}$, $y = 0$, $x = 0$, $x = 2$

(a) 3 (b) -5 (c) 10 (d) 7 (e) 20

0. $y = \arctan x$, $y = 0$, $x = 0$, $x = 1$

(a) 10 (b) $\frac{3}{4}$ (c) 5 (d) -6 (e) 15

Getting at the Concept

41. Give the integration formula for finding the volumes of solids using (a) the disk method and (b) the washer method.

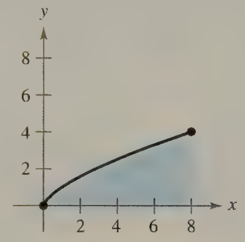
42. Give the integration formula for finding the volumes of solids of known cross sections.

Getting at the Concept *(continued)*

43. A region bounded by the parabola $y = 4x - x^2$ and the x-axis is revolved about the x-axis. A second region bounded by the parabola $y = 4 - x^2$ and the x-axis is revolved about the x-axis. Without integrating, how do the volumes of the two solids compare? Explain.

44. The region in the figure is revolved about the indicated axes and line. Order the volumes of the resulting solids from least to greatest. Explain your reasoning.

(a) x-axis (b) y-axis (c) $x = 8$



45. If the portion of the line $y = \frac{1}{2}x$ lying in the first quadrant is revolved about the x-axis, a cone is generated. Find the volume of the cone extending from $x = 0$ to $x = 6$.

46. Use the disk method to verify that the volume of a right circular cone is $\frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.

47. Use the disk method to verify that the volume of a sphere is $\frac{4}{3}\pi r^3$.

48. A sphere of radius r is cut by a plane h $(h < r)$ units above the equator. Find the volume of the solid (spherical segment) above the plane.

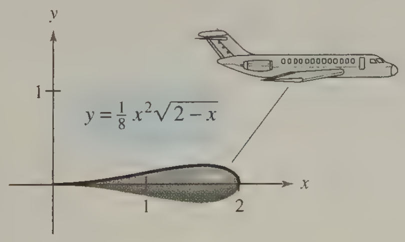
49. A cone of height H with a base of radius r is cut by a plane parallel to and h units above the base. Find the volume of the solid (frustum of a cone) below the plane.

50. The region bounded by $y = \sqrt{x}$, $y = 0$, $x = 0$, and $x = 4$ is revolved about the x-axis.

(a) Find the value of x in the interval $[0, 4]$ that divides the solid into two parts of equal volume.

(b) Find the values of x in the interval $[0, 4]$ that divide the solid into three parts of equal volume.

51. ***Volume of a Fuel Tank*** A tank on the wing of a jet aircraft is formed by revolving the region bounded by the graph of $y = \frac{1}{8}x^2\sqrt{2 - x}$ and the x-axis about the x-axis (see figure), where x and y are measured in meters. Find the tank's volume.



52. ***Volume of a Lab Glass*** A glass container can be modeled by revolving the graph of

$$y = \begin{cases} \sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}, & 0 \le x \le 11.5 \\ 2.95, & 11.5 < x \le 15 \end{cases}$$

about the x-axis, where x and y are measured in centimeters. Use a graphing utility to graph the function and find the volume of the container.

53. Find the volume of the solid generated if the upper half of the ellipse $9x^2 + 25y^2 = 225$ is revolved about

(a) the x-axis to form a prolate spheroid (shaped like a football).

(b) the y-axis to form an oblate spheroid (shaped like half of a candy).

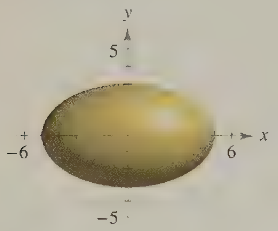


Figure for 53(a)

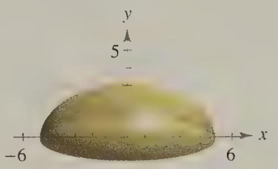


Figure for 53(b)

54. ***Minimum Volume*** The arc of $y = 4 - (x^2/4)$ on the interval $[0, 4]$ is revolved about the line $y = b$ (see figure).

(a) Find the volume of the resulting solid as a function of b.

(b) Use a graphing utility to graph the function in part (a), and use the graph to approximate the value of b that minimizes the volume of the solid.

(c) Use calculus to find the value of b that minimizes the volume of the solid, and compare the result with the answer to part (b).

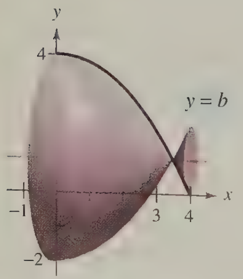


Figure for 54

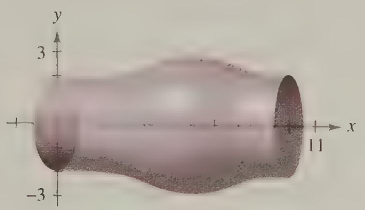


Figure for 56

55. ***Water Depth in a Tank*** A tank on a water tower is a sphere of radius 50 feet. Determine the depths of the water when the tank is filled to one-fourth and three-fourths of its total capacity. (*Note:* Use the root-finding capabilities of a graphing utility after evaluating the definite integral.)

56. ***Modeling Data*** A draftsman is asked to determine th amount of material required to produce a machine part (se figure in first column). The diameters d of the part at equall spaced points x are listed in the table. The measurements ar listed in centimeters.

x	0	1	2	3	4	5
d	4.2	3.8	4.2	4.7	5.2	5.7

x	6	7	8	9	10
d	5.8	5.4	4.9	4.4	4.6

(a) Use these data with Simpson's Rule to approximate th volume of the part.

(b) Use the regression capabilities of a graphing utility to fin a fourth-degree polynomial through the points representin the radius of the solid. Plot the data and graph the model.

(c) Use a graphing utility to approximate the definite integra yielding the volume of the part. Compare the result with th answer to part (a).

57. ***Think About It*** Match each integral with the solid whos volume it represents, and give the dimensions of each solid.

(a) Right circular cylinder (b) Ellipsoid

(c) Sphere (d) Right circular cone (e) Torus

(i) $\pi \int_0^h \left(\frac{rx}{h}\right)^2 dx$

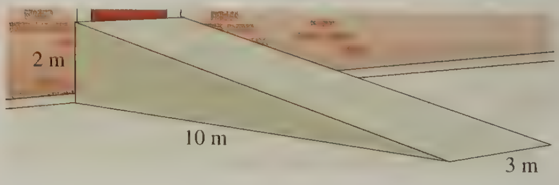
(ii) $\pi \int_0^h r^2\, dx$

(iii) $\pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx$

(iv) $\pi \int_{-b}^{b} \left(\sqrt{1 - \frac{x^2}{b^2}}\right)^2 dx$

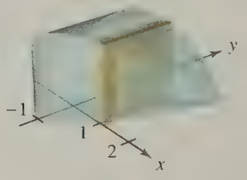
(v) $\pi \int_{-r}^{r} \left[\left(R + \sqrt{r^2 - x^2}\right)^2 - \left(R - \sqrt{r^2 - x^2}\right)^2\right] dx$

58. Find the volume of concrete in a ramp that is 3 meters wide an whose cross sections are right triangles with base 10 meters an height 2 meters (see figure).

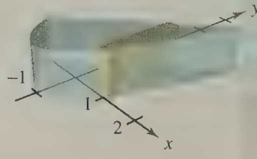


59. Find the volume of the solid whose base is bounded by th graphs of $y = x + 1$ and $y = x^2 - 1$, with the indicated cros sections taken perpendicular to the x-axis.

(a) Squares

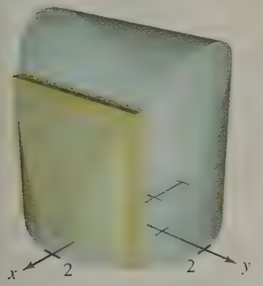


(b) Rectangles of height 1

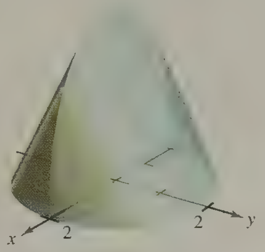


0. Find the volume of the solid whose base is bounded by the circle $x^2 + y^2 = 4$, with the indicated cross sections taken perpendicular to the x-axis.

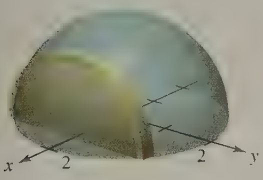
(a) Squares



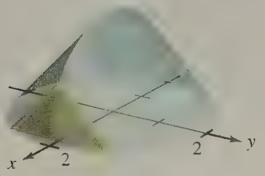
(b) Equilateral triangles



(c) Semicircles

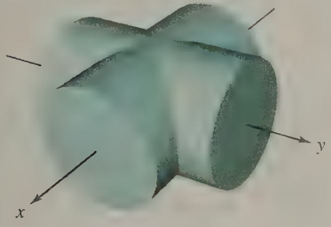


(d) Isosceles right triangles

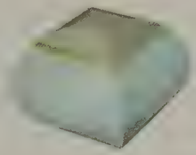


1. The base of a solid is bounded by $y = x^3$, $y = 0$, and $x = 1$. Find the volume of the solid for each of the following cross sections (taken perpendicular to the y-axis): (a) squares, (b) semicircles, (c) equilateral triangles, and (d) semiellipses whose heights are twice the lengths of their bases.

2. Find the volume of the solid of intersection (the solid common to both) of the two right circular cylinders of radius r whose axes meet at right angles (see figure).



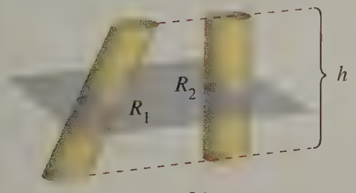
Two intersecting cylinders



Solid of intersection

OR FURTHER INFORMATION For more information on this roblem, see the article "Estimating the Volumes of Solid Figures ith Curved Surfaces" by Donald Cohen in *Mathematics Teacher*. o view this article, go to the website *www.matharticles.com*.

3. ***Cavalieri's Theorem*** Prove that if two solids have equal altitudes and all plane sections parallel to their bases and at equal distances from their bases have equal areas, then the solids have the same volume (see figure).



Area of R_1 = area of R_2

64. A manufacturer drills a hole through the center of a metal sphere of radius R. The hole has a radius r. Find the volume of the resulting ring.

65. For the metal sphere in Exercise 64, let $R = 5$. What value of r will produce a ring whose volume is exactly half the volume of the sphere?

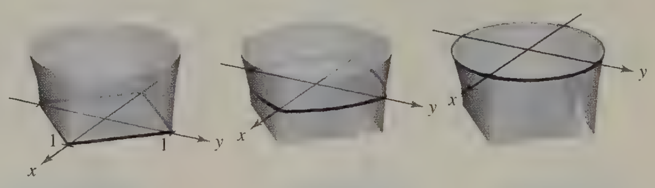
66. The solid shown in the figure has cross sections bounded by the graph of

$$|x|^a + |y|^a = 1$$

where $1 \le a \le 2$.

(a) Describe the cross section when $a = 1$ and $a = 2$.

(b) Describe a procedure for approximating the volume of the solid.

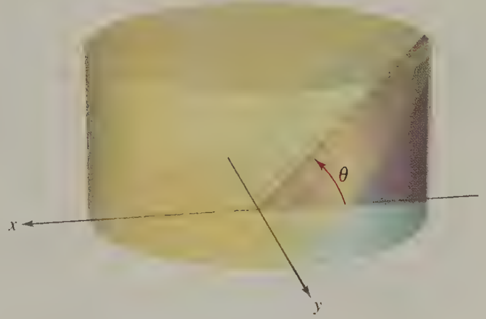


$|x|^1 + |y|^1 = 1$ $|x|^a + |y|^a = 1$ $|x|^2 + |y|^2 = 1$

67. Two planes cut a right circular cylinder to form a wedge. One plane is perpendicular to the axis of the cylinder and the second makes an angle of θ degrees with the first (see figure).

(a) Find the volume of the wedge if $\theta = 45°$.

(b) Find the volume of the wedge for an arbitrary angle θ. Assuming that the cylinder has sufficient length, how does the volume of the wedge change as θ increases from 0° to 90°?



Section 6.3 Volume: The Shell Method

- Find the volume of a solid of revolution using the shell method.
- Compare the uses of the disk method and the shell method.

The Shell Method

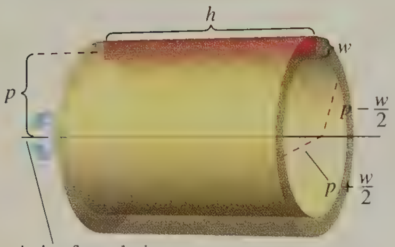


Figure 6.26

In this section, you will study an alternative method for finding the volume of a solid of revolution. This method is called the **shell method** because it uses cylindrica shells. We will compare the advantages of the disk and shell methods later in this section.

To begin, consider a representative rectangle as shown in Figure 6.26, where w is the width of the rectangle, h is the height of the rectangle, and p is the distance between the axis of revolution and the *center* of the rectangle. When this rectangle is revolved about its axis of revolution, it forms a cylindrical shell (or tube) of thickness w. To find the volume of this shell, consider two cylinders. The radius of the larger cylinder corresponds to the outer radius of the shell, and the radius of the smaller cylinder corresponds to the inner radius of the shell. Because p is the average radius of the shell, you know the outer radius is $p + (w/2)$ and the inner radius is $p - (w/2)$

$$p + \frac{w}{2} \qquad \text{Outer radius}$$

$$p - \frac{w}{2} \qquad \text{Inner radius}$$

So, the volume of the shell is

$$\begin{aligned}\text{Volume of shell} &= (\text{volume of cylinder}) - (\text{volume of hole}) \\ &= \pi\left(p + \frac{w}{2}\right)^2 h - \pi\left(p - \frac{w}{2}\right)^2 h \\ &= 2\pi phw \\ &= 2\pi(\text{average radius})(\text{height})(\text{thickness}).\end{aligned}$$

You can use this formula to find the volume of a solid of revolution. Assume tha the plane region in Figure 6.27 is revolved about a line to form the indicated solid. I you consider a horizontal rectangle of width Δy, then, as the plane region is revolved about a line parallel to the x-axis, the rectangle generates a representative shell whose volume is

$$\Delta V = 2\pi[p(y)h(y)]\,\Delta y.$$

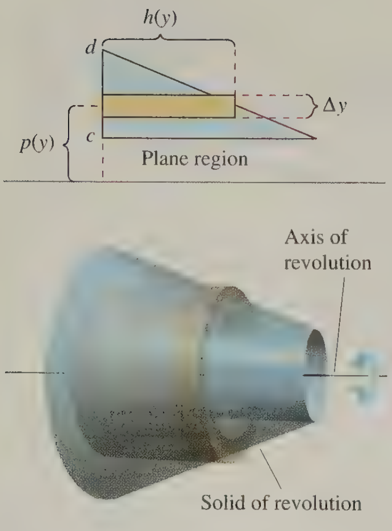


Figure 6.27

You can approximate the volume of the solid by n such shells of thickness Δy, heigh $h(y_i)$, and average radius $p(y_i)$.

$$\text{Volume of solid} \approx \sum_{i=1}^{n} 2\pi[p(y_i)h(y_i)]\,\Delta y = 2\pi\sum_{i=1}^{n}[p(y_i)h(y_i)]\,\Delta y$$

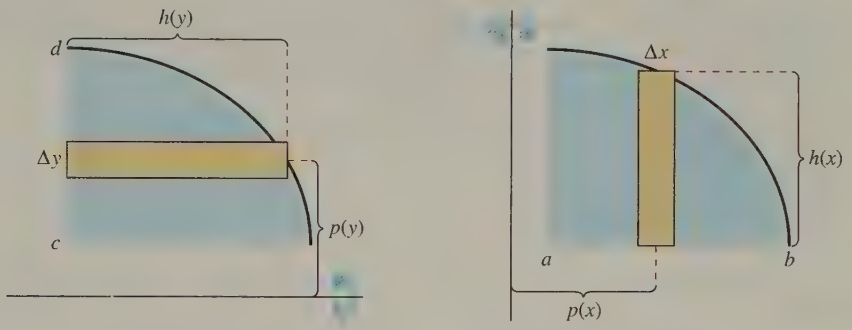
This approximation appears to become better and better as $\|\Delta\| \to 0$ $(n \to \infty)$. Therefore, we define the volume of the solid to be

$$\begin{aligned}\text{Volume of solid} &= \lim_{\|\Delta\|\to 0} 2\pi\sum_{i=1}^{n}[p(y_i)h(y_i)]\,\Delta y \\ &= 2\pi\int_c^d [p(y)h(y)]\,dy.\end{aligned}$$

The Shell Method

To find the volume of a solid of revolution with the **shell method,** use one of the following, as shown in Figure 6.28.

Horizontal Axis of Revolution	*Vertical Axis of Revolution*
$\text{Volume} = V = 2\pi \int_c^d p(y)h(y)\,dy$	$\text{Volume} = V = 2\pi \int_a^b p(x)h(x)\,dx$



Horizontal axis of revolution

Vertical axis of revolution

Figure 6.28

Example 1 Using the Shell Method to Find Volume

Find the volume of the solid of revolution formed by revolving the region bounded by

$$y = x - x^3$$

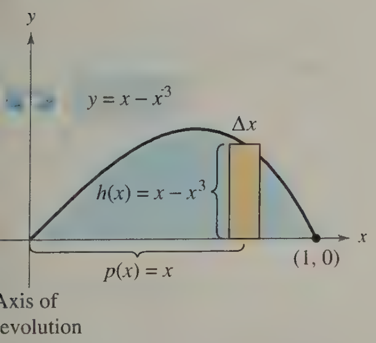
and the x-axis $(0 \le x \le 1)$ about the y-axis.

Solution Because the axis of revolution is vertical, use a vertical representative rectangle, as shown in Figure 6.29. The width Δx indicates that x is the variable of integration. The distance from the center of the rectangle to the axis of revolution is $p(x) = x$, and the height of the rectangle is

$$h(x) = x - x^3.$$

Because x ranges from 0 to 1, the volume of the solid is

$$\begin{aligned} V = 2\pi \int_a^b p(x)h(x)\,dx &= 2\pi \int_0^1 x(x - x^3)\,dx && \text{Apply shell method.} \\ &= 2\pi \int_0^1 (-x^4 + x^2)\,dx && \text{Simplify.} \\ &= 2\pi \left[-\frac{x^5}{5} + \frac{x^3}{3} \right]_0^1 && \text{Integrate.} \\ &= 2\pi \left(-\frac{1}{5} + \frac{1}{3} \right) \\ &= \frac{4\pi}{15}. \end{aligned}$$



gure 6.29

Example 2 Using the Shell Method to Find Volume

Find the volume of the solid of revolution formed by revolving the region bounded by the graph of

$$x = e^{-y^2}$$

and the y-axis $(0 \le y \le 1)$ about the x-axis.

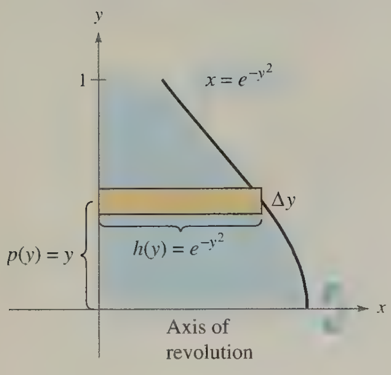


Figure 6.30

Solution Because the axis of revolution is horizontal, use a horizontal representative rectangle, as shown in Figure 6.30. The width Δy indicates that y is the variable of integration. The distance from the center of the rectangle to the axis of revolution is $p(y) = y$, and the height of the rectangle is $h(y) = e^{-y^2}$. Because y ranges from 0 to 1, the volume of the solid is

$$V = 2\pi \int_c^d p(y)h(y)\,dy = 2\pi \int_0^1 ye^{-y^2}\,dy \qquad \text{Apply shell method.}$$

$$= -\pi \Big[e^{-y^2} \Big]_0^1 \qquad \text{Integrate.}$$

$$= \pi \left(1 - \frac{1}{e} \right)$$

$$\approx 1.986.$$

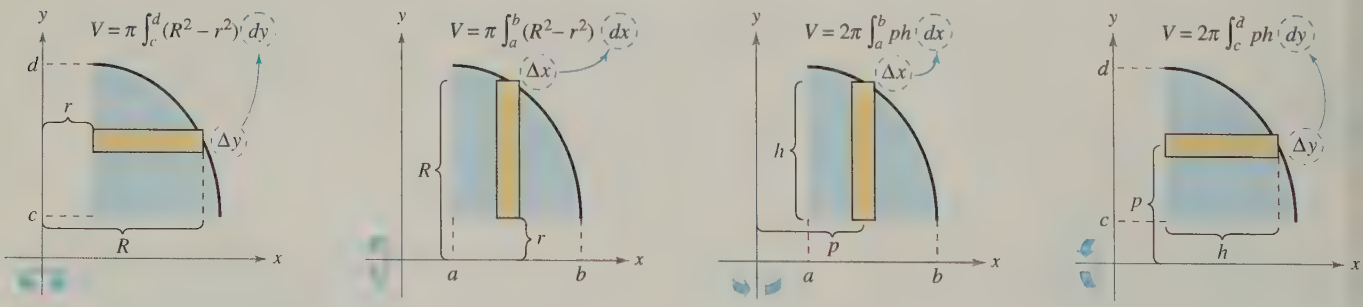
NOTE To see the advantage of using the shell method in Example 2, solve the equation $x = e^{-y^2}$ for y.

$$y = \begin{cases} 1, & 0 \le x \le 1/e \\ \sqrt{-\ln x}, & 1/e < x \le 1 \end{cases}$$

Then use this equation to find the volume using the disk method.

Comparison of Disk and Shell Methods

The disk and shell methods can be distinguished as follows. For the disk method, the representative rectangle is always *perpendicular* to the axis of revolution, whereas for the shell method, the representative rectangle is always *parallel* to the axis of revolution, as shown in Figure 6.31.



y
$V = \pi \int_c^d (R^2 - r^2)\,dy$
d
r
Δy
c
R
x
y
$V = \pi \int_a^b (R^2 - r^2)\,dx$
Δx
R
r
a
b
x
y
$V = 2\pi \int_a^b ph\,dx$
Δx
h
p
a
b
x
y
$V = 2\pi \int_c^d ph\,dy$
d
Δy
p
c
h
x

Vertical axis of revolution

Horizontal axis of revolution

Vertical axis of revolution

Horizontal axis of revolution

Disk method: Representative rectangle is perpendicular to the axis of revolution.

Shell method: Representative rectangle is parallel to the axis of revolution.

Figure 6.31

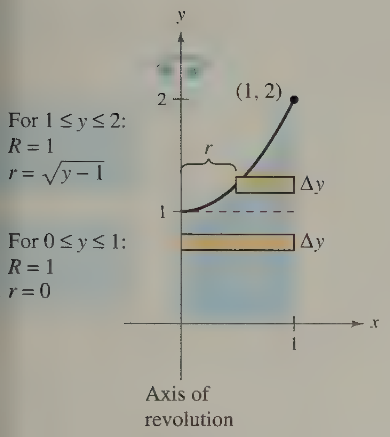
Often, one method is more convenient to use than the other. The following example illustrates a case in which the shell method is preferable.

Example 3 Shell Method Preferable

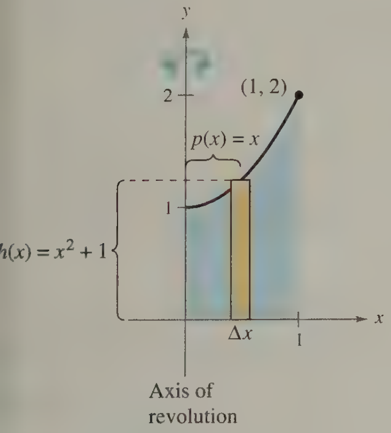
Find the volume of the solid formed by revolving the region bounded by the graphs of

$$y = x^2 + 1, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 1$$

about the y-axis.



a) Disk method



b) Shell method

Figure 6.32

Solution In Example 4 in the preceding section, you saw that the disk method requires two integrals to determine the volume of this solid. See Figure 6.32(a).

$$\begin{aligned}
V &= \pi \int_0^1 (1^2 - 0^2)\,dy + \pi \int_1^2 \left[1^2 - \left(\sqrt{y-1}\right)^2\right] dy && \text{Apply disk method.}\\
&= \pi \int_0^1 1\,dy + \pi \int_1^2 (2-y)\,dy && \text{Simplify.}\\
&= \pi \Big[y\Big]_0^1 + \pi \left[2y - \frac{y^2}{2}\right]_1^2 && \text{Integrate.}\\
&= \pi + \pi\left(4 - 2 - 2 + \frac{1}{2}\right)\\
&= \frac{3\pi}{2}
\end{aligned}$$

In Figure 6.32(b), you can see that the shell method requires only one integral to find the volume.

$$\begin{aligned}
V &= 2\pi \int_a^b p(x)h(x)\,dx && \text{Apply shell method.}\\
&= 2\pi \int_0^1 x(x^2+1)\,dx\\
&= 2\pi \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^1 && \text{Integrate.}\\
&= 2\pi\left(\frac{3}{4}\right)\\
&= \frac{3\pi}{2}
\end{aligned}$$

Suppose the region in Example 3 were revolved about the vertical line $x = 1$. Would the resulting solid of revolution have a greater volume or a smaller volume than the solid in Example 3? Without integrating, you should be able to reason that the resulting solid would have a smaller volume because "more" of the revolved region would be closer to the axis of revolution. To confirm this, try solving the following integral, which gives the volume of the solid.

$$V = 2\pi \int_0^1 (1-x)(x^2+1)\,dx \qquad p(x) = 1 - x$$

FOR FURTHER INFORMATION To learn more about the disk and shell methods, see the article "The Disc and Shell Method" by Charles A. Cable in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

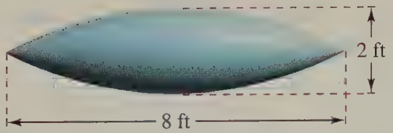


Figure 6.33

Example 4 Volume of a Pontoon

A pontoon is to be made in the shape shown in Figure 6.33. The pontoon is designed by rotating the graph of

$$y = 1 - \frac{x^2}{16}, \qquad -4 \le x \le 4$$

about the x-axis, where x and y are measured in feet. Find the volume of the pontoon

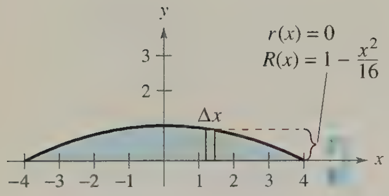
Solution Refer to Figure 6.34(a) and use the disk method as follows.

$$V = \pi \int_{-4}^{4} \left(1 - \frac{x^2}{16}\right)^2 dx \qquad \text{Apply disk method.}$$

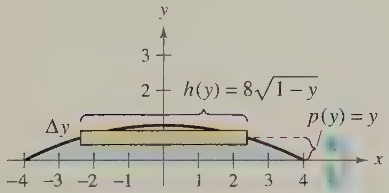
$$= \pi \int_{-4}^{4} \left(1 - \frac{x^2}{8} + \frac{x^4}{256}\right) dx \qquad \text{Simplify.}$$

$$= \pi \left[x - \frac{x^3}{24} + \frac{x^5}{1280}\right]_{-4}^{4} \qquad \text{Integrate.}$$

$$= \frac{64\pi}{15} \approx 13.4 \text{ cubic feet}$$



(a) Disk method



(b) Shell method

Figure 6.34

Try using Figure 6.34(b) to set up the integral for the volume using the shell method Does the integral seem more complicated?

For the shell method in Example 4, you would have to solve for x in terms of y in the equation

$$y = 1 - (x^2/16).$$

Sometimes, solving for x is very difficult (or even impossible). In such cases you must use a vertical rectangle (of width Δx), thus making x the variable of integration The position (horizontal or vertical) of the axis of revolution then determines the method to be used. This is illustrated in Example 5.

Example 5 Shell Method Necessary

Find the volume of the solid formed by revolving the region bounded by the graphs o $y = x^3 + x + 1$, $y = 1$, and $x = 1$ about the line $x = 2$, as shown in Figure 6.35.

Solution In the equation $y = x^3 + x + 1$, you cannot easily solve for x in terms o y. (See the section on Newton's Method.) Therefore, the variable of integration mus be x, and you should choose a vertical representative rectangle. Because the rectangl is parallel to the axis of revolution, use the shell method and obtain

$$V = 2\pi \int_a^b p(x)h(x)\,dx = 2\pi \int_0^1 (2 - x)(x^3 + x + 1 - 1)\,dx \qquad \text{Apply shell method}$$

$$= 2\pi \int_0^1 (-x^4 + 2x^3 - x^2 + 2x)\,dx \qquad \text{Simplify.}$$

$$= 2\pi \left[-\frac{x^5}{5} + \frac{x^4}{2} - \frac{x^3}{3} + x^2\right]_0^1 \qquad \text{Integrate.}$$

$$= 2\pi\left(-\frac{1}{5} + \frac{1}{2} - \frac{1}{3} + 1\right)$$

$$= \frac{29\pi}{15}.$$

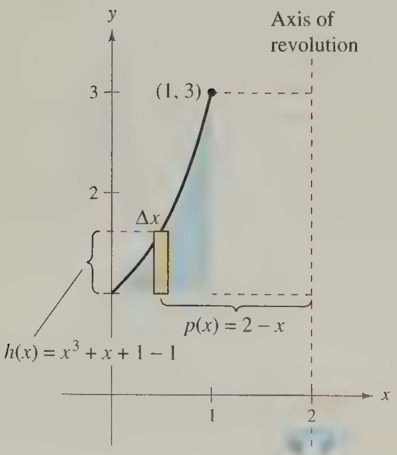
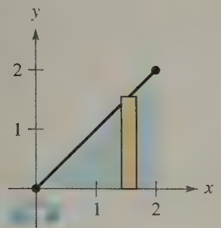


Figure 6.35

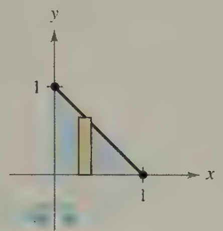
EXERCISES FOR SECTION 6.3

n Exercises 1–12, use the shell method to set up and evaluate he integral that gives the volume of the solid generated by evolving the plane region about the y-axis.

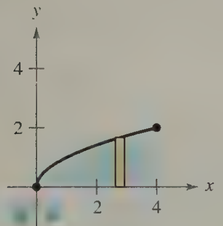
1. $y = x$



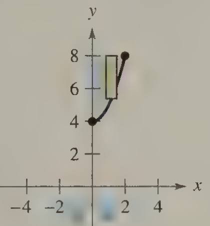
2. $y = 1 - x$



3. $y = \sqrt{x}$



4. $y = x^2 + 4$



5. $y = x^2,\ y = 0,\ x = 2$

6. $y = \frac{1}{2}x^2,\ y = 0,\ x = 6$

7. $y = x^2,\ y = 4x - x^2$

8. $y = 4 - x^2,\ y = 0$

9. $y = 4x - x^2,\ x = 0,\ y = 4$

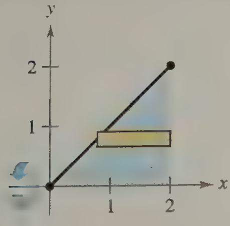
0. $y = 2x,\ y = 4,\ x = 0$

1. $y = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2},\ y = 0,\ x = 0,\ x = 1$

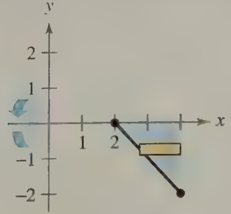
2. $y = \begin{cases} \dfrac{\sin x}{x}, & x > 0 \\ 1, & x = 0 \end{cases},\ y = 0,\ x = 0,\ x = \pi$

n Exercises 13–16, use the shell method to set up and evaluate he integral that gives the volume of the solid generated by evolving the plane region about the x-axis.

3. $y = x$



14. $y = 2 - x$



5. $y = \dfrac{1}{x}, x = 1, x = 2, y = 0$

16. $x + y^2 = 16, x = 0$

In Exercises 17–20, use the shell method to find the volume of the solid generated by revolving the plane region about the indicated line.

17. $y = x^2,\ y = 4x - x^2$, about the line $x = 4$

18. $y = x^2,\ y = 4x - x^2$, about the line $x = 2$

19. $y = 4x - x^2,\ y = 0$, about the line $x = 5$

20. $y = \sqrt{x},\ y = 0,\ x = 4$, about the line $x = 6$

In Exercises 21–24, use the disk *or* the shell method to find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the indicated line.

21. $y = x^3,\ y = 0,\ x = 2$

(a) the x-axis (b) the y-axis (c) the line $x = 4$

22. $y = \dfrac{10}{x^2},\ y = 0,\ x = 1,\ x = 5$

(a) the x-axis (b) the y-axis (c) the line $y = 10$

23. $x^{1/2} + y^{1/2} = a^{1/2},\ x = 0,\ y = 0$

(a) the x-axis (b) the y-axis (c) the line $x = a$

24. $x^{2/3} + y^{2/3} = a^{2/3},\quad a > 0$ (hypocycloid)

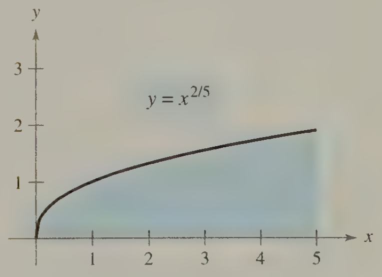
(a) the x-axis (b) the y-axis

Getting at the Concept

25. Give the integration formula for finding the volume of a solid using the shell method.

26. The region in the figure is revolved about the indicated axes and line. Order the volumes of the resulting solids from least to greatest. Explain your reasoning.

(a) x-axis (b) y-axis (c) $x = 5$



In Exercises 27 and 28, give a geometric argument that explains why the integrals have equal values.

27. $\pi \displaystyle\int_1^5 (x - 1)\,dx = 2\pi \int_0^2 y[5 - (y^2 + 1)]\,dy$

28. $\pi \displaystyle\int_0^2 [16 - (2y)^2]\,dy = 2\pi \int_0^4 x\left(\frac{x}{2}\right)dx$

In Exercises 29–32, (a) use a graphing utility to graph the plane region bounded by the graphs of the equations, and (b) use the integration capabilities of the graphing utility to approximate the volume of the solid generated by revolving the region about the y-axis.

29. $x^{4/3} + y^{4/3} = 1,\ x = 0,\ y = 0$, first quadrant

30. $y = \sqrt{1 - x^3},\ y = 0,\ x = 0$

31. $y = \sqrt[3]{(x-2)^2(x-6)^2},\ y = 0,\ x = 2,\ x = 6$

32. $y = \dfrac{2}{1 + e^{1/x}},\ y = 0,\ x = 1,\ x = 3$

Think About It **In Exercises 33 and 34, determine which value best approximates the volume of the solid generated by revolving the region bounded by the graphs of the equations about the y-axis. (Make your selection on the basis of a sketch of the solid and *not* by performing any calculations.)**

33. $y = 2e^{-x},\ y = 0,\ x = 0,\ x = 2$

(a) $\frac{3}{2}$ (b) -2 (c) 4 (d) 7.5 (e) 15

34. $y = \tan x,\ y = 0,\ x = 0,\ x = \dfrac{\pi}{4}$

(a) 3.5 (b) $-\frac{9}{4}$ (c) 8 (d) 10 (e) 1

35. ***Machine Part*** A solid is generated by revolving the region bounded by $y = \frac{1}{2}x^2$ and $y = 2$ about the y-axis. A hole, centered along the axis of revolution, is drilled through this solid so that one fourth of the volume is removed. Find the diameter of the hole.

36. ***Machine Part*** A solid is generated by revolving the region bounded by $y = \sqrt{9 - x^2}$ and $y = 0$ about the y-axis. A hole, centered along the axis of revolution, is drilled through this solid so that one third of the volume is removed. Find the diameter of the hole.

37. ***Volume of a Torus*** A torus is formed by revolving the region bounded by the circle $x^2 + y^2 = 1$ about the line $x = 2$, as shown in the figure. Find the volume of this "doughnut-shaped" solid. (*Hint:* The integral $\int_{-1}^{1} \sqrt{1 - x^2}\,dx$ represents the area of a semicircle.)

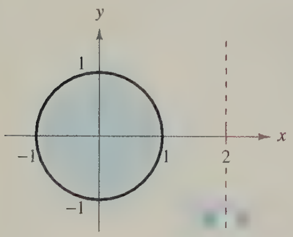


Figure for 37

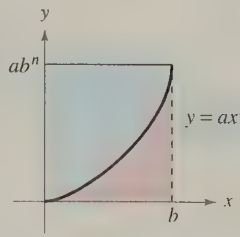


Figure for 40

38. ***Volume of a Torus*** Repeat Exercise 37 for a torus formed by revolving the region bounded by the circle $x^2 + y^2 = r^2$ about the line $x = R$, where $r < R$.

39. ***Volume of a Segment of a Sphere*** Let a sphere of radius r be cut by a plane, thus forming a segment of height h. Show that the volume of this segment is $\frac{1}{3}\pi h^2(3r - h)$.

40. ***Exploration*** Consider the region bounded by the graphs o $y = ax^n$, $y = ab^n$, and $x = 0$ (see figure in first column).

(a) Find the ratio $R_1(n)$ of the area of the region to the area o the circumscribed rectangle.

(b) Find $\lim\limits_{n\to\infty} R_1(n)$ and compare the result with the area of the circumscribed rectangle.

(c) Find the volume of the solid of revolution formed by revolving the region about the y-axis. Find the ratio $R_2(n$ of this volume to the volume of the circumscribed righ circular cylinder.

(d) Find $\lim\limits_{n\to\infty} R_2(n)$ and compare the result with the volume o the circumscribed cylinder.

(e) Use the results of parts (b) and (d) to make a conjecture about the shape of the graph of $y = ax^n\ (0 \le x \le b)$ as $n \to \infty$.

41. ***Think About It*** Match each of the integrals with the soli whose volume it represents, and give the dimensions of eac solid.

(a) Right circular cone (b) Torus (c) Sphere

(d) Right circular cylinder (e) Ellipsoid

(i) $2\pi \displaystyle\int_0^r hx\,dx$

(ii) $2\pi \displaystyle\int_0^r hx\left(1 - \frac{x}{r}\right) dx$

(iii) $2\pi \displaystyle\int_0^r 2x\sqrt{r^2 - x^2}\,dx$

(iv) $2\pi \displaystyle\int_0^b 2ax\sqrt{1 - \frac{x^2}{b^2}}\,dx$

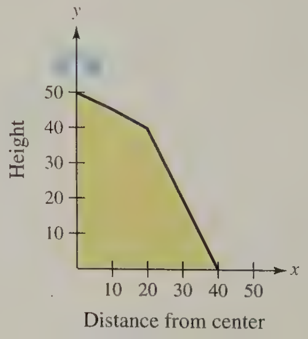
(v) $2\pi \displaystyle\int_{-r}^r (R - x)\left(2\sqrt{r^2 - x^2}\right) dx$

42. ***Volume of a Storage Shed*** A storage shed has a circular base of diameter 80 feet (see figure). Starting at the center, the interior height is measured every 10 feet and recorded in the table.

x	0	10	20	30	40
Height	50	45	40	20	0

(a) Use Simpson's Rule to approximate the volume of the shed.

(b) Note that the roof line consists of two line segments. Fin the equations of the line segments and use integration t find the volume of the shed.



43. ***Modeling Data*** A pond is approximately circular, with a diameter of 400 feet (see figure). Starting at the center, the depth of the water is measured every 25 feet and recorded in the table.

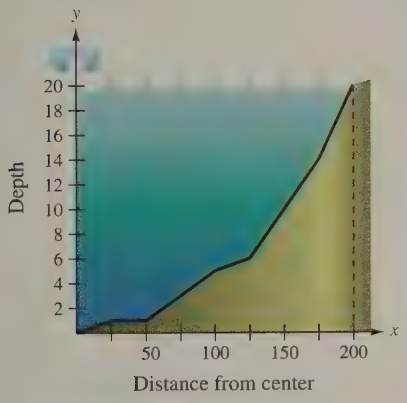
x	0	25	50	75	100	125	150	175	200
Depth	20	19	19	17	15	14	10	6	0

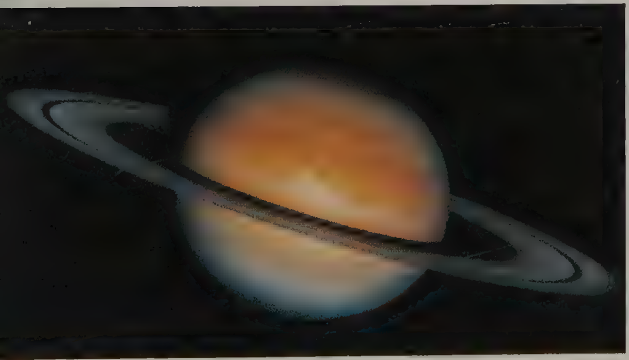
(a) Use Simpson's Rule to approximate the volume of water in the pond.

(b) Use the regression capabilities of a graphing utility to find a quadratic model for the depths recorded in the table. Use the graphing utility to plot the depths and graph the model.

(c) Use the integration capabilities of a graphing utility and the model in part (b) to approximate the volume of water in the pond.

(d) Use the result in part (c) to approximate the number of gallons of water in the pond if 1 cubic foot of water is approximately 7.48 gallons.





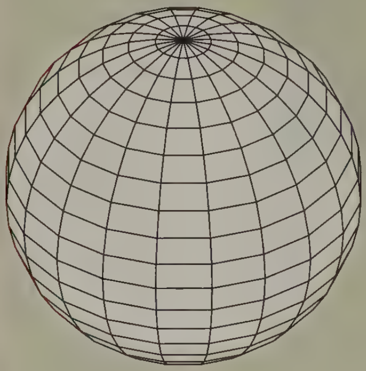
The color-enhanced photo of Saturn was taken by Voyager 1. In the photograph, the oblateness of Saturn is clearly visible.

SECTION PROJECT SATURN

The Oblateness of Saturn Saturn is the most oblate of the nine planets in our solar system. Its equatorial radius is 60,268 kilometers and its polar radius is 54,364 kilometers.

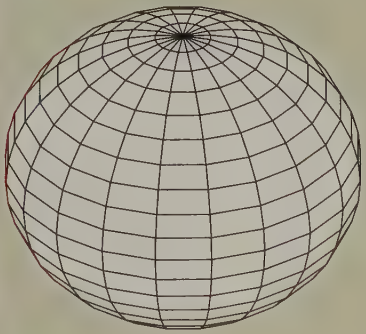
(a) Find the ratio of the volumes of the sphere and the oblate ellipsoid shown below.

(b) If a planet were spherical and had the same volume as Saturn, what would its radius be?



Computer model of "spherical Saturn," whose equatorial radius is equal to its polar radius. The equation of the cross section passing through the pole is

$x^2 + y^2 = 60{,}268^2.$



Computer model of "oblate Saturn," whose equatorial radius is greater than its polar radius. The equation of the cross section passing through the pole is

$$\frac{x^2}{60{,}268^2} + \frac{y^2}{54{,}364^2} = 1.$$

Section 6.4 Arc Length and Surfaces of Revolution

- Find the arc length of a smooth curve.
- Find the area of a surface of revolution.

Bettmann/Corbis

Christian Huygens (1629–1695)

The Dutch mathematician Christian Huygens, who invented the pendulum clock, and James Gregory (1638–1675), a Scottish mathematician, both made early contributions to the problem of finding the length of a rectifiable curve.

Arc Length

In this section, definite integrals are used to find the arc length of a curve and the area of a surface of revolution. In both cases, we approximate an arc (a segment of a curve) by straight line segments whose lengths are given by the familiar distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

A **rectifiable** curve is one that has a finite arc length. You will see that a sufficient condition for the graph of a function f to be rectifiable between $(a, f(a))$ and $(b, f(b))$ is that f' be continuous on $[a, b]$. Such a function is **continuously differentiable** on $[a, b]$, and its graph on the interval $[a, b]$ is a **smooth curve.**

Consider a function $y = f(x)$ that is continuously differentiable on the interval $[a, b]$. You can approximate the graph of f by n line segments whose endpoints are determined by the partition

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_n = b$$

as shown in Figure 6.36. By letting $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, you can approximate the length of the graph by

$$\begin{aligned} s &\approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2 (\Delta x_i)^2} \\ &= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,(\Delta x_i). \end{aligned}$$

This approximation appears to become better and better as $\|\Delta\| \to 0$ $(n \to \infty)$. Therefore, we define the length of the graph to be

$$s = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,(\Delta x_i).$$

Because $f'(x)$ exists for each x in (x_{i-1}, x_i), the Mean Value Theorem guarantees the existence of c_i in (x_{i-1}, x_i) such that

$$\begin{aligned} f(x_i) - f(x_{i-1}) &= f'(c_i)(x_i - x_{i-1}) \\ \frac{\Delta y_i}{\Delta x_i} &= f'(c_i). \end{aligned}$$

Because f' is continuous on $[a, b]$, it follows that $\sqrt{1 + [f'(x)]^2}$ is also continuous (and hence integrable) on $[a, b]$, which implies that

$$\begin{aligned} s &= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\,(\Delta x_i) \\ &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \end{aligned}$$

We call s the **arc length** of f between a and b.

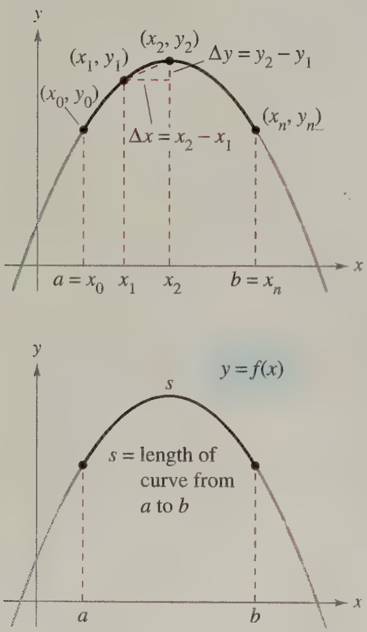


Figure 6.36

Definition of Arc Length

Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

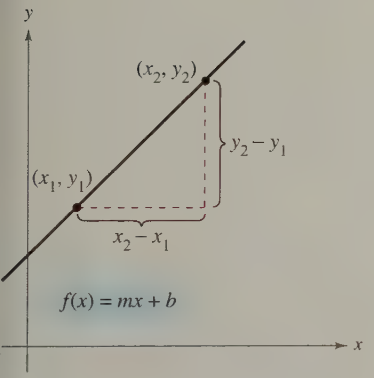
Similarly, for a smooth curve given by $x = g(y)$, the **arc length** of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Because the definition of arc length can be applied to a linear function, you can check to see that this new definition agrees with the standard distance formula for the length of a line segment. This is done in Example 1.

Example 1 The Length of a Line Segment

Find the arc length from (x_1, y_1) to (x_2, y_2) on the graph of $f(x) = mx + b$, as shown in Figure 6.37.



The arc length of the graph of f from (x_1, y_1) to (x_2, y_2) is the same as the standard distance formula.
Figure 6.37

Solution Because

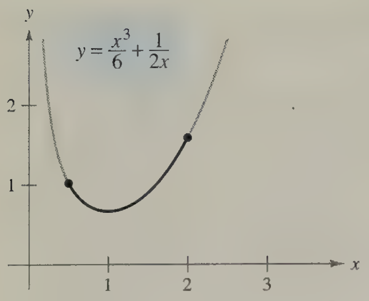
$$m = f'(x) = \frac{y_2 - y_1}{x_2 - x_1}$$

it follows that

$$\begin{aligned} s &= \int_{x_1}^{x_2} \sqrt{1 + [f'(x)]^2}\, dx && \text{Formula for arc length} \\ &= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2}\, dx \\ &= \sqrt{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{(x_2 - x_1)^2}}\,(x)\Bigg]_{x_1}^{x_2} && \text{Integrate and simplify.} \\ &= \sqrt{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{(x_2 - x_1)^2}}\,(x_2 - x_1) \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$

which is the formula for the distance between two points in the plane. ▱

TECHNOLOGY Definite integrals representing arc length often are very difficult to evaluate. In this section we present a few examples. In the next chapter, with more advanced integration techniques, you will be able to tackle more difficult arc length problems. In the meantime, remember that you can always use a numerical integration program to approximate an arc length. For instance, try using the numerical integration feature of a graphing utility to approximate the arc lengths in Examples 2 and 3.



The arc length of the graph of y on $\left[\frac{1}{2}, 2\right]$ is $\frac{33}{16}$.
Figure 6.38

Example 2 Finding Arc Length

Find the arc length of the graph of

$$y = x^3/6 + 1/(2x)$$

on the interval $\left[\frac{1}{2}, 2\right]$, as shown in Figure 6.38.

Solution Using

$$\frac{dy}{dx} = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$$

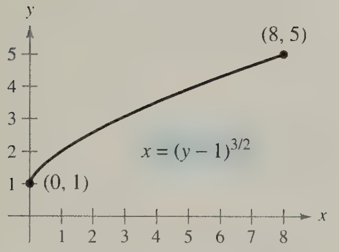
yields an arc length of

$$\begin{aligned} s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx &= \int_{1/2}^{2} \sqrt{1 + \left[\frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)\right]^2}\,dx && \text{Formula for arc length} \\ &= \int_{1/2}^{2} \sqrt{\frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right)}\,dx \\ &= \int_{1/2}^{2} \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)dx && \text{Simplify.} \\ &= \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_{1/2}^{2} && \text{Integrate.} \\ &= \frac{1}{2}\left(\frac{13}{6} + \frac{47}{24}\right) \\ &= \frac{33}{16}. \end{aligned}$$

FOR FURTHER INFORMATION To see how arc length can be used to define trigonometric functions, see the article "Trigonometry Requires Calculus, Not Vice Versa" by Yves Nievergelt in *UMAP Modules*. To view this article, go to the website *www.matharticles.com*.

Example 3 Finding Arc Length

Find the arc length of the graph of $(y - 1)^3 = x^2$ on the interval $[0, 8]$, as shown in Figure 6.39.



The arc length of the graph of y on $[0, 8]$ is approximately 9.0734.
Figure 6.39

Solution Begin by solving for x in terms of y: $x = \pm(y - 1)^{3/2}$. Choosing the positive value of x produces

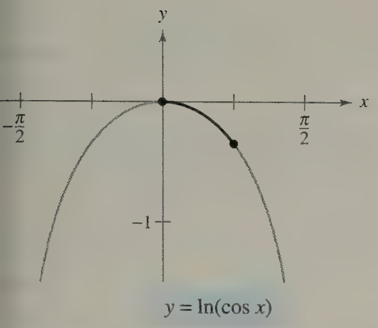
$$\frac{dx}{dy} = \frac{3}{2}(y - 1)^{1/2}.$$

The x-interval $[0, 8]$ corresponds to the y-interval $[1, 5]$, and the arc length is

$$\begin{aligned} s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy &= \int_1^5 \sqrt{1 + \left[\frac{3}{2}(y - 1)^{1/2}\right]^2}\,dy && \text{Formula for arc length} \\ &= \int_1^5 \sqrt{\frac{9}{4}y - \frac{5}{4}}\,dy \\ &= \frac{1}{2}\int_1^5 \sqrt{9y - 5}\,dy && \text{Simplify.} \\ &= \frac{1}{18}\left[\frac{(9y - 5)^{3/2}}{3/2}\right]_1^5 && \text{Integrate.} \\ &= \frac{1}{27}(40^{3/2} - 4^{3/2}) \\ &\approx 9.0734. \end{aligned}$$

Example 4 Finding Arc Length

Find the arc length of the graph of $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$, as shown in Figure 6.40.



he arc length of the graph of y on $\left[0, \frac{\pi}{4}\right]$ is pproximately 0.8814.
igure 6.40

Solution Using

$$\frac{dy}{dx} = -\frac{\sin x}{\cos x} = -\tan x$$

yields an arc length of

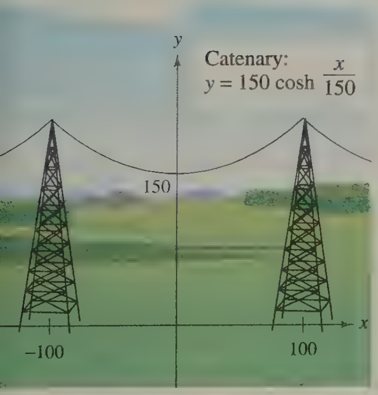
$$\begin{aligned}
s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx &= \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx && \text{Formula for arc length} \\
&= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx && \text{Trigonometric identity} \\
&= \int_0^{\pi/4} \sec x\, dx && \text{Simplify.} \\
&= \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4} && \text{Integrate.} \\
&= \ln\left(\sqrt{2} + 1\right) - \ln 1 \\
&\approx 0.8814.
\end{aligned}$$

Example 5 Length of a Cable

An electric cable is hung between two towers that are 200 feet apart, as shown in Figure 6.41. The cable takes the shape of a catenary whose equation is

$$y = 75(e^{x/150} + e^{-x/150}) = 150 \cosh \frac{x}{150}.$$

Find the arc length of the cable between the two towers.



he arc length of the cable is approximately 5 feet.
gure 6.41

Solution Because $y' = \frac{1}{2}(e^{x/150} - e^{-x/150})$, you can write

$$(y')^2 = \frac{1}{4}(e^{x/75} - 2 + e^{-x/75})$$

and

$$1 + (y')^2 = \frac{1}{4}(e^{x/75} + 2 + e^{-x/75}) = \left[\frac{1}{2}(e^{x/150} + e^{-x/150})\right]^2.$$

Therefore, the arc length of the cable is

$$\begin{aligned}
s = \int_a^b \sqrt{1 + (y')^2}\, dx &= \frac{1}{2}\int_{-100}^{100} (e^{x/150} + e^{-x/150})\, dx && \text{Formula for arc length} \\
&= 75\Big[e^{x/150} - e^{-x/150}\Big]_{-100}^{100} && \text{Integrate.} \\
&= 150(e^{2/3} - e^{-2/3}) \\
&\approx 215 \text{ feet.}
\end{aligned}$$

Area of a Surface of Revolution

In Sections 6.2 and 6.3, integration was used to calculate the volume of a solid c revolution. We now look at a procedure for finding the area of a surface of revolutio

Definition of a Surface of Revolution

If the graph of a continuous function is revolved about a line, the resulting surface is a **surface of revolution.**

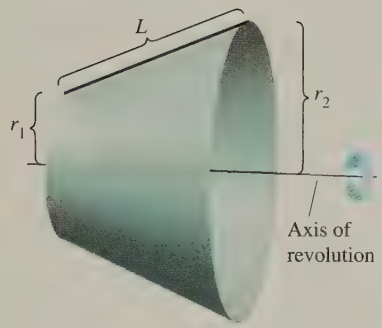


Figure 6.42

The area of a surface of revolution is derived from the formula for the latera surface area of the frustum of a right circular cone. Consider the line segment i Figure 6.42, where L is the length of the line segment, r_1 is the radius at the left en of the line segment, and r_2 is the radius at the right end of the line segment. When th line segment is revolved about its axis of revolution, it forms a frustum of a right ci cular cone, with

$$S = 2\pi r L$$ Lateral surface area of frustum

where

$$r = \frac{1}{2}(r_1 + r_2).$$ Average radius of frustum

(In Exercise 55, you are asked to verify the formula for S.)

Suppose the graph of a function f, having a continuous derivative on the interv $[a, b]$, is revolved about the x-axis to form a surface of revolution, as shown in Figu 6.43. Let Δ be a partition of $[a, b]$, with subintervals of width Δx_i. Then the lin segment of length

$$\Delta L_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

generates a frustum of a cone. Let r_i be the average radius of this frustum. By th Intermediate Value Theorem, a point d_i exists (in the ith subinterval) such th $r_i = f(d_i)$. The lateral surface area ΔS_i of the frustum is

$$\begin{aligned}\Delta S_i &= 2\pi r_i \Delta L_i \\ &= 2\pi f(d_i)\sqrt{\Delta x_i^2 + \Delta y_i^2} \\ &= 2\pi f(d_i)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i.\end{aligned}$$

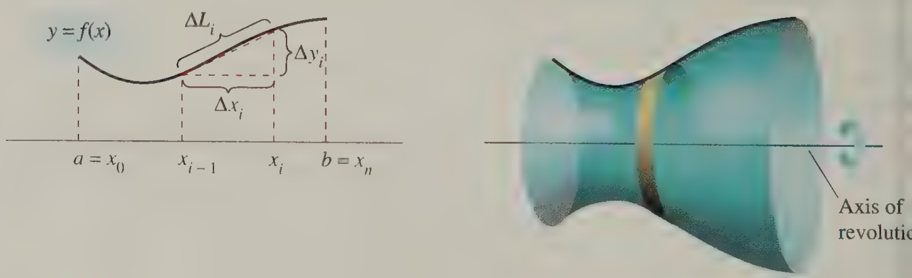


Figure 6.43

By the Mean Value Theorem, a point c_i exists in (x_{i-1}, x_i) such that

$$f'(c_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{\Delta y_i}{\Delta x_i}.$$

Therefore, $\Delta S_i = 2\pi f(d_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x_i$, and the total surface area can be approximated by

$$S \approx 2\pi \sum_{i=1}^{n} f(d_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x_i.$$

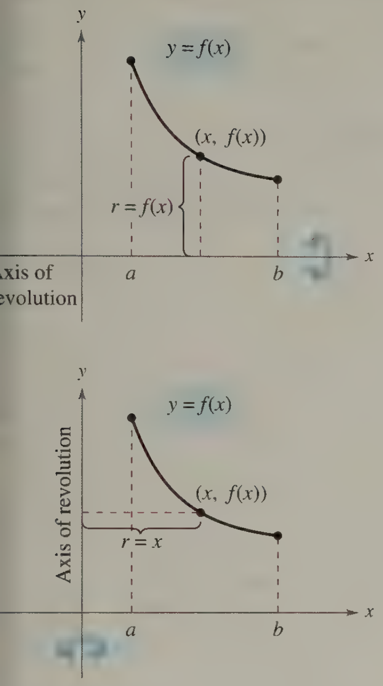
It can be shown that the limit of the right side as $\|\Delta\| \to 0$ $(n \to \infty)$, is

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

In a similar manner, if the graph of f is revolved about the y-axis, then S is

$$S = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}\,dx.$$

In both formulas for S, you can regard the products $2\pi f(x)$ and $2\pi x$ as the circumference of the circle traced by a point (x, y) on the graph of f as it is revolved about the x- or y-axis (Figure 6.44). In one case the radius is $r = f(x)$, and in the other case the radius is $r = x$. Moreover, by appropriately adjusting r, you can generalize the formula for surface area to cover *any* horizontal or vertical axis of revolution, as indicated in the following definition.



igure 6.44

Definition of the Area of a Surface of Revolution

Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\,dx \qquad y \text{ is a function of } x.$$

where $r(x)$ is the distance between the graph of f and the axis of revolution. If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_c^d r(y)\sqrt{1 + [g'(y)]^2}\,dy \qquad x \text{ is a function of } y.$$

where $r(y)$ is the distance between the graph of g and the axis of revolution.

The formulas in this definition are sometimes written as

$$S = 2\pi \int_a^b r(x)\,ds \qquad y \text{ is a function of } x.$$

and

$$S = 2\pi \int_c^d r(y)\,ds \qquad x \text{ is a function of } y.$$

where $ds = \sqrt{1 + [f'(x)]^2}\,dx$ and $ds = \sqrt{1 + [g'(y)]^2}\,dy$, respectively.

Example 6 The Area of a Surface of Revolution

Find the area of the surface formed by revolving the graph of

$$f(x) = x^3$$

on the interval $[0, 1]$ about the x-axis, as shown in Figure 6.45.

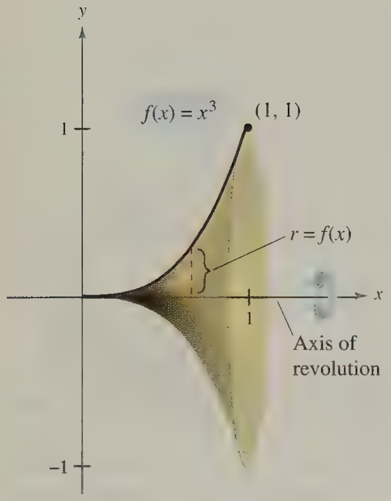


Figure 6.45

Solution The distance between the x-axis and the graph of f is $r(x) = f(x)$, an because $f'(x) = 3x^2$, the surface area is

$$\begin{aligned} S &= 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\,dx && \text{Formula for surface area} \\ &= 2\pi \int_0^1 x^3\sqrt{1 + (3x^2)^2}\,dx \\ &= \frac{2\pi}{36}\int_0^1 (36x^3)(1 + 9x^4)^{1/2}\,dx && \text{Simplify.} \\ &= \frac{\pi}{18}\left[\frac{(1 + 9x^4)^{3/2}}{3/2}\right]_0^1 && \text{Integrate.} \\ &= \frac{\pi}{27}(10^{3/2} - 1) \\ &\approx 3.563. \end{aligned}$$

Example 7 The Area of a Surface of Revolution

Find the area of the surface formed by revolving the graph of

$$f(x) = x^2$$

on the interval $[0, \sqrt{2}]$ about the y-axis, as shown in Figure 6.46.

Solution In this case, the distance between the graph of f and the y-axis is $r(x) =$ Using $f'(x) = 2x$, you can determine that the surface area is

$$\begin{aligned} S &= 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\,dx && \text{Formula for surface area} \\ &= 2\pi \int_0^{\sqrt{2}} x\sqrt{1 + (2x)^2}\,dx \\ &= \frac{2\pi}{8}\int_0^{\sqrt{2}} (1 + 4x^2)^{1/2}(8x)\,dx && \text{Simplify.} \\ &= \frac{\pi}{4}\left[\frac{(1 + 4x^2)^{3/2}}{3/2}\right]_0^{\sqrt{2}} && \text{Integrate.} \\ &= \frac{\pi}{6}[(1 + 8)^{3/2} - 1] \\ &= \frac{13\pi}{3}. \end{aligned}$$

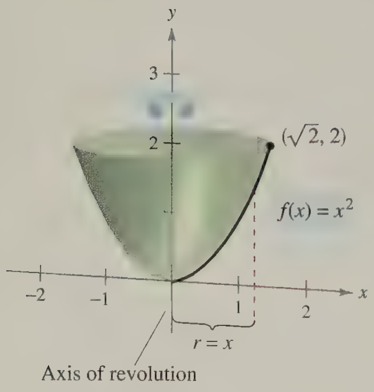


Figure 6.46

EXERCISES FOR SECTION 6.4

n Exercises 1 and 2, find the distance between the points by sing (a) the Distance Formula and (b) integration.

1. $(0, 0),\quad (5, 12)$ **2.** $(1, 2),\quad (7, 10)$

n Exercises 3–10, find the arc length of the graph of the funcion over the indicated interval.

3. $y = \frac{2}{3}x^{3/2} + 1,\quad [0, 1]$ **4.** $y = 2x^{3/2} + 3,\quad [0, 9]$

5. $y = \frac{3}{2}x^{2/3},\quad [1, 8]$ **6.** $y = \frac{3}{2}x^{2/3} + 4,\quad [1, 27]$

7. $y = \dfrac{x^4}{8} + \dfrac{1}{4x^2},\quad [1, 2]$ **8.** $y = \dfrac{x^5}{10} + \dfrac{1}{6x^3},\quad [1, 2]$

9. $y = \ln(\sin x),\quad \left[\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right]$ **10.** $y = \dfrac{1}{2}(e^x + e^{-x}),\quad [0, 2]$

n Exercises 11–20, (a) graph the function, highlighting the part ndicated by the given interval, (b) find a definite integral that epresents the arc length of the curve over the indicated interval nd observe that the integral cannot be evaluated with the echniques studied thus far, and (c) use the integration capabilties of a graphing utility to approximate the arc length.

1. $y = 4 - x^2,\quad [0, 2]$ **12.** $y = x^2 + x - 2,\quad [-2, 1]$

3. $y = \dfrac{1}{x},\quad [1, 3]$ **14.** $y = \dfrac{1}{x + 1},\quad [0, 1]$

5. $y = \sin x,\quad [0, \pi]$ **16.** $y = \cos x,\quad \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

7. $x = e^{-y},\quad [0, 2]$ **18.** $y = \ln x,\quad [1, 5]$

9. $y = 2 \arctan x,\quad [0, 1]$ **20.** $x = \sqrt{36 - y^2},\quad [0, 3]$

pproximation **In Exercises 21 and 22, determine which value est approximates the length of the arc represented by the inteeral. (Make your selection on the basis of a sketch of the arc and *ot* by performing any calculations.)**

1. $$\int_0^2 \sqrt{1 + \left[\frac{d}{dx}\left(\frac{5}{x^2 + 1}\right)\right]^2}\, dx$$

(a) 25 (b) 5 (c) 2 (d) -4 (e) 3

2. $$\int_0^{\pi/4} \sqrt{1 + \left[\frac{d}{dx}(\tan x)\right]^2}\, dx$$

(a) 3 (b) -2 (c) 4 (d) $\dfrac{4\pi}{3}$ (e) 1

pproximation **In Exercises 23 and 24, approximate the arc ength of the graph of the function over the interval [0, 4] in four ays. (a) Use the Distance Formula to find the distance between ne endpoints of the arc. (b) Use the Distance Formula to find ne lengths of the four line segments connecting the points on ne arc when $x = 0$, $x = 1$, $x = 2$, $x = 3$, and $x = 4$. Find the um of the four lengths. (c) Use Simpson's Rule with $n = 10$ to pproximate the integral yielding the indicated arc length. l) Use the integration capabilities of a graphing utility to pproximate the integral yielding the indicated arc length.**

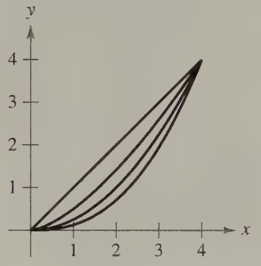
3. $f(x) = x^3$ **24.** $f(x) = (x^2 - 4)^2$

25. ***Think About It*** The figure shows the graphs of the functions $y_1 = x$, $y_2 = \frac{1}{2}x^{3/2}$, $y_3 = \frac{1}{4}x^2$, and $y_4 = \frac{1}{8}x^{5/2}$ on the interval [0, 4]. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*

(a) Label the functions.

(b) List the functions in order of increasing arc length.

(c) Verify your answer in part (b) by approximating each arc length accurate to three decimal places.



26. ***Think About It*** Explain why the two integrals are equal.

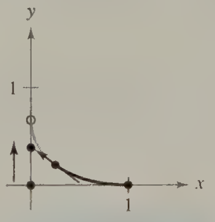
$$\int_1^e \sqrt{1 + \frac{1}{x^2}}\, dx = \int_0^1 \sqrt{1 + e^{2x}}\, dx$$

Use the integration capabilities of a graphing utility to verify that the integrals are equal.

27. ***Length of Pursuit*** A fleeing object leaves the origin and moves up the y-axis (see figure). At the same time, a pursuer leaves the point (1, 0) and always moves toward the fleeing object. If the pursuer's speed is twice that of the fleeing object, the equation of the path is

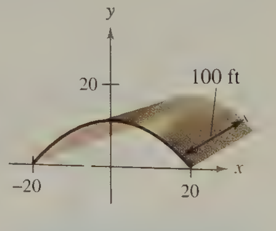
$$y = \frac{1}{3}(x^{3/2} - 3x^{1/2} + 2).$$

How far has the fleeing object traveled when it is caught? Show that the pursuer has traveled twice as far.



$y = \frac{1}{3}(x^{3/2} - 3x^{1/2} + 2)$

Figure for 27



$y = 31 - 10(e^{x/20} + e^{-x/20})$

Figure for 28

28. ***Roof Area*** A barn is 100 feet long and 40 feet wide (see figure). A cross section of the roof is the inverted catenary

$$y = 31 - 10(e^{x/20} + e^{-x/20}).$$

Find the number of square feet of roofing on the barn.

29. ***Length of a Catenary*** Electrical wires suspended between two towers form a catenary (see figure) modeled by the equation

$$y = 20\cosh\frac{x}{20}, \qquad -20 \le x \le 20$$

where x and y are measured in meters. Find the length of the suspended cable if the towers are 40 meters apart.

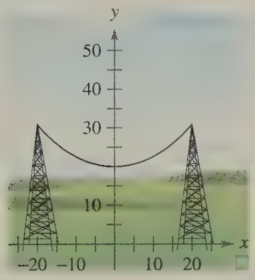


Figure for 29

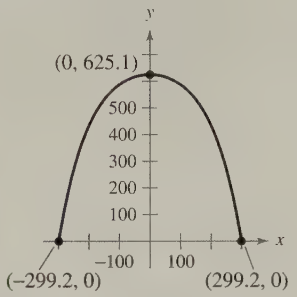


Figure for 30

30. ***Length of Gateway Arch*** The Gateway Arch in St. Louis, Missouri, is modeled by

$$y = 693.8597 - 68.7672\cosh 0.0100333x,$$
$$-299.2239 \le x \le 299.2239.$$

(See the Section Project: St. Louis Arch.) Find the length of this curve (see figure).

31. Find the arc length from $(0, 3)$ clockwise to $(2, \sqrt{5})$ along the circle $x^2 + y^2 = 9$.

32. Find the arc length from $(-3, 4)$ clockwise to $(4, 3)$ along the circle $x^2 + y^2 = 25$. Show that the result is one-fourth the circumference of the circle.

In Exercises 33–36, set up and evaluate the definite integral for the area of the surface generated by revolving the curve about the x-axis.

33. $y = \frac{1}{3}x^3$, $[0, 3]$

34. $y = 2\sqrt{x}$, $[4, 9]$

35. $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$, $[1, 2]$

36. $y = \dfrac{x}{2}$, $[0, 6]$

In Exercises 37 and 38, set up and evaluate the definite integral that gives the area of the surface of revolution generated by revolving the curve about the y-axis.

	Function	*Interval*
37.	$y = \sqrt[3]{x} + 2$	$[1, 8]$
38.	$y = 9 - x^2$	$[0, 3]$

In Exercises 39 and 40, use the integration capabilities of a graphing utility to approximate the surface area of the solid of revolution.

	Function	*Interval*
39.	$y = \sin x$ revolved about the x-axis	$[0, \pi]$
40.	$y = \ln x$ revolved about the y-axis	$[1, e]$

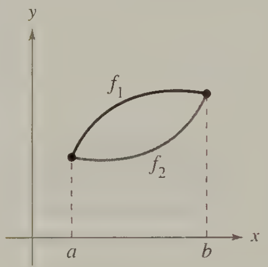
Getting at the Concept

41. Define a rectifiable curve.

42. What precalculus formula and representative element are used to develop the integration formula for arc length?

43. What precalculus formula and representative element are used to develop the integration formula for the area of a surface of revolution?

44. The graphs of the functions f_1 and f_2 on the interval $[a, b]$ are shown in the figure. The graph of each is revolved about the x-axis. Which surface of revolution has the greater surface area? Explain.



45. A right circular cone is generated by revolving the regio bounded by $y = hx/r$, $y = h$, and $x = 0$ about the y-axi Verify that the lateral surface area of the cone is

$$S = \pi r\sqrt{r^2 + h^2}.$$

46. A sphere of radius r is generated by revolving the graph $y = \sqrt{r^2 - x^2}$ about the x-axis. Verify that the surface area the sphere is $4\pi r^2$.

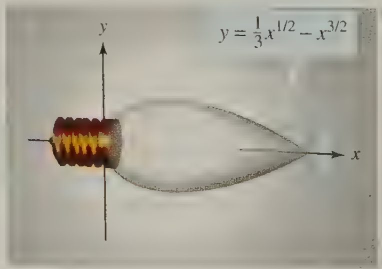
47. Find the area of the zone of a sphere formed by revolving th graph of $y = \sqrt{9 - x^2}$, $0 \le x \le 2$, about the y-axis.

48. Find the area of the zone of a sphere formed by revolving th graph of $y = \sqrt{r^2 - x^2}$, $0 \le x \le a$, about the y-axis. Assum that $a < r$.

49. ***Bulb Design*** An ornamental light bulb is designed by revolv ing the graph of

$$y = \tfrac{1}{3}x^{1/2} - x^{3/2}, \qquad 0 \le x \le \tfrac{1}{3}$$

about the x-axis, where x and y are measured in feet (se figure). Find the surface area of the bulb and use the result t approximate the amount of glass needed to make the bul (Assume that the glass is 0.015 inch thick.)



0. ***Modeling Data*** The circumference C (in inches) of a vase is measured at 3-inch intervals starting at its base. The measurements are shown in the table, where y is the vertical distance in inches from the base.

y	0	3	6	9	12	15	18
C	50	65.5	70	66	58	51	48

(a) Use the data to approximate the volume of the vase by summing the volumes of approximating disks.

(b) Use the data to approximate the outside surface area (excluding the base) of the vase by summing the outside surface areas of approximating frustums of right circular cones.

(c) Use the regression capabilities of a graphing utility to find a cubic model for the points (y, r) where $r = C/(2\pi)$. Use the graphing utility to plot the points and graph the model.

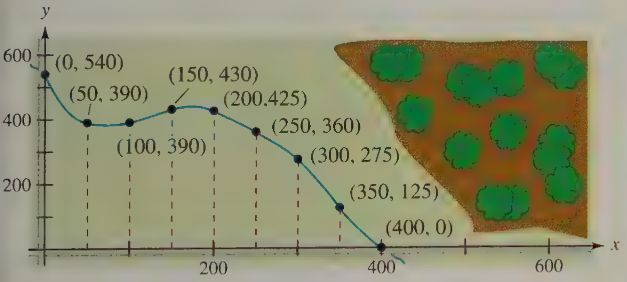
(d) Use the model in part (c) and the integration capabilities of a graphing utility to approximate the volume and outside surface area of the vase. Compare the results with your answers in parts (a) and (b).

1. ***Modeling Data*** Property bounded by two perpendicular roads and a stream is shown in the figure. All distances are measured in feet.

(a) Use the regression capabilities of a graphing utility to fit a fourth-degree polynomial to the path of the stream.

(b) Use the model of part (a) to approximate the area of the property in acres.

(c) Use the integration capabilities of a graphing utility to find the length of the stream that bounds the property.



2. ***Individual Project*** Select a solid of revolution from everyday life. Measure the radius of the solid at a minimum of seven points along its axis. Use the data to approximate the volume of the solid and the surface area of the lateral sides of the solid.

3. Let R be the region bounded by $y = 1/x$, the x-axis, $x = 1$, and $x = b$, where $b > 1$. Let D be the solid formed when R is revolved about the x-axis.

(a) Find the volume V of D.

(b) Express the surface area S as an integral.

(c) Show that V approaches a finite limit as $b \to \infty$.

(d) Show that $S \to \infty$ as $b \to \infty$.

54. ***Think About It*** Consider the equation $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.

(a) Use a graphing utility to graph the equation.

(b) Set up the definite integral for finding the first quadrant arc length of the graph in part (a).

(c) Compare the interval of integration in part (b) and the domain of the integrand. Is it possible to evaluate the definite integral? Is it possible to use Simpson's Rule to evaluate the definite integral? Explain. (You will learn how to evaluate this type of integral in Section 7.8.)

55. (a) Given a circular sector with radius L and central angle θ (see figure), show that the area of the sector is given by

$$S = \frac{1}{2}L^2\theta.$$

(b) By joining the straight line edges of the sector in part (a), a right circular cone is formed (see figure) and the lateral surface area of the cone is the same as the area of the sector. Show that the area is

$$S = \pi r L$$

where r is the radius of the base of the cone. (*Hint:* The arc length of the sector equals the circumference of the base of the cone.)

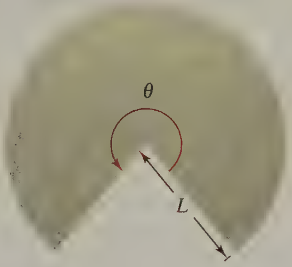


Figure for 55(a)

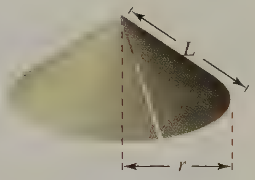
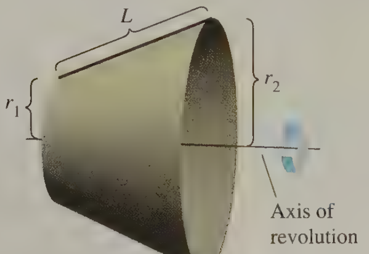


Figure for 55(b)

(c) Use the result in part (b) to verify that the formula for the lateral surface area of the frustum of a cone with slant height L and radii r_1 and r_2 (see figure) is

$$S = \pi(r_1 + r_2)L.$$

(*Note:* This formula was used to develop the integral for finding the surface area of a surface of revolution.)



56. ***Writing*** Read the article "Arc Length, Area and the Arcsine Function" by Andrew M. Rockett in *Mathematics Magazine*. Then write a paragraph explaining how the arcsine function can be defined in terms of an arc length. (To view this article, go to the website *www.matharticles.com*.)

Section 6.5 Work

- Find the work done by a constant force.
- Find the work done by a variable force.

Work Done by a Constant Force

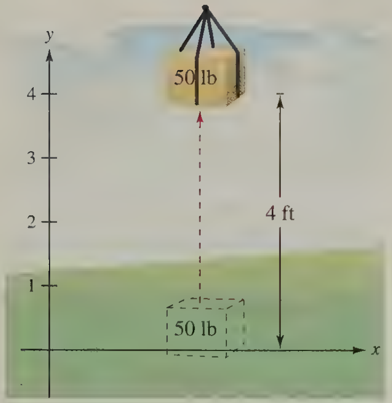
The concept of work is important to scientists and engineers for determining the ene gy needed to perform various jobs. For instance, it is useful to know the amount c work done when a crane lifts a steel girder, when a spring is compressed, when a rocl et is propelled into the air, or when a truck pulls a load along a highway.

In general, we say that **work** is done by a force when it moves an object. If th force applied to the object is *constant*, we have the following definition of work.

Definition of Work Done by a Constant Force

If an object is moved a distance D in the direction of an applied constant force F, then the **work** W done by the force is defined as $W = FD$.

There are many types of forces—centrifugal, electromotive, and gravitational, t name a few. A **force** can be thought of as a *push* or a *pull*; a force changes the state c rest or state of motion of a body. For gravitational forces on earth, it is common to us units of measure corresponding to the weight of an object.



The work done in lifting a 50-pound object 4 feet is 200 foot-pounds.
Figure 6.47

Example 1 **Lifting an Object**

Determine the work done in lifting a 50-pound object 4 feet.

Solution The magnitude of the required force F is the weight of the object, as show in Figure 6.47. So, the work done in lifting the object 4 feet is

$$\begin{aligned} W &= FD && \text{Work} = \text{(force)(distance)} \\ &= 50(4) && \text{Force} = 50 \text{ pounds, distance} = 4 \text{ feet} \\ &= 200 \text{ foot-pounds.} \end{aligned}$$

In the U.S. measurement system, work is typically expressed in foot-pounc (ft • lb), inch-pounds, or foot-tons. In the centimeter-gram-second (C-G-S) systen the basic unit of force is the **dyne**—the force required to produce an acceleration c 1 centimeter per second per second on a mass of 1 gram. In this system, work typically expressed in dyne-centimeters (ergs) or newton-meters (joules), whe 1 joule $= 10^7$ ergs.

EXPLORATION

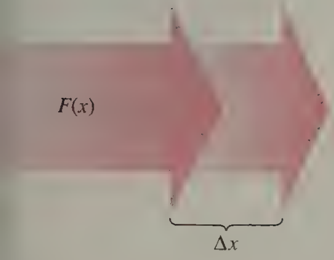
How Much Work? In Example 1, 200 foot-pounds of work were needed to lift the 50-pound object 4 feet vertically off the ground. Suppose that once you lifted the object, you held it and walked a horizontal distance of 4 feet. Would this require an additional 200 foot-pounds of work? Explain your reasoning.

Work Done by a Variable Force

In Example 1, the force involved is *constant*. If a *variable* force is applied to an object, calculus is needed to determine the work done, because the amount of force changes as the object changes position. For instance, the force required to compress a spring increases as the spring is compressed.

Suppose that an object is moved along a straight line from $x = a$ to $x = b$ by a continuously varying force $F(x)$. Let Δ be a partition that divides the interval $[a, b]$ into n subintervals determined by

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_n = b$$



he amount of force changes as an object anges position (Δx).
igure 6.48

and let $\Delta x_i = x_i - x_{i-1}$. For each i, choose c_i such that $x_{i-1} \le c_i \le x_i$. Then at c_i the force is given by $F(c_i)$. Because F is continuous, you can approximate the work done in moving the object through the ith subinterval by the increment

$$\Delta W_i = F(c_i)\,\Delta x_i$$

as shown in Figure 6.48. So, the total work done as the object moves from a to b is approximated by

$$\begin{aligned} W &\approx \sum_{i=1}^{n} \Delta W_i \\ &= \sum_{i=1}^{n} F(c_i)\,\Delta x_i. \end{aligned}$$

This approximation appears to become better and better as $\|\Delta\| \to 0$ $(n \to \infty)$. Therefore, we define the work to be

$$\begin{aligned} W &= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} F(c_i)\,\Delta x_i \\ &= \int_a^b F(x)\,dx. \end{aligned}$$

Definition of Work Done by a Variable Force

If an object is moved along a straight line by a continuously varying force $F(x)$, then the **work** W done by the force as the object is moved from $x = a$ to $x = b$ is

$$\begin{aligned} W &= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \Delta W_i \\ &= \int_a^b F(x)\,dx. \end{aligned}$$

Bettmann/Corbis

Emilie de Breteuil (1706–1749)

Another major work by de Breteuil was the translation of Newton's "Philosophiae Naturalis Principia Mathematica" into French. Her translation and commentary greatly contributed to the acceptance of Newtonian science in Europe.

The remaining examples in this section use some well-known physical laws. The discoveries of many of these laws occurred during the same period in which calculus was being developed. In fact, during the seventeenth and eighteenth centuries, there was little difference between physicists and mathematicians. One such physicist-mathematician was Emilie de Breteuil. Breteuil was instrumental in synthesizing the work of many other scientists, including Newton, Leibniz, Huygens, Kepler, and Descartes. Her physics text *Institutions* was widely used for many years.

The following three laws of physics were developed by Robert Hook (1635–1703), Isaac Newton (1642–1727), and Charles Coulomb (1736–1806).

1. **Hooke's Law:** The force F required to compress or stretch a spring (within it elastic limits) is proportional to the distance d that the spring is compressed o stretched from its original length. That is,

   $$F = kd$$

   where the constant of proportionality k (the spring constant) depends on th specific nature of the spring.

2. **Newton's Law of Universal Gravitation:** The force F of attraction between tw particles of masses m_1 and m_2 is proportional to the product of the masses an inversely proportional to the square of the distance d between the two particles That is,

   $$F = k\frac{m_1 m_2}{d^2}.$$

   If m_1 and m_2 are given in grams and d in centimeters, F will be in dynes for a valu of $k = 6.670 \times 10^{-8}$ cubic centimeter per gram-second squared.

3. **Coulomb's Law:** The force between two charges q_1 and q_2 in a vacuum i proportional to the product of the charges and inversely proportional to the squar of the distance d between the two charges. That is,

   $$F = k\frac{q_1 q_2}{d^2}.$$

   If q_1 and q_2 are given in electrostatic units and d in centimeters, F will be in dyne for a value of $k = 1$.

EXPLORATION

The work done in compressing the spring in Example 2 from $x = 3$ inches to $x = 6$ inches is 3375 inch-pounds. Should the work done in compressing the spring from $x = 0$ inches to $x = 3$ inches be more than, the same as, or less than this? Explain.

Example 2 Compressing a Spring

A force of 750 pounds compresses a spring 3 inches from its natural length of 1 inches. Find the work done in compressing the spring an additional 3 inches.

Solution By Hooke's Law, the force $F(x)$ required to compress the spring x unit (from its natural length) is $F(x) = kx$. Using the given data, it follows that $F(3) =$ $750 = (k)(3)$ and thus $k = 250$ and $F(x) = 250x$, as shown in Figure 6.49. To find th increment of work, assume that the force required to compress the spring over a sma increment Δx is nearly constant. So, the increment of work is

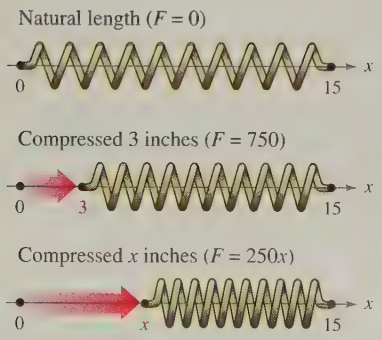
$$\Delta W = (\text{force})(\text{distance increment}) = (250x)\,\Delta x.$$

Because the spring is compressed from $x = 3$ to $x = 6$ inches less than its natur length, the work required is

$$W = \int_a^b F(x)\,dx = \int_3^6 250x\,dx \qquad \text{Formula for work}$$

$$= 125x^2\Big]_3^6 = 4500 - 1125 = 3375 \text{ inch-pounds.}$$

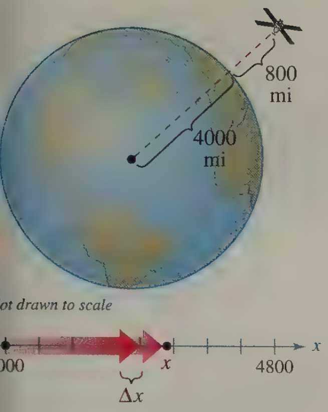
Note that you do *not* integrate from $x = 0$ to $x = 6$ because you were asked to dete mine the work done in compressing the spring an *additional* 3 inches (not includir the first 3 inches).


Natural length ($F = 0$)

Compressed 3 inches ($F = 750$)

Compressed x inches ($F = 250x$)

Figure 6.49



he work required to move a space module 00 miles above earth is approximately 056 × 10^{11} foot-pounds.

igure 6.50

Example 3 Moving a Space Module into Orbit

A space module weighs 15 tons on the surface of earth. How much work is done in propelling the module to a height of 800 miles above earth, as shown in Figure 6.50? (Use 4000 miles as the radius of earth. Do not consider the effect of air resistance or the weight of the propellant.)

Solution Because the weight of a body varies inversely as the square of its distance from the center of earth, the force $F(x)$ exerted by gravity is

$$F(x) = \frac{C}{x^2}. \qquad C \text{ is the constant of proportionality.}$$

Because the module weighs 15 tons on the surface of earth and the radius of earth is approximately 4000 miles, you have

$$\begin{aligned} 15 &= \frac{C}{(4000)^2} \\ 240{,}000{,}000 &= C. \end{aligned}$$

So, the increment of work is

$$\begin{aligned} \Delta W &= (\text{force})(\text{distance increment}) \\ &= \frac{240{,}000{,}000}{x^2}\,\Delta x. \end{aligned}$$

Finally, because the module is propelled from $x = 4000$ to $x = 4800$ miles, the total work done is

$$\begin{aligned} W &= \int_a^b F(x)\,dx = \int_{4000}^{4800} \frac{240{,}000{,}000}{x^2}\,dx && \text{Formula for work} \\ &= \frac{-240{,}000{,}000}{x}\Bigg]_{4000}^{4800} && \text{Integrate.} \\ &= -50{,}000 + 60{,}000 \\ &= 10{,}000 \text{ mile-tons} \\ &\approx 1.056 \times 10^{11} \text{ foot-pounds.} \end{aligned}$$

In the C-G-S system, using a conversion factor of 1 foot-pound ≈ 1.35582 joules, the work done is

$$W \approx 1.432 \times 10^{11} \text{ joules.}$$

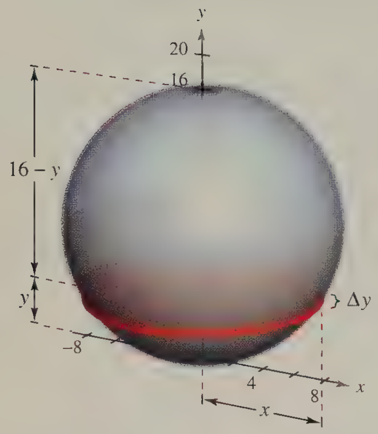
The solutions to Examples 2 and 3 conform to our development of work as the summation of increments in the form

$$\Delta W = (\text{force})(\text{distance increment}) = (F)(\Delta x).$$

Another way to formulate the increment of work is

$$\Delta W = (\text{force increment})(\text{distance}) = (\Delta F)(x).$$

This second interpretation of ΔW is useful in problems involving the movement of nonrigid substances such as fluids and chains.



The work required to pump oil out through a hole in the top of the tank is approximately 589,782 foot-pounds.
Figure 6.51

Example 4 Emptying a Tank of Oil

A spherical tank of radius 8 feet is half full of oil that weighs 50 pounds per cubi
foot. Find the work required to pump oil out through a hole in the top of the tank.

Solution Consider the oil to be subdivided into disks of thickness Δy and radius x
as shown in Figure 6.51. Because the increment of force for each disk is given by it
weight, you have

$$\begin{aligned}\Delta F &= \text{weight}\\ &= \left(\frac{50 \text{ pounds}}{\text{cubic foot}}\right)(\text{volume})\\ &= 50(\pi x^2 \Delta y) \text{ pounds.}\end{aligned}$$

For a circle of radius 8 and center at (0, 8), you have

$$\begin{aligned}x^2 + (y - 8)^2 &= 8^2\\ x^2 &= 16y - y^2\end{aligned}$$

and you can write the force increment as

$$\begin{aligned}\Delta F &= 50(\pi x^2 \Delta y)\\ &= 50\pi(16y - y^2)\,\Delta y.\end{aligned}$$

In Figure 6.51, note that a disk y feet from the bottom of the tank must be moved
distance of $(16 - y)$ feet. Therefore, the increment of work is

$$\begin{aligned}\Delta W &= \Delta F(16 - y)\\ &= 50\pi(16y - y^2)\,\Delta y(16 - y)\\ &= 50\pi(256y - 32y^2 + y^3)\,\Delta y.\end{aligned}$$

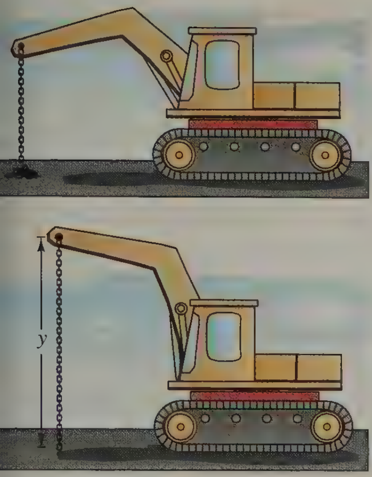
Because the tank is half full, y ranges from 0 to 8, and the work required to empty th
tank is

$$\begin{aligned}W &= \int_0^8 50\pi(256y - 32y^2 + y^3)\,dy\\ &= 50\pi\left[128y^2 - \frac{32}{3}y^3 + \frac{y^4}{4}\right]_0^8\\ &= 50\pi\left(\frac{11{,}264}{3}\right)\\ &\approx 589{,}782 \text{ foot-pounds.}\end{aligned}$$

To estimate the reasonableness of the result in Example 4, consider that th
weight of the oil in the tank is

$$\begin{aligned}\left(\frac{1}{2}\right)(\text{volume})(\text{density}) &= \frac{1}{2}\left(\frac{4}{3}\pi 8^3\right)(50)\\ &\approx 53{,}616.5 \text{ pounds.}\end{aligned}$$

Lifting the entire half-tank of oil 8 feet would involve work of $8(53{,}616.5) \approx$ 428,93
foot-pounds. Because the oil is actually lifted between 8 and 16 feet, it seems reaso
able that the work done is 589,782 foot-pounds.



he work required to raise one end of the
ain 20 feet is 1000 foot-pounds.
igure 6.52

Example 5 Lifting a Chain

A 20-foot chain weighing 5 pounds per foot is lying coiled on the ground. How much work is required to raise one end of the chain to a height of 20 feet so that it is fully extended, as shown in Figure 6.52?

Solution Imagine that the chain is divided into small sections, each of length Δy. Then the weight of each section is the increment of force

$$\Delta F = (\text{weight}) = \left(\frac{5 \text{ pounds}}{\text{foot}}\right)(\text{length}) = 5\Delta y.$$

Because a typical section (initially on the ground) is raised to a height of y, the increment of work is

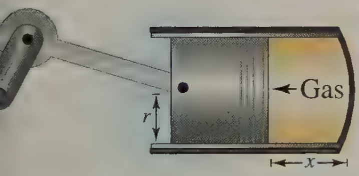
$$\Delta W = (\text{force increment})(\text{distance}) = (5\ \Delta y)y = 5y\ \Delta y.$$

Because y ranges from 0 to 20, the total work is

$$W = \int_0^{20} 5y\,dy = \frac{5y^2}{2}\bigg]_0^{20} = \frac{5(400)}{2} = 1000 \text{ foot-pounds.}$$

In the next example we consider a piston of radius r in a cylindrical casing, as shown in Figure 6.53. As the gas in the cylinder expands, the piston moves and work is done. If p represents the pressure of the gas (in pounds per square foot) against the piston head and V represents the volume of the gas (in cubic feet), the work increment involved in moving the piston Δx feet is

$$\Delta W = (\text{force})(\text{distance increment}) = F(\Delta x) = p(\pi r^2)\,\Delta x = p\,\Delta V.$$



ork done by expanding gas
igure 6.53

So, as the volume of the gas expands from V_0 to V_1, the work done in moving the piston is

$$W = \int_{V_0}^{V_1} p\,dV.$$

Assuming the pressure of the gas to be inversely proportional to its volume, we have $p = k/V$ and the integral for work becomes

$$W = \int_{V_0}^{V_1} \frac{k}{V}\,dV.$$

Example 6 Work Done by an Expanding Gas

A quantity of gas with an initial volume of 1 cubic foot and a pressure of 500 pounds per square foot expands to a volume of 2 cubic feet. Find the work done by the gas. (Assume that the pressure is inversely proportional to the volume.)

Solution Because $p = k/V$ and $p = 500$ when $V = 1$, we have $k = 500$. So, the work is

$$W = \int_{V_0}^{V_1} \frac{k}{V}\,dV = \int_1^2 \frac{500}{V}\,dV = 500 \ln|V|\bigg]_1^2 \approx 346.6 \text{ foot-pounds.}$$

Lab Series LAB 8

EXERCISES FOR SECTION 6.5

Constant Force **In Exercises 1–4, determine the work done by the constant force.**

1. A 100-pound bag of sugar is lifted 10 feet.

2. An electric hoist lifts a 2800-pound car 4 feet.

3. A force of 112 newtons is required to slide a cement block 4 meters in a construction project.

4. The locomotive of a freight train pulls its cars with a constant force of 9 tons a distance of one-half mile.

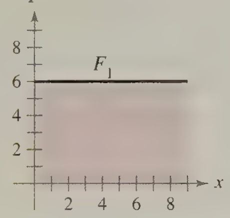
Getting at the Concept

5. State the definition of work done by a constant force.

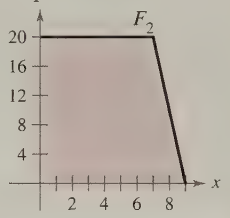
6. State the definition of work done by a variable force.

7. The graphs show the force F_i (in pounds) required to move an object 9 feet along the x-axis. Order the force functions from the one that yields the least work to the one that yields the most work without doing any calculations.

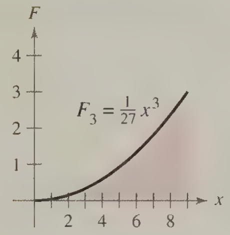
(a)



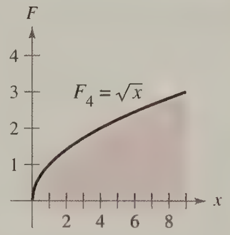
(b)



(c)



(d)



8. Verify your answer to Exercise 7 by calculating the work for each force function.

Hooke's Law **In Exercises 9–16, use Hooke's Law to determine the variable force in the spring problem.**

9. A force of 5 pounds compresses a 15-inch spring a total of 4 inches. How much work is done in compressing the spring 7 inches?

10. How much work is done in compressing the spring in Exercise 9 from a length of 10 inches to a length of 6 inches?

11. A force of 250 newtons stretches a spring 30 centimeters. How much work is done in stretching the spring from 20 centimeters to 50 centimeters?

12. A force of 800 newtons stretches a spring 70 centimeters on a mechanical device for driving fence posts. Find the work done in stretching the spring the required 70 centimeters.

13. A force of 20 pounds stretches a spring 9 inches in an exercis machine. Find the work done in stretching the spring 1 fo from its natural position.

14. An overhead garage door has two springs, one on each side the door. A force of 15 pounds is required to stretch each sprin 1 foot. Because of the pulley system, the springs stretch onl one-half the distance the door travels. Find the work done b the pair of springs if the door moves a total of 8 feet and th springs are at their natural length when the door is open.

15. Eighteen foot-pounds of work is required to stretch a spring inches from its natural length. Find the work required to stretc the spring an additional 3 inches.

16. Seven and one-half foot-pounds of work is required to con press a spring 2 inches from its natural length. Find the wor required to compress the spring an additional one-half inch.

17. ***Propulsion*** Neglecting air resistance and the weight of the pr pellant, determine the work done in propelling a 5-ton satellite a height of

(a) 100 miles above earth.

(b) 300 miles above earth.

18. ***Propulsion*** Use the information in Exercise 17 to write tl work W of the propulsion system as a function of the height of the satellite above earth. Find the limit (if it exists) of W h approaches infinity.

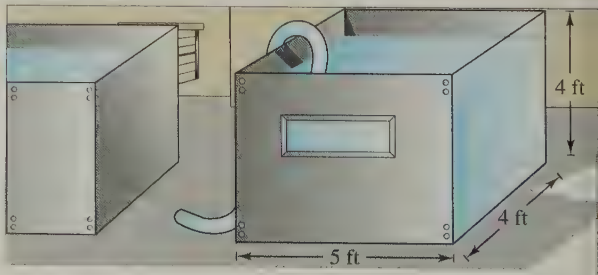
19. ***Propulsion*** Neglecting air resistance and the weight of tl propellant, determine the work done in propelling a 10-tc satellite to a height of

(a) 11,000 miles above earth.

(b) 22,000 miles above earth.

20. ***Propulsion*** A lunar module weighs 12 tons on the surface earth. How much work is done in propelling the module fro the surface of the moon to a height of 50 miles? Consider tl radius of the moon to be 1100 miles and its force of gravity be one-sixth that of earth.

21. ***Pumping Water*** A rectangular tank with a base 4 feet by feet and a height of 4 feet is full of water (see figure). The wat weighs 62.4 pounds per cubic foot. How much work is done pumping water out over the top edge in order to empty (a) ha of the tank? (b) all of the tank?



22. ***Think About It*** Explain why the answer in part (b) Exercise 21 is not twice the answer in part (a).

Pumping Water A cylindrical water tank 4 meters high with a radius of 2 meters is buried so that the top of the tank is 1 meter below ground level (see figure). How much work is done in pumping a full tank of water up to ground level? (The water weighs 9800 newtons per cubic meter.)

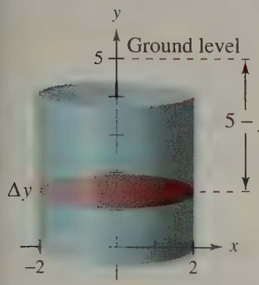


Figure for 23

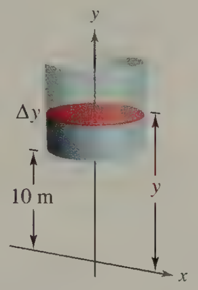


Figure for 24

Pumping Water Suppose the tank in Exercise 23 is located on a tower so that the bottom of the tank is 10 meters above the level of a stream (see figure). How much work is done in filling the tank half full of water through a hole in the bottom, using water from the stream?

Pumping Water An open tank has the shape of a right circular cone (see figure). The tank is 8 feet across the top and 6 feet high. How much work is done in emptying the tank by pumping the water over the top edge?

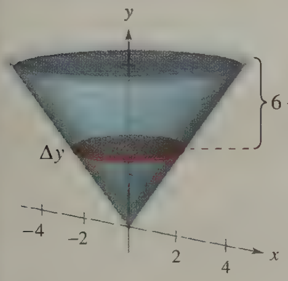


Figure for 25

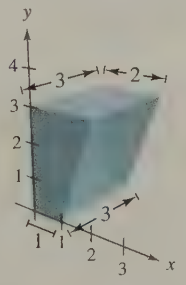


Figure for 28

Pumping Water If water is pumped in through the bottom of the tank in Exercise 25, how much work is done to fill the tank

(a) to a depth of 2 feet?

(b) from a depth of 4 feet to a depth of 6 feet?

Pumping Water A hemispherical tank of radius 6 feet is positioned so that its base is circular. How much work is required to fill the tank with water through a hole in the base if the water source is at the base?

Pumping Diesel Fuel The fuel tank on a truck has trapezoidal cross sections with dimensions (in feet) shown in the figure. Assume that an engine is approximately 3 feet above the top of the fuel tank and that diesel fuel weighs approximately 55.6 pounds per cubic foot. Find the work done by the fuel pump in raising a full tank of fuel to the level of the engine.

Pumping Gasoline **In Exercises 29 and 30, find the work done in pumping gasoline that weighs 42 pounds per cubic foot. (*Hint:* Evaluate one integral by a geometric formula and the other by observing that the integrand is an odd function.)**

29. A cylindrical gasoline tank 3 feet in diameter and 4 feet long is carried on the back of a truck and is used to fuel tractors. The axis of the tank is horizontal. Find the work done in pumping the entire contents of the fuel tank into a tractor if the opening on the tractor tank is 5 feet above the top of the tank in the truck.

30. The top of a cylindrical storage tank for gasoline at a service station is 4 feet below ground level. The axis of the tank is horizontal and its diameter and length are 5 feet and 12 feet, respectively. Find the work done in pumping the entire contents of the full tank to a height of 3 feet above ground level.

Lifting a Chain **In Exercises 31–34, consider a 15-foot chain hanging from a winch 15 feet above ground level. Find the work done by the winch in winding up the specified amount of chain, if the chain weighs 3 pounds per foot.**

31. Wind up the entire chain.

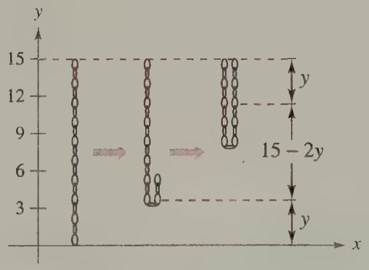
32. Wind up one-third of the chain.

33. Run the winch until the bottom of the chain is at the 10-foot level.

34. Wind up the entire chain with a 500-pound load attached to it.

Lifting a Chain **In Exercises 35 and 36, consider a 15-foot hanging chain that weighs 3 pounds per foot. Find the work done in lifting the chain vertically to the indicated position.**

35. Take the bottom of the chain and raise it to the 15-foot level, leaving the chain doubled and still hanging vertically (see figure).



36. Repeat Exercise 35 raising the bottom of the chain to the 12-foot level.

Demolition Crane **In Exercises 37 and 38, consider a demolition crane with a 500-pound ball suspended from a 40-foot cable that weighs 1 pound per foot.**

37. Find the work required to wind up 15 feet of the apparatus.

38. Find the work required to wind up all 40 feet of the apparatus.

Boyle's Law **In Exercises 39 and 40, find the work done by the gas for the given volume and pressure. Assume that the pressure is inversely proportional to the volume. (See Example 6.)**

39. A quantity of gas with an initial volume of 2 cubic feet and a pressure of 1000 pounds per square foot expands to a volume of 3 cubic feet.

40. A quantity of gas with an initial volume of 1 cubic foot and a pressure of 2500 pounds per square foot expands to a volume of 3 cubic feet.

41. ***Electric Force*** Two electrons repel each other with a force that varies inversely as the square of the distance between them. If one electron is fixed at the point (2, 4), find the work done in moving the second electron from (−2, 4) to (1, 4).

42. ***Modeling Data*** The hydraulic cylinder on a woodsplitter has a 4-inch bore (diameter) and a stroke of 2 feet. The hydraulic pump creates a maximum pressure of 2000 pounds per square inch. Therefore, the maximum force created by the cylinder is $2000(\pi 2^2) = 8000\pi$ pounds.

(a) Find the work done through one extension of the cylinder given that the maximum force is required.

(b) The force exerted in splitting a piece of wood is variable. Measurements of the force obtained when a piece of wood was split are shown in the table. The variable x measures the extension of the cylinder in feet, and F is the force in pounds. Use Simpson's Rule to approximate the work done in splitting the piece of wood.

x	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2
$F(x)$	0	20,000	22,000	15,000	10,000	5000	0

Table for 42(b)

(c) Use the regression capabilities of a graphing utility to fir a fourth-degree polynomial model for the data. Plot the da and graph the model.

(d) Use the model in part (c) to approximate the extension the cylinder when the force is maximum.

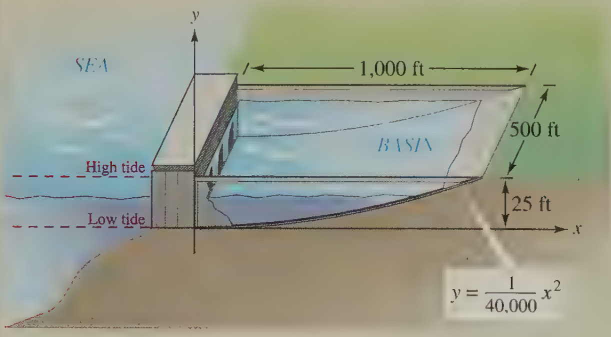
(e) Use the model in part (c) to approximate the work done splitting the piece of wood.

Hydraulic Press **In Exercises 43–46, use the integratio capabilities of a graphing utility to approximate the work don by a press in a manufacturing process. A model for the variab force F (in pounds) and the distance x (in feet) the press mov is given.**

	Force	*Interval*
43.	$F(x) = 1000[1.8 - \ln(x + 1)]$	$0 \le x \le 5$
44.	$F(x) = \dfrac{e^{x^2} - 1}{100}$	$0 \le x \le 4$
45.	$F(x) = 100x\sqrt{125 - x^3}$	$0 \le x \le 5$
46.	$F(x) = 1000 \sinh x$	$0 \le x \le 2$

SECTION PROJECT TIDAL ENERGY

Tidal power plants use "tidal energy" to produce electrical energy. To construct a tidal power plant, a dam is built to separate a bay from the sea. Electrical energy is produced as the water flows back and forth between the bay and the sea. The amount of "natural energy" produced depends on the volume of the bay and the tidal range—the vertical distance between high and low tides. (Throughout the world, several natural bays have tidal ranges in excess of 15 feet; the Bay of Fundy in Nova Scotia has a tidal range of 47.5 feet.)



(a) Consider a basin with a rectangular base, as shown in the figure. The basin has a tidal range of 25 feet, with low tide corresponding to $y = 0$. How much water does the basin hold at high tide?

(b) The amount of energy produced during the filling (or the emptying) of the basin is proportional to the amount of work required to fill (or empty) the basin. How much work is required to fill the basin with seawater? (Use a seawater density of 64 pounds per cubic foot.)

Francois Gohier/Photo Researchers, Inc.

Francois Gohier/Photo Researchers, Inc.

The Bay of Fundy in Nova Scotia has an extreme tidal range, as displayed in the greatly contrasting photos above.

FOR FURTHER INFORMATION For more information on tidal power, see the article "LaRance: Six Years of Operating a Tidal Power Plant in France" by J. Cotillon in *Water Power Magazine.*

Moments, Centers of Mass, and Centroids

- Understand the definition of mass.
- Find the center of mass in a one-dimensional system.
- Find the center of mass in a two-dimensional system.
- Find the center of mass of a planar lamina.
- Use the Theorem of Pappus to find the volume of a solid of revolution.

Mass

In this section you will study several important applications of integration that are related to **mass.** Mass is a measure of a body's resistance to changes in motion, and is independent of the particular gravitational system in which the body is located. However, because so many applications involving mass occur on earth's surface, we tend to equate an object's mass with its weight. This is not technically correct. Weight is a type of force and as such is dependent on gravity. Force and mass are related by the equation

$$\text{Force} = (\text{mass})(\text{acceleration}).$$

The table below lists some commonly used measures of mass and force, together with their conversion factors.

System of Measurement	Measure of Mass	Measure of Force
U.S.	Slug	Pound $= (\text{slug})(\text{ft/sec}^2)$
International	Kilogram	Newton $= (\text{kilogram})(\text{m/sec}^2)$
C-G-S	Gram	Dyne $= (\text{gram})(\text{cm/sec}^2)$

Conversions:

1 pound = 4.448 newtons	1 slug = 14.59 kilograms
1 newton = 0.2248 pound	1 kilogram = 0.06854 slug
1 dyne = 0.000002247 pound	1 gram = 0.00006854 slug
1 dyne = 0.00001 newton	1 meter = 0.3048 foot

Example 1 Mass on the Surface of Earth

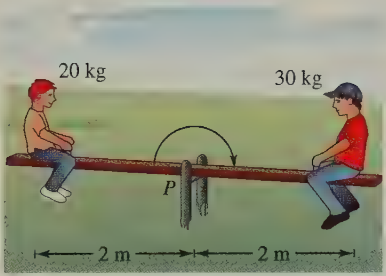
Find the mass (in slugs) of an object whose weight at sea level is 1 pound.

Solution Using 32 feet per second per second as the acceleration due to gravity produces

$$\begin{aligned}\text{Mass} &= \frac{\text{force}}{\text{acceleration}} \qquad \text{Force} = (\text{mass})(\text{acceleration})\\ &= \frac{1 \text{ pound}}{32 \text{ feet per second per second}}\\ &= 0.03125 \frac{\text{pound}}{\text{foot per second per second}}\\ &= 0.03125 \text{ slug.}\end{aligned}$$

Because many applications involving mass occur on earth's surface, this amount of mass is called a **pound mass.**

Center of Mass in a One-Dimensional System



The seesaw will balance when the left and the right moments are equal.
Figure 6.54

We will consider two types of moments of a mass—the **moment about a point** and the **moment about a line.** To define these two moments, consider an idealized situation in which a mass m is concentrated at a point. If x is the distance between this point mass and another point P, the **moment of m about the point P** is

$$\text{Moment} = mx$$

and x is the **length of the moment arm.**

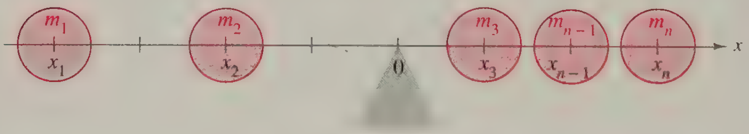
The concept of moment can be demonstrated simply by a seesaw, as illustrated in Figure 6.54. Suppose a child of mass 20 kilograms sits 2 meters to the left of fulcrum P, and an older child of mass 30 kilograms sits 2 meters to the right of P. From experience, you know that the seesaw will begin to rotate clockwise, moving the larger child down. This rotation occurs because the moment produced by the child on the left is less than the moment produced by the child on the right.

$$\begin{aligned}\text{Left moment} &= (20)(2) = 40 \text{ kilogram-meters}\\ \text{Right moment} &= (30)(2) = 60 \text{ kilogram-meters}\end{aligned}$$

To balance the seesaw, the two moments must be equal. For example, if the larger child moved to a position $\frac{4}{3}$ meters from the fulcrum, the seesaw would balance, because each child would produce a moment of 40 kilogram-meters.

To generalize this, you can introduce a coordinate line on which the origin corresponds to the fulcrum, as shown in Figure 6.55. Suppose several point masses are located on the x-axis. The measure of the tendency of this system to rotate about the origin is the **moment about the origin,** and it is defined as the sum of the n products m_ix_i.

$$M_0 = m_1x_1 + m_2x_2 + \cdot \cdot \cdot + m_nx_n$$



If $m_1x_1 + m_2x_2 + \cdot \cdot \cdot + m_nx_n = 0$, the system is in equilibrium.
Figure 6.55

If M_0 is 0, the system is said to be in **equilibrium.**

For a system that is not in equilibrium, the **center of mass** is defined as the point $\bar{x}$ at which the fulcrum could be relocated to attain equilibrium. If the system were translated $\bar{x}$ units, each coordinate x_i would become $(x_i - \bar{x})$, and because the moment of the translated system is 0, you have

$$\sum_{i=1}^{n} m_i(x_i - \bar{x}) = \sum_{i=1}^{n} m_ix_i - \sum_{i=1}^{n} m_i\bar{x} = 0.$$

Solving for $\bar{x}$ produces

$$\bar{x} = \frac{\displaystyle\sum_{i=1}^{n} m_ix_i}{\displaystyle\sum_{i=1}^{n} m_i} = \frac{\text{moment of system about origin}}{\text{total mass of system}}.$$

If $m_1x_1 + m_2x_2 + \cdot \cdot \cdot + m_nx_n = 0$, the system is in equilibrium.

Moments and Center of Mass: One-Dimensional System

Let the point masses $m_1, m_2, \ldots, m_n$ be located at $x_1, x_2, \ldots, x_n$.

1. The **moment about the origin** is $M_0 = m_1x_1 + m_2x_2 + \cdots + m_nx_n$.
2. The **center of mass** is $\bar{x} = \dfrac{M_0}{m}$, where $m = m_1 + m_2 + \cdots + m_n$ is the **total mass** of the system.

Example 2 The Center of Mass of a Linear System

Find the center of mass of the linear system shown in Figure 6.56.

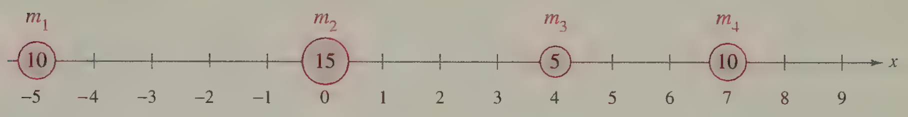


Figure 6.56

Solution The moment about the origin is

$$\begin{aligned} M_0 &= m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 \\ &= 10(-5) + 15(0) + 5(4) + 10(7) \\ &= -50 + 0 + 20 + 70 \\ &= 40. \end{aligned}$$

Because the total mass of the system is $m = 10 + 15 + 5 + 10 = 40$, the center of mass is

$$\bar{x} = \frac{M_0}{m} = \frac{40}{40} = 1.$$

NOTE In Example 2, where should you locate the fulcrum so that the point masses will be in equilibrium?

Rather than define the moment of a mass, you could define the moment of a *force*. In this context, the center of mass is called the **center of gravity.** Suppose that a system of point masses $m_1, m_2, \ldots, m_n$ is located at $x_1, x_2, \ldots, x_n$. Then, because force = (mass)(acceleration), the total force of the system is

$$\begin{aligned} F &= m_1a + m_2a + \cdots + m_na \\ &= ma. \end{aligned}$$

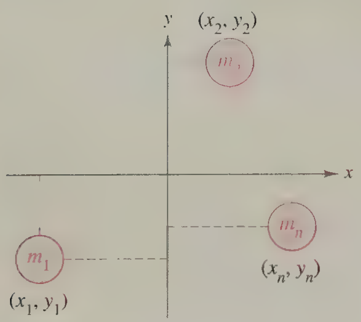
The **torque** (moment) about the origin is

$$\begin{aligned} T_0 &= (m_1a)x_1 + (m_2a)x_2 + \cdots + (m_na)x_n \\ &= M_0a \end{aligned}$$

and the **center of gravity** is

$$\frac{T_0}{F} = \frac{M_0a}{ma} = \frac{M_0}{m} = \bar{x}.$$

Therefore, the center of gravity and the center of mass have the same location.



In a two-dimensional system, there is a moment about the y-axis, M_y, and a moment about the x-axis, M_x.
Figure 6.57

Center of Mass in a Two-Dimensional System

You can extend the concept of moment to two dimensions by considering a system o masses located in the xy-plane at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, as shown i Figure 6.57. Rather than defining a single moment (with respect to the origin), w define two moments—one with respect to the x-axis and one with respect to th y-axis.

Moments and Center of Mass: Two-Dimensional System

Let the point masses $m_1, m_2, \ldots, m_n$ be located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

1. The **moment about the y-axis** is $M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n$.
2. The **moment about the x-axis** is $M_x = m_1y_1 + m_2y_2 + \cdots + m_ny_n$.
3. The **center of mass** $(\bar{x}, \bar{y})$ (or **center of gravity**) is

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

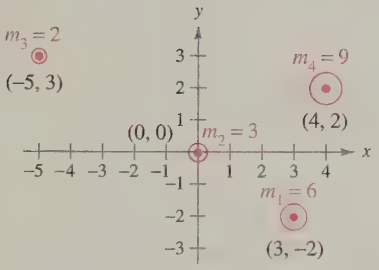
where $m = m_1 + m_2 + \cdots + m_n$ is the **total mass** of the system.

The moment of a system of masses in the plane can be taken about any horizontal or vertical line. In general, the moment about a line is the sum of the product of th masses and the *directed distances* from the points to the line.

$$\text{Moment} = m_1(y_1 - b) + m_2(y_2 - b) + \cdots + m_n(y_n - b) \quad \text{Horizontal line } y =$$

$$\text{Moment} = m_1(x_1 - a) + m_2(x_2 - a) + \cdots + m_n(x_n - a) \quad \text{Vertical line } x = a$$

Example 3 The Center of Mass of a Two-Dimensional System



The center of mass of the system is $\left(\frac{11}{5}, \frac{3}{5}\right)$.
Figure 6.58

Find the center of mass of a system of point masses $m_1 = 6$, $m_2 = 3$, $m_3 = 2$, an $m_4 = 9$, located at

$$(3, -2), (0, 0), (-5, 3), \text{ and } (4, 2)$$

as shown in Figure 6.58.

Solution

$$m = 6 + 3 + 2 + 9 = 20 \qquad \text{Mass}$$

$$M_y = 6(3) + 3(0) + 2(-5) + 9(4) = 44 \qquad \text{Moment about } y\text{-axis}$$

$$M_x = 6(-2) + 3(0) + 2(3) + 9(2) = 12 \qquad \text{Moment about } x\text{-axis}$$

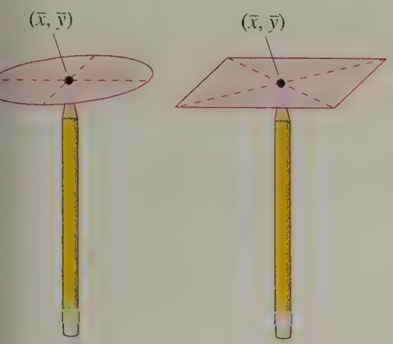
Therefore,

$$\bar{x} = \frac{M_y}{m} = \frac{44}{20} = \frac{11}{5}$$

and

$$\bar{y} = \frac{M_x}{m} = \frac{12}{20} = \frac{3}{5}$$

and thus the center of mass is $\left(\frac{11}{5}, \frac{3}{5}\right)$.



ou can think of the center of mass $(\bar{x}, \bar{y})$ f a lamina as its balancing point. For a cir- ılar lamina, the center of mass is the center f the circle. For a rectangular lamina, the enter of mass is the center of the rectangle.
igure 6.59

Center of Mass of a Planar Lamina

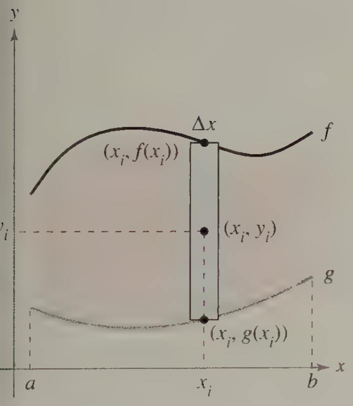
So far in this section we have assumed the total mass of a system to be distributed at discrete points in a plane or on a line. We now consider a thin, flat plate of material of constant density called a **planar lamina** (see Figure 6.59). **Density** is a measure of mass per unit of volume, such as grams per cubic centimeter. For planar laminas, however, density is considered to be a measure of mass per unit of area. Density is denoted by ρ, the lowercase Greek letter rho.

Consider an irregularly shaped planar lamina of uniform density ρ, bounded by the graphs of $y = f(x)$, $y = g(x)$, and $a \le x \le b$, as shown in Figure 6.60. The mass of this region is given by

$$\begin{aligned} m &= (\text{density})(\text{area}) \\ &= \rho \int_a^b [f(x) - g(x)]\,dx \\ &= \rho A \end{aligned}$$

where A is the area of the region. To find the center of mass of this lamina, partition the interval $[a, b]$ into n subintervals of equal width Δx. Let x_i be the center of the ith subinterval. You can approximate the portion of the lamina lying in the ith subinterval by a rectangle whose height is $h = f(x_i) - g(x_i)$. Because the density of the rectangle is ρ, its mass is

$$\begin{aligned} m_i &= (\text{density})(\text{area}) \\ &= \underbrace{\rho}_{\text{Density}}\,\underbrace{[f(x_i) - g(x_i)]}_{\text{Height}}\,\underbrace{\Delta x}_{\text{Width}}. \end{aligned}$$



lanar lamina of uniform density ρ
igure 6.60

Now, considering this mass to be located at the center (x_i, y_i) of the rectangle, the directed distance from the x-axis to (x_i, y_i) is $y_i = [f(x_i) + g(x_i)]/2$. So, the moment of m_i about the x-axis is

$$\begin{aligned} \text{Moment} &= (\text{mass})(\text{distance}) \\ &= m_i y_i = \rho[f(x_i) - g(x_i)]\,\Delta x\left[\frac{f(x_i) + g(x_i)}{2}\right]. \end{aligned}$$

Summing the moments and taking the limit as $n \to \infty$ suggest the definitions below.

Moments and Center of Mass of a Planar Lamina

Let f and g be continuous functions such that $f(x) \ge g(x)$ on $[a, b]$, and consider the planar lamina of uniform density ρ bounded by the graphs of $y = f(x)$, $y = g(x)$, and $a \le x \le b$.

1. The **moments about the x- and y-axes** are

$$M_x = \rho \int_a^b \left[\frac{f(x) + g(x)}{2}\right][f(x) - g(x)]\,dx$$

$$M_y = \rho \int_a^b x[f(x) - g(x)]\,dx.$$

2. The **center of mass** $(\bar{x}, \bar{y})$ is given by $\bar{x} = \dfrac{M_y}{m}$ and $\bar{y} = \dfrac{M_x}{m}$, where $m = \rho \int_a^b [f(x) - g(x)]\,dx$ is the mass of the lamina.

Example 4 The Center of Mass of a Planar Lamina

Find the center of mass of the lamina of uniform density ρ bounded by the graph c $f(x) = 4 - x^2$ and the x-axis.

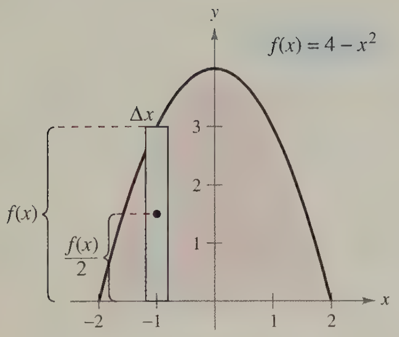


Figure 6.61

Solution Because the center of mass lies on the axis of symmetry, you know tha $\bar{x} = 0$. Moreover, the mass of the lamina is

$$\begin{aligned} m &= \rho \int_{-2}^{2} (4 - x^2)\,dx \\ &= \rho\left[4x - \frac{x^3}{3}\right]_{-2}^{2} \\ &= \frac{32\rho}{3}. \end{aligned}$$

To find the moment about the x-axis, place a representative rectangle in the region, a shown in Figure 6.61. The distance from the x-axis to the center of this rectangle is

$$y_i = \frac{f(x)}{2} = \frac{4 - x^2}{2}.$$

Because the mass of the representative rectangle is

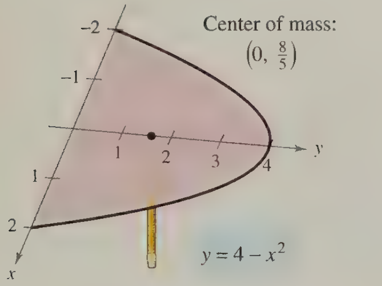
$$\rho f(x)\,\Delta x = \rho(4 - x^2)\,\Delta x$$

you have

$$\begin{aligned} M_x &= \rho \int_{-2}^{2} \frac{4 - x^2}{2}(4 - x^2)\,dx \\ &= \frac{\rho}{2} \int_{-2}^{2} (16 - 8x^2 + x^4)\,dx \\ &= \frac{\rho}{2}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2} \\ &= \frac{256\rho}{15} \end{aligned}$$

and $\bar{y}$ is given by

$$\bar{y} = \frac{M_x}{m} = \frac{256\rho/15}{32\rho/3} = \frac{8}{5}.$$



The center of mass is the balancing point.
Figure 6.62

So, the center of mass (the balancing point) of the lamina is $\left(0, \frac{8}{5}\right)$, as shown i Figure 6.62.

The density ρ in Example 4 is a common factor of both the moments and th mass, and as such cancels out of the quotients representing the coordinates of th center of mass. So, the center of mass of a lamina of *uniform* density depends only c the shape of the lamina and not on its density. For this reason, the point

$(\bar{x}, \bar{y})$ Center of mass or centroid

is sometimes called the center of mass of a *region* in the plane, or the **centroid** of th region. In other words, to find the centroid of a region in the plane, you simply assun that the region has a constant density of $\rho = 1$ and compute the corresponding cent of mass.

Example 5 The Centroid of a Plane Region

Find the centroid of the region bounded by the graphs of $f(x) = 4 - x^2$ and $g(x) = x + 2$.

Solution The two graphs intersect at the points $(-2, 0)$ and $(1, 3)$, as shown in Figure 6.63. So, the area of the region is

$$A = \int_{-2}^{1} [f(x) - g(x)]\, dx = \int_{-2}^{1} (2 - x - x^2)\, dx = \frac{9}{2}.$$

The centroid $(\bar{x}, \bar{y})$ of the region has the following coordinates.

$$\bar{x} = \frac{1}{A}\int_{-2}^{1} x[(4 - x^2) - (x + 2)]\, dx = \frac{2}{9}\int_{-2}^{1} (-x^3 - x^2 + 2x)\, dx$$

$$= \frac{2}{9}\left[-\frac{x^4}{4} - \frac{x^3}{3} + x^2\right]_{-2}^{1} = -\frac{1}{2}$$

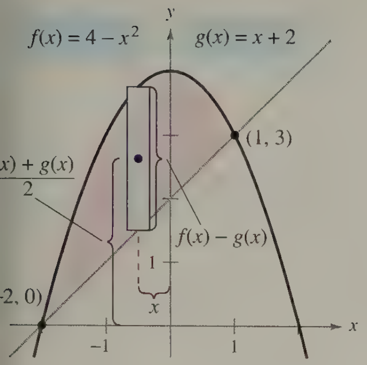
$$\bar{y} = \frac{1}{A}\int_{-2}^{1} \left[\frac{(4 - x^2) + (x + 2)}{2}\right][(4 - x^2) - (x + 2)]\, dx$$

$$= \frac{2}{9}\left(\frac{1}{2}\right)\int_{-2}^{1} (-x^2 + x + 6)(-x^2 - x + 2)\, dx$$

$$= \frac{1}{9}\int_{-2}^{1} (x^4 - 9x^2 - 4x + 12)\, dx$$

$$= \frac{1}{9}\left[\frac{x^5}{5} - 3x^3 - 2x^2 + 12x\right]_{-2}^{1} = \frac{12}{5}.$$

So, the centroid of the region is $(\bar{x}, \bar{y}) = \left(-\frac{1}{2}, \frac{12}{5}\right)$.



he centroid of the region is $\left(-\frac{1}{2}, \frac{12}{5}\right)$.
igure 6.63

EXPLORATION

Cut an irregular shape from a piece of cardboard.

a. Hold a pencil vertically and move the object on the pencil point until the centroid is located.

b. Divide the object into representative elements. Make the necessary measurements and numerically approximate the centroid. Compare your result with the result in part (a).

For simple plane regions, you may be able to find the centroid without resorting to integration.

Example 6 The Centroid of a Simple Plane Region

Find the centroid of the region shown in Figure 6.64(a).

Solution By superimposing a coordinate system on the region, as shown in Figure 6.64(b), you can locate the centroids of the three rectangles at

$$\left(\frac{1}{2}, \frac{3}{2}\right), \quad \left(\frac{5}{2}, \frac{1}{2}\right), \quad \text{and} \quad (5, 1).$$

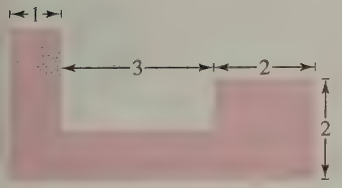
Using these three points, you can find the centroid of the region.

$$A = \text{area of region} = 3 + 3 + 4 = 10$$

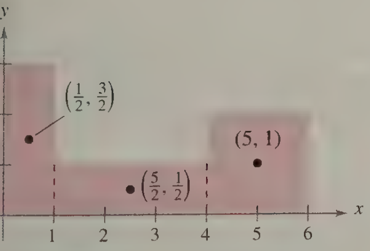
$$\bar{x} = \frac{(1/2)(3) + (5/2)(3) + (5)(4)}{10} = \frac{29}{10} = 2.9$$

$$\bar{y} = \frac{(3/2)(3) + (1/2)(3) + (1)(4)}{10} = \frac{10}{10} = 1$$

So, the centroid of the region is (2.9, 1).

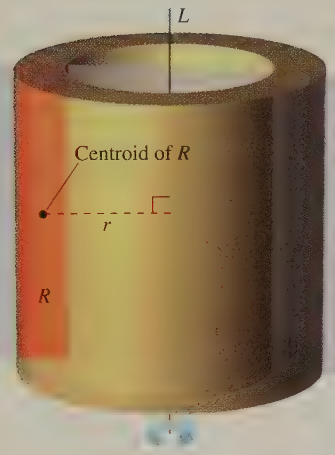


Original region


y
$\left(\frac{1}{2}, \frac{3}{2}\right)$
(5, 1)
$\left(\frac{5}{2}, \frac{1}{2}\right)$
1
2
3
4
5
6
x

The centroid of the region is (2.9, 1).
gure 6.64

NOTE In Example 6, notice that (2.9, 1) is not the “average” of $\left(\frac{1}{2}, \frac{3}{2}\right)$, $\left(\frac{5}{2}, \frac{1}{2}\right)$, and (5, 1).



The volume V is $2\pi rA$ where A is the area of region R.
Figure 6.65

Theorem of Pappus

The final topic in this section is a useful theorem credited to Pappus of Alexandria (c 300 A.D.), a Greek mathematician whose eight-volume *Mathematical Collection* is record of much of classical Greek mathematics. We delay the proof of this theore until Section 13.4 (Exercise 54).

THEOREM 6.1 The Theorem of Pappus

Let R be a region in a plane and let L be a line in the same plane such that L does not intersect the interior of R, as shown in Figure 6.65. If r is the distance between the centroid of R and the line, then the volume V of the solid of revolution formed by revolving R about the line is

$$V = 2\pi rA$$

where A is the area of R. (Note that $2\pi r$ is the distance traveled by the centroid as the region is revolved about the line.)

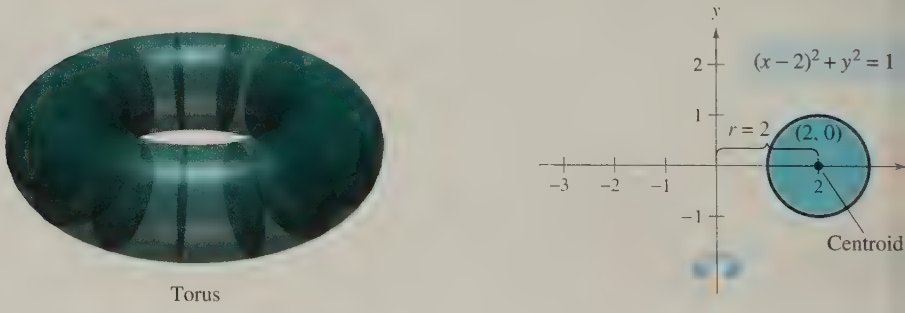
The Theorem of Pappus can be used to find the volume of a torus, as shown the following example. Recall that a torus is a doughnut-shaped solid formed b revolving a circular region about a line that lies in the same plane as the circle (b does not intersect the circle).

Example 7 Finding Volume by the Theorem of Pappus

Find the volume of the torus formed by revolving the circular region bounded by

$$(x-2)^2 + y^2 = 1$$

about the y-axis, as shown in Figure 6.66(a).



(a) (b)
Figure 6.66

Solution In Figure 6.66(b), you can see that the centroid of the circular region (2, 0). So, the distance between the centroid and the axis of revolution is $r =$ Because the area of the circular region is $A = \pi$, the volume of the torus is

$$\begin{aligned} V &= 2\pi rA \\ &= 2\pi(2)(\pi) \\ &= 4\pi^2 \\ &\approx 39.5. \end{aligned}$$

EXPLORATION

Use the shell method to show that the volume of the torus is given by

$$V = \int_1^3 4\pi x\sqrt{1-(x-2)^2}\,dx.$$

Evaluate this integral using a graphing utility. Does your answer agree with the one in Example 7?

EXERCISES FOR SECTION 6.6

Exercises 1–4, find the center of mass of the point masses ing on the x-axis.

1. $m_1 = 6, m_2 = 3, m_3 = 5$
$x_1 = -5, x_2 = 1, x_3 = 3$

2. $m_1 = 7, m_2 = 4, m_3 = 3, m_4 = 8$
$x_1 = -3, x_2 = -2, x_3 = 5, x_4 = 6$

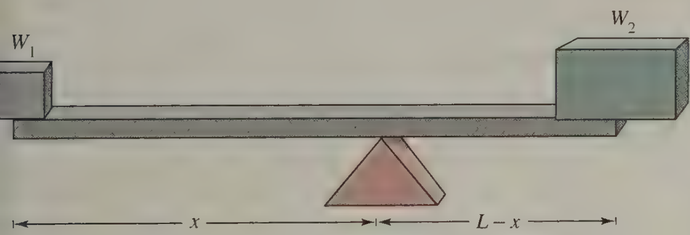
3. $m_1 = 1, m_2 = 1, m_3 = 1, m_4 = 1, m_5 = 1$
$x_1 = 7, x_2 = 8, x_3 = 12, x_4 = 15, x_5 = 18$

4. $m_1 = 12, m_2 = 1, m_3 = 6, m_4 = 3, m_5 = 11$
$x_1 = -6, x_2 = -4, x_3 = -2, x_4 = 0, x_5 = 8$

5. *Graphical Reasoning*
   (a) Translate each point mass in Exercise 3 to the right 5 units and determine the resulting center of mass.
   (b) Translate each point mass in Exercise 4 to the left 3 units and determine the resulting center of mass.

6. *Conjecture* Use the result of Exercise 5 to make a conjecture about the change in the center of mass that results when each point mass is translated k units horizontally.

atics Problems **In Exercises 7 and 8, consider a beam of ngth L with a fulcrum x feet from one end (see figure). If there re objects with weights W_1 and W_2 placed on opposite ends of e beam, find x such that the system is in equilibrium.**



7. Two children weighing 50 pounds and 75 pounds are going to play on a seesaw that is 10 feet long.

8. In order to move a 550-pound rock, a person weighing 200 pounds wants to balance it on a beam that is 5 feet long.

Exercise 9–12, find the center of mass of the given system of oint masses.

m_i	5	1	3
(x_i, y_i)	(2, 2)	(−3, 1)	(1, −4)

m_i	10	2	5
(x_i, y_i)	(1, −1)	(5, 5)	(−4, 0)

11.

m_i	3	4
(x_i, y_i)	(−2, −3)	(−1, 0)

m_i	2	1	6
(x_i, y_i)	(7, 1)	(0, 0)	(−3, 0)

12.

m_i	12	6	$\frac{15}{2}$	15
(x_i, y_i)	(2, 3)	(−1, 5)	(6, 8)	(2, −2)

In Exercises 13–24, find M_x, M_y, and $(\bar{x}, \bar{y})$ for the laminas of uniform density ρ bounded by the graphs of the equations.

13. $y = \sqrt{x}, y = 0, x = 4$
14. $y = \frac{1}{2}x^2, y = 0, x = 2$
15. $y = x^2, y = x^3$
16. $y = \sqrt{x}, y = x$
17. $y = -x^2 + 4x + 2, y = x + 2$
18. $y = \sqrt{x} + 1, y = \frac{1}{3}x + 1$
19. $y = x^{2/3}, y = 0, x = 8$
20. $y = x^{2/3}, y = 4$
21. $x = 4 - y^2, x = 0$
22. $x = 2y - y^2, x = 0$
23. $x = -y, x = 2y - y^2$
24. $x = y + 2, x = y^2$

In Exercises 25–28, set up and evaluate the integrals for finding the area and moments about the x- and y-axes for the region bounded by the graphs of the equations. (Assume $\rho = 1$.)

25. $y = x^2, y = x$
26. $y = \dfrac{1}{x}, y = 0, 1 \le x \le 4$
27. $y = 2x + 4, y = 0, 0 \le x \le 3$
28. $y = x^2 - 4, y = 0$

In Exercises 29–32, use a graphing utility to graph the region bounded by the graphs of the equations. Use the integration capabilities of the graphing utility to approximate the centroid of the region.

29. $y = 10x\sqrt{125 - x^3}, y = 0$
30. $y = xe^{-x/2}, y = 0, x = 0, x = 4$
31. *Prefabricated End Section of a Building*
$y = 5\sqrt[3]{400 - x^2}, y = 0$
32. *Witch of Agnesi*
$y = 8/(x^2 + 4), y = 0, x = -2, x = 2$

In Exercises 33–38, find and/or verify the centroid of the common region used in engineering.

33. ***Triangle*** Show that the centroid of the triangle with vertices $(-a, 0)$, $(a, 0)$, and (b, c) is the point of intersection of the medians (see figure).

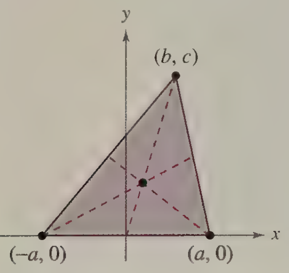


Figure for 33

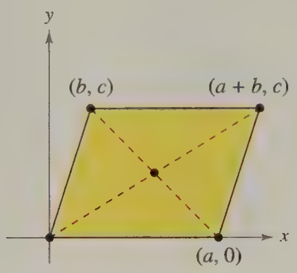


Figure for 34

34. ***Parallelogram*** Show that the centroid of the parallelogram with vertices $(0, 0)$, $(a, 0)$, (b, c), and $(a + b, c)$ is the point of intersection of the diagonals (see figure).

35. ***Trapezoid*** Find the centroid of the trapezoid with vertices $(0, 0)$, $(0, a)$, (c, b), and $(c, 0)$. Show that it is the intersection of the line connecting the midpoints of the parallel sides and the line connecting the extended parallel sides, as shown in the figure.

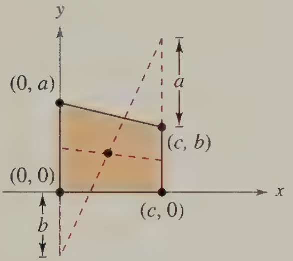


Figure for 35

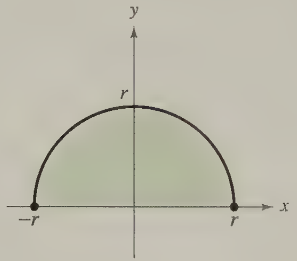


Figure for 36

36. ***Semicircle*** Find the centroid of the region bounded by the graphs of $y = \sqrt{r^2 - x^2}$ and $y = 0$ (see figure).

37. ***Semiellipse*** Find the centroid of the region bounded by the graphs of $y = \frac{b}{a}\sqrt{a^2 - x^2}$ and $y = 0$ (see figure).

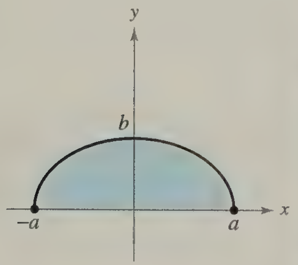


Figure for 37

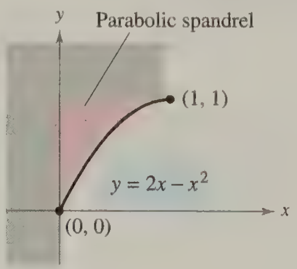


Figure for 38

38. ***Parabolic Spandrel*** Find the centroid of the **parabolic spandrel** shown in the figure.

39. ***Graphical Reasoning*** Consider the region bounded by t graphs of $y = x^2$ and $y = b$, where $b > 0$.

(a) Sketch a graph of the region.

(b) Use the graph in part (a) to determine $\bar{x}$. Explain.

(c) Set up the integral for finding M_y. Because of the form of t integrand, the value of the integral can be obtained witho integrating. What is the form of the integrand and what is t value of the integral? Compare with the result in part (b).

(d) Use the graph in part (a) to determine whether $\bar{y} > \frac{b}{2}$ or $\bar{y} < \frac{b}{2}$. Explain.

(e) Use integration to verify your answer in part (d).

40. ***Graphical and Numerical Reasoning*** Consider the regi bounded by the graphs of $y = x^{2n}$ and $y = b$, where $b > 0$ a n is a positive integer.

(a) Set up the integral for finding M_y. Because of the form the integrand, the value of the integral can be obtain without integrating. What is the form of the integrand a what is the value of the integral? Compare with the result part (b).

(b) Is $\bar{y} > \frac{b}{2}$ or $\bar{y} < \frac{b}{2}$? Explain.

(c) Use integration to find $\bar{y}$ as a function of n.

(d) Use the result in part (c) to complete the table.

n	1	2	3	4
$\bar{y}$				

(e) Find $\lim_{n \to \infty} \bar{y}$.

(f) Give a geometric explanation of the result in part (e).

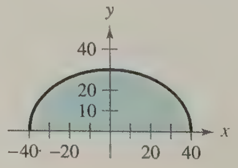
41. ***Modeling Data*** The manufacturer of glass for a window i conversion van needs to approximate its center of mass. A coo dinate system is superimposed on a prototype of the glass (s figure). The measurements (in centimeters) for the right half the symmetric piece of glass are shown in the table.

x	0	10	20	30	40
y	30	29	26	20	0

(a) Use Simpson's Rule to approximate the center of mass the glass.

(b) Use the regression capabilities of a graphing utility to fi a fourth-degree polynomial model for the data.

(c) Use the integration capabilities of a graphing utility and t model to approximate the center of mass of the gla Compare with the result in part (a).



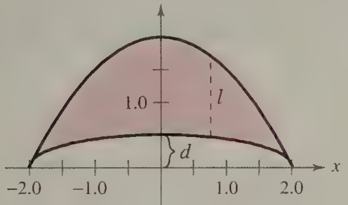
2. ***Modeling Data*** The manufacturer of a boat needs to approximate the center of mass of a section of the hull. A coordinate system is superimposed on a prototype (see figure). The measurements (in feet) for the right half of the symmetric prototype are listed in the table.

x	0	0.5	1.0	1.5	2
l	1.50	1.45	1.30	0.99	0
d	0.50	0.48	0.43	0.33	0

(a) Use Simpson's Rule to approximate the center of mass of the hull section.

(b) Use the regression capabilities of a graphing utility to find fourth-degree polynomial models for both curves shown in the figure. Plot the data and graph the models.

(c) Use the integration capabilities of a graphing utility and the model to approximate the center of mass of the hull section. Compare with the result in part (a).



n Exercises 43–46, introduce an appropriate coordinate ystem and find the coordinates of the center of mass of the lanar lamina. (The answer depends on the position of the oordinate system.)

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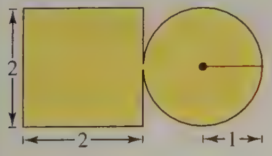

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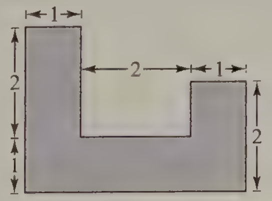

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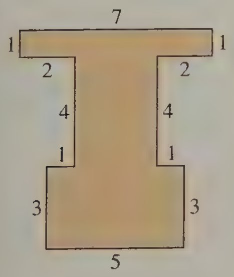
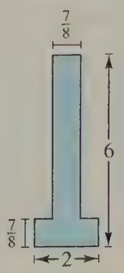


46.



7. Find the center of mass of the lamina in Exercise 43 if the circular portion of the lamina has twice the density of the square portion of the lamina.

3. Find the center of mass of the lamina in Exercise 43 if the square portion of the lamina has twice the density of the circular portion of the lamina.

In Exercises 49–52, use the Theorem of Pappus to find the volume of the solid of revolution.

49. The torus formed by revolving the circle $(x - 5)^2 + y^2 = 16$ about the y-axis

50. The torus formed by revolving the circle $x^2 + (y - 3)^2 = 4$ about the x-axis

51. The solid formed by revolving the region bounded by the graphs of $y = x$, $y = 4$, and $x = 0$ about the x-axis

52. The solid formed by revolving the region bounded by the graphs of $y = 2\sqrt{x - 2}$, $y = 0$, and $x = 6$ about the y-axis

Getting at the Concept

53. Let the point masses $m_1, m_2, \ldots, m_n$ be located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. Define the center of mass $(\bar{x}, \bar{y})$.

54. What is meant by a planar lamina? Describe what is meant by the center of mass $(\bar{x}, \bar{y})$ of a planar lamina.

55. The centroid of the plane region bounded by the graphs of $y = f(x)$, $y = 0$, $x = 0$, and $x = 1$ is $\left(\frac{5}{6}, \frac{5}{18}\right)$. Is it possible to find the centroid of each of the following regions bounded by the graphs of the equations? If so, identify the centroid and explain your answer.

(a) $y = f(x) + 2$, $y = 2$, $x = 0$, and $x = 1$

(b) $y = f(x - 2)$, $y = 0$, $x = 2$, and $x = 3$

(c) $y = -f(x)$, $y = 0$, $x = 0$, and $x = 1$

(d) $y = f(x)$, $y = 0$, $x = -1$, and $x = 1$

56. State the Theorem of Pappus.

In Exercises 57 and 58, use the *Second Theorem of Pappus*, which is stated as follows. If a segment of a plane curve C is revolved about an axis that does not intersect the curve (except possibly at its endpoints), the area S of the resulting surface of revolution is given by the product of the length of C times the distance d traveled by the centroid of C.

57. A sphere is formed by revolving the graph of

$$y = \sqrt{r^2 - x^2}$$

about the x-axis. Use the formula for surface area, $S = 4\pi r^2$, to find the centroid of the semicircle $y = \sqrt{r^2 - x^2}$.

58. A torus is formed by revolving the graph of

$$(x - 1)^2 + y^2 = 1$$

about the y-axis. Find the surface area of the torus.

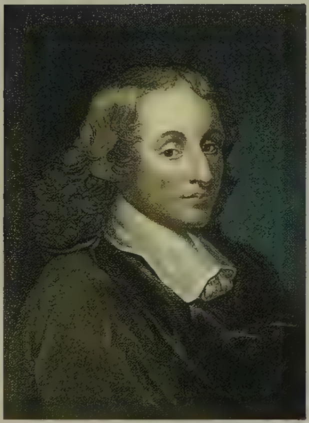
59. Let $n \geq 1$ be constant, and consider the region bounded by $f(x) = x^n$, the x-axis, and $x = 1$. Find the centroid of this region. As $n \to \infty$, what does the region look like, and where is its centroid?

Section 6.7 Fluid Pressure and Fluid Force

- Find fluid pressure and fluid force.

Fluid Pressure and Fluid Force

Swimmers know that the deeper an object is submerged in a fluid, the greater the pre sure on the object. **Pressure** is defined as the force per unit of area over the surface a body. For example, because a column of water that is 10 feet in height and 1 inc square weighs 4.3 pounds, the *fluid pressure* at a depth of 10 feet of water is 4 pounds per square inch.* At 20 feet, this would increase to 8.6 pounds per square inc and in general the pressure is proportional to the depth of the object in the fluid.


The Granger Collection

Blaise Pascal (1623–1662)

Pascal is well known for his work in many areas of mathematics and physics, and also for his influence on Leibniz. Although much of Pascal's work in calculus was intuitive and lacked the rigor of modern mathematics, he nevertheless anticipated many important results.

Definition of Fluid Pressure

The **pressure** on an object at depth h in a liquid is

$$\text{Pressure} = P = wh$$

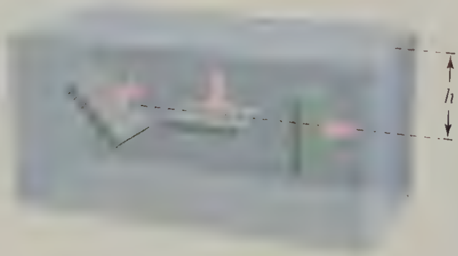
where w is the weight-density of the liquid per unit of volume.

Below are some common weight-densities of fluids in pounds per cubic foot.

Ethyl alcohol	49.4
Gasoline	41.0–43.0
Glycerin	78.6
Kerosene	51.2
Mercury	849.0
Seawater	64.0
Water	62.4

When calculating fluid pressure, you can use an important (and rather surprisin physical law called **Pascal's Principle**, named after the French mathematician Blai Pascal. Pascal's Principle states that the pressure exerted by a fluid at a depth h transmitted equally *in all directions*. For example, in Figure 6.67, the pressure at th indicated depth is the same for all three objects. Because fluid pressure is given terms of force per unit area ($P = F/A$), the fluid force on a *submerged horizont* surface of area A is

$$\text{Fluid force} = F = PA = (\text{pressure})(\text{area}).$$



The pressure at h is the same for all three objects.
Figure 6.67

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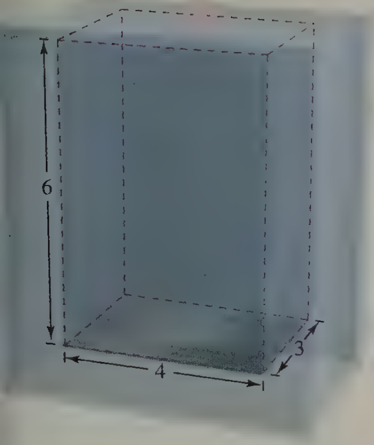
[*] *The total pressure on an object in 10 feet of water would also include the pressure due earth's atmosphere. At sea level, atmospheric pressure is approximately 14.7 pounds per squa inch.*



he fluid force on a horizontal metal sheet is qual to the fluid pressure times the area.
igure 6.68

Example 1 Fluid Force on a Submerged Sheet

Find the fluid force on a rectangular metal sheet measuring 3 feet by 4 feet that is submerged in 6 feet of water, as shown in Figure 6.68.

Solution Because the weight-density of water is 62.4 pounds per cubic foot and the sheet is submerged in 6 feet of water, the fluid pressure is

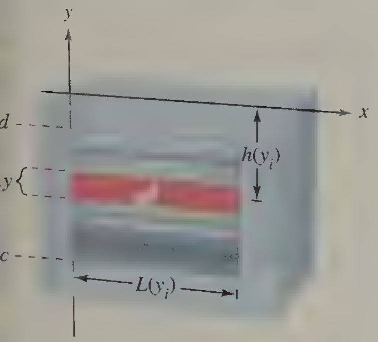
$$\begin{aligned} P &= (62.4)(6) && P = wh \\ &= 374.4 \text{ pounds per square foot.} \end{aligned}$$

Because the total area of the sheet is $A = (3)(4) = 12$ square feet, the fluid force is

$$\begin{aligned} F = PA &= \left(374.4 \frac{\text{pounds}}{\text{square foot}}\right)(12 \text{ square feet}) \\ &= 4492.8 \text{ pounds.} \end{aligned}$$

This result is independent of the size of the body of water. The fluid force would be the same in a swimming pool or lake.

In Example 1, the fact that the sheet is rectangular and horizontal means that you do not need the methods of calculus to solve the problem. We now look at a surface that is submerged vertically in a fluid. The problem is more difficult because the pressure is not constant over the surface.



alculus methods must be used to find the uid force on a vertical metal plate.
igure 6.69

Suppose a vertical plate is submerged in a fluid of weight-density w (per unit of volume), as shown in Figure 6.69. To determine the total force against *one side* of the region from depth c to depth d, you can subdivide the interval $[c, d]$ into n subintervals, each of width Δy. Next, consider the representative rectangle of width Δy and length $L(y_i)$, where y_i is in the ith subinterval. The force against this representative rectangle is

$$\begin{aligned} \Delta F_i &= w(\text{depth})(\text{area}) \\ &= wh(y_i)L(y_i)\,\Delta y. \end{aligned}$$

The force against n such rectangles is

$$\sum_{i=1}^{n} \Delta F_i = w \sum_{i=1}^{n} h(y_i)L(y_i)\,\Delta y.$$

Note that w is considered to be constant and is factored out of the summation. Therefore, taking the limit as $\|\Delta\| \to 0$ $(n \to \infty)$ suggests the following definition.

Definition of Force Exerted by a Fluid

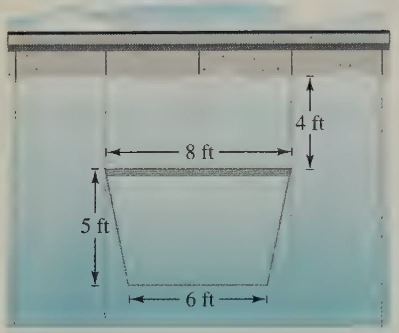
The **force F exerted by a fluid** of constant weight-density w (per unit of volume) against a submerged vertical plane region from $y = c$ to $y = d$ is

$$F = w \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} h(y_i)L(y_i)\,\Delta y = w \int_c^d h(y)L(y)\,dy$$

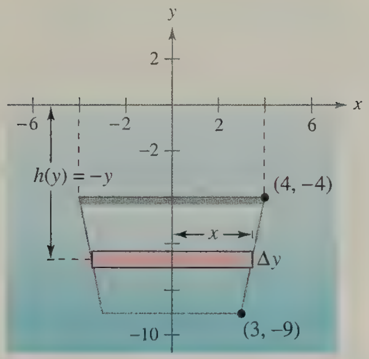
where $h(y)$ is the depth of the fluid at y and $L(y)$ is the horizontal length of the region at y.

Example 2 Fluid Force on a Vertical Surface

A vertical gate in a dam has the shape of an isosceles trapezoid 8 feet across the t
and 6 feet across the bottom, with a height of 5 feet, as shown in Figure 6.70(
What is the fluid force on the gate if the top of the gate is 4 feet below the surfa
of the water?



(a) Water gate in a dam



(b) The fluid force against the gate is 13,936 pounds.

Figure 6.70

Solution In setting up a mathematical model for this problem, you are at liberty
locate the x- and y-axis in several different ways. A convenient approach is to let t
y-axis bisect the gate and place the x-axis at the surface of the water, as shown
Figure 6.70(b). So, the depth of the water at y in feet is

$$\text{Depth} = h(y) = -y.$$

To find the length $L(y)$ of the region at y, we find the equation of the line forming t
right side of the gate. Because this line passes through the points $(3, -9)$ and $(4, -$
its equation is

$$\begin{aligned} y - (-9) &= \frac{-4 - (-9)}{4 - 3}(x - 3) \\ y + 9 &= 5(x - 3) \\ y &= 5x - 24 \\ x &= \frac{y + 24}{5}. \end{aligned}$$

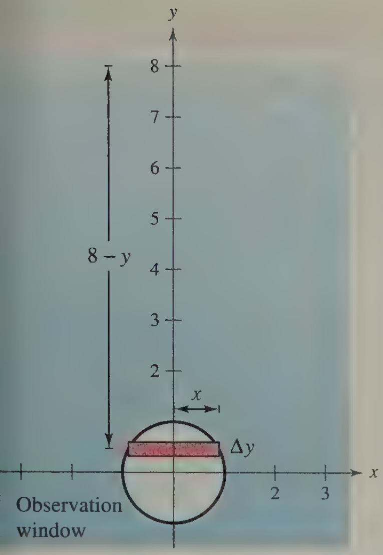
In Figure 6.70(b) you can see that the length of the region at y is

$$\begin{aligned} \text{Length} &= 2x \\ &= \frac{2}{5}(y + 24) \\ &= L(y). \end{aligned}$$

Finally, by integrating from $y = -9$ to $y = -4$, you can calculate the fluid force to

$$\begin{aligned} F &= w\int_c^d h(y)L(y)\,dy \\ &= 62.4\int_{-9}^{-4} (-y)\left(\frac{2}{5}\right)(y + 24)\,dy \\ &= -62.4\left(\frac{2}{5}\right)\int_{-9}^{-4} (y^2 + 24y)\,dy \\ &= -62.4\left(\frac{2}{5}\right)\left[\frac{y^3}{3} + 12y^2\right]_{-9}^{-4} \\ &= -62.4\left(\frac{2}{5}\right)\left(\frac{-1675}{3}\right) \\ &= 13{,}936 \text{ pounds.} \end{aligned}$$

NOTE In Example 2, we let the x-axis coincide with the surface of the water. This
convenient, but arbitrary. In choosing a coordinate system to represent a physical situation,
should consider various possibilities. Often you can simplify the calculations in a problem
locating the coordinate system to take advantage of special characteristics of the problem, s
as symmetry.



The fluid force on the window is 1608.5 pounds.
Figure 6.71

Example 3 Fluid Force on a Vertical Surface

A circular observation window on a marine science ship has a radius of 1 foot, and the center of the window is 8 feet below water level, as shown in Figure 6.71. What is the fluid force on the window?

Solution To take advantage of symmetry, locate a coordinate system such that the origin coincides with the center of the window, as shown in Figure 6.71. The depth at y is then

$$\text{Depth} = h(y) = 8 - y.$$

The horizontal length of the window is $2x$, and you can use the equation for the circle, $x^2 + y^2 = 1$, to solve for x as follows.

$$\text{Length} = 2x = 2\sqrt{1 - y^2} = L(y)$$

Finally, because y ranges from -1 to 1, and using 64 pounds per cubic foot as the weight-density of seawater, you have

$$\begin{aligned} F &= w\int_c^d h(y)L(y)\,dy \\ &= 64\int_{-1}^{1} (8 - y)(2)\sqrt{1 - y^2}\,dy. \end{aligned}$$

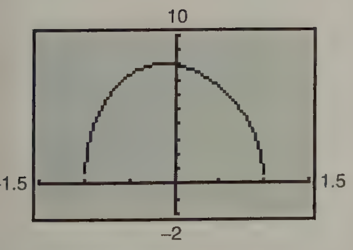
Initially it looks as if this integral would be difficult to solve. However, if you break the integral into two parts and apply symmetry, the solution is simple.

$$F = 64(16)\int_{-1}^{1} \sqrt{1 - y^2}\,dy - 64(2)\int_{-1}^{1} y\sqrt{1 - y^2}\,dy$$

The second integral is 0 (because the integrand is odd and the limits of integration are symmetric to the origin). Moreover, by recognizing that the first integral represents the area of a semicircle of radius 1, you obtain

$$\begin{aligned} F &= 64(16)\left(\frac{\pi}{2}\right) - 64(2)(0) \\ &= 512\pi \\ &\approx 1608.5 \text{ pounds.} \end{aligned}$$

So, the fluid force on the window is 1608.5 pounds.



is not differentiable at $x = \pm 1$.
Figure 6.72

TECHNOLOGY To confirm the result obtained in Example 3, you might have considered using Simpson's Rule to approximate the value of

$$128\int_{-1}^{1} (8 - x)\sqrt{1 - x^2}\,dx.$$

From the graph of

$$f(x) = (8 - x)\sqrt{1 - x^2}$$

however, you can see that f is not differentiable when $x = \pm 1$ (see Figure 6.72). This means that you cannot apply Theorem 4.19 from Section 4.6 to determine the potential error in Simpson's Rule. Without knowing the potential error, the approximation is of little value. Try using a graphing utility to approximate the integral.

EXERCISES FOR SECTION 6.7

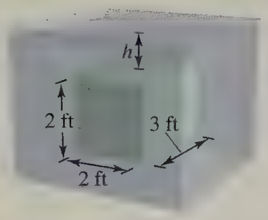
Force on a Submerged Sheet **In Exercises 1 and 2, the area of the top side of a piece of sheet metal is given. The sheet metal is submerged horizontally in 5 feet of water. Find the fluid force on the top side.**

1. 3 square feet

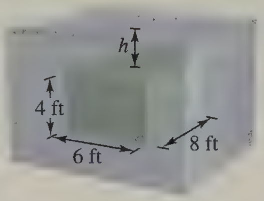
2. 16 square feet

Buoyant Force **In Exercises 3 and 4, find the buoyant force of a rectangular solid of the given dimensions submerged in water so that the top side is parallel to the surface of the water. The buoyant force is the difference between the fluid forces on the top and bottom sides of the solid.**

3.

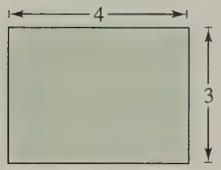


4.

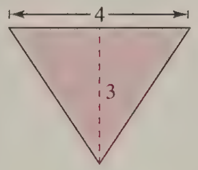


Fluid Force on a Tank Wall **In Exercises 5–10, find the fluid force on the vertical side of the tank, where the dimensions are given in feet. Assume that the tank is full of water.**

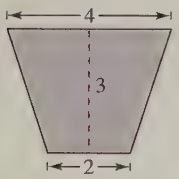
5. Rectangle



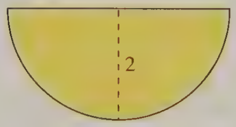
6. Triangle



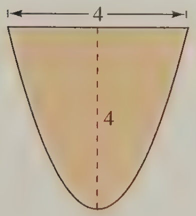
7. Trapezoid



8. Semicircle

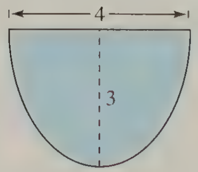


9. Parabola, $y = x^2$



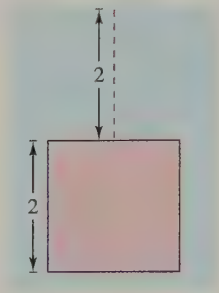
10. Semiellipse,

$y = -\frac{1}{2}\sqrt{36 - 9x^2}$

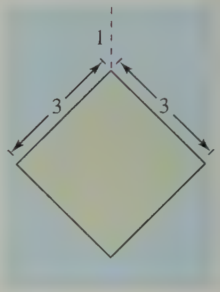


Fluid Force of Water **In Exercises 11–14, find the fluid force o the vertical plate submerged in water, where the dimensions ar given in meters and the weight-density of water is 9800 newton per cubic meter.**

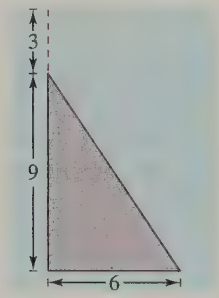
11. Square



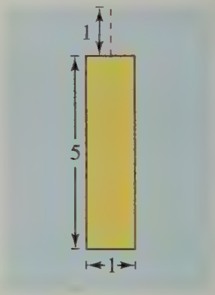
12. Square



13. Triangle

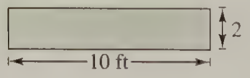


14. Rectangle



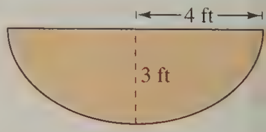
Force on a Concrete Form **In Exercises 15–18, the figure is th vertical side of a form for poured concrete that weighs 140. pounds per cubic foot. Determine the force on this part of th concrete form.**

15. Rectangle

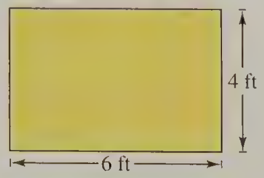


16. Semiellipse,

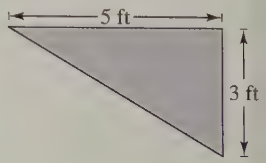
$y = -\frac{3}{4}\sqrt{16 - x^2}$



17. Rectangle



18. Triangle



19. ***Fluid Force of Gasoline*** A cylindrical gasoline tank is place so that the axis of the cylinder is horizontal. Find the fluid forc on a circular end of the tank if the tank is half full, assumin that the diameter is 3 feet and the gasoline weighs 42 pounc per cubic foot.

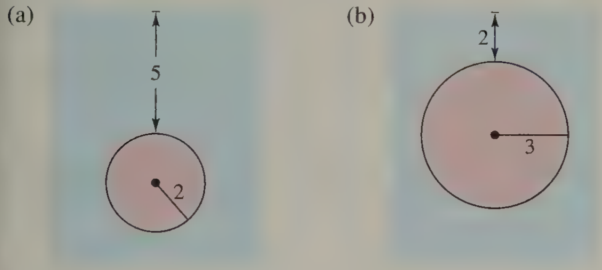
Fluid Force of Gasoline Repeat Exercise 19 for a tank that is full. (Evaluate one integral by a geometric formula and the other by observing that the integrand is an odd function.)

Fluid Force on a Circular Plate A circular plate of radius r feet is submerged vertically in a tank of fluid that weighs w pounds per cubic foot. The center of the circle is k $(k > r)$ feet below the surface of the fluid. Show that the fluid force on the surface of the plate is

$$F = wk(\pi r^2).$$

(Evaluate one integral by a geometric formula and the other by observing that the integrand is an odd function.)

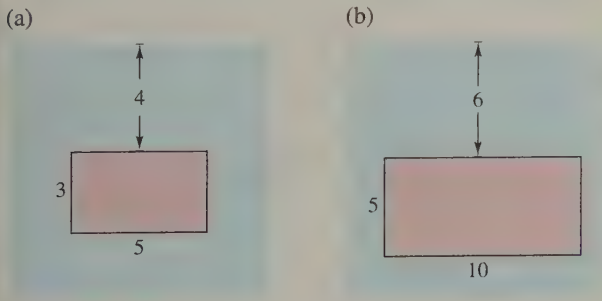
Fluid Force on a Circular Plate Use the result of Exercise 21 to find the fluid force on each of the circular plates shown in the figure. Assume the plates are in the wall of a tank filled with water and the measurements are given in feet.



Fluid Force on a Rectangular Plate A rectangular plate of height h feet and base b feet is submerged vertically in a tank of fluid that weighs w pounds per cubic foot. The center is k feet below the surface of the fluid, where $h \leq k/2$. Show that the fluid force on the surface of the plate is

$$F = wkhb.$$

Fluid Force on a Rectangular Plate Use the result of Exercise 23 to find the fluid force on each of the rectangular plates shown in the figure. Assume the plates are in the wall of a tank filled with water and the measurements are given in feet.

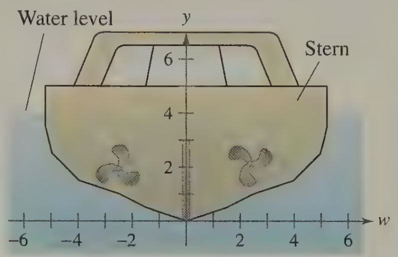


Submarine Porthole A porthole on a vertical side of a submarine (submerged in seawater) is 1 foot square. Find the fluid force on the porthole, assuming that the center of the square is 15 feet below the surface.

Submarine Porthole Repeat Exercise 25 for a circular porthole that has a diameter of 1 foot. The center is 15 feet below the surface.

27. ***Modeling Data*** The vertical stern of a boat with a superimposed coordinate system is shown in the figure. The table shows the width w of the stern at indicated values of y. Find the fluid force against the stern if the measurements are given in feet.

y	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
w	0	3	5	8	9	10	10.25	10.5	10.5



28. ***Irrigation Canal Gate*** The vertical cross section of an irrigation canal is modeled by

$$f(x) = \frac{5x^2}{x^2 + 4}$$

where x is measured in feet and $x = 0$ corresponds to the center of the canal. Use the integration capabilities of a graphing utility to approximate the fluid force against a vertical gate used to stop the flow of water if the water is 3 feet deep.

In Exercises 29 and 30, use the integration capabilities of a graphing utility to approximate the fluid force on the vertical plate bounded by the x-axis and the top half of the graph of the equation. Assume that the base of the plate is 12 feet beneath the surface of the water.

29. $x^{2/3} + y^{2/3} = 4^{2/3}$ **30.** $\dfrac{x^2}{28} + \dfrac{y^2}{16} = 1$

31. ***Think About It***

(a) Approximate the depth of the water in the tank in Exercise 5 if the fluid force is one-half as great as when the tank is full.

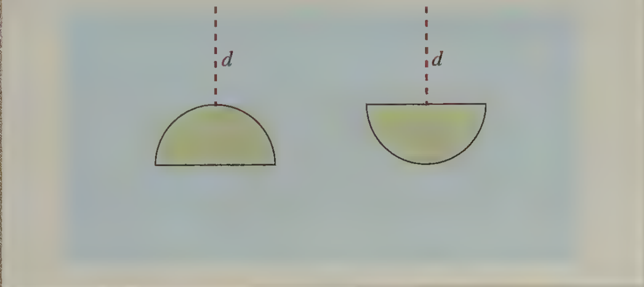
(b) Explain why the answer in part (a) is not $\frac{3}{2}$.

Getting at the Concept

32. Define fluid pressure.

33. Define fluid force against a submerged vertical plane region.

34. Two identical semicircular windows are placed at the same depth in the vertical wall of an aquarium (see figure). Which has the greater fluid force? Explain.



REVIEW EXERCISES FOR CHAPTER 6

6.1 ***Area*** **In Exercises 1–10, sketch the region bounded by the graphs of the equations, and determine the area of the region.**

1. $y = \frac{1}{x^2}$, $y = 0$, $x = 1$, $x = 5$ **2.** $y = \frac{1}{x^2}$, $y = 4$, $x = 5$

3. $y = \frac{1}{x^2 + 1}$, $y = 0$, $x = -1$, $x = 1$

4. $x = y^2 - 2y$, $x = -1$, $y = 0$

5. $y = x$, $y = x^3$

6. $x = y^2 + 1$, $x = y + 3$

7. $y = e^x$, $y = e^2$, $x = 0$

8. $y = \csc x$, $y = 2$ (one region)

9. $y = \sin x$, $y = \cos x$, $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$

10. $x = \cos y$, $x = \frac{1}{2}$, $\frac{\pi}{3} \le y \le \frac{7\pi}{3}$

In Exercises 11–14, use a graphing utility to graph the region bounded by the graphs of the functions, and use the integration capabilities of the graphing utility to find the area of the region.

11. $y = x^2 - 8x + 3$, $y = 3 + 8x - x^2$

12. $y = x^2 - 4x + 3$, $y = x^3$, $x = 0$

13. $\sqrt{x} + \sqrt{y} = 1$, $y = 0$, $x = 0$

14. $y = x^4 - 2x^2$, $y = 2x^2$

In Exercises 15–18, use vertical and horizontal representative rectangles to set up integrals for finding the area of the region bounded by the graphs of the equations. Find the area of the region by evaluating the easier of the two integrals.

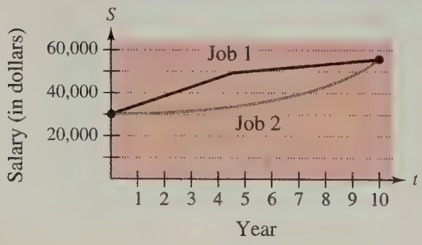
15. $x = y^2 - 2y$, $x = 0$

16. $y = \sqrt{x - 1}$, $y = \frac{x - 1}{2}$

17. $y = 1 - \frac{x}{2}$, $y = x - 2$, $y = 1$

18. $y = \sqrt{x - 1}$, $y = 2$, $y = 0$, $x = 0$

19. ***Think About It*** A person has two job offers. The starting salary for each is \$30,000, and after 10 years of service each will pay \$56,000. The salary increases for each offer are shown in the figure. From a strictly monetary viewpoint, which is the better offer? Explain.



20. ***Modeling Data*** The table shows the annual service revenue in billions of dollars for the cellular telephone industry for years 1992 through 1998. *(Source: Cellular Telecommunicati Industry Association)*

Year	1992	1993	1994	1995	1996	1997	199
R_1	7.8	10.9	14.2	19.1	23.6	27.5	33.1

(a) Use the regression capabilities of a graphing utility to fit exponential model to the data. Let t be time in years, w $t = 2$ corresponding to 1992. Use the graphing utility plot the data and graph the model.

(b) A financial consultant believes that a model for serv revenue for the years 2000 through 2005 is

$$R_2 = 10 + 5.28e^{0.2t}.$$

What is the difference in total service revenue between two models for the years 2000 through 2005?

6.2, 6.3 **In Exercises 21–28, find the volume of the solid ge erated by revolving the plane region bounded by the equatio about the indicated lines.**

21. $y = x$, $y = 0$, $x = 4$

(a) the x-axis (b) the y-axis

(c) the line $x = 4$ (d) the line $x = 6$

22. $y = \sqrt{x}$, $y = 2$, $x = 0$

(a) the x-axis (b) the line $y = 2$

(c) the y-axis (d) the line $x = -1$

23. $\frac{x^2}{16} + \frac{y^2}{9} = 1$ (a) the y-axis (oblate spheroid)

(b) the x-axis (prolate spheroid)

24. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a) the y-axis (oblate spheroid)

(b) the x-axis (prolate spheroid)

25. $y = \frac{1}{x^4 + 1}$, $y = 0$, $x = 0$, $x = 1$

revolved about the y-axis

26. $y = \frac{1}{\sqrt{1 + x^2}}$, $y = 0$, $x = -1$, $x = 1$

revolved about the x-axis

27. $y = 1/(1 + \sqrt{x - 2})$, $y = 0$, $x = 2$, $x = 6$

revolved about the y-axis

28. $y = e^{-x}$, $y = 0$, $x = 0$, $x = 1$

revolved about the x-axis

In Exercises 29 and 30, consider the region bounded by graphs of the equations $y = x\sqrt{x + 1}$ and $y = 0$.

29. ***Area*** Find the area of the region.

30. ***Volume*** Find the volume of the solid generated by revolv the region about (a) the x-axis and (b) the y-axis.

1. ***Depth of Gasoline in a Tank*** A gasoline tank is an oblate spheroid generated by revolving the region bounded by the graph of $(x^2/16) + (y^2/9) = 1$ about the y-axis, where x and y are measured in feet. Find the depth of the gasoline in the tank when it is filled to one-fourth its capacity.

2. ***Magnitude of a Base*** The base of a solid is a circle of radius a, and its vertical cross sections are equilateral triangles. Find the radius of the circle if the volume of the solid is 10 cubic meters.

6.4 ***Arc Length*** **In Exercises 33 and 34, find the arc length f the graph of the function over the indicated interval.**

3. $f(x) = \frac{4}{5}x^{5/4}$, $[0, 4]$

34. $y = \frac{1}{6}x^3 + \frac{1}{2x}$, $[1, 3]$

5. ***Length of a Catenary*** A cable of a suspension bridge forms a catenary modeled by the equation

$$y = 300 \cosh\left(\frac{x}{2000}\right) - 280, \quad -2000 \le x \le 2000$$

where x and y are measured in feet. Use a graphing utility to approximate the length of the cable.

6. ***Approximation*** Determine which value best approximates the length of the arc represented by the integral

$$\int_0^{\pi/4} \sqrt{1 + (\sec^2 x)^2}\, dx.$$

(Make your selection on the basis of a sketch of the arc and *not* by performing any calculations.)

(a) -2 (b) 1 (c) π (d) 4 (e) 3

7. ***Surface Area*** Use integration to find the lateral surface area of a right circular cone of height 4 and radius 3.

3. ***Surface Area*** The region bounded by the graphs of $y = 2\sqrt{x}$, $y = 0$, and $x = 3$ is revolved about the x-axis. Find the surface area of the solid generated.

6.5

9. ***Work*** Find the work done in stretching a spring from its natural length of 10 inches to a length of 15 inches, if a force of 4 pounds is needed to stretch it 1 inch from its natural position.

0. ***Work*** Find the work done in stretching a spring from its natural length of 9 inches to double that length. The force required to stretch the spring is 50 pounds.

1. ***Work*** A water well has an 8-inch casing (diameter) and is 175 feet deep. If the water is 25 feet from the top of the well, determine the amount of work done in pumping it dry, assuming that no water enters the well while it is being pumped.

2. ***Work*** Repeat Exercise 41, assuming that water enters the well at a rate of 4 gallons per minute and the pump works at a rate of 12 gallons per minute. How many gallons are pumped in this case?

3. ***Work*** A chain 10 feet long weighs 5 pounds per foot and is hung from a platform 20 feet above the ground. How much work is required to raise the entire chain to the 20-foot level?

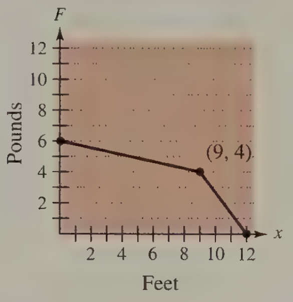
44. ***Work*** A windlass, 200 feet above ground level on the top of a building, uses a cable weighing 4 pounds per foot. Find the work done in winding up the cable if

(a) one end is at ground level.

(b) there is a 300-pound load attached to the end of the cable.

45. ***Work*** The work done by a variable force in a press is 80 foot-pounds. The press moves a distance of 4 feet and the force is a quadratic of the form $F = ax^2$. Find a.

46. ***Work*** Find the work done by the force F shown in the figure.



6.6 **In Exercises 47–50, find the centroid of the region bounded by the graphs of the equations.**

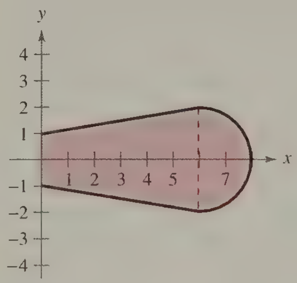
47. $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x = 0$, $y = 0$

48. $y = x^2$, $y = 2x + 3$

49. $y = a^2 - x^2$, $y = 0$

50. $y = x^{2/3}$, $y = \frac{1}{2}x$

51. ***Centroid*** A blade on an industrial fan has the configuration of a semicircle attached to a trapezoid (see figure). Find the centroid of the blade.



6.7

52. ***Fluid Force*** A swimming pool is 5 feet deep at one end and 10 feet deep at the other, and the bottom is an inclined plane. The length and width of the pool are 40 feet and 20 feet. If the pool is full of water, what is the fluid force on each of the vertical walls?

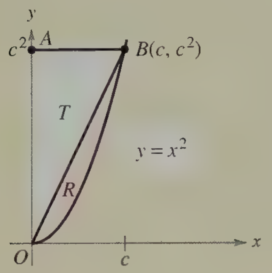
53. ***Fluid Force*** Show that the fluid force against any vertical region in a liquid is the product of the weight per cubic volume of the liquid, the area of the region, and the depth of the centroid of the region.

54. ***Fluid Force*** Using the result of Exercise 53, find the fluid force on one side of a vertical circular plate of radius 4 feet that is submerged in water so that its center is 5 feet below the surface.

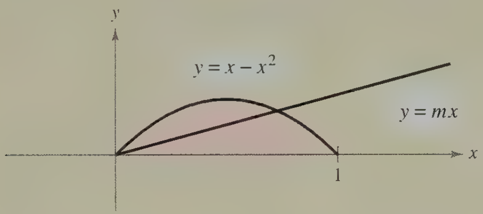
P.S. Problem Solving

1. Let R be the area of the region in the first quadrant bounded by the parabola $y = x^2$ and the line $y = cx$, $c > 0$. Let T be the area of the triangle AOB. Calculate the limit

$$\lim_{c \to 0^+} \frac{T}{R}.$$



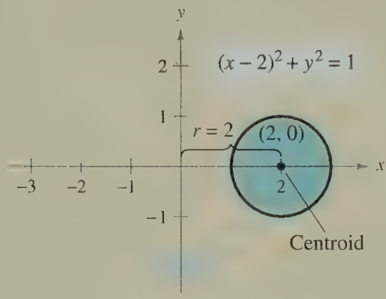
2. Let R be the region bounded by the parabola $y = x - x^2$ and the x-axis. Find the equation of the line $y = mx$ that divides this region into two regions of equal area.



3. (a) A torus is formed by revolving the region bounded by the circle

$$(x - 2)^2 + y^2 = 1$$

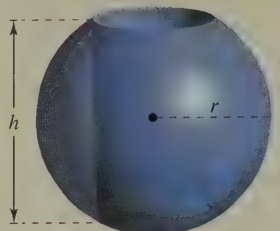
about the y-axis (see figure). Use the disk method to calculate the volume of the torus.



(b) Use the disk method to find the volume of the general torus if the circle has radius r and its center is R units from the axis of rotation.

4. Graph the curve $8y^2 = x^2(1 - x^2)$. Use a computer algebra system to find the surface area of the solid of revolution obtained by revolving the curve about the x-axis.

5. A hole is cut through the center of a sphere of radius r (s figure). The height of the remaining spherical ring is h. Find t volume of the ring and show that it is independent of the radi of the sphere.



6. A rectangle R of length l and width w is revolved about t line L (see figure). Find the volume of the resulting solid revolution.

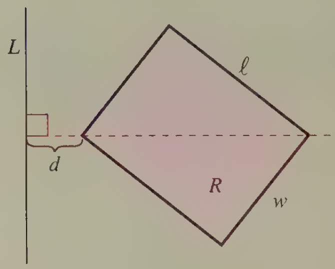


Figure for 6

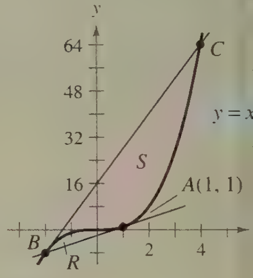


Figure for 7

7. (a) The tangent line to the curve $y = x^3$ at the point $A(1,$ intersects the curve at another point B. Let R be the area the region bounded by the curve and the tangent line. T tangent line at B intersects the curve at another point C (s figure). Let S be the area of the region bounded by the cur and this second tangent line. How are the areas R and related?

(b) Repeat the above construction by selecting an arbitra point A on the curve $y = x^3$. Show that the two areas and S are always related in the same way.

8. The graph of $y = f(x)$ passes through the origin. The arc leng of the curve from $(0, 0)$ to $(x, f(x))$ is given by

$$s(x) = \int_0^x \sqrt{1 + e^t}\, dt.$$

Identify the function f.

9. Let f be rectifiable on the interval $[a, b]$, and let

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

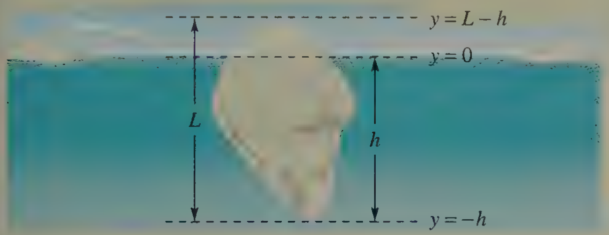
(a) Find $\dfrac{ds}{dx}$.

(b) Find ds and $(ds)^2$.

(c) If $f(t) = t^{3/2}$, find $s(x)$ on $[1, 3]$.

(d) Calculate $s(2)$ and describe what it signifies.

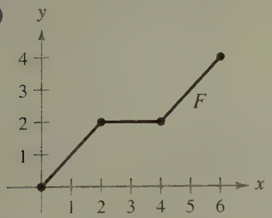
0. The Archimedes Principle states that the upward or buoyant force on an object within a fluid is equal to the weight of the fluid that the object displaces. For a partially submerged object, you can obtain information about the relative densities of the floating object and the fluid by observing how much of the object is above and below the surface. You can also determine the size of a floating object if you know the amount that is above the surface and the relative densities. Suppose you can see the top of a floating iceberg. The density of ocean water is 1.03×10^3 kg/m^3, and that of ice is 0.92×10^3 kg/m^3. What percent of the total iceberg is below the surface?

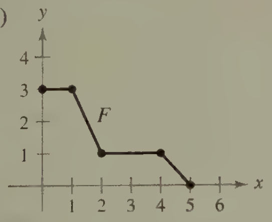


1. Sketch the region bounded on the left by $x = 1$, bounded above by $y = 1/x^3$, and bounded below by $y = -1/x^3$.
   (a) Find the centroid of the region for $1 \le x \le 6$.
   (b) Find the centroid of the region for $1 \le x \le b$.
   (c) Where is the centroid as $b \to \infty$?

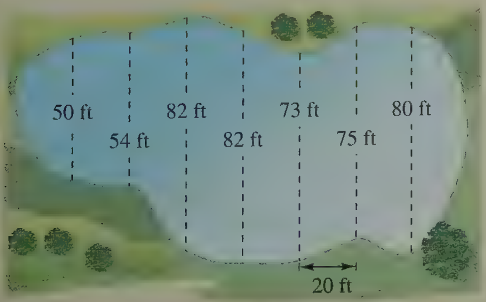
2. Sketch the region to the right of the y-axis, bounded above by $y = 1/x^4$ and bounded below by $y = -1/x^4$.
   (a) Find the centroid of the region for $1 \le x \le 6$.
   (b) Find the centroid of the region for $1 \le x \le b$.
   (c) Where is the centroid as $b \to \infty$?

3. Find the work done by each force F.

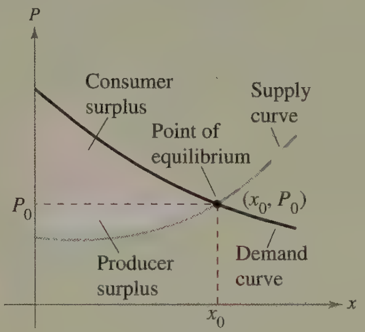
(a)


(b)


4. To estimate the surface area of a pond, a surveyor takes several measurements, as shown in the figure. Estimate the surface area of the pond using (a) the Trapezoidal Rule and (b) Simpson's Rule.

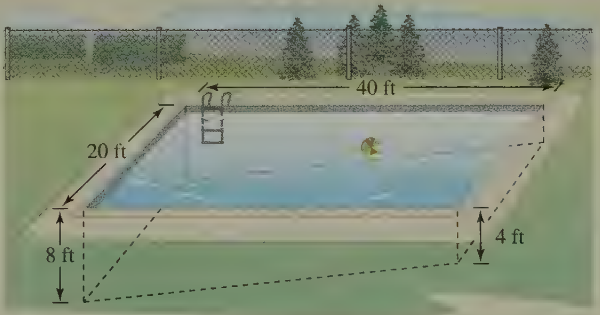


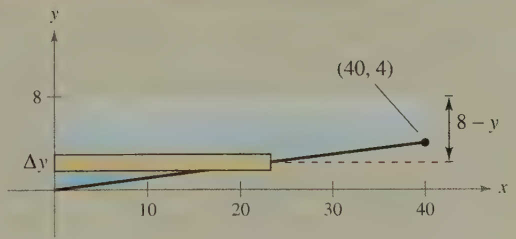
In Exercises 15 and 16, find the consumer surplus and producer surplus for the supply and demand curves. The consumer surplus and producer surplus are represented by the areas shown in the figure.



	Demand Function	*Supply Function*
15.	$p_1(x) = 50 - 0.5x$	$p_2(x) = 0.125x$
16.	$p_1(x) = 1000 - 0.4x^2$	$p_2(x) = 42x$

17. A swimming pool is 20 feet wide, 40 feet long, 4 feet deep at one end, and 8 feet deep at the other end (see figure). The bottom is an inclined plane. Find the fluid force on each of the vertical walls.





Making a Mercator Map

When flying or sailing, pilots expect to be given a steady compass course to follow. On a standard flat map, this is difficult because a steady compass course results in a curved line, as shown in the lower left and middle figures on the facing page.

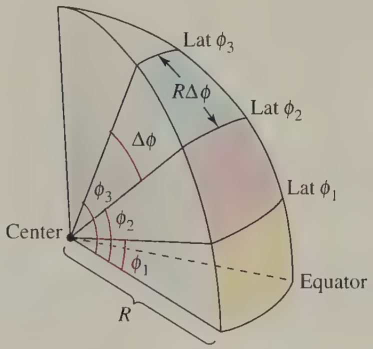
For curved lines to appear as straight lines on a flat map, Flemish geographer Gerardus Mercator (1512–1594) realized that latitude lines must be stretched horizontally by a scaling factor of sec ϕ, where ϕ is the angle of the latitude line. For the map to preserve the angles between latitude and longitude lines, the lengths of longitude lines are also stretched by a scaling factor of sec ϕ at latitude ϕ. The Mercator map has latitude lines that are not equidistant, as shown in the lower left figure on the facing page.

To calculate these vertical lengths, imagine a globe with latitude lines marked at angles of every $\Delta\phi$ radians, with $\Delta\phi = \phi_i - \phi_{i-1}$. The arc length of consecutive latitude lines is $R\Delta\phi$. On the Mercator map, the vertical distance between the equator and the first latitude line is $R\Delta\phi \sec \phi_1$. The vertical distance between the first and second latitude lines is $R\Delta\phi \sec \phi_2$. The vertical distance between the second and third latitude lines is $R\Delta\phi \sec \phi_3$, and so on, as shown in the figure on the right below.

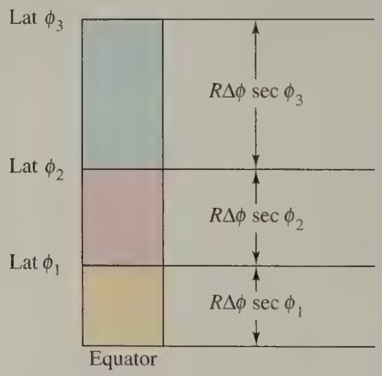
On a globe, the angle between consecutive latitude lines is $\Delta\phi$, and the arc length between them is $R\Delta\phi$ (see the left-hand figure below). On a Mercator map, the vertical distance between the ith and $(i - 1)$st latitude lines is $R\Delta\phi \sec \phi_i$, and the distance from the equator to the ith latitude line is approximately

$$R\Delta\phi \sec \phi_1 + R\Delta\phi \sec \phi_2 + \cdot \cdot \cdot + R\Delta\phi \sec \phi_i$$

(see right-hand figure below).



Globe


Lat ϕ_3
$R\Delta\phi$ sec ϕ_3
Lat ϕ_2
$R\Delta\phi$ sec ϕ_2
Lat ϕ_1
$R\Delta\phi$ sec ϕ_1
Equator

Mercator map

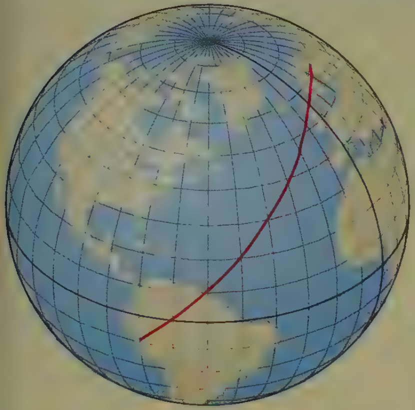
QUESTIONS

1. Use summation notation to write an expression to calculate how far from the equator to draw the line representing latitude ϕ_n.
2. In the calculations above, Mercator realized that the smaller the value used for $\Delta\phi$, the better the map became (better in the sense that straight lines could be used to plot steady compass courses). From your knowledge of calculus, how could you use Mercator's observation to calculate the total vertical distance of a latitude line from the equator?
3. Use the result of Question 2 to find how far from the equator to place latitude lines whose angles are 10°, 20°, 30°, 40°, and 50°. (Use a globe radius of $R = 6$ inches.)
4. What problem do you encounter when you attempt to calculate how far from the equator to place the North Pole?

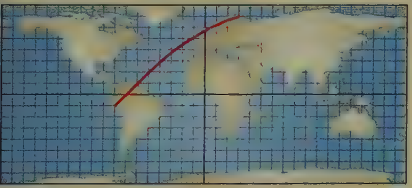
The concepts presented here will be explored further in this chapter. For an extension of this application, see Lab 10 in the lab series that accompanies this text at college.hmco.com.

Integration Techniques, L'Hôpital's Rule, and Improper Integrals

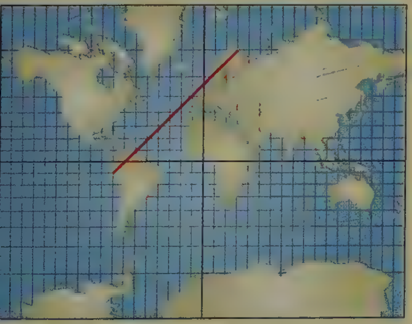
7



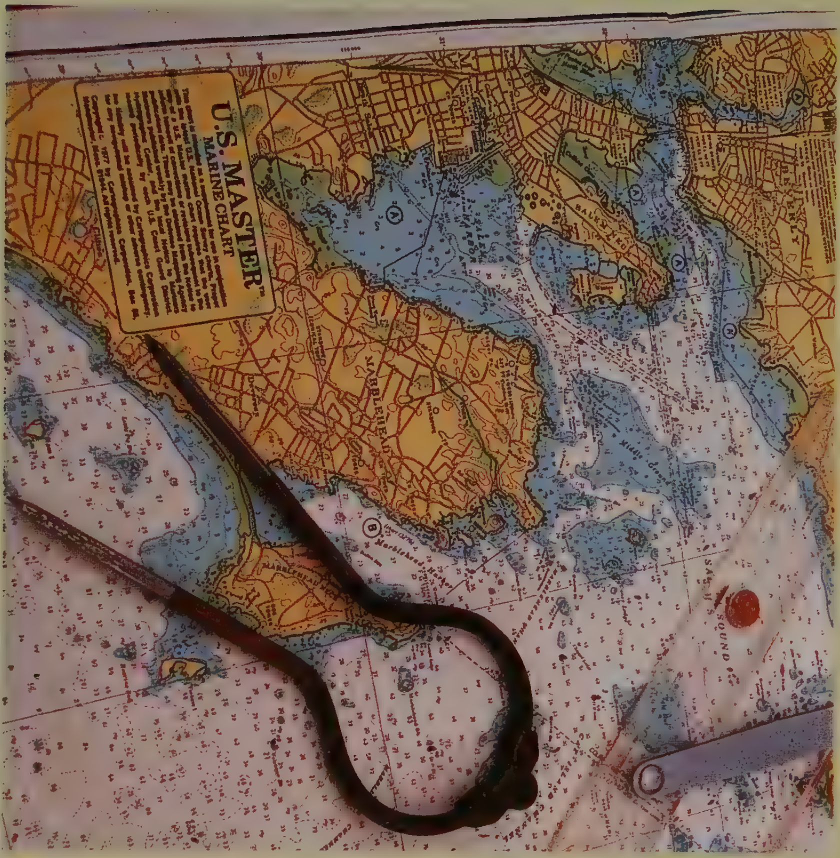
Globe: flight with constant 45° bearing



Standard flat map: flight with constant 45° bearing



Mercator map: flight with constant 45° bearing



Richard Pasley/Stock Boston

Gerardus Mercator was known as one of the best geographers of the Renaissance. He was also the first to refer to a collection of maps as an "atlas."

Bettmann/Corbis

Section 7.1 Basic Integration Rules

- Review procedures for fitting an integrand to one of the basic integration rules.

Fitting Integrands to Basic Rules

In this chapter, you will study several integration techniques that greatly expand the s of integrals to which the basic integration rules can be applied. The formulas a reviewed on page 484 and on the inside front cover. A major step in solving any int gration problem is recognizing the proper basic integration rule to be used. This is n easy. As demonstrated in Example 1, slight differences in the integrand can lead to ve different solution techniques.

Example 1 A Comparison of Three Similar Integrals

Evaluate each of the integrals.

a. $\displaystyle\int \frac{4}{x^2+9}\,dx$ **b.** $\displaystyle\int \frac{4x}{x^2+9}\,dx$ **c.** $\displaystyle\int \frac{4x^2}{x^2+9}\,dx$

Solution

a. Use the Arctangent Rule and let $u = x$ and $a = 3$.

$$\int \frac{4}{x^2+9}\,dx = 4\int \frac{1}{x^2+3^2}\,dx \qquad \text{Constant Multiple Rule}$$

$$= 4\left(\frac{1}{3}\arctan\frac{x}{3}\right) + C \qquad \text{Arctangent Rule}$$

$$= \frac{4}{3}\arctan\frac{x}{3} + C \qquad \text{Simplify.}$$

b. Here the Arctangent Rule does not apply because the numerator contains a fact of x. Consider the Log Rule and let $u = x^2 + 9$. Then $du = 2x\,dx$, and you hav

$$\int \frac{4x}{x^2+9}\,dx = 2\int \frac{2x\,dx}{x^2+9} \qquad \text{Constant Multiple Rule}$$

$$= 2\int \frac{du}{u} \qquad \text{Substitution: } u = x^2 + 9$$

$$= 2\ln|u| + C \qquad \text{Log Rule}$$

$$= 2\ln(x^2+9) + C. \qquad \text{Rewrite as a function of } x.$$

c. Because the degree of the numerator is equal to the degree of the denominator, yo should first use division to rewrite the improper rational function as the sum of polynomial and a proper rational function.

$$\int \frac{4x^2}{x^2+9}\,dx = \int \left(4 - \frac{36}{x^2+9}\right)dx \qquad \text{Rewrite using long division.}$$

$$= \int 4\,dx - 36\int \frac{1}{x^2+9}\,dx \qquad \text{Write as two integrals.}$$

$$= 4x - 36\left(\frac{1}{3}\arctan\frac{x}{3}\right) + C \qquad \text{Integrate.}$$

$$= 4x - 12\arctan\frac{x}{3} + C \qquad \text{Simplify.}$$

EXPLORATION

A Comparison of Three Similar Integrals Which, if any, of the following integrals can be evaluated using the 20 basic integration rules? For any that can be evaluated, do so. For any that can't, explain why.

a. $\displaystyle\int \frac{3}{\sqrt{1-x^2}}\,dx$

b. $\displaystyle\int \frac{3x}{\sqrt{1-x^2}}\,dx$

c. $\displaystyle\int \frac{3x^2}{\sqrt{1-x^2}}\,dx$

NOTE Notice in Example 1c that some preliminary algebra was required before applying the rules for integration, and that subsequently more than one rule was needed to evaluate the resulting integral.

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indicates that in the Interactive 3.0 *CD-ROM and* Internet 3.0 *versions of this te (available at* college.hmco.com) *you will find an Open Exploration, which further explores th example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

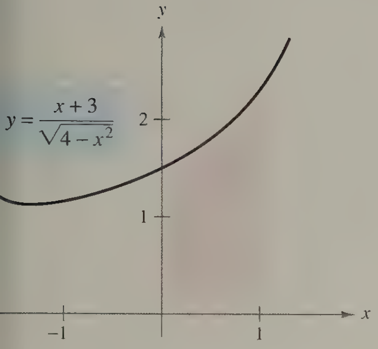
Example 2 Using Two Basic Rules to Solve a Single Integral

Evaluate $\displaystyle\int_0^1 \frac{x+3}{\sqrt{4-x^2}}\,dx$.

Solution Begin by writing the integral as the sum of two integrals. Then apply the Power Rule and the Arcsine Rule as follows.

$$\begin{aligned}
\int_0^1 \frac{x+3}{\sqrt{4-x^2}}\,dx &= \int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx + \int_0^1 \frac{3}{\sqrt{4-x^2}}\,dx \\
&= -\frac{1}{2}\int_0^1 (4-x^2)^{-1/2}(-2x)\,dx + 3\int_0^1 \frac{1}{\sqrt{2^2-x^2}}\,dx \\
&= \left[-(4-x^2)^{1/2} + 3\arcsin\frac{x}{2}\right]_0^1 \\
&= \left(-\sqrt{3}+\frac{1}{2}\pi\right) - (-2+0) \\
&\approx 1.839
\end{aligned}$$

(See Figure 7.1.)



ne area of the region is approximately 839.

gure 7.1

TECHNOLOGY Simpson's Rule can be used to give a good approximation of the value of the integral in Example 2 (for $n = 10$, the approximation is 1.839). When using numerical integration, however, you should be aware that Simpson's Rule does not give good approximations when one or both of the limits of integration are near a vertical asymptote. For instance, using the Fundamental Theorem of Calculus, you can obtain

$$\int_0^{1.99} \frac{x+3}{\sqrt{4-x^2}}\,dx \approx 6.213.$$

Applying Simpson's Rule (with $n = 10$) to this integral produces an approximation of 6.889.

Example 3 A Substitution Involving $a^2 - u^2$

Evaluate $\displaystyle\int \frac{x^2}{\sqrt{16-x^6}}\,dx$.

Solution Because the radical in the denominator can be written in the form

$$\sqrt{a^2-u^2} = \sqrt{4^2-(x^3)^2}$$

you can try the substitution $u = x^3$. Then $du = 3x^2\,dx$, and you have

$$\begin{aligned}
\int \frac{x^2}{\sqrt{16-x^6}}\,dx &= \frac{1}{3}\int \frac{3x^2\,dx}{\sqrt{16-(x^3)^2}} && \text{Rewrite integral.} \\
&= \frac{1}{3}\int \frac{du}{\sqrt{4^2-u^2}} && \text{Substitution: } u = x^3 \\
&= \frac{1}{3}\arcsin\frac{u}{4} + C && \text{Arcsine Rule} \\
&= \frac{1}{3}\arcsin\frac{x^3}{4} + C. && \text{Rewrite as a function of } x.
\end{aligned}$$

TUDY TIP Rules 18, 19, and 20 of e basic integration rules (see page 484) have expressions involving the sum or ference of two squares:

$a^2 - u^2$

$a^2 + u^2$

$u^2 - a^2$

th such an expression, consider the bstitution $u = f(x)$, as in Example 3.

Surprisingly, two of the most commonly overlooked integration rules are the Lo Rule and the Power Rule. Notice in the next two examples how these two integratio rules can be disguised.

Example 4 A Disguised Form of the Log Rule

Evaluate $\displaystyle\int \frac{1}{1+e^x}\,dx$.

Solution The integral does not appear to fit any of the basic rules. However, th quotient form suggests the Log Rule. If you let $u = 1 + e^x$, then $du = e^x\,dx$. You ca obtain the required du by adding and subtracting e^x in the numerator, as follows.

$$
\begin{aligned}
\int \frac{1}{1+e^x}\,dx &= \int \frac{1+e^x-e^x}{1+e^x}\,dx && \text{Add and subtract } e^x \text{ in numerator.}\\
&= \int \left(\frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x}\right) dx && \text{Rewrite as two fractions.}\\
&= \int dx - \int \frac{e^x\,dx}{1+e^x} && \text{Rewrite as two integrals.}\\
&= x - \ln(1+e^x) + C && \text{Integrate.}
\end{aligned}
$$

NOTE There is usually more than one way to solve an integration problem. For instance, Example 4, try integrating by multiplying the numerator and denominator by e^{-x} to obtain integral of the form $-\int du/u$. See if you can get the same answer by this procedure. (B careful: the answer will appear in a different form.)

Example 5 A Disguised Form of the Power Rule

Evaluate $\int (\cot x)[\ln(\sin x)]\,dx$.

Solution Again, the integral does not appear to fit any of the basic rules. Howeve considering the two primary choices for u ($u = \cot x$ and $u = \ln \sin x$), you can se that the second choice is the appropriate one because

$$u = \ln \sin x \quad \text{and} \quad du = \frac{\cos x}{\sin x}\,dx = \cot x\,dx.$$

So, you have

$$
\begin{aligned}
\int (\cot x)[\ln(\sin x)]\,dx &= \int u\,du && \text{Substitution: } u = \ln \sin x\\
&= \frac{u^2}{2} + C && \text{Integrate.}\\
&= \frac{1}{2}[\ln(\sin x)]^2 + C. && \text{Rewrite as a function of } x.
\end{aligned}
$$

NOTE In Example 5, try *checking* that the derivative of

$$\frac{1}{2}[\ln(\sin x)]^2 + C$$

is the integrand of the original integral.

Review of Basic Integration Rules ($a > 0$)

1. $\displaystyle\int kf(u)\,du = k\int f(u)\,du$
2. $\displaystyle\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\displaystyle\int du = u + C$
4. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$
5. $\displaystyle\int \frac{du}{u} = \ln|u| + C$
6. $\displaystyle\int e^u\,du = e^u + C$
7. $\displaystyle\int a^u\,du = \left(\frac{1}{\ln a}\right)a^u + C$
8. $\displaystyle\int \sin u\,du = -\cos u + C$
9. $\displaystyle\int \cos u\,du = \sin u + C$
10. $\displaystyle\int \tan u\,du = -\ln|\cos u| + C$
11. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$
12. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\displaystyle\int \csc u\,du = -\ln|\csc u + \cot u| + C$
14. $\displaystyle\int \sec^2 u\,du = \tan u + C$
15. $\displaystyle\int \csc^2 u\,du = -\cot u + C$
16. $\displaystyle\int \sec u \tan u\,du = \sec u + C$
17. $\displaystyle\int \csc u \cot u\,du = -\csc u + C$
18. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$
19. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$
20. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

Trigonometric identities can often be used to fit integrals to one of the basic integration rules.

Example 6 Using Trigonometric Identities

Evaluate $\int \tan^2 2x \, dx$.

Solution Note that $\tan^2 u$ is not in the list of basic integration rules. However, $\sec^2 u$ is in the list. This suggests the trigonometric identity $\tan^2 u = \sec^2 u - 1$. If you let $u = 2x$, then $du = 2\,dx$ and

$$\begin{aligned}
\int \tan^2 2x \, dx &= \frac{1}{2}\int \tan^2 u \, du && \text{Substitution: } u = 2x \\
&= \frac{1}{2}\int (\sec^2 u - 1) \, du && \text{Trigonometric identity} \\
&= \frac{1}{2}\int \sec^2 u \, du - \frac{1}{2}\int du && \text{Rewrite as two integrals.} \\
&= \frac{1}{2}\tan u - \frac{u}{2} + C && \text{Integrate.} \\
&= \frac{1}{2}\tan 2x - x + C. && \text{Rewrite as a function of } x.
\end{aligned}$$

TECHNOLOGY If you have access to a computer algebra system, try using it to evaluate the integrals in this section. Compare the *form* of the antiderivative given by the software with the form obtained by hand. Sometimes the forms will be the same, but often they will differ. For instance, why is the antiderivative $\ln 2x + C$ equivalent to the antiderivative $\ln x + C$?

We conclude this section with a summary of the common procedures for fitting integrands to the basic integration rules.

Procedures for Fitting Integrands to Basic Rules

Technique	*Example*
Expand (numerator).	$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$
Separate numerator.	$\dfrac{1 + x}{x^2 + 1} = \dfrac{1}{x^2 + 1} + \dfrac{x}{x^2 + 1}$
Complete the square.	$\dfrac{1}{\sqrt{2x - x^2}} = \dfrac{1}{\sqrt{1 - (x - 1)^2}}$
Divide improper rational function.	$\dfrac{x^2}{x^2 + 1} = 1 - \dfrac{1}{x^2 + 1}$
Add and subtract terms in numerator.	$\dfrac{2x}{x^2 + 2x + 1} = \dfrac{2x + 2 - 2}{x^2 + 2x + 1} = \dfrac{2x + 2}{x^2 + 2x + 1} - \dfrac{2}{(x + 1)^2}$
Use trigonometric identities.	$\cot^2 x = \csc^2 x - 1$
Multiply and divide by Pythagorean conjugate.	$\dfrac{1}{1 + \sin x} = \left(\dfrac{1}{1 + \sin x}\right)\left(\dfrac{1 - \sin x}{1 - \sin x}\right) = \dfrac{1 - \sin x}{1 - \sin^2 x}$ $= \dfrac{1 - \sin x}{\cos^2 x} = \sec^2 x - \dfrac{\sin x}{\cos^2 x}$

NOTE Remember that you can separate numerators but not denominators. Watch out for this common error when fitting integrands to basic rules.

$$\frac{1}{x^2 + 1} \neq \frac{1}{x^2} + \frac{1}{1} \qquad \text{Do not separate denominators.}$$

EXERCISES FOR SECTION 7.1

In Exercises 1–4, select the correct antiderivative.

1. $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2+1}}$

(a) $2\sqrt{x^2+1} + C$ (b) $\sqrt{x^2+1} + C$

(c) $\frac{1}{2}\sqrt{x^2+1} + C$ (d) $\ln(x^2+1) + C$

2. $\dfrac{dy}{dx} = \dfrac{x}{x^2+1}$

(a) $\ln\sqrt{x^2+1} + C$ (b) $\dfrac{2x}{(x^2+1)^2} + C$

(c) $\arctan x + C$ (d) $\ln(x^2+1) + C$

3. $\dfrac{dy}{dx} = \dfrac{1}{x^2+1}$

(a) $\ln\sqrt{x^2+1} + C$ (b) $\dfrac{2x}{(x^2+1)^2} + C$

(c) $\arctan x + C$ (d) $\ln(x^2+1) + C$

4. $\dfrac{dy}{dx} = x\cos(x^2+1)$

(a) $2x\sin(x^2+1) + C$ (b) $-\frac{1}{2}\sin(x^2+1) + C$

(c) $\frac{1}{2}\sin(x^2+1) + C$ (d) $-2x\sin(x^2+1) + C$

In Exercises 5–14, select the basic integration formula you can use to evaluate the integral, and identify *u* and *a* when appropriate.

5. $\displaystyle\int (3x-2)^4\,dx$

6. $\displaystyle\int \frac{2t-1}{t^2-t+2}\,dt$

7. $\displaystyle\int \frac{1}{\sqrt{x}(1-2\sqrt{x})}\,dx$

8. $\displaystyle\int \frac{2}{(2t-1)^2+4}\,dt$

9. $\displaystyle\int \frac{3}{\sqrt{1-t^2}}\,dt$

10. $\displaystyle\int \frac{-2x}{\sqrt{x^2-4}}\,dx$

11. $\displaystyle\int t\sin t^2\,dt$

12. $\displaystyle\int \sec 3x\tan 3x\,dx$

13. $\displaystyle\int \cos x e^{\sin x}\,dx$

14. $\displaystyle\int \frac{1}{x\sqrt{x^2-4}}\,dx$

In Exercises 15–54, evaluate the indefinite integral.

15. $\displaystyle\int (-2x+5)^{3/2}\,dx$

16. $\displaystyle\int 6(x-4)^5\,dx$

17. $\displaystyle\int \frac{5}{(z-4)^5}\,dz$

18. $\displaystyle\int \frac{2}{(t-9)^2}\,dt$

19. $\displaystyle\int t^2\sqrt[3]{t^3-1}\,dt$

20. $\displaystyle\int x\sqrt{4-2x^2}\,dx$

21. $\displaystyle\int \left[v + \frac{1}{(3v-1)^3}\right]dv$

22. $\displaystyle\int \left[x - \frac{3}{(2x+3)^2}\right]dx$

23. $\displaystyle\int \frac{t^2-3}{-t^3+9t+1}\,dt$

24. $\displaystyle\int \frac{x+1}{\sqrt{x^2+2x-4}}\,dx$

25. $\displaystyle\int \frac{x^2}{x-1}\,dx$

26. $\displaystyle\int \frac{2x}{x-4}\,dx$

27. $\displaystyle\int \frac{e^x}{1+e^x}\,dx$

28. $\displaystyle\int \left(\frac{1}{3x-1} - \frac{1}{3x+1}\right)dx$

29. $\displaystyle\int (1+2x^2)^2\,dx$

30. $\displaystyle\int x\left(1+\frac{1}{x}\right)^3\,dx$

31. $\displaystyle\int x\cos 2\pi x^2\,dx$

32. $\displaystyle\int \sec 4x\,dx$

33. $\displaystyle\int \csc \pi x\cot \pi x\,dx$

34. $\displaystyle\int \frac{\sin x}{\sqrt{\cos x}}\,dx$

35. $\displaystyle\int e^{5x}\,dx$

36. $\displaystyle\int \csc^2 x e^{\cot x}\,dx$

37. $\displaystyle\int \frac{2}{e^{-x}+1}\,dx$

38. $\displaystyle\int \frac{5}{3e^x-2}\,dx$

39. $\displaystyle\int \frac{\ln x^2}{x}\,dx$

40. $\displaystyle\int (\tan x)[\ln(\cos x)]\,dx$

41. $\displaystyle\int \frac{1+\sin x}{\cos x}\,dx$

42. $\displaystyle\int \frac{1+\cos\alpha}{\sin\alpha}\,d\alpha$

43. $\displaystyle\int \frac{1}{\cos\theta - 1}\,d\theta$

44. $\displaystyle\int \frac{2}{3(\sec x - 1)}\,dx$

45. $\displaystyle\int \frac{3z+2}{z^2+9}\,dz$

46. $\displaystyle\int \frac{3}{t^2+1}\,dt$

47. $\displaystyle\int \frac{-1}{\sqrt{1-(2t-1)^2}}\,dt$

48. $\displaystyle\int \frac{1}{4+3x^2}\,dx$

49. $\displaystyle\int \frac{\tan(2/t)}{t^2}\,dt$

50. $\displaystyle\int \frac{e^{1/t}}{t^2}\,dt$

51. $\displaystyle\int \frac{3}{\sqrt{6x-x^2}}\,dx$

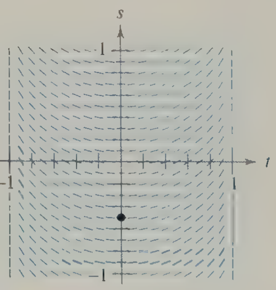
52. $\displaystyle\int \frac{1}{(x-1)\sqrt{4x^2-8x+3}}$

53. $\displaystyle\int \frac{4}{4x^2+4x+65}\,dx$

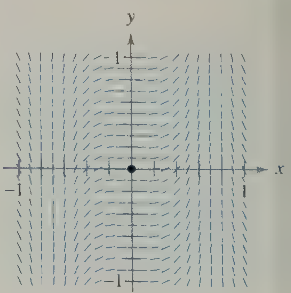
54. $\displaystyle\int \frac{1}{\sqrt{1-4x-x^2}}\,dx$

Slope Fields **In Exercises 55 and 56, a differential equation, point, and a slope field are given. (a) Sketch two approxima solutions of the differential equation on the slope field, one which passes through the indicated point. (b) Use integration find the particular solution of the differential equation and use graphing utility to graph the solution. Compare the result wi the sketches in part (a). To print an enlarged copy of the grap go the the website *www.mathgraphs.com*.**

55. $\dfrac{ds}{dt} = \dfrac{t}{\sqrt{1-t^4}}, \quad \left(0, -\frac{1}{2}\right)$



56. $\dfrac{dy}{dx} = \tan^2(2x), \quad (0, 0)$



In Exercises 57 and 58, use a computer algebra system to sket the slope field for the differential equation and graph the soluti through the specified initial condition.

57. $\dfrac{dy}{dx} = 0.2y, \quad y(0) = 3$

58. $\dfrac{dy}{dx} = 5 - y, \quad y(0) = 1$

Exercises 59–62, solve the differential equation.

$\frac{dy}{dx} = (1 + e^x)^2$

60. $\frac{dr}{dt} = \frac{(1 + e^t)^2}{e^t}$

$(4 + \tan^2 x)y' = \sec^2 x$

62. $y' = \frac{1}{x\sqrt{4x^2 - 1}}$

Exercises 63–70, evaluate the definite integral. Use the inte- ation capabilities of a graphing utility to verify your result.

$\int_0^{\pi/4} \cos 2x\, dx$

64. $\int_0^{\pi} \sin^2 t \cos t\, dt$

$\int_0^1 xe^{-x^2}\, dx$

66. $\int_1^e \frac{1 - \ln x}{x}\, dx$

$\int_0^4 \frac{2x}{\sqrt{x^2 + 9}}\, dx$

68. $\int_1^2 \frac{x - 2}{x}\, dx$

$\int_0^{2/\sqrt{3}} \frac{1}{4 + 9x^2}\, dx$

70. $\int_0^4 \frac{1}{\sqrt{25 - x^2}}\, dx$

Exercises 71–74, use a computer algebra system to evaluate e integral. Use the computer algebra system to graph two tiderivatives. Describe the relationship between the two aphs of the antiderivatives.

$\int \frac{1}{x^2 + 4x + 13}\, dx$

72. $\int \frac{x - 2}{x^2 + 4x + 13}\, dx$

$\int \frac{1}{1 + \sin \theta}\, d\theta$

74. $\int \left(\frac{e^x + e^{-x}}{2}\right)^3 dx$

etting at the Concept

n Exercises 75–78, state the integration formula you would se to perform the integration. Do not integrate.

5. $\int x(x^2 + 1)^3\, dx$

76. $\int x \sec(x^2 + 1) \tan(x^2 + 1)\, dx$

7. $\int \frac{x}{x^2 + 1}\, dx$

78. $\int \frac{1}{x^2 + 1}\, dx$

9. Explain why the antiderivative $y_1 = e^{x+C_1}$ is equivalent to the antiderivative $y_2 = Ce^x$.

0. Explain why the antiderivative $y_1 = \sec^2 x + C_1$ is equivalent to the antiderivative $y_2 = \tan^2 x + C$.

Determine the constants a and b such that

$\sin x + \cos x = a \sin(x + b)$.

Use this result to integrate $\int \frac{dx}{\sin x + \cos x}$.

Think About It Use a graphing utility to graph the function $f(x) = \frac{1}{5}(x^3 - 7x^2 + 10x)$. Use the graph to determine whether

$\int_0^5 f(x)\, dx$

is positive or negative. Explain.

Approximation **In Exercises 83 and 84, determine which value best approximates the area of the region between the x-axis and the function over the given interval. (Make your selection on the basis of a sketch of the region and *not* by integrating.)**

83. $f(x) = \frac{4x}{x^2 + 1}$, $[0, 2]$

(a) 3 (b) 1 (c) −8 (d) 8 (e) 10

84. $f(x) = \frac{4}{x^2 + 1}$, $[0, 2]$

(a) 3 (b) 1 (c) −4 (d) 4 (e) 10

Area **In Exercises 85 and 86, find the area of the region bounded by the graph(s) of the equation(s).**

85. $y^2 = x^2(1 - x^2)$

86. $y = \sin 2x, \quad y = 0, \quad x = 0, \quad x = \pi/2$

87. ***Area*** The graphs of $f(x) = x$ and $g(x) = ax^2$ intersect at the points $(0, 0)$ and $(1/a, 1/a)$. Find a $(a > 0)$ such that the area of the region bounded by the graphs of these two functions is $\frac{2}{3}$.

88. ***Interpreting an Integral*** You are given the integral

$$\int_0^2 2\pi x^2\, dx$$

but are not told what it represents. (There is more than one correct answer for each part.)

(a) Sketch the region whose area is given by the integral.

(b) Sketch the solid whose volume is given by the integral if the disk method is used.

(c) Sketch the solid whose volume is given by the integral if the shell method is used.

89. ***Volume*** The region bounded by $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = b$ $(b > 0)$ is revolved about the y-axis.

(a) Find the volume of the solid generated if $b = 1$.

(b) Find b such that the volume of the generated solid is $\frac{4}{3}$ cubic units.

90. ***Average Value*** Compute the average value of each of the functions over the indicated interval.

(a) $f(x) = \sin nx$, $0 \le x \le \pi/n$, n is a positive integer

(b) $f(x) = \frac{1}{1 + x^2}$, $-3 \le x \le 3$

91. ***Centroid*** Find the x-coordinate of the centroid of the region bounded by the graphs of

$$y = \frac{5}{\sqrt{25 - x^2}}, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 4.$$

92. ***Surface Area*** Find the area of the surface formed by revolving the graph of $y = 2\sqrt{x}$ on the interval $[0, 9]$ about the x-axis.

Arc Length **In Exercises 93 and 94, use the integration capabilities of a graphing utility to approximate the arc length of the curve over the indicated interval.**

93. $y = \tan \pi x$, $\left[0, \frac{1}{4}\right]$

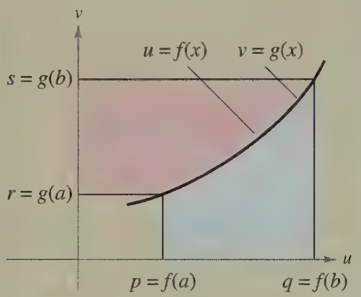
94. $y = x^{2/3}$, $[1, 8]$

Section 7.2 Integration by Parts

- Find an antiderivative using integration by parts.
- Use a tabular method to perform integration by parts.

EXPLORATION

Proof Without Words Here is a different approach to proving the formula for integration by parts. Exercise taken from "Proof Without Words: Integration by Parts" by Roger B. Nelsen, *Mathematics Magazine*, April 1991. Used by permission of the author.



Area + Area = $qs - pr$

$$\int_r^s u\,dv + \int_q^p v\,du = \Big[uv\Big]_{(p,r)}^{(q,s)}$$

$$\int_r^s u\,dv = \Big[uv\Big]_{(p,r)}^{(q,s)} - \int_q^p v\,du$$

Explain how this graph proves the theorem. Which notation in this proof is unfamiliar? What do you think it means?

Integration by Parts

In this section you will study an important integration technique called **integration** **parts.** This technique can be applied to a wide variety of functions and is particular useful for integrands involving *products* of algebraic and transcendental functions. F instance, integration by parts works well with integrals such as

$$\int x \ln x\,dx, \quad \int x^2 e^x\,dx, \quad \text{and} \quad \int e^x \sin x\,dx.$$

Integration by parts is based on the formula for the derivative of a product

$$\begin{aligned}\frac{d}{dx}[uv] &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= uv' + vu'\end{aligned}$$

where both u and v are differentiable functions of x. If u' and v' are continuous, y can integrate both sides of this equation to obtain

$$\begin{aligned}uv &= \int uv'\,dx + \int vu'\,dx\\ &= \int u\,dv + \int v\,du.\end{aligned}$$

By rewriting this equation, you obtain the following theorem.

THEOREM 7.1 Integration by Parts

If u and v are functions of x and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du.$$

This formula expresses the original integral in terms of another integral. Depen ing on the choices of u and dv, it may be easier to evaluate the second integral th the original one. Because the choices of u and dv are critical in the integration by pa process, the following guidelines are provided.

Guidelines for Integration by Parts

1. Try letting dv be the most complicated portion of the integrand that fits a basic integration rule. Then u will be the remaining factor(s) of the integrand.
2. Try letting u be the portion of the integrand whose derivative is a function simpler than u. Then dv will be the remaining factor(s) of the integrand.

Example 1 Integration by Parts

Evaluate $\int xe^x\,dx$.

Solution To apply integration by parts, you need to write the integral in the form $\int u\,dv$. There are several ways to do this.

$$\int \underbrace{(x)}_{u}\underbrace{(e^x dx)}_{dv}, \quad \int \underbrace{(e^x)}_{u}\underbrace{(x\,dx)}_{dv}, \quad \int \underbrace{(1)}_{u}\underbrace{(xe^x\,dx)}_{dv}, \quad \int \underbrace{(xe^x)}_{u}\underbrace{(dx)}_{dv}$$

The guidelines on page 488 suggest choosing the first option because the derivative of $u = x$ is simpler than x, and $dv = e^x\,dx$ is the most complicated portion of the integrand that fits a basic integration formula.

$$dv = e^x\,dx \quad \Longrightarrow \quad v = \int dv = \int e^x\,dx = e^x$$

$$u = x \quad \Longrightarrow \quad du = dx$$

Now, integration by parts produces the following.

$$\int u\,dv = uv - \int v\,du \qquad \text{Integration by parts formula}$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx \qquad \text{Substitute.}$$

$$= xe^x - e^x + C \qquad \text{Integrate.}$$

To check this, differentiate $xe^x - e^x + C$ to see that you obtain the original integrand.

TE In Example 1, note that it is necessary to include a constant of gration when solving

$$v = \int e^x\,dx = e^x + C_1.$$

llustrate this, replace $v = e^x$ by $e^x + C_1$ and apply integration by s to see that you obtain the same lt.

Example 2 Integration by Parts

Evaluate $\int x^2 \ln x\,dx$.

Solution In this case, x^2 is more easily integrated than $\ln x$. Furthermore, the derivative of $\ln x$ is simpler than $\ln x$. Therefore, you should let $dv = x^2\,dx$.

$$dv = x^2\,dx \quad \Longrightarrow \quad v = \int x^2\,dx = \frac{x^3}{3}$$

$$u = \ln x \quad \Longrightarrow \quad du = \frac{1}{x}\,dx$$

Integration by parts produces the following.

$$\int u\,dv = uv - \int v\,du \qquad \text{Integration by parts formula}$$

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int \left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right)dx \qquad \text{Substitute.}$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx \qquad \text{Simplify.}$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C \qquad \text{Integrate.}$$

You can check this result by differentiating.

$$\frac{d}{dx}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right] = \frac{x^3}{3}\left(\frac{1}{x}\right) + (\ln x)(x^2) - \frac{x^2}{3} = x^2 \ln x$$

R FURTHER INFORMATION To see integration by parts is used to prove ling's approximation

$$\ln(n!) = n \ln n - n,$$

the article "The Validity of Stirling's roximation: A Physical Chemistry ject" by A. S. Wallner and K. A. ndt in *Journal of Chemical cation*. To view this article, go to the site *www.matharticles.com*.

ECHNOLOGY Try graphing

$$\int x^2 \ln x\,dx \quad \text{and} \quad \frac{x^3}{3}\ln x - \frac{x^3}{9}$$

n your graphing utility. Do you get e same graph? (This will take a hile, so be patient.)

One surprising application of integration by parts involves integrands consist of a single factor, such as $\int \ln x\,dx$ or $\int \arcsin x\,dx$. In such cases, you should $dv = dx$, as illustrated in the next example.

Example 3 An Integrand with a Single Term

Evaluate $\displaystyle\int_0^1 \arcsin x\,dx.$

Solution Let $dv = dx$.

$$dv = dx \quad \Rightarrow \quad v = \int dx = x$$

$$u = \arcsin x \quad \Rightarrow \quad du = \frac{1}{\sqrt{1 - x^2}}\,dx$$

Integration by parts now produces the following.

$$\int u\,dv = uv - \int v\,du \qquad \text{Integration by parts formula}$$

$$\int \arcsin x\,dx = x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}}\,dx \qquad \text{Substitute.}$$

$$= x \arcsin x + \frac{1}{2}\int (1 - x^2)^{-1/2}(-2x)\,dx \qquad \text{Rewrite.}$$

$$= x \arcsin x + \sqrt{1 - x^2} + C \qquad \text{Integrate.}$$

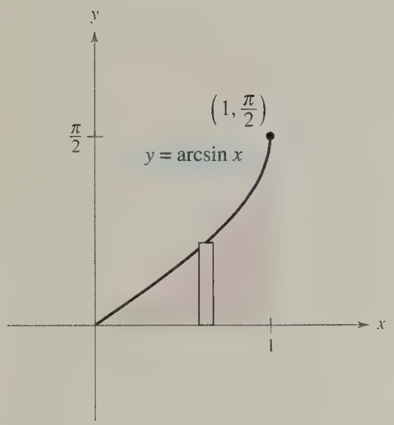
Using this antiderivative, you can evaluate the definite integral as follows.

$$\int_0^1 \arcsin x\,dx = \Big[x \arcsin x + \sqrt{1 - x^2}\Big]_0^1$$

$$= \frac{\pi}{2} - 1$$

$$\approx 0.571$$

The area represented by this definite integral is shown in Figure 7.2.



The area of the region is approximately 0.571.

Figure 7.2

TECHNOLOGY Remember that there are two ways to use technology to evaluat definite integral: (1) you can use a numerical approximation such as the Trapezoi Rule or Simpson's Rule, or (2) you can use a computer algebra system to find antiderivative and then apply the Fundamental Theorem of Calculus. Both meth have shortcomings. To find the possible error when using a numerical method, integrand must have a second derivative (Trapezoidal Rule) or a fourth derivat (Simpson's Rule) in the interval of integration: the integrand in Example 3 fail meet this requirement. To apply the Fundamental Theorem of Calculus, symbolic integration utility must be able to find the antiderivative.

Which method would you use to evaluate

$$\int_0^1 \arctan x\,dx?$$

Which method would you use to evaluate

$$\int_0^1 \arctan x^2\,dx?$$

Some integrals require repeated use of the integration by parts formula.

Example 4 Repeated Use of Integration by Parts

Evaluate $\int x^2 \sin x\,dx$.

Solution The factors x^2 and $\sin x$ are equally easy to integrate. However, the derivative of x^2 becomes simpler, whereas the derivative of $\sin x$ does not. Therefore, you should let $u = x^2$.

$$dv = \sin x\,dx \quad \Rightarrow \quad v = \int \sin x\,dx = -\cos x$$

$$u = x^2 \quad \Rightarrow \quad du = 2x\,dx$$

Now, integration by parts produces the following.

$$\int x^2 \sin x\,dx = -x^2 \cos x + \int 2x \cos x\,dx \qquad \text{First use of integration by parts}$$

This first use of integration by parts has succeeded in simplifying the original integral, but the integral on the right still doesn't fit a basic integration rule. To evaluate that integral, you can apply integration by parts again. This time, let $u = 2x$.

$$dv = \cos x\,dx \quad \Rightarrow \quad v = \int \cos x\,dx = \sin x$$

$$u = 2x \quad \Rightarrow \quad du = 2\,dx$$

Now, integration by parts produces

$$\begin{aligned}\int 2x \cos x\,dx &= 2x \sin x - \int 2 \sin x\,dx \qquad \text{Second use of integration by parts}\\ &= 2x \sin x + 2 \cos x + C.\end{aligned}$$

Combining these two results, you can write

$$\int x^2 \sin x\,dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

When making repeated applications of integration by parts, you need to be careful not to interchange the substitutions in successive applications. For instance, in Example 4, the first substitution was $u = x^2$ and $dv = \sin x\,dx$. If, in the second application, you had switched the substitution to $u = \cos x$ and $dv = 2x$, you would have obtained

$$\begin{aligned}\int x^2 \sin x\,dx &= -x^2 \cos x + \int 2x \cos x\,dx\\ &= -x^2 \cos x + x^2 \cos x + \int x^2 \sin x\,dx\\ &= \int x^2 \sin x\,dx\end{aligned}$$

thus undoing the previous integration and returning to the *original* integral. When making repeated applications of integration by parts, you should also watch for the appearance of a *constant multiple* of the original integral. For instance, this occurs when you use integration by parts to evaluate $\int e^x \cos 2x\,dx$, and also occurs in the next example.

EXPLORATION

ry to evaluate

$$\int e^x \cos 2x\,dx$$

letting $u = \cos 2x$ and $dv = e^x\,dx$ the first substitution. For the cond substitution, let $u = \sin 2x$ d $dv = e^x\,dx$.

NOTE The integral in Example 5 is an important one. In Section 7.4 (Example 5), you will see that it is used to find the arc length of a parabolic segment.

Example 5 Integration by Parts

Evaluate $\int \sec^3 x\, dx$.

Solution The most complicated portion of the integrand that can be easily integrat is $\sec^2 x$, so you should let $dv = \sec^2 x\, dx$ and $u = \sec x$.

$$dv = \sec^2 x\, dx \quad \Rightarrow \quad v = \int \sec^2 x\, dx = \tan x$$

$$u = \sec x \quad \Rightarrow \quad du = \sec x \tan x\, dx$$

Integration by parts produces the following.

$$\int u\, dv = uv - \int v\, du$$ Integration by parts formula

$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec x \tan^2 x\, dx$$ Substitute.

$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)\, dx$$ Trigonometric identit

$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$ Rewrite.

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$ Collect like integrals.

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$ Integrate and divide by 2.

STUDY TIP The trigonometric identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

play an important role in this chapter.

Example 6 Finding a Centroid

A machine part is modeled by the region bounded by the graph of $y = \sin x$ and t x-axis, $0 \le x \le \pi/2$, as shown in Figure 7.3. Find the centroid of this region.

Solution Begin by finding the area of the region.

$$A = \int_0^{\pi/2} \sin x\, dx = \Big[-\cos x\Big]_0^{\pi/2} = 1$$

Now, you can find the coordinates of the centroid as follows.

$$\bar{y} = \frac{1}{A}\int_0^{\pi/2} \frac{\sin x}{2}(\sin x)\, dx = \frac{1}{4}\int_0^{\pi/2} (1 - \cos 2x)\, dx = \frac{1}{4}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} =$$

You can evaluate the integral for $\bar{x}$, $(1/A)\int_0^{\pi/2} x \sin x\, dx$, with integration by parts. do this, let $dv = \sin x\, dx$ and $u = x$. This produces $v = -\cos x$ and $du = dx$, and y can write

$$\int x \sin x\, dx = -x\cos x + \int \cos x\, dx$$
$$= -x\cos x + \sin x + C.$$

Finally, you can determine $\bar{x}$ to be

$$\bar{x} = \frac{1}{A}\int_0^{\pi/2} x \sin x\, dx = \Big[-x\cos x + \sin x\Big]_0^{\pi/2} = 1.$$

Therefore, the centroid of the region is $(1, \pi/8)$.

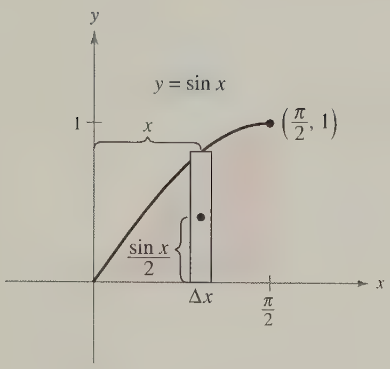


Figure 7.3

As you gain experience in using integration by parts, your skill in determining u and dv will increase. The following summary lists several common integrals with suggestions for the choices of u and dv.

TUDY TIP You can use the acronym ATE as a guideline for choosing u in egration by parts. In order, check the egrand for the following.

there a **L**ogarithmic part?
there an **I**nverse trigonometric part?
there an **A**lgebraic part?
there a **T**rigonometric part?
there an **E**xponential part?

Summary of Common Integrals Using Integration by Parts

1. For integrals of the form

$$\int x^n e^{ax}\,dx, \qquad \int x^n \sin ax\,dx, \qquad \text{or} \qquad \int x^n \cos ax\,dx$$

let $u = x^n$ and let $dv = e^{ax}\,dx$, $\sin ax\,dx$, or $\cos ax\,dx$.

2. For integrals of the form

$$\int x^n \ln x\,dx, \qquad \int x^n \arcsin ax\,dx, \qquad \text{or} \qquad \int x^n \arctan ax\,dx$$

let $u = \ln x$, $\arcsin ax$, or $\arctan ax$ and let $dv = x^n\,dx$.

3. For integrals of the form

$$\int e^{ax} \sin bx\,dx \qquad \text{or} \qquad \int e^{ax} \cos bx\,dx$$

let $u = \sin bx$ or $\cos bx$ and let $dv = e^{ax}\,dx$.

Tabular Method

In problems involving repeated applications of integration by parts, a tabular method, illustrated in Example 7, can help to organize the work. This method works well for integrals of the form $\int x^n \sin ax\,dx$, $\int x^n \cos ax\,dx$, and $\int x^n e^{ax}\,dx$.

Example 7 Using the Tabular Method

Evaluate $\int x^2 \sin 4x\,dx$.

Solution Begin as usual by letting $u = x^2$ and $dv = v'\,dx = \sin 4x\,dx$. Next, create a table consisting of three columns, as follows.

Alternate Signs	*u and Its Derivatives*	*v′ and Its Antiderivatives*
$+$	x^2	$\sin 4x$
$-$	$2x$	$-\frac{1}{4}\cos 4x$
$+$	2	$-\frac{1}{16}\sin 4x$
$-$	0	$\frac{1}{64}\cos 4x$

Differentiate until you obtain 0 as a derivative.

The solution is obtained by adding the signed products of the diagonal entries:

$$\int x^2 \sin 4x\,dx = -\frac{1}{4}x^2 \cos 4x + \frac{1}{8}x \sin 4x + \frac{1}{32}\cos 4x + C.$$

R FURTHER INFORMATION For re information on the tabular thod, see the article "Tabular egration by Parts" by David Horowitz *The College Mathematics Journal*, l the article "More on Tabular egration by Parts" by Leonard lman in *The College Mathematics rnal*. To view these articles, go to the bsite *www.matharticles.com*.

EXERCISES FOR SECTION 7.2

In Exercises 1–4, match the antiderivative with the correct integral. [Integrals are labeled (a), (b), (c), and (d).]

(a) $\int \ln x\,dx$ **(b)** $\int x \sin x\,dx$

(c) $\int x^2 e^x\,dx$ **(d)** $\int x^2 \cos x\,dx$

1. $y = \sin x - x \cos x$

2. $y = x^2 \sin x + 2x \cos x - 2 \sin x$

3. $y = x^2 e^x - 2xe^x + 2e^x$

4. $y = -x + x \ln x$

In Exercises 5–10, identify *u* and *dv* for evaluating the integral using integration by parts. (Do not evaluate the integral.)

5. $\int xe^{2x}\,dx$ **6.** $\int x^2 e^{2x}\,dx$

7. $\int (\ln x)^2\,dx$ **8.** $\int \ln 3x\,dx$

9. $\int x \sec^2 x\,dx$ **10.** $\int x^2 \cos x\,dx$

In Exercises 11–36, evaluate the integral. (*Note:* Solve by the simplest method—not all require integration by parts.)

11. $\int xe^{-2x}\,dx$ **12.** $\int \frac{2x}{e^x}\,dx$

13. $\int x^3 e^x\,dx$ **14.** $\int \frac{e^{1/t}}{t^2}\,dt$

15. $\int x^2 e^{x^3}\,dx$ **16.** $\int x^4 \ln x\,dx$

17. $\int t \ln(t+1)\,dt$ **18.** $\int \frac{1}{x(\ln x)^3}\,dx$

19. $\int \frac{(\ln x)^2}{x}\,dx$ **20.** $\int \frac{\ln x}{x^2}\,dx$

21. $\int \frac{xe^{2x}}{(2x+1)^2}\,dx$ **22.** $\int \frac{x^3 e^{x^2}}{(x^2+1)^2}\,dx$

23. $\int (x^2 - 1)e^x\,dx$ **24.** $\int \frac{\ln 2x}{x^2}\,dx$

25. $\int x\sqrt{x-1}\,dx$ **26.** $\int \frac{x}{\sqrt{2+3x}}\,dx$

27. $\int x \cos x\,dx$ **28.** $\int x \sin x\,dx$

29. $\int x^3 \sin x\,dx$ **30.** $\int x^2 \cos x\,dx$

31. $\int t \csc t \cot t\,dt$ **32.** $\int \theta \sec \theta \tan \theta\,d\theta$

33. $\int \arctan x\,dx$ **34.** $\int 4 \arccos x\,dx$

35. $\int e^{2x} \sin x\,dx$ **36.** $\int e^x \cos 2x\,dx$

In Exercises 37–42, solve the differential equation.

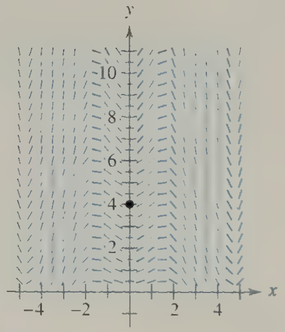
37. $y' = xe^{x^2}$ **38.** $y' = \ln x$

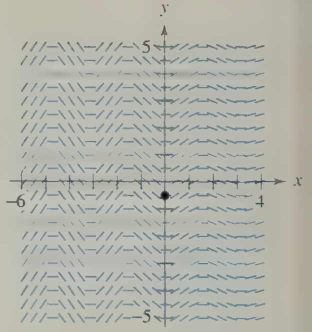
39. $\frac{dy}{dt} = \frac{t^2}{\sqrt{2+3t}}$ **40.** $\frac{dy}{dx} = x^2\sqrt{x-1}$

41. $(\cos y)y' = 2x$ **42.** $y' = \arctan \frac{x}{2}$

Slope Fields **In Exercises 43 and 44, a differential equation, point, and a slope field are given. (a) Sketch two approximat solutions of the differential equation on the slope field, one which passes through the indicated point. (b) Use integration find the particular solution of the differential equation and us a graphing utility to graph the solution. Compare the resu with the sketches in part (a). To print an enlarged copy of th graph, go the the website *www.mathgraphs.com*.**

43. $\frac{dy}{dx} = x\sqrt{y}\cos x,\ (0, 4)$ **44.** $\frac{dy}{dx} = e^{-x/3} \sin 2x,\ \left(0, -\frac{18}{37}\right.$





In Exercises 45 and 46, use a computer algebra system to sketc the slope field for the differential equation and graph the sol tion through the specified initial condition.

45. $\frac{dy}{dx} = \frac{x}{y}e^{x/8}$ **46.** $\frac{dy}{dx} = \frac{x}{y}\sin x$

$y(0) = 2$ $y(0) = 4$

In Exercises 47–58, evaluate the definite integral. Use a grap ing utility to confirm your result.

47. $\int_0^4 xe^{-x/2}\,dx$ **48.** $\int_0^1 x^2 e^x\,dx$

49. $\int_0^{\pi/2} x \cos x\,dx$ **50.** $\int_0^{\pi} x \sin 2x\,dx$

51. $\int_0^{1/2} \arccos x\,dx$ **52.** $\int_0^1 x \arcsin x^2\,dx$

53. $\int_0^1 e^x \sin x\,dx$ **54.** $\int_0^2 e^{-x} \cos x\,dx$

55. $\int_1^2 x^2 \ln x\,dx$ **56.** $\int_0^1 \ln(1+x^2)\,dx$

57. $\int_2^4 x\,\text{arcsec}\,x\,dx$ **58.** $\int_0^{\pi/4} x \sec^2 x\,dx$

Exercises 59–64, use the tabular method to evaluate the tegral.

. $\int x^2e^{2x}\,dx$ 60. $\int x^3e^{-2x}\,dx$

. $\int x^3 \sin x\,dx$ 62. $\int x^3 \cos 2x\,dx$

. $\int x \sec^2 x\,dx$ 64. $\int x^2(x-2)^{3/2}\,dx$

Getting at the Concept

65. Integration by parts is based on what differentiation rule?

66. In your own words, state guidelines for integration by parts.

In Exercises 67–72, state whether you would use integration by parts to evaluate the integral. If so, identify what you would use for *u* and *dv*.

67. $\int \frac{\ln x}{x}\,dx$ 68. $\int x \ln x\,dx$

69. $\int x^2e^{2x}\,dx$ 70. $\int 2x\,e^{x^2}\,dx$

71. $\int \frac{x}{\sqrt{x+1}}\,dx$ 72. $\int \frac{x}{\sqrt{x^2+1}}\,dx$

Exercises 73–76, use a computer algebra system to evaluate e integral.

. $\int t^3\,e^{-4t}\,dt$ 74. $\int \alpha^4 \sin \pi\alpha\,d\alpha$

. $\int_0^{\pi/2} e^{-2x} \sin 3x\,dx$ 76. $\int_0^5 x^4\,(25-x^2)^{3/2}\,dx$

. Integrate $\int 2x\sqrt{2x-3}\,dx$

(a) by parts, letting $dv = \sqrt{2x-3}\,dx$.

(b) by substitution, letting $u = 2x-3$.

. Integrate $\int x\sqrt{4+x}\,dx$

(a) by parts, letting $dv = \sqrt{4+x}\,dx$.

(b) by substitution, letting $u = 4+x$.

. Integrate $\int \frac{x^3}{\sqrt{4+x^2}}\,dx$

(a) by parts, letting $dv = \left(x/\sqrt{4+x^2}\right)dx$.

(b) by substitution, letting $u = 4+x^2$.

. Integrate $\int x\sqrt{4-x}\,dx$

(a) by parts, letting $dv = \sqrt{4-x}\,dx$.

(b) by substitution, letting $u = 4-x$.

In Exercises 81 and 82, use a computer algebra system to evaluate the integral for $n = 0$, 1, 2, and 3. Use the result to obtain a general rule for the integral for any positive integer n and test your results for $n = 4$.

81. $\int x^n \ln x\,dx$ 82. $\int x^ne^x\,dx$

In Exercises 83–88, use integration by parts to verify the formula. (For Exercises 83–86, assume that n is a positive integer.)

83. $\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1} \cos x\,dx$

84. $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1} \sin x\,dx$

85. $\int x^n \ln x\,dx = \frac{x^{n+1}}{(n+1)^2}[-1 + (n+1)\ln x] + C$

86. $\int x^n\,e^{ax}\,dx = \frac{x^n\,e^{ax}}{a} - \frac{n}{a}\int x^{n-1}\,e^{ax}\,dx$

87. $\int e^{ax} \sin bx\,dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2} + C$

88. $\int e^{ax} \cos bx\,dx = \frac{e^{ax}\,(a \cos bx + b \sin bx)}{a^2+b^2} + C$

In Exercises 89–92, evaluate the integral by using the appropriate formula from Exercises 83–88.

89. $\int x^3 \ln x\,dx$ 90. $\int x^2 \cos x\,dx$

91. $\int e^{2x} \cos 3x\,dx$ 92. $\int x^3e^{2x}\,dx$

Area **In Exercises 93–96, use a graphing utility to sketch the region bounded by the graphs of the equations, and find the area of the region.**

93. $y = xe^{-x}$, $y = 0$, $x = 4$

94. $y = \frac{1}{9}\,xe^{-x/3}$, $y = 0$, $x = 0$, $x = 3$

95. $y = e^{-x} \sin \pi x$, $y = 0$, $x = 0$, $x = 1$

96. $y = x \sin x$, $y = 0$, $x = 0$, $x = \pi$

97. ***Area, Volume, and Centroid*** Given the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$, find

(a) the area of the region.

(b) the volume of the solid generated by revolving the region about the x-axis.

(c) the volume of the solid generated by revolving the region about the y-axis.

(d) the centroid of the region.

98. ***Centroid*** Find the centroid of the region bounded by the graphs of $y = \arcsin x$, $x = 0$, and $y = \pi/2$. How is this problem related to Example 6 in this section?

99. *Average Displacement* A damping force affects the vibration of a spring so that the displacement of the spring is

$$y = e^{-4t}(\cos 2t + 5 \sin 2t).$$

Find the average value of y on the interval from $t = 0$ to $t = \pi$.

100. *Memory Model* A model for the ability M of a child to memorize, measured on a scale from 0 to 10, is

$$M = 1 + 1.6t \ln t, \quad 0 < t \leq 4$$

where t is the child's age in years. Find the average value of this function

(a) between the child's first and second birthdays.

(b) between the child's third and fourth birthdays.

Present Value **In Exercises 101 and 102, find the present value P of a continuous income flow of $c(t)$ dollars per year if**

$$P = \int_0^{t_1} c(t)e^{-rt}\,dt$$

where t_1 is the time in years and r is the annual interest rate compounded continuously.

101. $c(t) = 100{,}000 + 4000t,\ r = 5\%,\ t_1 = 10$

102. $c(t) = 30{,}000 + 500t,\ r = 7\%,\ t_1 = 5$

Integrals Used to Find Fourier Coefficients **In Exercises 103 and 104, verify the value of the definite integral, where n is a positive integer.**

103. $$\int_{-\pi}^{\pi} x \sin nx\,dx = \begin{cases} \dfrac{2\pi}{n}, & n \text{ is odd} \\ -\dfrac{2\pi}{n}, & n \text{ is even} \end{cases}$$

104. $$\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{(-1)^n 4\pi}{n^2}$$

105. *Vibrating String* A string stretched between the two points (0, 0) and (2, 0) is plucked by displacing the string h units at its midpoint. The motion of the string is modeled by a **Fourier Sine Series** whose coefficients are given by

$$b_n = h\int_0^1 x \sin\frac{n\pi x}{2}\,dx + h\int_1^2 (-x + 2)\sin\frac{n\pi x}{2}\,dx.$$

Find b_n.

106. Find the fallacy in the following argument that $0 = 1$.

$dv = dx$ ⇨ $v = \int dx = x$

$u = \dfrac{1}{x}$ $du = -\dfrac{1}{x^2}\,dx$

$$0 + \int\frac{dx}{x} = \left(\frac{1}{x}\right)(x) - \int\left(-\frac{1}{x^2}\right)(x)\,dx = 1 + \int\frac{dx}{x}$$

So, $0 = 1$.

107. Let $y = f(x)$ be positive and strictly increasing on the interv $0 < a \leq x \leq b$. Consider the region R bounded by the grapl of $y = f(x)$, $y = 0$, $x = a$, and $x = b$. If R is revolved abo the y-axis, show that the disk method and shell method yie the same volume.

108. *Think About It* Explain why

$$\int_0^{\pi/2} x \sin x\,dx \leq \int_0^{\pi/2} x\,dx.$$

Evaluate the integrals to verify the inequality.

109. Consider the differential equation $f'(x) = xe^{-x}$ with the initi condition $f(0) = 0$.

(a) Use integration to solve the differential equation.

(b) Use a graphing utility to graph the solution of the diffe ential equation.

(c) *Euler's Method* From the definition of the derivative follows that for "small" Δx

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f(x + \Delta x) \approx f(x) + [f'(x)]\,\Delta x.$$

Consider points of the form

$$(x_n, y_n) = (n\,\Delta x, y_{n-1} + f'(x_{n-1})\,\Delta x)$$

where $(x_0, y_0) = (0, 0)$. Starting with $n = 0$, use tl recursive capabilities of a graphing utility to generate tl next 80 points of this form when $\Delta x = 0.05$. Use tl graphing utility to plot the points and compare the resu with the graph in part (b).

(d) Starting with $n = 0$, repeat part (c) by generating the ne 40 points when $\Delta x = 0.1$.

(e) Give a geometric explanation of the process described part (c). Why do you think the result in part (c) is a bett approximation of the solution than the result in part (d)'

110. *Euler's Method* Consider the differential equation

$$f'(x) = \cos\sqrt{x}$$

with the initial condition $f(0) = 2$.

(a) Try solving the differential equation by integration. C you perform the integration?

(b) Starting with $n = 0$, use the recursive capabilities of graphing utility to generate 80 points of the form shown part (c) of Exercise 109 when $\Delta x = 0.05$. Plot the poin for an approximation of the graph of the solution of tl differential equation.

Section 7.3 Trigonometric Integrals

- Solve trigonometric integrals involving powers of sine and cosine.
- Solve trigonometric integrals involving powers of secant and tangent.
- Solve trigonometric integrals involving sine-cosine products with different angles.

Sheila Scott Macintyre (1910–1960)

Sheila Scott Macintyre published her first paper on the asymptotic periods of integral functions in 1935. She completed her doctorate work at Aberdeen University, where she taught. In 1958 she accepted a visiting research fellowship at the University of Cincinnati.

Integrals Involving Powers of Sine and Cosine

In this section you will study techniques for evaluating integrals of the form

$$\int \sin^m x \cos^n x\,dx \qquad \text{and} \qquad \int \sec^m x \tan^n x\,dx$$

where either m or n is a positive integer. To find antiderivatives for these forms, try to break them into combinations of trigonometric integrals to which you can apply the Power Rule.

For instance, you can evaluate $\int \sin^5 x \cos x\,dx$ with the Power Rule by letting $u = \sin x$. Then, $du = \cos x\,dx$ and you have

$$\int \sin^5 x \cos x\,dx = \int u^5\,du = \frac{u^6}{6} + C = \frac{\sin^6 x}{6} + C.$$

To break up $\int \sin^m x \cos^n x\,dx$ into forms to which you can apply the Power Rule, use the following identities.

$\sin^2 x + \cos^2 x = 1$ Pythagorean identity

$\sin^2 x = \dfrac{1 - \cos 2x}{2}$ Half-angle identity for $\sin^2 x$

$\cos^2 x = \dfrac{1 + \cos 2x}{2}$ Half-angle identity for $\cos^2 x$

Guidelines for Evaluating Integrals Involving Sine and Cosine

1. If the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then, expand and integrate.

$$\int \sin^{\overbrace{2k+1}^{\text{Odd}}} x \cos^n x\,dx = \int \overbrace{(\sin^2 x)^k}^{\text{Convert to cosines}} \cos^n x \overbrace{\sin x\,dx}^{\text{Save for } du} = \int (1 - \cos^2 x)^k \cos^n x \sin x\,dx$$

2. If the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then, expand and integrate.

$$\int \sin^m x \cos^{\overbrace{2k+1}^{\text{Odd}}} x\,dx = \int \sin^m x \overbrace{(\cos^2 x)^k}^{\text{Convert to sines}} \overbrace{\cos x\,dx}^{\text{Save for } du} = \int \sin^m x\,(1 - \sin^2 x)^k \cos x\,dx$$

3. If the powers of both the sine and cosine are even and nonnegative, make repeated use of the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \text{and} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to convert the integrand to odd powers of the cosine. Then proceed as in guideline 2.

TECHNOLOGY Try using a computer algebra system to evaluate the integral in Example 1. When we did this, we obtained

$$\int \sin^3 x \cos^4 x\, dx =$$
$$-\cos^5 x\left(\frac{1}{7}\sin^2 x + \frac{2}{35}\right) + C.$$

Is this equivalent to the result obtained in Example 1?

Example 1 Power of Sine Is Odd and Positive

Evaluate $\int \sin^3 x \cos^4 x\, dx$.

Solution Because you expect to use the Power Rule with $u = \cos x$, *save one sin factor* to form du and convert the remaining sine factors to cosines.

$$\begin{aligned}\int \sin^3 x \cos^4 x\, dx &= \int \sin^2 x \cos^4 x (\sin x)\, dx && \text{Rewrite.}\\ &= \int (1-\cos^2 x)\cos^4 x \sin x\, dx && \text{Trigonometric identity}\\ &= \int (\cos^4 x - \cos^6 x)\sin x\, dx && \text{Multiply.}\\ &= \int \cos^4 x \sin x\, dx - \int \cos^6 x \sin x\, dx\\ &= -\int \cos^4 x(-\sin x)\, dx + \int \cos^6 x(-\sin x)\, dx\\ &= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C && \text{Integrate.}\end{aligned}$$

In Example 1, *both* of the powers m and n happened to be positive integers However, the same strategy will work as long as either m or n is odd and positive. Fo instance, in the next example the power of the cosine is 3, but the power of the sin is $-\frac{1}{2}$.

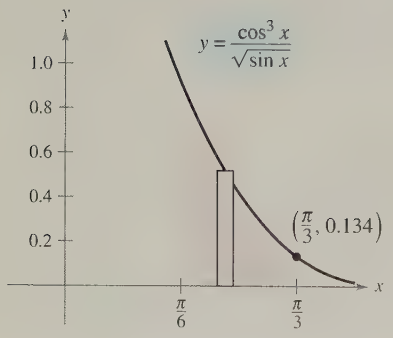
Example 2 Power of Cosine Is Odd and Positive

Evaluate $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx$.

Solution Because you expect to use the Power Rule with $u = \sin x$, *save one cosin factor* to form du and convert the remaining cosine factors to sines.

$$\begin{aligned}\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx &= \int_{\pi/6}^{\pi/3} \frac{\cos^2 x \cos x}{\sqrt{\sin x}}\, dx\\ &= \int_{\pi/6}^{\pi/3} \frac{(1-\sin^2 x)(\cos x)}{\sqrt{\sin x}}\, dx\\ &= \int_{\pi/6}^{\pi/3} [(\sin x)^{-1/2}\cos x - (\sin x)^{3/2}\cos x]\, dx\\ &= \left[\frac{(\sin x)^{1/2}}{1/2} - \frac{(\sin x)^{5/2}}{5/2}\right]_{\pi/6}^{\pi/3}\\ &= 2\left(\frac{\sqrt{3}}{2}\right)^{1/2} - \frac{2}{5}\left(\frac{\sqrt{3}}{2}\right)^{5/2} - \sqrt{2} + \frac{\sqrt{32}}{80}\\ &\approx 0.239\end{aligned}$$

Figure 7.4 shows the region whose area is represented by this integral.



The area of the region is approximately 0.239.
Figure 7.4

Bettmann/Corbis

JOHN WALLIS (1616–1703)

Wallis did much of his work in calculus prior to Newton and Leibniz, and he influenced the thinking of both of these men. Wallis is also credited with introducing the present symbol (∞) for infinity.

Example 3 Power of Cosine Is Even and Nonnegative

Evaluate $\int \cos^4 x\,dx$.

Solution Because m and n are both even and nonnegative ($m = 0$), you can replace $\cos^4 x$ by $[(1 + \cos 2x)/2]^2$.

$$\begin{aligned}\int \cos^4 x\,dx &= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx\\ &= \int \left(\frac{1}{4} + \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}\right) dx\\ &= \int \left[\frac{1}{4} + \frac{\cos 2x}{2} + \frac{1}{4}\left(\frac{1+\cos 4x}{2}\right)\right] dx\\ &= \frac{3}{8}\int dx + \frac{1}{4}\int 2\cos 2x\,dx + \frac{1}{32}\int 4\cos 4x\,dx\\ &= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C\end{aligned}$$

Try using a symbolic differentiation utility to verify this. Can you simplify the derivative to obtain the original integrand?

In Example 3, if you were to evaluate the definite integral from 0 to $\pi/2$, you would obtain

$$\begin{aligned}\int_0^{\pi/2} \cos^4 x\,dx &= \left[\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right]_0^{\pi/2}\\ &= \left(\frac{3\pi}{16} + 0 + 0\right) - (0 + 0 + 0)\\ &= \frac{3\pi}{16}.\end{aligned}$$

Note that the only term that contributes to the solution is $3x/8$. This observation is generalized in the following formulas developed by John Wallis.

Wallis's Formulas

1. If n is odd ($n \geq 3$), then

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right).$$

2. If n is even ($n \geq 2$), then

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right).$$

These formulas are also valid if $\cos^n x$ is replaced by $\sin^n x$. (You are asked to prove both formulas in Exercise 96.)

Integrals Involving Powers of Secant and Tangent

The following guidelines can help you evaluate integrals of the form $\int \sec^m x \tan^n x\ d$

Guidelines for Evaluating Integrals Involving Secant and Tangent

1. If the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then expand and integrate.

$$\int \sec^{\overbrace{2k}^{\text{Even}}} x \tan^n x\, dx = \int \overbrace{(\sec^2 x)^{k-1}}^{\text{Convert to tangents}} \tan^n x \overbrace{\sec^2 x\, dx}^{\text{Save for } du} = \int (1 + \tan^2 x)^{k-1} \tan^n x \sec^2 x\, dx$$

2. If the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then expand and integrate.

$$\int \sec^m x \tan^{\overbrace{2k+1}^{\text{Odd}}} x\, dx = \int \sec^{m-1} x \overbrace{(\tan^2 x)^k}^{\text{Convert to secants}} \overbrace{\sec x \tan x\, dx}^{\text{Save for } du} = \int \sec^{m-1} x(\sec^2 x - 1)^k \sec x \tan x\, dx$$

3. If there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

$$\int \tan^n x\, dx = \int \tan^{n-2} x \overbrace{(\tan^2 x)}^{\text{Convert to secants}}\, dx = \int \tan^{n-2} x(\sec^2 x - 1)\, dx$$

4. If the integral is of the form $\int \sec^m x\, dx$, where m is odd and positive, use integration by parts, as illustrated in Example 5 in the preceding section.
5. If none of the first four guidelines applies, try converting to sines and cosines.

Example 4 Power of Tangent Is Odd and Positive

Evaluate $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx.$

Solution Because you expect to use the Power Rule with $u = \sec x$, *save a factor o* *(sec x tan x)* to form du and convert the remaining tangent factors to secants.

$$\begin{aligned}\int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx &= \int (\sec x)^{-1/2} \tan^3 x\, dx\\ &= \int (\sec x)^{-3/2}(\tan^2 x)(\sec x \tan x)\, dx\\ &= \int (\sec x)^{-3/2}(\sec^2 x - 1)(\sec x \tan x)\, dx\\ &= \int [(\sec x)^{1/2} - (\sec x)^{-3/2}](\sec x \tan x)\, dx\\ &= \frac{2}{3}(\sec x)^{3/2} + 2(\sec x)^{-1/2} + C\end{aligned}$$

OTE In Example 5, the power of the ngent is odd and positive. So, you uld also evaluate the integral with the ocedure described in guideline 2 on ge 500. In Exercise 81, you are asked show that the results obtained by these o procedures differ only by a constant.

Example 5 Power of Secant Is Even and Positive

Evaluate $\int \sec^4 3x \tan^3 3x \, dx$.

Solution Let $u = \tan 3x$, then $du = 3 \sec^2 3x \, dx$ and you can write

$$\begin{aligned}\int \sec^4 3x \tan^3 3x \, dx &= \int \sec^2 3x \tan^3 3x(\sec^2 3x) \, dx \\ &= \int (1 + \tan^2 3x) \tan^3 3x \, (\sec^2 3x) \, dx \\ &= \frac{1}{3}\int (\tan^3 3x + \tan^5 3x)(3 \sec^2 3x) \, dx \\ &= \frac{1}{3}\left(\frac{\tan^4 3x}{4} + \frac{\tan^6 3x}{6}\right) + C \\ &= \frac{\tan^4 3x}{12} + \frac{\tan^6 3x}{18} + C.\end{aligned}$$

Example 6 Power of Tangent Is Even

Evaluate $\int_0^{\pi/4} \tan^4 x \, dx$.

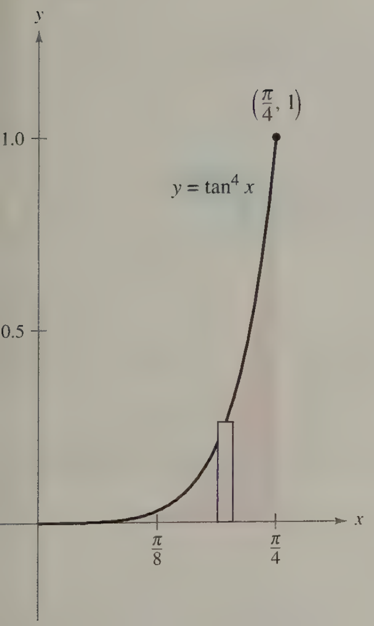
Solution Because there are no secant factors, you can begin by converting a tangent-squared factor to a secant-squared factor.

$$\begin{aligned}\int \tan^4 x \, dx &= \int \tan^2 x(\tan^2 x) \, dx \\ &= \int \tan^2 x(\sec^2 x - 1) \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx \\ &= \frac{\tan^3 x}{3} - \tan x + x + C\end{aligned}$$

You can evaluate the definite integral as follows.

$$\begin{aligned}\int_0^{\pi/4} \tan^4 x \, dx &= \left[\frac{\tan^3 x}{3} - \tan x + x\right]_0^{\pi/4} \\ &= \frac{\pi}{4} - \frac{2}{3} \\ &\approx 0.119\end{aligned}$$

The area represented by the definite integral is shown in Figure 7.5. Try using Simpson's Rule to approximate this integral. With $n = 10$, you should obtain an approximation that is within 0.00001 of the actual value.



e area of the region is approximately 119.

gure 7.5

For integrals involving powers of cotangents and cosecants, you can follow strategy similar to that used for powers of tangents and secants. Also, when integra ing trigonometric functions, remember that it sometimes helps to convert the enti integrand to powers of sines and cosines.

Example 7 Converting to Sines and Cosines

Evaluate $\displaystyle\int \frac{\sec x}{\tan^2 x}\,dx$.

Solution Because the first four guidelines on page 500 do not apply, try convertir the integrand to sines and cosines. In this case, you are able to integrate the resultir powers of sine and cosine as follows.

$$\begin{aligned}
\int \frac{\sec x}{\tan^2 x}\,dx &= \int \left(\frac{1}{\cos x}\right)\left(\frac{\cos x}{\sin x}\right)^2 dx \\
&= \int (\sin x)^{-2}(\cos x)\,dx \\
&= -(\sin x)^{-1} + C \\
&= -\csc x + C
\end{aligned}$$

Integrals Involving Sine-Cosine Products with Different Angles

Integrals involving the products of sines and cosines of two *different* angles occur i many applications. In such instances you can use the following product-to-sur identities.

FOR FURTHER INFORMATION To learn more about integrals involving sine-cosine products with different angles, see the article "Integrals of Products of Sine and Cosine with Different Arguments" by Sherrie J. Nicol in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

$$\begin{aligned}
\sin mx \sin nx &= \frac{1}{2}(\cos[(m-n)x] - \cos[(m+n)x]) \\
\sin mx \cos nx &= \frac{1}{2}(\sin[(m-n)x] + \sin[(m+n)x]) \\
\cos mx \cos nx &= \frac{1}{2}(\cos[(m-n)x] + \cos[(m+n)x])
\end{aligned}$$

Example 8 Using Product-to-Sum Identities

Evaluate $\int \sin 5x \cos 4x\,dx$.

Solution Considering the second product-to-sum identity above, you can write tl following.

$$\begin{aligned}
\int \sin 5x \cos 4x\,dx &= \frac{1}{2}\int (\sin x + \sin 9x)\,dx \\
&= \frac{1}{2}\left(-\cos x - \frac{\cos 9x}{9}\right) + C \\
&= -\frac{\cos x}{2} - \frac{\cos 9x}{18} + C
\end{aligned}$$

EXERCISES FOR SECTION 7.3

. Consider the function $f(x) = \sin^4 x + \cos^4 x$.

(a) Use the power-reducing formulas to write the function in terms of the first power of the cosine.

(b) Determine another way of rewriting the function. Use a graphing utility to verify your result.

(c) Determine a trigonometric expression to add to the function so that it becomes a perfect square trinomial. Rewrite the function as a perfect square trinomial minus the term that you added. Use a graphing utility to verify your result.

(d) Rewrite the result in part (c) in terms of the sine of a double angle. Use a graphing utility to verify your result.

(e) In how many ways have you rewritten the trigonometric function? When rewriting a trigonometric expression, your result may not be the same as another person's result. Does this mean that one of you is wrong? Explain.

. Match the antiderivative in the left column with the correct integral in the right column.

(a) $y = \sec x$

(b) $y = \cos x + \sec x$

(c) $y = x - \tan x + \frac{1}{3}\tan^3 x$

(d) $y = 3x + 2\sin x\cos^3 x + 3\sin x\cos x$

(i) $\int \sin x \tan^2 x\,dx$

(ii) $8\int \cos^4 x\,dx$

(iii) $\int \sin x \sec^2 x\,dx$

(iv) $\int \tan^4 x\,dx$

Exercises 3–16, evaluate the integral.

. $\int \cos^3 x \sin x\,dx$

4. $\int \cos^3 x \sin^4 x\,dx$

. $\int \sin^5 2x \cos 2x\,dx$

6. $\int \sin^3 x\,dx$

. $\int \sin^5 x \cos^2 x\,dx$

8. $\int \cos^3 \frac{x}{3}\,dx$

. $\int \cos^3 \theta\sqrt{\sin\theta}\,d\theta$

10. $\int \frac{\sin^5 t}{\sqrt{\cos t}}\,dt$

. $\int \cos^2 3x\,dx$

12. $\int \sin^2 2x\,dx$

. $\int \sin^2 \alpha\cos^2 \alpha\,d\alpha$

14. $\int \sin^4 2\theta\,d\theta$

. $\int x\sin^2 x\,dx$

16. $\int x^2\sin^2 x\,dx$

Exercises 17–20, verify Wallis's Formulas by evaluating the tegral.

. $\int_0^{\pi/2} \cos^3 x\,dx = \frac{2}{3}$

18. $\int_0^{\pi/2} \cos^5 x\,dx = \frac{8}{15}$

. $\int_0^{\pi/2} \cos^7 x\,dx = \frac{16}{35}$

20. $\int_0^{\pi/2} \sin^2 x\,dx = \frac{\pi}{4}$

In Exercises 21–38, evaluate the integral involving secant and tangent.

21. $\int \sec 3x\,dx$

22. $\int \sec^2(2x - 1)\,dx$

23. $\int \sec^4 5x\,dx$

24. $\int \sec^6 3x\,dx$

25. $\int \sec^3 \pi x\,dx$

26. $\int \tan^2 x\,dx$

27. $\int \tan^5 \frac{x}{4}\,dx$

28. $\int \tan^3 \frac{\pi x}{2}\sec^2 \frac{\pi x}{2}\,dx$

29. $\int \sec^2 x\tan x\,dx$

30. $\int \tan^3 2t\sec^3 2t\,dt$

31. $\int \tan^2 x\sec^2 x\,dx$

32. $\int \tan^5 2x\sec^2 2x\,dx$

33. $\int \sec^6 4x\tan 4x\,dx$

34. $\int \sec^2 \frac{x}{2}\tan \frac{x}{2}\,dx$

35. $\int \sec^3 x\tan x\,dx$

36. $\int \tan^3 3x\,dx$

37. $\int \frac{\tan^2 x}{\sec x}\,dx$

38. $\int \frac{\tan^2 x}{\sec^5 x}\,dx$

In Exercises 39–42, solve the differential equation.

39. $\frac{dr}{d\theta} = \sin^4 \pi\theta$

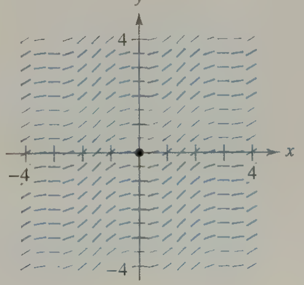
40. $\frac{ds}{d\alpha} = \sin^2 \frac{\alpha}{2}\cos^2 \frac{\alpha}{2}$

41. $y' = \tan^3 3x\sec 3x$

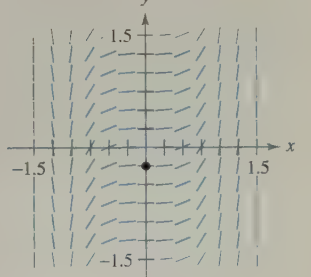
42. $y' = \sqrt{\tan x}\sec^4 x$

Slope Fields **In Exercises 43 and 44, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the indicated point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

43. $\frac{dy}{dx} = \sin^2 x,\ (0, 0)$



44. $\frac{dy}{dx} = \sec^2 x\tan^2 x,\ \left(0, -\frac{1}{4}\right)$



In Exercises 45 and 46, use a computer algebra system to sketch the slope field for the differential equation, and graph the solution through the specified initial condition.

45. $\dfrac{dy}{dx} = \dfrac{3 \sin x}{y}, \ y(0) = 2$

46. $\dfrac{dy}{dx} = 3\sqrt{y} \tan^2 x, \ y(0) = 3$

In Exercises 47–50, evaluate the integral.

47. $\displaystyle\int \sin 3x \cos 2x \, dx$

48. $\displaystyle\int \cos 4\theta \cos(-3\theta) \, d\theta$

49. $\displaystyle\int \sin\theta \sin 3\theta \, d\theta$

50. $\displaystyle\int \sin(-4x) \cos 3x \, dx$

In Exercises 51–60, evaluate the integral. Use a computer algebra system to confirm your result.

51. $\displaystyle\int \cot^3 2x \, dx$

52. $\displaystyle\int \tan^4 \frac{x}{2} \sec^4 \frac{x}{2} \, dx$

53. $\displaystyle\int \csc^4 \theta \, d\theta$

54. $\displaystyle\int \csc^2 3x \cot 3x \, dx$

55. $\displaystyle\int \frac{\cot^2 t}{\csc t} \, dt$

56. $\displaystyle\int \frac{\cot^3 t}{\csc t} \, dt$

57. $\displaystyle\int \frac{1}{\sec x \tan x} \, dx$

58. $\displaystyle\int \frac{\sin^2 x - \cos^2 x}{\cos x} \, dx$

59. $\displaystyle\int (\tan^4 t - \sec^4 t) \, dt$

60. $\displaystyle\int \frac{1 - \sec t}{\cos t - 1} \, dt$

In Exercises 61–68, evaluate the definite integral.

61. $\displaystyle\int_{-\pi}^{\pi} \sin^2 x \, dx$

62. $\displaystyle\int_{0}^{\pi/3} \tan^2 x \, dx$

63. $\displaystyle\int_{0}^{\pi/4} \tan^3 x \, dx$

64. $\displaystyle\int_{0}^{\pi/4} \sec^2 t \sqrt{\tan t} \, dt$

65. $\displaystyle\int_{0}^{\pi/2} \frac{\cos t}{1 + \sin t} \, dt$

66. $\displaystyle\int_{-\pi}^{\pi} \sin 3\theta \cos\theta \, d\theta$

67. $\displaystyle\int_{-\pi/2}^{\pi/2} \cos^3 x \, dx$

68. $\displaystyle\int_{-\pi/2}^{\pi/2} (\sin^2 x + 1) \, dx$

In Exercises 69–74, use a computer algebra system to evaluate the integral. Graph the antiderivatives for two different values of the constant of integration.

69. $\displaystyle\int \cos^4 \frac{x}{2} \, dx$

70. $\displaystyle\int \sin^2 x \cos^2 x \, dx$

71. $\displaystyle\int \sec^5 \pi x \, dx$

72. $\displaystyle\int \tan^3(1 - x) \, dx$

73. $\displaystyle\int \sec^5 \pi x \tan \pi x \, dx$

74. $\displaystyle\int \sec^4(1 - x) \tan(1 - x) \, dx$

In Exercises 75–78, use a computer algebra system to evaluate the definite integral.

75. $\displaystyle\int_{0}^{\pi/4} \sin 2\theta \sin 3\theta \, d\theta$

76. $\displaystyle\int_{0}^{\pi/2} (1 - \cos \theta)^2 \, d\theta$

77. $\displaystyle\int_{0}^{\pi/2} \sin^4 x \, dx$

78. $\displaystyle\int_{0}^{\pi/2} \sin^6 x \, dx$

Getting at the Concept

79. In your own words, describe how you would integrate $\int \sin^m x \cos^n x \, dx$ for each of the following.

(a) m is positive and odd.

(b) n is positive and odd.

(c) m and n are both positive and even.

80. In your own words, describe how you would integrate $\int \sec^m x \tan^n x \, dx$ for each of the following.

(a) m is positive and even.

(b) n is positive and odd.

(c) n is positive and even, and there are no secant factors.

(d) m is positive and odd, and there are no tangent factors.

In Exercises 81 and 82, (a) find the indefinite integral in tw different ways. (b) Use a graphing utility to graph the ant derivative (without the constant of integration) obtained b each method to show that the results differ only by a constan (c) Verify analytically that the results differ only by a constan

81. $\displaystyle\int \sec^4 3x \tan^3 3x \, dx$

82. $\displaystyle\int \sec^2 x \tan x \, dx$

83. ***Area*** Find the area of the region bounded by the graphs of th equations $y = \sin^2 \pi x$, $y = 0$, $x = 0$, and $x = 1$.

84. ***Volume*** Find the volume of the solid generated by revolvir the region bounded by the graphs of the equations $y = \tan$ $y = 0$, $x = -\pi/4$, and $x = \pi/4$ about the x-axis.

Volume and Centroid **In Exercises 85 and 86, for the regio bounded by the graphs of the equations, find (a) the volume the solid formed by revolving the region about the x-axis and (the centroid of the region.**

85. $y = \sin x, \ y = 0, \ x = 0, \ x = \pi$

86. $y = \cos x, \ y = 0, \ x = 0, \ x = \dfrac{\pi}{2}$

In Exercises 87–90, use integration by parts to verify th reduction formula.

87. $\displaystyle\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$

88. $\displaystyle\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$

89. $\displaystyle\int \cos^m x \sin^n x \, dx = -\frac{\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x \, dx$

90. $\displaystyle\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$

Exercises 91–94, use the results of Exercises 87–90 to evaluate e integral.

. $\int \sin^5 x\, dx$

92. $\int \cos^4 x\, dx$

. $\int \sec^4 \frac{2\pi x}{5}\, dx$

94. $\int \sin^4 x \cos^2 x\, dx$

. ***Modeling Data*** The table shows the normal maximum (high) and minimum (low) temperatures for Erie, Pennsylvania for each month of a year. *(Source: NOAA)*

Month	Jan	Feb	Mar	Apr	May	Jun
Max	30.9	32.2	41.1	53.7	64.6	74.0
Min	18.0	17.7	25.8	36.1	45.4	55.2

Month	Jul	Aug	Sep	Oct	Nov	Dec
Max	78.2	77.0	71.0	60.1	47.1	35.7
Min	59.9	59.4	53.1	43.2	34.3	24.2

The maximum and minimum temperatures can be modeled by

$$f(t) = a_0 + a_1 \cos \frac{\pi t}{6} + b_1 \sin \frac{\pi t}{6}$$

where a_0, a_1, and b_1 are as follows.

$$a_0 = \frac{1}{12}\int_0^{12} f(t)\, dt$$

$$a_1 = \frac{1}{6}\int_0^{12} f(t) \cos \frac{\pi t}{6}\, dt$$

$$b_1 = \frac{1}{6}\int_0^{12} f(t) \sin \frac{\pi t}{6}\, dt$$

(a) Approximate the model $H(t)$ for the maximum temperatures. Let $t = 0$ correspond to January. (*Hint:* Use Simpson's Rule to approximate the integrals and use the January data twice.)

(b) Repeat part (a) for a model $L(t)$ for the minimum temperature data.

(c) Use a graphing utility to compare each model with the actual data. During what part of the year is the difference between the maximum and minimum temperatures greatest?

. ***Wallis's Formulas*** Use the result of Exercise 88 to prove the following versions of Wallis's Formulas.

(a) If n is odd $(n \geq 3)$, then

$$\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) \cdots \left(\frac{n-1}{n}\right).$$

(b) If n is even $(n \geq 2)$, then

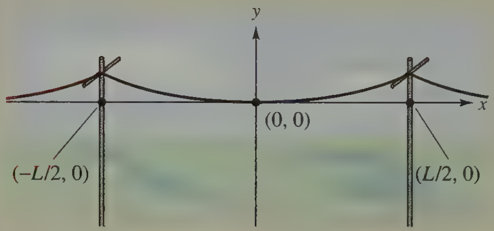
$$\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \cdots \left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right).$$

97. The **inner product** of two functions f and g on $[a, b]$ is given by $\langle f, g\rangle = \int_a^b f(x)g(x)\, dx$. Two distinct functions f and g are said to be **orthogonal** if $\langle f, g\rangle = 0$. Show that the following set of functions is orthogonal on $[-\pi, \pi]$.

$$\{\sin x, \sin 2x, \sin 3x, \ldots, \cos x, \cos 2x, \cos 3x, \ldots\}$$

SECTION PROJECT POWER LINES

Power lines are constructed by stringing wire between supports and adjusting the tension on each span. The wire hangs between supports in the shape of a catenary, as shown in the figure.



Let T be the tension (in pounds) on a span of wire, let u be the density (in pounds per foot), let $g \approx 32.2$ be the acceleration due to gravity (in feet per second per second), and let L be the distance (in feet) between the supports. Then the equation of the catenary is

$$y = \frac{T}{ug}\left(\cosh \frac{ugx}{T} - 1\right)$$

where x and y are measured in feet.

(a) Find the length of the wire between two spans.

(b) To measure the tension in a span, power line workers use the *return wave method.* The wire is struck at one support, creating a wave in the line, and the time t (in seconds) it takes for the wave to make a round trip is measured. The velocity v (in feet per second) is given by $v = \sqrt{T/u}$. How long does it take the wave to make a round trip between supports?

(c) The sag s (in inches) can be obtained by evaluating y when $x = L/2$ in the equation for the catenary (and multiplying by 12). In practice, however, power line workers use the "lineman's equation" given by $s \approx 12.075t^2$. Use the fact that $[\cosh(ugL/2T) + 1] \approx 2$ to derive this equation.

FOR FURTHER INFORMATION To learn more about the mathematics of power lines, see the article "Constructing Power Lines" by Thomas O'Neil in *The UMAP Journal*. To view this article, go to the website *www.matharticles.com*.

Section 7.4 Trigonometric Substitution

- Use trigonometric substitution to solve an integral.
- Use integrals to model and solve real-life applications.

EXPLORATION

Integrating a Radical Function
Up to this point in the text, you have not evaluated the following integral.

$$\int_{-1}^{1} \sqrt{1 - x^2}\, dx$$

From geometry, you should be able to find the exact value of this integral—what is it? Using numerical integration, with Simpson's Rule or the Trapezoidal Rule, you can't be sure of the accuracy of the approximation. Why?

Try finding the exact value using the substitution

$$x = \sin\theta \quad \text{and} \quad dx = \cos\theta\, d\theta.$$

Does your answer agree with the value you obtained using geometry?

Trigonometric Substitution

Now that you can evaluate integrals involving powers of trigonometric functions, y can use **trigonometric substitution** to evaluate integrals involving the radicals

$$\sqrt{a^2 - u^2}, \qquad \sqrt{a^2 + u^2}, \qquad \text{and} \qquad \sqrt{u^2 - a^2}.$$

The objective with trigonometric substitution is to eliminate the radical in the int grand. You do this with the Pythagorean identities

$$\cos^2\theta = 1 - \sin^2\theta, \quad \sec^2\theta = 1 + \tan^2\theta, \quad \text{and} \quad \tan^2\theta = \sec^2\theta - 1.$$

For example, if $a > 0$, let $u = a\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then

$$\begin{aligned}\sqrt{a^2 - u^2} &= \sqrt{a^2 - a^2\sin^2\theta}\\ &= \sqrt{a^2(1 - \sin^2\theta)}\\ &= \sqrt{a^2\cos^2\theta}\\ &= a\cos\theta.\end{aligned}$$

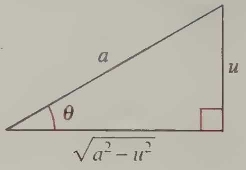
Note that $\cos\theta \ge 0$, because $-\pi/2 \le \theta \le \pi/2$.

Trigonometric Substitution ($a > 0$)

1. For integrals involving $\sqrt{a^2 - u^2}$, let

$$u = a\sin\theta.$$

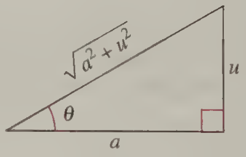
Then $\sqrt{a^2 - u^2} = a\cos\theta$, where $-\pi/2 \le \theta \le \pi/2$.



2. For integrals involving $\sqrt{a^2 + u^2}$, let

$$u = a\tan\theta.$$

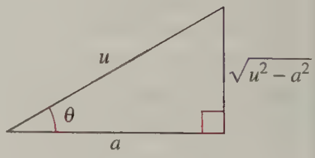
Then $\sqrt{a^2 + u^2} = a\sec\theta$, where $-\pi/2 < \theta < \pi/2$.



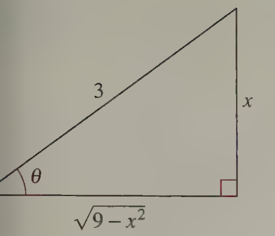
3. For integrals involving $\sqrt{u^2 - a^2}$, let

$$u = a\sec\theta.$$

Then $\sqrt{u^2 - a^2} = \pm a\tan\theta$, where $0 \le \theta < \pi/2$ or $\pi/2 < \theta \le \pi$. Use the positive value if $u > a$ and the negative value if $u < -a$.



NOTE The restrictions on θ ensure that the function that defines the substitution is one-to-o In fact, these are the same intervals over which the arcsine, arctangent, and arcsecant are defin



n $\theta = \frac{x}{3}$, $\cot\theta = \frac{\sqrt{9-x^2}}{x}$

igure 7.6

Example 1 Trigonometric Substitution: $u = a\sin\theta$

Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$.

Solution First, note that none of the basic integration rules applies. To use trigonometric substitution, you should observe that $\sqrt{9-x^2}$ is of the form $\sqrt{a^2-u^2}$. So, you can use the substitution

$$x = a\sin\theta = 3\sin\theta.$$

Using differentiation and the triangle shown in Figure 7.6, you obtain

$$dx = 3\cos\theta\,d\theta, \qquad \sqrt{9-x^2} = 3\cos\theta, \qquad \text{and} \qquad x^2 = 9\sin^2\theta.$$

Therefore, trigonometric substitution yields the following.

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{9-x^2}} &= \int \frac{3\cos\theta\,d\theta}{(9\sin^2\theta)(3\cos\theta)} && \text{Substitute.}\\
&= \frac{1}{9}\int \frac{d\theta}{\sin^2\theta} && \text{Simplify.}\\
&= \frac{1}{9}\int \csc^2\theta\,d\theta && \text{Trigonometric identity}\\
&= -\frac{1}{9}\cot\theta + C && \text{Apply Cosecant Rule.}\\
&= -\frac{1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right) + C && \text{Substitute for } \cot\theta.\\
&= -\frac{\sqrt{9-x^2}}{9x} + C
\end{aligned}$$

Note that the triangle in Figure 7.6 can be used to convert the θ's back to x's as follows.

$$\cot\theta = \frac{\text{adj.}}{\text{opp.}} = \frac{\sqrt{9-x^2}}{x}$$

TECHNOLOGY Use a computer algebra system to integrate each of the following.

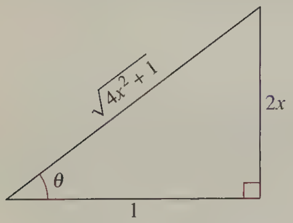
$$\int \frac{dx}{\sqrt{9-x^2}} \qquad \int \frac{dx}{x\sqrt{9-x^2}} \qquad \int \frac{dx}{x^2\sqrt{9-x^2}} \qquad \int \frac{dx}{x^3\sqrt{9-x^2}}$$

Then use trigonometric substitution to duplicate the results obtained with the computer algebra system.

In an earlier chapter, you saw how the inverse hyperbolic functions can be used to evaluate the integrals

$$\int \frac{du}{\sqrt{u^2 \pm a^2}}, \qquad \int \frac{du}{a^2-u^2}, \qquad \text{and} \qquad \int \frac{du}{u\sqrt{a^2 \pm u^2}}.$$

You can also evaluate these integrals using trigonometric substitution. This is illustrated in the next example.



$\tan\theta = 2x$, $\sec\theta = \sqrt{4x^2+1}$
Figure 7.7

Example 2 Trigonometric Substitution: $u = a\tan\theta$

Evaluate $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}$.

Solution Let $u = 2x$, $a = 1$, and $2x = \tan\theta$, as shown in Figure 7.7. Then,

$$dx = \frac{1}{2}\sec^2\theta\, d\theta \qquad \text{and} \qquad \sqrt{4x^2+1} = \sec\theta.$$

Trigonometric substitution produces the following.

$$\int \frac{1}{\sqrt{4x^2+1}}\,dx = \frac{1}{2}\int \frac{\sec^2\theta\, d\theta}{\sec\theta} \qquad \text{Substitute.}$$

$$= \frac{1}{2}\int \sec\theta\, d\theta \qquad \text{Simplify.}$$

$$= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C \qquad \text{Apply Secant Rule.}$$

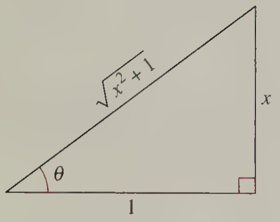
$$= \frac{1}{2}\ln\left|\sqrt{4x^2+1} + 2x\right| + C \qquad \text{Back-substitute.}$$

Try checking this result with a computer algebra system. Is the result given in th form or in the form of an inverse hyperbolic function?

You can extend the use of trigonometric substitution to cover integrals involvir expressions such as $(a^2 - u^2)^{n/2}$ by writing the expression as

$$(a^2 - u^2)^{n/2} = \left(\sqrt{a^2 - u^2}\right)^n.$$

Example 3 Trigonometric Substitution: Rational Powers



$\tan\theta = x$, $\sin\theta = \dfrac{x}{\sqrt{x^2+1}}$

Figure 7.8

Evaluate $\displaystyle\int \frac{dx}{(x^2+1)^{3/2}}$.

Solution Begin by writing $(x^2+1)^{3/2}$ as $\left(\sqrt{x^2+1}\right)^3$. Then, let $a = 1$ ar $u = x = \tan\theta$, as shown in Figure 7.8. Using

$$dx = \sec^2\theta\, d\theta \qquad \text{and} \qquad \sqrt{x^2+1} = \sec\theta$$

you can apply trigonometric substitution as follows.

$$\int \frac{dx}{(x^2+1)^{3/2}} = \int \frac{dx}{\left(\sqrt{x^2+1}\right)^3} \qquad \text{Rewrite denominator.}$$

$$= \int \frac{\sec^2\theta\, d\theta}{\sec^3\theta} \qquad \text{Substitute.}$$

$$= \int \frac{d\theta}{\sec\theta} \qquad \text{Simplify.}$$

$$= \int \cos\theta\, d\theta \qquad \text{Trigonometric identity}$$

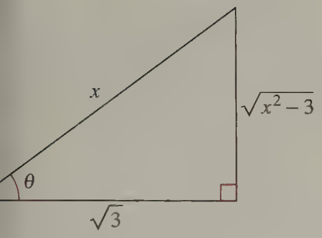
$$= \sin\theta + C \qquad \text{Apply Cosine Rule.}$$

$$= \frac{x}{\sqrt{x^2+1}} + C \qquad \text{Back-substitute.}$$

For definite integrals, it is often convenient to determine the integration limits for θ that avoid converting back to x. You might want to review this procedure in Section 4.5, Examples 8 and 9.

Example 4 Converting the Limits of Integration

Evaluate $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx.$



$\theta = \dfrac{x}{\sqrt{3}}$, $\tan\theta = \dfrac{\sqrt{x^2-3}}{\sqrt{3}}$

gure 7.9

Solution Because $\sqrt{x^2-3}$ has the form $\sqrt{u^2-a^2}$, you can consider

$$u = x, \quad a = \sqrt{3}, \quad \text{and} \quad x = \sqrt{3}\sec\theta$$

as shown in Figure 7.9. Then,

$$dx = \sqrt{3}\sec\theta\tan\theta\,d\theta \qquad \text{and} \qquad \sqrt{x^2-3} = \sqrt{3}\tan\theta.$$

To determine the upper and lower limits of integration, use the substitution $x = \sqrt{3}\sec\theta$, as follows.

Lower Limit

When $x = \sqrt{3}$, $\sec\theta = 1$

and $\theta = 0$.

Upper Limit

When $x = 2$, $\sec\theta = \dfrac{2}{\sqrt{3}}$

and $\theta = \dfrac{\pi}{6}$.

Therefore, you have

Integration limits for x

Integration limits for θ

$$\begin{aligned}
\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx &= \int_{0}^{\pi/6} \frac{(\sqrt{3}\tan\theta)(\sqrt{3}\sec\theta\tan\theta)\,d\theta}{\sqrt{3}\sec\theta}\\
&= \int_{0}^{\pi/6} \sqrt{3}\tan^2\theta\,d\theta\\
&= \sqrt{3}\int_{0}^{\pi/6} (\sec^2\theta - 1)\,d\theta\\
&= \sqrt{3}\Big[\tan\theta - \theta\Big]_{0}^{\pi/6}\\
&= \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right)\\
&= 1 - \frac{\sqrt{3}\pi}{6}\\
&\approx 0.0931.
\end{aligned}$$

In Example 4, try converting back to the variable x and evaluating the antiderivative at the original limits of integration. You should obtain

$$\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx = \sqrt{3}\,\frac{\sqrt{x^2-3}}{\sqrt{3}} - \operatorname{arcsec}\frac{x}{\sqrt{3}}\Bigg]_{\sqrt{3}}^{2}.$$

When using trigonometric substitution to evaluate definite integrals, you must careful to check that the values of θ lie in the intervals discussed at the beginning this section. For instance, if in Example 4 you had been asked to evaluate the defini integral

$$\int_{-2}^{-\sqrt{3}} \frac{\sqrt{x^2-3}}{x}\,dx$$

then using $u = x$ and $a = \sqrt{3}$ in the interval $\left[-2, -\sqrt{3}\right]$ would imply that $u <$ So, when determining the upper and lower limits of integration, you would have choose θ such that $\pi/2 < \theta \leq \pi$. In this case the integral would be evaluated follows.

$$\begin{aligned}
\int_{-2}^{-\sqrt{3}} \frac{\sqrt{x^2-3}}{x}\,dx &= \int_{5\pi/6}^{\pi} \frac{\left(-\sqrt{3}\tan\theta\right)\left(\sqrt{3}\sec\theta\tan\theta\right)d\theta}{\sqrt{3}\sec\theta} \\
&= \int_{5\pi/6}^{\pi} -\sqrt{3}\tan^2\theta\,d\theta \\
&= -\sqrt{3}\int_{5\pi/6}^{\pi} (\sec^2\theta - 1)\,d\theta \\
&= -\sqrt{3}\Big[\tan\theta - \theta\Big]_{5\pi/6}^{\pi} \\
&= -\sqrt{3}\left[(0-\pi) - \left(-\frac{1}{\sqrt{3}} - \frac{5\pi}{6}\right)\right] \\
&= -1 + \frac{\sqrt{3}\pi}{6} \\
&\approx -0.0931
\end{aligned}$$

Trigonometric substitution can be used with completing the square. For instanc try evaluating the following integral.

$$\int \sqrt{x^2-2x}\,dx$$

To begin, you could complete the square and write the integral as

$$\int \sqrt{(x-1)^2-1^2}\,dx.$$

Trigonometric substitution can be used to evaluate the three integrals listed in t following theorem. These integrals will be encountered several times in the remaind of the text. When this happens, we will simply refer to this theorem. (In Exercise you are asked to verify the formulas given in the theorem.)

THEOREM 7.2 Special Integration Formulas ($a > 0$)

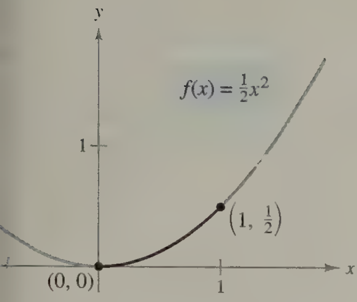
1. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{1}{2}\left(a^2 \arcsin\frac{u}{a} + u\sqrt{a^2-u^2}\right) + C$

2. $\displaystyle\int \sqrt{u^2-a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2-a^2} - a^2 \ln\left|u + \sqrt{u^2+a^2}\right|\right) + C,\ u > a$

3. $\displaystyle\int \sqrt{u^2+a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2+a^2} + a^2 \ln\left|u + \sqrt{u^2+a^2}\right|\right) + C$

Applications

Example 5 Finding Arc Length



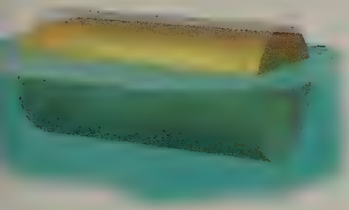
e arc length of the curve from (0, 0) to
$\frac{1}{2}$) is approximately 1.148.
gure 7.10

Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$ (see Figure 7.10).

Solution (Refer to the arc length formula in Section 6.4.)

$$\begin{aligned}
s &= \int_0^1 \sqrt{1 + [f'(x)]^2}\,dx && \text{Formula for arc length}\\
&= \int_0^1 \sqrt{1 + x^2}\,dx && f'(x) = x\\
&= \int_0^{\pi/4} \sec^3 \theta\,d\theta && \text{Let } a = 1 \text{ and } x = \tan\theta.\\
&= \frac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} && \text{Example 5, Section 7.2}\\
&= \frac{1}{2}\big[\sqrt{2} + \ln\big(\sqrt{2} + 1\big)\big] \approx 1.148
\end{aligned}$$

Example 6 Comparing Two Fluid Forces



e barrel is not quite full of oil—the top
foot of the barrel is empty.
gure 7.11

A sealed barrel of oil (weighing 48 pounds per cubic foot) is floating in seawater (weighing 64 pounds per cubic foot), as shown in Figures 7.11 and 7.12. (The barrel is not completely full of oil—on its side, the top 0.2 foot of the barrel is empty.) Compare the fluid forces against one end of the barrel from the inside and from the outside.

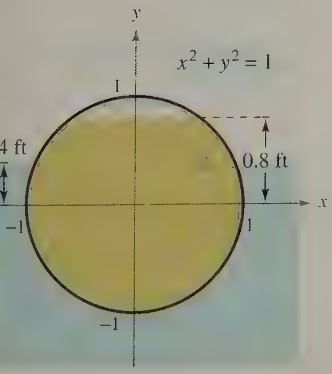
Solution In Figure 7.12, locate the coordinate system with the origin at the center of the circle given by $x^2 + y^2 = 1$. To find the fluid force against an end of the barrel *from the inside*, integrate between -1 and 0.8 (using a weight of $w = 48$).

$$F = w\int_c^d h(y)L(y)\,dy \qquad \text{General equation (see Section 6.7)}$$

$$\begin{aligned}
F_{\text{inside}} &= 48\int_{-1}^{0.8} (0.8 - y)(2)\sqrt{1 - y^2}\,dy\\
&= 76.8\int_{-1}^{0.8} \sqrt{1 - y^2}\,dy - 96\int_{-1}^{0.8} y\sqrt{1 - y^2}\,dy
\end{aligned}$$

To find the fluid force *from the outside*, integrate between -1 and 0.4 (using a weight of $w = 64$).

$$\begin{aligned}
F_{\text{outside}} &= 64\int_{-1}^{0.4} (0.4 - y)(2)\sqrt{1 - y^2}\,dy\\
&= 51.2\int_{-1}^{0.4} \sqrt{1 - y^2}\,dy - 128\int_{-1}^{0.4} y\sqrt{1 - y^2}\,dy
\end{aligned}$$



ure 7.12

We leave the details of integration for you to complete in Exercise 74. Intuitively, would you say that the force from the oil (the inside) or the force from the seawater (the outside) is greater? By evaluating these two integrals, you can determine that

$$F_{\text{inside}} \approx 121.3 \text{ pounds} \qquad \text{and} \qquad F_{\text{outside}} \approx 93.0 \text{ pounds.}$$

EXERCISES FOR SECTION 7.4

In Exercises 1–4, match the antiderivative with the correct integral. [Integrals are labeled (a), (b), (c), and (d).]

(a) $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

(b) $\displaystyle\int \frac{\sqrt{x^2+16}}{x}\,dx$

(c) $\displaystyle\int \sqrt{7+6x-x^2}\,dx$

(d) $\displaystyle\int \frac{x^2}{\sqrt{x^2-16}}\,dx$

1. $4\ln\left|\dfrac{\sqrt{x^2+16}-4}{x}\right| + \sqrt{x^2+16} + C$

2. $8\ln\left|\sqrt{x^2-16}+x\right| + \dfrac{x\sqrt{x^2-16}}{2} + C$

3. $8\arcsin\dfrac{x}{4} - \dfrac{x\sqrt{16-x^2}}{2} + C$

4. $8\arcsin\dfrac{x-3}{4} + \dfrac{(x-3)\sqrt{7+6x-x^2}}{2} + C$

In Exercises 5–8, evaluate the indefinite integral using the substitution $x = 5\sin\theta$.

5. $\displaystyle\int \frac{1}{(25-x^2)^{3/2}}\,dx$

6. $\displaystyle\int \frac{10}{x^2\sqrt{25-x^2}}\,dx$

7. $\displaystyle\int \frac{\sqrt{25-x^2}}{x}\,dx$

8. $\displaystyle\int \frac{x^2}{\sqrt{25-x^2}}\,dx$

In Exercises 9–12, evaluate the indefinite integral using the substitution $x = 2\sec\theta$.

9. $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx$

10. $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$

11. $\displaystyle\int x^3\sqrt{x^2-4}\,dx$

12. $\displaystyle\int \frac{x^3}{\sqrt{x^2-4}}\,dx$

In Exercises 13–16, evaluate the indefinite integral using the substitution $x = \tan\theta$.

13. $\displaystyle\int x\sqrt{1+x^2}\,dx$

14. $\displaystyle\int \frac{9x^3}{\sqrt{1+x^2}}\,dx$

15. $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx$

16. $\displaystyle\int \frac{x^2}{(1+x^2)^2}\,dx$

In Exercises 17 and 18, use Theorem 7.2 to evaluate the integral.

17. $\displaystyle\int \sqrt{4+9x^2}\,dx$

18. $\displaystyle\int \sqrt{1+x^2}\,dx$

In Exercises 19–40, evaluate the integral.

19. $\displaystyle\int \frac{x}{\sqrt{x^2+9}}\,dx$

20. $\displaystyle\int \frac{x}{\sqrt{9-x^2}}\,dx$

21. $\displaystyle\int \frac{1}{\sqrt{16-x^2}}\,dx$

22. $\displaystyle\int \frac{1}{\sqrt{25-x^2}}\,dx$

23. $\displaystyle\int \sqrt{16-4x^2}\,dx$

24. $\displaystyle\int x\sqrt{16-4x^2}\,dx$

25. $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx$

26. $\displaystyle\int \frac{t}{(1-t^2)^{3/2}}\,dt$

27. $\displaystyle\int \frac{\sqrt{1-x^2}}{x^4}\,dx$

28. $\displaystyle\int \frac{\sqrt{4x^2+9}}{x^4}\,dx$

29. $\displaystyle\int \frac{1}{x\sqrt{4x^2+9}}\,dx$

30. $\displaystyle\int \frac{1}{x\sqrt{4x^2+16}}\,dx$

31. $\displaystyle\int \frac{-5x}{(x^2+5)^{3/2}}\,dx$

32. $\displaystyle\int \frac{1}{(x^2+3)^{3/2}}\,dx$

33. $\displaystyle\int e^{2x}\sqrt{1+e^{2x}}\,dx$

34. $\displaystyle\int (x+1)\sqrt{x^2+2x+2}\;d$

35. $\displaystyle\int e^{x}\sqrt{1-e^{2x}}\,dx$

36. $\displaystyle\int \frac{\sqrt{1-x}}{\sqrt{x}}\,dx$

37. $\displaystyle\int \frac{1}{4+4x^2+x^4}\,dx$

38. $\displaystyle\int \frac{x^3+x+1}{x^4+2x^2+1}\,dx$

39. $\displaystyle\int \operatorname{arcsec} 2x\,dx,\quad x > \tfrac{1}{2}$

40. $\displaystyle\int x\arcsin x\,dx$

In Exercises 41–44, complete the square and evaluate t integral.

41. $\displaystyle\int \frac{1}{\sqrt{4x-x^2}}\,dx$

42. $\displaystyle\int \frac{x^2}{\sqrt{2x-x^2}}\,dx$

43. $\displaystyle\int \frac{x}{\sqrt{x^2+4x+8}}\,dx$

44. $\displaystyle\int \frac{x}{\sqrt{x^2-6x+5}}\,dx$

In Exercises 45–50, evaluate the integral using (a) the giv integration limits and (b) the limits obtained by trigonomet substitution.

45. $\displaystyle\int_0^{\sqrt{3}/2} \frac{t^2}{(1-t^2)^{3/2}}\,dt$

46. $\displaystyle\int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}}\,dt$

47. $\displaystyle\int_0^{3} \frac{x^3}{\sqrt{x^2+9}}\,dx$

48. $\displaystyle\int_0^{3/5} \sqrt{9-25x^2}\,dx$

49. $\displaystyle\int_4^{6} \frac{x^2}{\sqrt{x^2-9}}\,dx$

50. $\displaystyle\int_3^{6} \frac{\sqrt{x^2-9}}{x^2}\,dx$

In Exercises 51–54, use a computer algebra system to evalua the integral. Verify the result by differentiation.

51. $\displaystyle\int \frac{x^2}{\sqrt{x^2+10x+9}}\,dx$

52. $\displaystyle\int (x^2+2x+11)^{3/2}\,dx$

53. $\displaystyle\int \frac{x^2}{\sqrt{x^2-1}}\,dx$

54. $\displaystyle\int x^2\sqrt{x^2-4}\,dx$

Getting at the Concept

5. State the substitution you would make if you used trigonometric substitution and the integral involved the given radical, where $a > 0$.

(a) $\sqrt{a^2 - u^2}$ (b) $\sqrt{a^2 + u^2}$

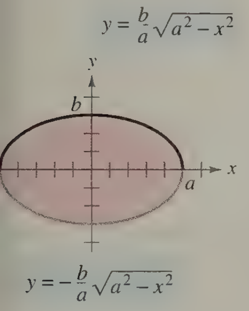
(c) $\sqrt{u^2 - a^2}$

6. State the method of integration you would use to perform the integration. Do not integrate.

(a) $\int x\sqrt{x^2 + 1}\, dx$ (b) $\int x^2\sqrt{x^2 - 1}\, dx$

Area Find the area enclosed by the ellipse shown in the figure.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$


$y = \frac{b}{a}\sqrt{a^2 - x^2}$
y
b
x
a
$y = -\frac{b}{a}\sqrt{a^2 - x^2}$

ure for 57

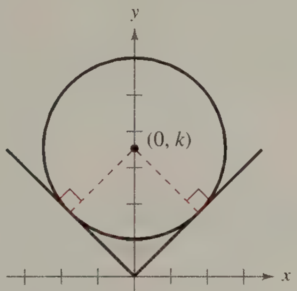


Figure for 58

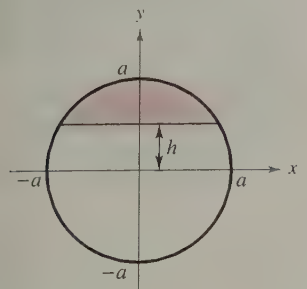
Mechanical Design The surface of a machine part is the region between the graphs of $y = |x|$ and $x^2 + (y - k)^2 = 25$ (see figure).

(a) Find k if the circle is tangent to the graph of $y = |x|$.

(b) Find the area of the surface of the machine part.

(c) Find the area of the surface of the machine part as a function of the radius of the circle r.

Area Find the area of the shaded region of the circle of radius a, if the chord is h units $(0 < h < a)$ from the center of the circle (see figure).



Volume The axis of a storage tank in the form of a right circular cylinder is horizontal (see figure). The radius and length of the tank are 1 meter and 3 meters.

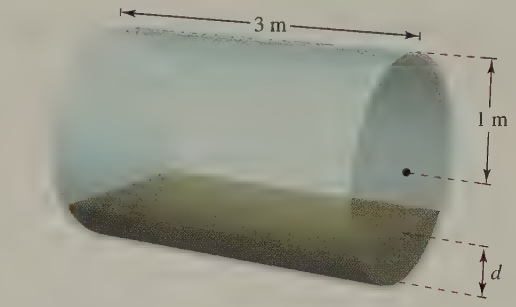
(a) Determine the volume of fluid in the tank as a function of its depth d.

(b) Use a graphing utility to graph the function in part (a).

(c) Design a dip stick for the tank with markings of $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.

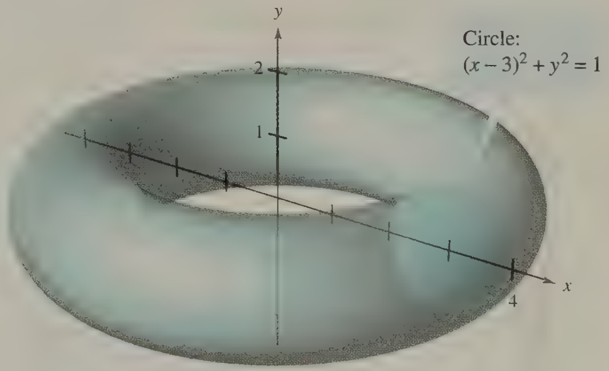
(d) If fluid is entering the tank at a rate of $\frac{1}{4}$ cubic meter per minute, determine the rate of change of depth of the fluid as a function of its depth d.

(e) Use a graphing utility to graph the function in part (d). When will the rate of change of depth be minimum? Does this agree with your intuition? Explain.



Volume of a Torus **In Exercises 61 and 62, find the volume of the torus generated by revolving the region bounded by the graph of the circle about the y-axis.**

61. $(x - 3)^2 + y^2 = 1$ (see figure)



62. $(x - h)^2 + y^2 = r^2$, $h > r$

Arc Length **In Exercises 63 and 64, find the arc length of the curve over the indicated interval.**

	Function	*Interval*
63.	$y = \ln x$	$[1, 5]$
64.	$y = \frac{1}{2}x^2$	$[0, 4]$

65. ***Arc Length*** Show that the length of one arch of the sine curve is equal to the length of one arch of the cosine curve.

66. ***Conjecture***

(a) Find formulas for the distance between $(0, 0)$ and (a, a^2) along the line and along the parabola $y = x^2$.

(b) Use the formulas from part (a) to find the distances for $a = 1$ and $a = 10$.

(c) Make a conjecture about the difference between the two distances as a increases.

Projectile Motion **In Exercises 67 and 68, (a) use a graphing utility to graph the path of a projectile that follows the path given by the graph of the equation, (b) determine the range of the projectile, and (c) use the integration capabilities of a graphing utility to determine the distance the projectile travels.**

67. $y = x - 0.005x^2$

68. $y = x - \dfrac{x^2}{72}$

Centroid **In Exercises 69 and 70, find the centroid of the region determined by the graphs of the inequalities.**

69. $y \le 3/\sqrt{x^2+9},\ y \ge 0,\ x \ge -4,\ x \le 4$

70. $y \le \frac{1}{4}x^2,\ (x-4)^2 + y^2 \le 16,\ y \ge 0$

71. ***Surface Area*** Find the surface area of the solid generated by revolving the region bounded by the graphs of $y = x^2$, $y = 0$, $x = 0$, and $x = \sqrt{2}$ about the x-axis.

72. ***Average Field Strength*** The field strength H of a magnet of length $2L$ on a particle r units from the center of the magnet is

$$H = \frac{2mL}{(r^2 + L^2)^{3/2}}$$

where $\pm m$ are the poles of the magnet (see figure). Find the average field strength as the particle moves from 0 to R units from the center by evaluating the integral

$$\frac{1}{R}\int_0^R \frac{2mL}{(r^2+L^2)^{3/2}}\,dr.$$

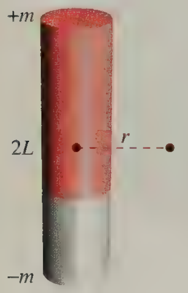


Figure for 72

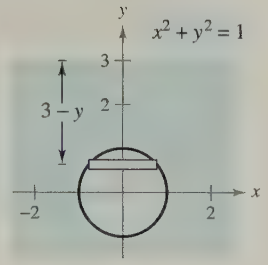


Figure for 73

73. ***Fluid Force*** Find the fluid force on a circular observation window of radius 1 foot in a vertical wall of a large water-filled tank at a fish hatchery for each of the indicated depths (see figure). Use trigonometric substitution to evaluate the one integral. (Recall that in Section 6.7 in a similar problem, you evaluated one integral by a geometric formula and the other by observing that the integrand was odd.)

(a) The center of the window is 3 feet below the water's surface.

(b) The center of the window is d feet below the water's surface ($d > 1$).

74. ***Fluid Force*** Evaluate the following two integrals, which yield the fluid forces given in Example 6.

(a) $F_{\text{inside}} = 48\displaystyle\int_{-1}^{0.8} (0.8 - y)(2)\sqrt{1 - y^2}\,dy$

(b) $F_{\text{outside}} = 64\displaystyle\int_{-1}^{0.4} (0.4 - y)(2)\sqrt{1 - y^2}\,dy$

75. ***Tractrix*** A person moves from the origin along the posit y-axis pulling a weight at the end of a 12-meter rope (figure). Initially, the weight is located at the point (12, 0).

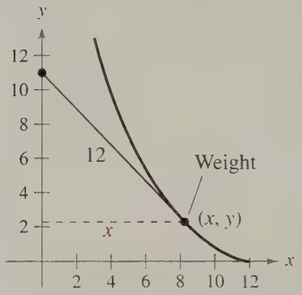
(a) Show that the slope of the tangent line of the path of weight is

$$\frac{dy}{dx} = -\frac{\sqrt{144 - x^2}}{x}.$$

(b) Use the result in part (a) to find the equation of the path the weight. Use a graphing utility to graph the path a compare it with the figure.

(c) Find any vertical asymptotes of the graph in part (b).

(d) When the person has reached the point (0, 12), how far the weight moved?



76. ***Modeling Data*** For the years 1990 through 1997, the aver size S (in thousands of dollars) of ordinary life insura policies in force in the United States is given in the tal *(Source: American Council of Life Insurance)*

Year	1990	1991	1992	1993	1994	1995	1996	199
S	37.9	41.5	43.0	45.8	45.9	49.1	52.3	56

A model for these data is

$$S = \sqrt{1520.4 + 111.2t + 15.8t^2}$$

where t is the time in years, with $t = 0$ corresponding to 19 Use a graphing utility to answer each of the following.

(a) Graph the model for $0 \le t \le 7$.

(b) Find the rate of increase in S when $t = 5$.

(c) Use the model and integration to predict the average va of S for the years 2000 through 2002.

True or False? **In Exercises 77–80, determine whether statement is true or false. If it is false, explain why or give example that shows it is false.**

77. If $x = \sin\theta$, then $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \int d\theta$.

78. If $x = \sec\theta$, then $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \sec\theta\tan\theta\,d\theta$.

79. If $x = \tan\theta$, then $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{(1+x^2)^{3/2}} = \int_0^{4\pi/3} \cos\theta\,d\theta$.

80. If $x = \sin\theta$, then $\displaystyle\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx = 2\int_0^{\pi/2} \sin^2\theta\cos^2\theta$

81. Use trigonometric substitution to verify the integration for las given in Theorem 7.2.

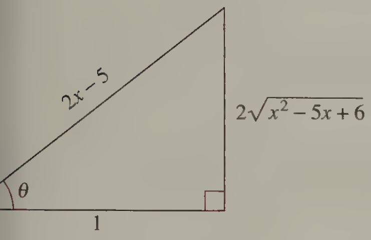
Section 7.5 Partial Fractions

- Understand the concept of a partial fraction decomposition.
- Use partial fraction decomposition with linear factors to integrate rational functions.
- Use partial fraction decomposition with quadratic factors to integrate rational functions.

Partial Fractions

This section examines a procedure for decomposing a rational function into simpler rational functions to which you can apply the basic integration formulas. This procedure is called the **method of partial fractions.** To see the benefit of the method of partial fractions, consider the integral

$$\int \frac{1}{x^2 - 5x + 6}\,dx.$$



$\theta = 2x - 5$
ure 7.13

To evaluate this integral *without* partial fractions, you can complete the square and use trigonometric substitution (see Figure 7.13) to obtain the following.

$$\begin{aligned}
\int \frac{1}{x^2 - 5x + 6}\,dx &= \int \frac{dx}{(x - 5/2)^2 - (1/2)^2} && a = \tfrac{1}{2},\ x - \tfrac{5}{2} = \tfrac{1}{2}\sec\theta \\
&= \int \frac{(1/2)\sec\theta\tan\theta\,d\theta}{(1/4)\tan^2\theta} && dx = \tfrac{1}{2}\sec\theta\tan\theta\,d\theta \\
&= 2\int \csc\theta\,d\theta \\
&= 2\ln|\csc\theta - \cot\theta| + C \\
&= 2\ln\left|\frac{2x - 5}{2\sqrt{x^2 - 5x + 6}} - \frac{1}{2\sqrt{x^2 - 5x + 6}}\right| + C \\
&= 2\ln\left|\frac{x - 3}{\sqrt{x^2 - 5x + 6}}\right| + C \\
&= 2\ln\left|\frac{\sqrt{x - 3}}{\sqrt{x - 2}}\right| + C \\
&= \ln\left|\frac{x - 3}{x - 2}\right| + C \\
&= \ln|x - 3| - \ln|x - 2| + C
\end{aligned}$$

JOHN BERNOULLI (1667–1748)

e method of partial fractions was intro-
ced by John Bernoulli, a Swiss mathemati-
n who was instrumental in the early
velopment of calculus. John Bernoulli was
rofessor at the University of Basel and
ight many outstanding students, the most
nous of whom was Leonhard Euler.

Now, suppose you had observed that

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x - 3} - \frac{1}{x - 2}. \qquad \text{Partial fraction decomposition}$$

Then you could evaluate the integral easily, as follows.

$$\begin{aligned}
\int \frac{1}{x^2 - 5x + 6}\,dx &= \int \left(\frac{1}{x - 3} - \frac{1}{x - 2}\right)dx \\
&= \ln|x - 3| - \ln|x - 2| + C
\end{aligned}$$

This method is clearly preferable to trigonometric substitution. However, its use depends on the ability to factor the denominator, $x^2 - 5x + 6$, and to find the **partial fractions**

$$\frac{1}{x - 3} \quad \text{and} \quad -\frac{1}{x - 2}.$$

In this section, you will study techniques for finding partial fraction decompositions.

STUDY TIP In precalculus you learned how to combine functions such as

$$\frac{1}{x-2}+\frac{-1}{x+3}=\frac{5}{(x-2)(x+3)}.$$

The method of partial fractions shows you how to reverse this process.

$$\frac{5}{(x-2)(x+3)}=\frac{?}{x-2}+\frac{?}{x+3}$$

Recall from algebra that every polynomial with real coefficients can be facto into linear and irreducible quadratic factors.* For instance, the polynomial

$$x^5+x^4-x-1$$

can be written as

$$\begin{aligned}x^5+x^4-x-1&=x^4(x+1)-(x+1)\\&=(x^4-1)(x+1)\\&=(x^2+1)(x^2-1)(x+1)\\&=(x^2+1)(x+1)(x-1)(x+1)\\&=(x-1)(x+1)^2(x^2+1)\end{aligned}$$

where $(x-1)$ is a linear factor, $(x+1)^2$ is a repeated linear factor, and (x^2+1) is irreducible quadratic factor. Using this factorization, you can write the partial fract decomposition of the rational expression

$$\frac{N(x)}{x^5+x^4-x-1}$$

where $N(x)$ is a polynomial of degree less than 5, as follows.

$$\frac{N(x)}{(x-1)(x+1)^2(x^2+1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{Dx+E}{x^2+1}$$

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. **Divide if improper:** If $N(x)/D(x)$ is an improper fraction (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)}=(\text{a polynomial})+\frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply steps 2 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. **Factor denominator:** Completely factor the denominator into factors of the form

$$(px+q)^m \qquad \text{and} \qquad (ax^2+bx+c)^n$$

where ax^2+bx+c is irreducible.

3. **Linear factors:** For each factor of the form $(px+q)^m$, the partial fractio decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px+q)}+\frac{A_2}{(px+q)^2}+\cdots+\frac{A_m}{(px+q)^m}$$

4. **Quadratic factors:** For each factor of the form $(ax^2+bx+c)^n$, the partia fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

---

* *For a review of factorization techniques, see* Precalculus, *5th edition, by Larson and Hoste or* Precalculus: A Graphing Approach, *3rd edition, by Larson, Hostetler, and Edwards (Bos Massachusetts: Houghton Mifflin, 2001).*

Linear Factors

Algebraic techniques for determining the constants in the numerators of a partial decomposition with linear or repeated linear factors are demonstrated in Examples 1 and 2.

Example 1 Distinct Linear Factors

Write the partial fraction decomposition for $\dfrac{1}{x^2 - 5x + 6}$.

Solution Because $x^2 - 5x + 6 = (x - 3)(x - 2)$, you should include one partial fraction for each factor and write

$$\frac{1}{x^2 - 5x + 6} = \frac{A}{x - 3} + \frac{B}{x - 2}$$

where A and B are to be determined. Multiplying this equation by the least common denominator $(x - 3)(x - 2)$ yields the **basic equation**

$$1 = A(x - 2) + B(x - 3). \qquad \text{Basic equation}$$

Because this equation is to be true for all x, you can substitute any *convenient* values for x to obtain equations in A and B. The most convenient values are the ones that make particular factors equal to 0.

TE Note that the substitutions for x
Example 1 are chosen for their conve-
nce in determining values for A and
$x = 2$ is chosen to eliminate the term
$- 2)$, and $x = 3$ is chosen to elimi-
e the term $B(x - 3)$. The goal is to
ke *convenient* substitutions whenever
sible.

To solve for A, let $x = 3$ and obtain

$$\begin{aligned} 1 &= A(3 - 2) + B(3 - 3) && \text{Let } x = 3 \text{ in basic equation.} \\ 1 &= A(1) + B(0) \\ A &= 1. \end{aligned}$$

To solve for B, let $x = 2$ and obtain

$$\begin{aligned} 1 &= A(2 - 2) + B(2 - 3) && \text{Let } x = 2 \text{ in basic equation.} \\ 1 &= A(0) + B(-1) \\ B &= -1. \end{aligned}$$

Therefore, the decomposition is

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x - 3} - \frac{1}{x - 2}$$

as indicated at the beginning of this section.

R FURTHER INFORMATION To learn
fferent method for finding the partial
ction decomposition, called the
avyside Method, see the article
lculus to Algebra Connections in
tial Fraction Decomposition" by
eph Wiener and Will Watkins in *The*
ATYC Review. To view this article,
o the website *www.matharticles.com*.

Be sure you see that the method of partial fractions is practical only for integrals of rational functions whose denominators factor "nicely." For instance, if the denominator in Example 1 were changed to $x^2 - 5x + 5$, its factorization as

$$x^2 - 5x + 5 = \left[x + \frac{5 + \sqrt{5}}{2}\right]\left[x - \frac{5 - \sqrt{5}}{2}\right]$$

would be too cumbersome to use with partial fractions. In such cases, you should use completing the square or a computer algebra system to perform the integration. If you do this, you should obtain

$$\int \frac{1}{x^2 - 5x + 5}\,dx = \frac{\sqrt{5}}{5}\ln\left|2x - \sqrt{5} - 5\right| - \frac{\sqrt{5}}{5}\ln\left|2x + \sqrt{5} - 5\right| + C.$$

Example 2 Repeated Linear Factors

Evaluate $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx.$

Solution Because

$$\begin{aligned} x^3+2x^2+x &= x(x^2+2x+1)\\ &= x(x+1)^2 \end{aligned}$$

FOR FURTHER INFORMATION For an alternative approach to using partial fractions, see the article " A Shortcut in Partial Fractions" by Xun-Cheng Huang in *The College Mathematics Journal.* To view this article, go to the website *www.matharticles.com.*

you should include one fraction for *each power* of x and $(x+1)$ and write

$$\frac{5x^2+20x+6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Multiplying by the least common denominator $x(x+1)^2$ yields the *basic equation*

$$5x^2+20x+6 = A(x+1)^2 + Bx(x+1) + Cx. \qquad \text{Basic equation}$$

To solve for A, let $x=0$. This eliminates the B and C terms and yields

$$\begin{aligned} 6 &= A(1)+0+0\\ A &= 6. \end{aligned}$$

To solve for C, let $x=-1$. This eliminates the A and B terms and yields

$$\begin{aligned} 5-20+6 &= 0+0-C\\ C &= 9. \end{aligned}$$

The most convenient choices for x have been used, so to find the value of B, you c use *any other value* of x along with the calculated values of A and C. Using $x =$ $A=6$, and $C=9$ produces

$$\begin{aligned} 5+20+6 &= A(4)+B(2)+C\\ 31 &= 6(4)+2B+9\\ -2 &= 2B\\ B &= -1. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} \int \frac{5x^2+20x+6}{x(x+1)^2}\,dx &= \int \left(\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right) dx\\ &= 6\ln|x| - \ln|x+1| + 9\frac{(x+1)^{-1}}{-1} + C\\ &= \ln\left|\frac{x^6}{x+1}\right| - \frac{9}{x+1} + C. \end{aligned}$$

Try checking this result by differentiating. Include algebra in your check, simplifyi the derivative until you have obtained the original integrand.

NOTE It is necessary to make as many substitutions for x as there are unknow $(A, B, C, \ldots)$ to be determined. For instance, in Example 2, we made three substituti $(x=-1, x=0,$ and $x=1)$ to solve for C, A, and B.

TECHNOLOGY Most computer algebra systems, such as *Derive, Maple, Mathcad, Mathematica,* and the *TI-89,* can be used to convert a rational function to its partial fraction decomposition. For instance, using *Maple,* you obtain the following.

$$> \text{convert}\left(\frac{5x^2+20x+6}{x^3+2x^2+x}, \text{parfrac}, x\right)$$

$$\frac{6}{x} + \frac{9}{(x+1)^2} - \frac{1}{x+1}$$

Quadratic Factors

When using the method of partial fractions with *linear* factors, a convenient choice of x immediately yields a value for one of the coefficients. With *quadratic* factors, a system of linear equations usually has to be solved, regardless of the choice of x.

Example 3 Distinct Linear and Quadratic Factors

Evaluate $\displaystyle\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)}\,dx.$

Solution Because

$$(x^2 - x)(x^2 + 4) = x(x - 1)(x^2 + 4)$$

you should include one partial fraction for each factor and write

$$\frac{2x^3 - 4x - 8}{x(x - 1)(x^2 + 4)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + 4}.$$

Multiplying by the least common denominator $x(x - 1)(x^2 + 4)$ yields the *basic equation*

$$2x^3 - 4x - 8 = A(x - 1)(x^2 + 4) + Bx(x^2 + 4) + (Cx + D)(x)(x - 1).$$

To solve for A, let $x = 0$ and obtain

$$-8 = A(-1)(4) + 0 + 0 \quad \Longrightarrow \quad 2 = A.$$

To solve for B, let $x = 1$ and obtain

$$-10 = 0 + B(5) + 0 \quad \Longrightarrow \quad -2 = B.$$

At this point, C and D are yet to be determined. You can find these remaining constants by choosing two other values for x and solving the resulting system of linear equations. If $x = -1$, then, using $A = 2$ and $B = -2$, you can write

$$-6 = (2)(-2)(5) + (-2)(-1)(5) + (-C + D)(-1)(-2)$$
$$2 = -C + D.$$

If $x = 2$, you have

$$0 = (2)(1)(8) + (-2)(2)(8) + (2C + D)(2)(1)$$
$$8 = 2C + D.$$

Solving the linear system by subtracting the first equation from the second

$$-C + D = 2$$
$$2C + D = 8$$

yields $C = 2$. Consequently, $D = 4$, and it follows that

$$\int \frac{2x^3 - 4x - 8}{x(x - 1)(x^2 + 4)}\,dx =$$
$$\int \left(\frac{2}{x} - \frac{2}{x - 1} + \frac{2x}{x^2 + 4} + \frac{4}{x^2 + 4}\right) dx =$$
$$2 \ln|x| - 2 \ln|x - 1| + \ln(x^2 + 4) + 2 \arctan \frac{x}{2} + C.$$

In Examples 1, 2, and 3, we began the solution of the basic equation by subs tuting values of x that made the linear factors equal to 0. This method works w when the partial fraction decomposition involves linear factors. However, if t decomposition involves only quadratic factors, an alternative procedure is often mo convenient.

Example 4 Repeated Quadratic Factors

Evaluate $\displaystyle\int \frac{8x^3 + 13x}{(x^2 + 2)^2}\,dx$.

Solution Include one partial fraction for each power of $(x^2 + 2)$ and write

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2}.$$

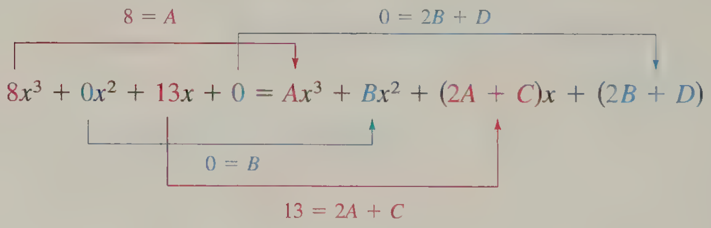
Multiplying by the least common denominator $(x^2 + 2)^2$ yields the *basic equation*

$$8x^3 + 13x = (Ax + B)(x^2 + 2) + Cx + D.$$

Expanding the basic equation and collecting like terms produces

$$8x^3 + 13x = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$$
$$8x^3 + 13x = Ax^3 + Bx^2 + (2A + C)x + (2B + D).$$

Now, you can equate the coefficients of like terms on opposite sides of the equatio


$8 = A$
$0 = 2B + D$
$8x^3 + 0x^2 + 13x + 0 = Ax^3 + Bx^2 + (2A + C)x + (2B + D)$
$0 = B$
$13 = 2A + C$

Using the known values $A = 8$ and $B = 0$, you can write the following.

$$13 = 2A + C = 2(8) + C \quad \Rightarrow \quad C = -3$$
$$0 = 2B + D = 2(0) + D \quad \Rightarrow \quad D = 0$$

Finally, you can conclude that

$$\int \frac{8x^3 + 13x}{(x^2 + 2)^2}\,dx = \int \left(\frac{8x}{x^2 + 2} + \frac{-3x}{(x^2 + 2)^2}\right) dx$$
$$= 4\ln(x^2 + 2) + \frac{3}{2(x^2 + 2)} + C.$$

TECHNOLOGY Use a computer algebra system to evaluate the integral Example 4—you might find that the form of the antiderivative is different. F instance, when you use a computer algebra system to work Example 4, you obta

$$\int \frac{8x^3 + 13x}{(x^2 + 2)^2}\,dx = \ln(x^8 + 8x^6 + 24x^4 + 32x^2 + 16) + \frac{3}{2(x^2 + 2)} + C.$$

Is this result equivalent to that obtained in Example 4?

When integrating rational expressions, keep in mind that for *improper* rational expressions such as

$$\frac{N(x)}{D(x)} = \frac{2x^3 + x^2 - 7x + 7}{x^2 + x - 2}$$

you must first divide to obtain

$$\frac{N(x)}{D(x)} = 2x - 1 + \frac{-2x + 5}{x^2 + x - 2}.$$

The proper rational expression is then decomposed into its partial fractions by the usual methods. Here are some guidelines for solving the basic equation that is obtained in a partial fraction decomposition.

Guidelines for Solving the Basic Equation

Linear Factors

1. Substitute the roots of the distinct linear factors into the basic equation.
2. For repeated linear factors, use the coefficients determined in guideline 1 to rewrite the basic equation. Then substitute other convenient values of x and solve for the remaining coefficients.

Quadratic Factors

1. Expand the basic equation.
2. Collect terms according to powers of x.
3. Equate the coefficients of like powers to obtain a system of linear equations involving A, B, C, and so on.
4. Solve the system of linear equations.

Before concluding this section, here are a few things you should remember. First, it is not necessary to use the partial fractions technique on all rational functions. For instance, the following integral is evaluated more easily by the Log Rule.

$$\int \frac{x^2 + 1}{x^3 + 3x - 4}\,dx = \frac{1}{3}\int \frac{3x^2 + 3}{x^3 + 3x - 4}\,dx = \frac{1}{3}\ln|x^3 + 3x - 4| + C$$

Second, if the integrand is not in reduced form, reducing it may eliminate the need for partial fractions, as shown in the following integral.

$$\begin{aligned}\int \frac{x^2 - x - 2}{x^3 - 2x - 4}\,dx &= \int \frac{(x + 1)(x - 2)}{(x - 2)(x^2 + 2x + 2)}\,dx \\ &= \int \frac{x + 1}{x^2 + 2x + 2}\,dx \\ &= \frac{1}{2}\ln|x^2 + 2x + 2| + C\end{aligned}$$

Finally, partial fractions can be used with some quotients involving transcendental functions. For instance, the substitution $u = \sin x$ allows you to write

$$\int \frac{\cos x}{\sin x(\sin x - 1)}\,dx = \int \frac{du}{u(u - 1)}. \qquad u = \sin x,\ du = \cos x\,dx$$

EXERCISES FOR SECTION 7.5

In Exercises 1–6, give the form of the partial fraction decomposition of the rational expression. Do not solve for the constants.

1. $\frac{5}{x^2-10x}$

2. $\frac{4x^2+3}{(x-5)^3}$

3. $\frac{2x-3}{x^3+10x}$

4. $\frac{x-2}{x^2+4x+3}$

5. $\frac{16x}{x^3-10x^2}$

6. $\frac{2x-1}{x(x^2+1)^2}$

In Exercises 7–28, use partial fractions to evaluate the integral.

7. $\int \frac{1}{x^2-1}\,dx$

8. $\int \frac{1}{4x^2-9}\,dx$

9. $\int \frac{3}{x^2+x-2}\,dx$

10. $\int \frac{x+1}{x^2+4x+3}\,dx$

11. $\int \frac{5-x}{2x^2+x-1}\,dx$

12. $\int \frac{5x^2-12x-12}{x^3-4x}\,dx$

13. $\int \frac{x^2+12x+12}{x^3-4x}\,dx$

14. $\int \frac{x^3-x+3}{x^2+x-2}\,dx$

15. $\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}\,dx$

16. $\int \frac{x+2}{x^2-4x}\,dx$

17. $\int \frac{4x^2+2x-1}{x^3+x^2}\,dx$

18. $\int \frac{2x-3}{(x-1)^2}\,dx$

19. $\int \frac{x^2+3x-4}{x^3-4x^2+4x}\,dx$

20. $\int \frac{4x^2}{x^3+x^2-x-1}\,dx$

21. $\int \frac{x^2-1}{x^3+x}\,dx$

22. $\int \frac{6x}{x^3-8}\,dx$

23. $\int \frac{x^2}{x^4-2x^2-8}\,dx$

24. $\int \frac{x^2-x+9}{(x^2+9)^2}\,dx$

25. $\int \frac{x}{16x^4-1}\,dx$

26. $\int \frac{x^2-4x+7}{x^3-x^2+x+3}\,dx$

27. $\int \frac{x^2+5}{x^3-x^2+x+3}\,dx$

28. $\int \frac{x^2+x+3}{x^4+6x^2+9}\,dx$

In Exercises 29–32, evaluate the definite integral. Use a graphing utility to verify your result.

29. $\int_0^1 \frac{3}{2x^2+5x+2}\,dx$

30. $\int_1^5 \frac{x-1}{x^2(x+1)}\,dx$

31. $\int_1^2 \frac{x+1}{x(x^2+1)}\,dx$

32. $\int_0^1 \frac{x^2-x}{x^2+x+1}\,dx$

In Exercises 33–40, use a computer algebra system to determine the antiderivative that passes through the indicated point. Use the system to graph the resulting antiderivative.

33. $\int \frac{3x}{x^2-6x+9}\,dx$, $(4, 0)$

34. $\int \frac{6x^2+1}{x^2(x-1)^3}\,dx$, $(2, 1)$

35. $\int \frac{x^2+x+2}{(x^2+2)^2}\,dx$, $(0, 1)$

36. $\int \frac{x^3}{(x^2-4)^2}\,dx$, $(3, 4)$

37. $\int \frac{2x^2-2x+3}{x^3-x^2-x-2}\,dx$, $(3, 10)$

38. $\int \frac{x(2x-9)}{x^3-6x^2+12x-8}\,dx$, $(3, 2)$

39. $\int \frac{1}{x^2-4}\,dx$, $(6, 4)$

40. $\int \frac{x^2-x+2}{x^3-x^2+x-1}\,dx$, (2,

In Exercises 41–46, use substitution to evaluate the integral.

41. $\int \frac{\sin x}{\cos x(\cos x-1)}\,dx$

42. $\int \frac{\sin x}{\cos x+\cos^2 x}\,dx$

43. $\int \frac{3\cos x}{\sin^2 x+\sin x-2}\,dx$

44. $\int \frac{\sec^2 x}{\tan x(\tan x+1)}\,dx$

45. $\int \frac{e^x}{(e^x-1)(e^x+4)}\,dx$

46. $\int \frac{e^x}{(e^{2x}+1)(e^x-1)}\,dx$

In Exercises 47–50, use the method of partial fractions to ver the integration formula.

47. $\int \frac{1}{x(a+bx)}\,dx = \frac{1}{a}\ln\left|\frac{x}{a+bx}\right| + C$

48. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

49. $\int \frac{x}{(a+bx)^2}\,dx = \frac{1}{b^2}\left(\frac{a}{a+bx} + \ln|a+bx|\right) + C$

50. $\int \frac{1}{x^2(a+bx)}\,dx = -\frac{1}{ax} - \frac{b}{a^2}\ln\left|\frac{x}{a+bx}\right| + C$

In Exercises 51 and 52, use a computer algebra system to sket the slope field for the differential equation, and graph the so tion through the specified initial condition.

51. $\frac{dy}{dx} = \frac{6}{4-x^2}$

$y(0) = 3$

52. $\frac{dy}{dx} = \frac{4}{x^2-2x-3}$

$y(0) = 5$

Getting at the Concept

53. What is the first step when integrating $\int \frac{x^3}{x-5}\,dx$?

54. Describe the decomposition of the proper rational functio $N(x)/D(x)$ (a) if $D(x) = (px+q)^m$, and (b) if $D(x) =$ $(ax^2+bx+c)^n$ where ax^2+bx+c is irreducible.

55. State the method you would use to evaluate each integra Do not integrate.

(a) $\int \frac{x+1}{x^2+2x-8}\,dx$

(b) $\int \frac{7x+4}{x^2+2x-8}\,dx$

(c) $\int \frac{4}{x^2+2x+5}\,dx$

Area Find the area of the region bounded by the graphs of $y = 7/(16 - x^2)$ and $y = 1$.

Modeling Data The predicted cost C (in 100,000s of dollars) for a company to remove $p\%$ of a chemical from its waste water is shown in the table.

p	0	10	20	30	40	50	60	70	80	90
C	0	0.7	1.0	1.3	1.7	2.0	2.7	3.6	5.5	11.2

A model for the data is

$$C = \frac{124p}{(10 + p)(100 - p)}, \quad 0 \le p < 100.$$

Use the model to find the average cost for removing between 75% and 80% of the chemical.

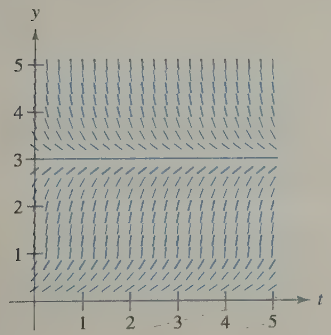
Logistics Growth In Chapter 5, the exponential growth equation was derived from the assumption that the rate of growth was proportional to the existing quantity. In practice, there often exists some upper limit L past which growth cannot occur. In such cases, we assume the rate of growth to be proportional not only to the existing quantity, but also to the difference between the existing quantity y and the upper limit L. That is,

$$\frac{dy}{dt} = ky(L - y).$$

In integral form, we can express this relationship as

$$\int \frac{dy}{y(L - y)} = \int k\,dt.$$

(a) A slope field for the differential equation $dy/dt = y(3 - y)$ is shown. Draw a possible solution to the differential equation if $y(0) = 5$, and another if $y(0) = \frac{1}{2}$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*



(b) Where $y(0)$ is greater than 3, what is the sign of the slope of the solution?

(c) For $y > 0$, find $\lim\limits_{t \to \infty} y(t)$.

(d) Evaluate the two integrals above and solve for y as a function of t, where y_0 is the initial quantity.

(e) Use the result in part (d) to find and graph the solutions in part (a). Use a graphing utility to graph the solutions and compare the results with the solutions in part (a).

(f) The graph of the function y is called a **logistics curve.** Show that the rate of growth is maximum at the point of inflection, and that this occurs when $y = L/2$.

59. ***Approximation*** Determine which value best approximates the area of the region between the x-axis and the graph of the function $10/[x(x^2 + 1)]$ over the interval $[1, 3]$. Make your selection on the basis of a sketch of the region and not by performing any calculations.

(a) -6 (b) 6 (c) 3 (d) 5 (e) 8

60. ***Volume and Centroid*** Consider the region bounded by the graphs of

$$y = 2x/(x^2 + 1),\ y = 0,\ x = 0, \text{ and } x = 3.$$

(a) Find the volume of the solid generated by revolving the region about the x-axis.

(b) Find the centroid of the region.

61. ***Epidemic Model*** A single infected individual enters a community of n susceptible individuals. Let x be the number of newly infected individuals at time t. The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So

$$\frac{dx}{dt} = k(x + 1)(n - x)$$

and you obtain

$$\int \frac{1}{(x + 1)(n - x)}\,dx = \int k\,dt.$$

Solve for x as a function of t.

62. ***Chemical Reactions*** In a chemical reaction, one unit of compound Y and one unit of compound Z are converted into a single unit of compound X. If x is the amount of compound X formed, and the rate of formation of X is proportional to the product of the amounts of unconverted compounds Y and Z, then

$$\frac{dx}{dt} = k(y_0 - x)(z_0 - x)$$

where y_0 and z_0 are the initial amounts of compounds Y and Z. From the above equation you obtain

$$\int \frac{1}{(y_0 - x)(z_0 - x)}\,dx = \int k\,dt.$$

(a) Perform the two integrations and solve for x in terms of t.

(b) Use the result in part (a) to find x as $t \to \infty$ if (1) $y_0 < z_0$, (2) $y_0 > z_0$, and (3) $y_0 = z_0$.

63. Evaluate

$$\int_0^1 \frac{x}{1 + x^4}\,dx$$

in two different ways, one of which is partial fractions.

Section 7.6 Integration by Tables and Other Integration Techniques

- Evaluate an indefinite integral using a table of integrals.
- Evaluate an indefinite integral using reduction formulas.
- Evaluate an indefinite integral involving rational functions of sine and cosine.

Integration by Tables

So far in this chapter you have studied several integration techniques that can be us with the basic integration rules. But merely knowing *how* to use the various techniqu is not enough. You also need to know *when* to use them. Integration is first and fo most a problem of recognition. That is, you must recognize which rule or technique apply to obtain an antiderivative. Frequently, a slight alteration of an integrand w require a different integration technique (or produce a function whose antiderivative not an elementary function), as shown below.

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$$ Integration by parts

$$\int \frac{\ln x}{x} \, dx = \frac{(\ln x)^2}{2} + C$$ Power Rule

$$\int \frac{1}{x \ln x} \, dx = \ln|\ln x| + C$$ Log Rule

$$\int \frac{x}{\ln x} \, dx = ?$$ Not an elementary functic

TECHNOLOGY A computer algebra system consists, in part, of a database of integration formulas. The primary difference between using a computer algebra system and using tables of integrals is that with a computer algebra system the computer searches through the database to find a fit. With integration tables, *you* must do the searching.

Many people find tables of integrals to be a valuable supplement to the integ tion techniques discussed in this chapter. Tables of common integrals can be found Appendix C. **Integration by tables** is not a "cure-all" for all of the difficulties tl can accompany integration—using tables of integrals requires considerable thoug and insight and often involves substitution.

Each integration formula in Appendix C can be developed using one or more the techniques in this chapter. You should try to verify several of the formulas. F instance, Formula 4.

$$\int \frac{u}{(a + bu)^2} \, du = \frac{1}{b^2} \left(\frac{a}{a + bu} + \ln|a + bu| \right) + C$$ Formula 4

can be verified using the method of partial fractions, and Formula 19

$$\int \frac{\sqrt{a + bu}}{u} \, du = 2\sqrt{a + bu} + a \int \frac{du}{u\sqrt{a + bu}}$$ Formula 19

can be verified using integration by parts. Note that the integrals in Appendix C : classified according to forms involving the following.

u^n	$(a + bu)$
$(a + bu + cu^2)$	$\sqrt{a + bu}$
$(a^2 \pm u^2)$	$\sqrt{u^2 \pm a^2}$
$\sqrt{a^2 - u^2}$	Trigonometric functions
Inverse trigonometric functions	Exponential functions
Logarithmic functions	

EXPLORATION

Use the tables of integrals in Appendix C and the substitution

$$u = \sqrt{x-1}$$

o evaluate the integral in Example 1. f you do this, you should obtain

$$\int \frac{dx}{x\sqrt{x-1}} = \int \frac{2\,du}{u^2+1}.$$

Does this produce the same result as hat obtained in Example 1?

Example 1 Integration by Tables

Evaluate $\displaystyle\int \frac{dx}{x\sqrt{x-1}}$.

Solution Because the expression inside the radical is linear, you should consider forms involving $\sqrt{a+bu}$.

$$\int \frac{du}{u\sqrt{a+bu}} = \frac{2}{\sqrt{-a}} \arctan \sqrt{\frac{a+bu}{-a}} + C \qquad \text{Formula 17 } (a < 0)$$

Let $a = -1$, $b = 1$, and $u = x$. Then $du = dx$, and you can write

$$\int \frac{dx}{x\sqrt{x-1}} = 2 \arctan \sqrt{x-1} + C.$$

Example 2 Integration by Tables

Evaluate $\int x\sqrt{x^4-9}\,dx$.

Solution Because the radical has the form $\sqrt{u^2-a^2}$, you should consider Formula 26.

$$\int \sqrt{u^2-a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2-a^2} - a^2 \ln\left|u + \sqrt{u^2-a^2}\right|\right) + C$$

Let $u = x^2$ and $a = 3$. Then $du = 2x\,dx$, and you have

$$\int x\sqrt{x^4-9}\,dx = \frac{1}{2}\int \sqrt{(x^2)^2 - 3^2}\,(2x)\,dx$$

$$= \frac{1}{4}\left(x^2\sqrt{x^4-9} - 9\ln\left|x^2 + \sqrt{x^4-9}\right|\right) + C.$$

Example 3 Integration by Tables

Evaluate $\displaystyle\int \frac{x}{1+e^{-x^2}}\,dx$.

TECHNOLOGY Example 3 shows the importance of having several solution techniques at your disposal. This integral is not difficult to solve with a table, but when we entered it into a well-known computer algebra system, the utility was unable to find the antiderivative.

Solution Of the forms involving e^u, consider the following.

$$\int \frac{du}{1+e^u} = u - \ln(1+e^u) + C \qquad \text{Formula 84}$$

Let $u = -x^2$. Then $du = -2x\,dx$, and you have

$$\int \frac{x}{1+e^{-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x\,dx}{1+e^{-x^2}}$$

$$= -\frac{1}{2}\left[-x^2 - \ln(1+e^{-x^2})\right] + C$$

$$= \frac{1}{2}\left[x^2 + \ln(1+e^{-x^2})\right] + C.$$

Reduction Formulas

Several of the integrals in the integration tables have the form $\int f(x)\,dx = g(x$ $\int h(x)\,dx$. Such integration formulas are called **reduction formulas** because t reduce a given integral to the sum of a function and a simpler integral.

Example 4 Using a Reduction Formula

Evaluate $\displaystyle\int x^3 \sin x\,dx$.

Solution Consider the following three formulas.

$$\int u \sin u\,du = \sin u - u\cos u + C \qquad \text{Formula 52}$$

$$\int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1}\cos u\,du \qquad \text{Formula 54}$$

$$\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1}\sin u\,du \qquad \text{Formula 55}$$

Using Formula 54, Formula 55, and then Formula 52 produces

$$\begin{aligned}\int x^3 \sin x\,dx &= -x^3\cos x + 3\int x^2\cos x\,dx\\ &= -x^3\cos x + 3\left(x^2\sin x - 2\int x\sin x\,dx\right)\\ &= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C.\end{aligned}$$

Example 5 Using a Reduction Formula

Evaluate $\displaystyle\int \frac{\sqrt{3-5x}}{2x}\,dx$.

Solution Consider the following two formulas.

$$\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C \qquad \text{Formula 17 } (a > 0)$$

$$\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}} \qquad \text{Formula 19}$$

Using Formula 19, with $a = 3$, $b = -5$, and $u = x$, produces

$$\begin{aligned}\frac{1}{2}\int \frac{\sqrt{3-5x}}{x}\,dx &= \frac{1}{2}\left(2\sqrt{3-5x} + 3\int \frac{dx}{x\sqrt{3-5x}}\right)\\ &= \sqrt{3-5x} + \frac{3}{2}\int \frac{dx}{x\sqrt{3-5x}}.\end{aligned}$$

Using Formula 17, with $a = 3$, $b = -5$, and $u = x$, produces

$$\begin{aligned}\int \frac{\sqrt{3-5x}}{2x}\,dx &= \sqrt{3-5x} + \frac{3}{2}\left(\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3-5x}-\sqrt{3}}{\sqrt{3-5x}+\sqrt{3}}\right|\right) + C\\ &= \sqrt{3-5x} + \frac{\sqrt{3}}{2}\ln\left|\frac{\sqrt{3-5x}-\sqrt{3}}{\sqrt{3-5x}+\sqrt{3}}\right| + C.\end{aligned}$$

TECHNOLOGY Sometimes when you use computer algebra systems you obtain results that look very different, but are actually equivalent. We used several to evaluate the integral in Example 5, as follows.

Maple

$\sqrt{3-5x}\ -$

$\sqrt{3}\ \text{arctanh}\left(\tfrac{1}{3}\sqrt{3-5x}\sqrt{3}\right)$

Derive

$\sqrt{3}\ \ln\left[\dfrac{\sqrt{(3-5x)}-\sqrt{3}}{\sqrt{x}}\right] +$

$\sqrt{(3-5x)}$

Mathematica

Sqrt[3 − 5x] −

$\text{Sqrt[3] ArcTanh}\left[\dfrac{\text{Sqrt[3 − 5x]}}{\text{Sqrt[3]}}\right]$

Mathcad

$\sqrt{3-5x}\ +$

$\tfrac{1}{2}\sqrt{3}\ \ln\left[-\tfrac{1}{5}\dfrac{(-6+5x+2\sqrt{3}\sqrt{3-5x})}{x}\right]$

Notice that computer algebra systems do not include a constant of integration.

Rational Functions of Sine and Cosine

Example 6 Integration by Tables

Evaluate $\displaystyle\int \frac{\sin 2x}{2 + \cos x}\, dx.$

Solution Substituting $2 \sin x \cos x$ for $\sin 2x$ produces

$$\int \frac{\sin 2x}{2 + \cos x}\, dx = 2 \int \frac{\sin x \cos x}{2 + \cos x}\, dx.$$

A check of the forms involving $\sin u$ or $\cos u$ in Appendix C shows that none of those listed applies. Therefore, you can consider forms involving $a + bu$. For example,

$$\int \frac{u\, du}{a + bu} = \frac{1}{b^2}(bu - a \ln|a + bu|) + C. \qquad \text{Formula 3}$$

Let $a = 2$, $b = 1$, and $u = \cos x$. Then $du = -\sin x\, dx$, and you have

$$\begin{aligned} 2 \int \frac{\sin x \cos x}{2 + \cos x}\, dx &= -2 \int \frac{\cos x(-\sin x\, dx)}{2 + \cos x} \\ &= -2(\cos x - 2 \ln|2 + \cos x|) + C \\ &= -2 \cos x + 4 \ln|2 + \cos x| + C. \end{aligned}$$

Example 6 involves a rational expression of $\sin x$ and $\cos x$. If you are unable to find an integral of this form in the integration tables, try using the following special substitution to convert the trigonometric expression to a standard rational expression.

Substitution for Rational Functions of Sine and Cosine

For integrals involving rational functions of sine and cosine, the substitution

$$u = \frac{\sin x}{1 + \cos x} = \tan \frac{x}{2}$$

yields

$$\cos x = \frac{1 - u^2}{1 + u^2}, \quad \sin x = \frac{2u}{1 + u^2}, \quad \text{and} \quad dx = \frac{2\, du}{1 + u^2}.$$

Proof From the substitution for u, it follows that

$$u^2 = \frac{\sin^2 x}{(1 + \cos x)^2} = \frac{1 - \cos^2 x}{(1 + \cos x)^2} = \frac{1 - \cos x}{1 + \cos x}.$$

Solving for $\cos x$ produces $\cos x = (1 - u^2)/(1 + u^2)$. To find $\sin x$, write $u = \sin x/(1 + \cos x)$ as

$$\sin x = u(1 + \cos x) = u\left(1 + \frac{1 - u^2}{1 + u^2}\right) = \frac{2u}{1 + u^2}.$$

Finally, to find dx, consider $u = \tan(x/2)$. Then you have $\arctan u = x/2$ and $dx = (2\, du)/(1 + u^2)$.

EXERCISES FOR SECTION 7.6

In Exercises 1 and 2, use a table of integrals with forms involving $a + bu$ to evaluate the integral.

1. $\displaystyle\int \frac{x^2}{1+x}\,dx$ **2.** $\displaystyle\int \frac{2}{3x^2(2x-5)^2}\,dx$

In Exercises 3 and 4, use a table of integrals with forms involving $\sqrt{u^2 \pm a^2}$ to evaluate the integral.

3. $\displaystyle\int e^x\sqrt{1+e^{2x}}\,dx$ **4.** $\displaystyle\int \frac{\sqrt{x^2-9}}{3x}\,dx$

In Exercises 5 and 6, use a table of integrals with forms involving $\sqrt{a^2 - u^2}$ to evaluate the integral.

5. $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$ **6.** $\displaystyle\int \frac{x}{\sqrt{9-x^4}}\,dx$

In Exercises 7–10, use a table of integrals with forms involving the trigonometric functions to evaluate the integral.

7. $\displaystyle\int \sin^4 2x\,dx$ **8.** $\displaystyle\int \frac{\cos^3\sqrt{x}}{\sqrt{x}}\,dx$

9. $\displaystyle\int \frac{1}{\sqrt{x}\left(1-\cos\sqrt{x}\right)}\,dx$ **10.** $\displaystyle\int \frac{1}{1-\tan 5x}\,dx$

In Exercises 11 and 12, use a table of integrals with forms involving e^u to evaluate the integral.

11. $\displaystyle\int \frac{1}{1+e^{2x}}\,dx$ **12.** $\displaystyle\int e^{-x/2}\sin 2x\,dx$

In Exercises 13 and 14, use a table of integrals with forms involving $\ln u$ to evaluate the integral.

13. $\displaystyle\int x^3\ln x\,dx$ **14.** $\displaystyle\int (\ln x)^3\,dx$

In Exercises 15–18, find the indefinite integral (a) using integration tables and (b) using the indicated method.

	Integral	*Method*
15.	$\displaystyle\int x^2e^x\,dx$	Integration by parts
16.	$\displaystyle\int x^4\ln x\,dx$	Integration by parts
17.	$\displaystyle\int \frac{1}{x^2(x+1)}\,dx$	Partial fractions
18.	$\displaystyle\int \frac{1}{x^2-75}\,dx$	Partial fractions

In Exercises 19–50, use integration tables to evaluate the integral.

19. $\displaystyle\int xe^{x^2}\,dx$ **20.** $\displaystyle\int \frac{x}{\sqrt{1+x}}\,dx$

21. $\displaystyle\int x\,\mathrm{arcsec}(x^2+1)\,dx$ **22.** $\displaystyle\int \mathrm{arcsec}\,2x\,dx$

23. $\displaystyle\int x^2\ln x\,dx$ **24.** $\displaystyle\int x\sin x\,dx$

25. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$ **26.** $\displaystyle\int \frac{x^2}{(3x-5)^2}\,dx$

27. $\displaystyle\int \frac{2x}{(1-3x)^2}\,dx$ **28.** $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$

29. $\displaystyle\int e^x\arccos e^x\,dx$ **30.** $\displaystyle\int \frac{\theta^2}{1-\sin\theta^3}\,d\theta$

31. $\displaystyle\int \frac{x}{1-\sec x^2}\,dx$ **32.** $\displaystyle\int \frac{e^x}{1-\tan e^x}\,dx$

33. $\displaystyle\int \frac{\cos x}{1+\sin^2 x}\,dx$ **34.** $\displaystyle\int \frac{1}{t[1+(\ln t)^2]}\,dt$

35. $\displaystyle\int \frac{\cos\theta}{3+2\sin\theta+\sin^2\theta}\,d\theta$ **36.** $\displaystyle\int \sqrt{3+x^2}\,dx$

37. $\displaystyle\int \frac{1}{x^2\sqrt{2+9x^2}}\,dx$ **38.** $\displaystyle\int x^2\sqrt{2+9x^2}\,dx$

39. $\displaystyle\int t^3\cos t\,dt$ **40.** $\displaystyle\int \sqrt{x}\arctan x^{3/2}\,d$

41. $\displaystyle\int \frac{\ln x}{x(3+2\ln x)}\,dx$ **42.** $\displaystyle\int \frac{e^x}{(1-e^{2x})^{3/2}}\,dx$

43. $\displaystyle\int \frac{x}{(x^2-6x+10)^2}\,dx$

44. $\displaystyle\int (2x-3)^2\sqrt{(2x-3)^2+4}\,dx$

45. $\displaystyle\int \frac{x}{\sqrt{x^4-6x^2+5}}\,dx$ **46.** $\displaystyle\int \frac{\cos x}{\sqrt{\sin^2 x+1}}\,dx$

47. $\displaystyle\int \frac{x^3}{\sqrt{4-x^2}}\,dx$ **48.** $\displaystyle\int \sqrt{\frac{3-x}{3+x}}\,dx$

49. $\displaystyle\int \frac{e^{3x}}{(1+e^x)^3}\,dx$ **50.** $\displaystyle\int \tan^3\theta\,d\theta$

In Exercises 51–56, verify the integration formula.

51. $\displaystyle\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) +$

52. $\displaystyle\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2}{(2n+1)b}\left(u^n\sqrt{a+bu} - na\int \frac{u^{n-1}}{\sqrt{a+bu}}\right.$

53. $\displaystyle\int \frac{1}{(u^2 \pm a^2)^{3/2}}\,du = \frac{\pm u}{a^2\sqrt{u^2 \pm a^2}} + C$

54. $\displaystyle\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

55. $\displaystyle\int \arctan u\,du = u\arctan u - \ln\sqrt{1+u^2} + C$

56. $\displaystyle\int (\ln u)^n\,du = u(\ln u)^n - n\int (\ln u)^{n-1}\,du$

Exercises 57–62, use a computer algebra system to determine e antiderivative that passes through the indicated point. Use e system to graph the resulting antiderivative.

$\int \frac{1}{x^{3/2}\sqrt{1-x}}\,dx,\ \left(\frac{1}{2}, 5\right)$

$\int x\sqrt{x^2+2x}\,dx,\ (0, 0)$

$\int \frac{1}{(x^2-6x+10)^2}\,dx,\ (3, 0)$

$\int \frac{\sqrt{2-2x-x^2}}{x+1}\,dx,\ \left(0, \sqrt{2}\right)$

$\int \frac{1}{\sin\theta\tan\theta}\,d\theta,\ \left(\frac{\pi}{4}, 2\right)$

$\int \frac{\sin\theta}{(\cos\theta)(1+\sin\theta)}\,d\theta,\ (0, 1)$

Exercises 63–70, evaluate the integral.

$\int \frac{1}{2-3\sin\theta}\,d\theta$

64. $\int \frac{\sin\theta}{1+\cos^2\theta}\,d\theta$

$\int_0^{\pi/2} \frac{1}{1+\sin\theta+\cos\theta}\,d\theta$

66. $\int_0^{\pi/2} \frac{1}{3-2\cos\theta}\,d\theta$

$\int \frac{\sin\theta}{3-2\cos\theta}\,d\theta$

68. $\int \frac{\cos\theta}{1+\cos\theta}\,d\theta$

$\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$

70. $\int \frac{1}{\sec\theta-\tan\theta}\,d\theta$

ea **In Exercises 71 and 72, find the area of the region bound- by the graphs of the equations.**

$y = \frac{x}{\sqrt{x+1}},\ y = 0,\ x = 8$

72. $y = \frac{x}{1+e^{x^2}},\ y = 0,\ x = 2$

Getting at the Concept

In Exercises 73–78, state (if possible) the method or integra- tion formula you would use to find the antiderivative. Do not integrate.

73. $\int \frac{e^x}{e^{2x}+1}\,dx$

74. $\int \frac{e^x}{e^x+1}\,dx$

75. $\int x\,e^{x^2}\,dx$

76. $\int x\,e^x\,dx$

77. $\int e^{x^2}\,dx$

78. $\int e^{2x}\sqrt{e^{2x}+1}\,dx$

79. Generate four integration problems that can be integrated from a table of integrals after an appropriate substitution. Use four different integration formulas from the table in the text.

0. Describe what is meant by a reduction formula. Give an example.

81. ***Work*** A hydraulic cylinder on an industrial machine pushes a steel block a distance of x feet ($0 \le x \le 5$), where the variable force required is

$$F(x) = 2000xe^{-x} \text{ pounds.}$$

Find the work done in pushing the block the full 5 feet through the machine.

82. ***Work*** Repeat Exercise 81, using a force of

$$F(x) = \frac{500x}{\sqrt{26-x^2}} \text{ pounds.}$$

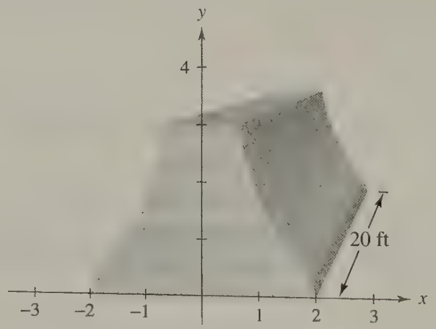
83. ***Building Design*** The cross section of a precast concrete beam for a building is bounded by the graphs of the equations

$$x = \frac{2}{\sqrt{1+y^2}},\ x = \frac{-2}{\sqrt{1+y^2}},\ y = 0, \text{ and } y = 3$$

where x and y are measured in feet. The length of the beam is 20 feet (see figure).

(a) Find the volume V and the weight W of the beam. Assume the concrete weighs 148 pounds per cubic foot.

(b) Find the centroid of a cross section of the beam.



84. ***Average Population Size*** A population is growing according to the logistics model

$$N = \frac{5000}{1+e^{4.8-1.9t}}$$

where t is the time in days. Find the average population over the interval $[0, 2]$.

In Exercises 85 and 86, use a graphing utility to (a) solve the integral equation for the constant k and (b) graph the region whose area is given by the integral.

85. $\int_0^4 \frac{k}{2+3x}\,dx = 10$

86. $\int_0^k 6x^2 e^{-x/2}\,dx = 50$

True or False **In Exercises 87 and 88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

87. To use a table of integrals, the integral you are evaluating must appear in the table.

88. When using a table of integrals, you may have to make substitutions to rewrite your integral in the form in which it appears in the table.

Section 7.7

Indeterminate Forms and L'Hôpital's Rule

- Recognize limits that produce indeterminate forms.
- Apply L'Hôpital's Rule to evaluate a limit.

Indeterminate Forms

Recall from Chapters 1 and 3 that the forms 0/0 and ∞/∞ are called *indetermin* because they do not guarantee that a limit exists, nor do they indicate what the limit if one does exist. When you encountered one of these indeterminate forms earlier in t text, you attempted to rewrite the expression by using various algebraic techniques.

Indeterminate Form	*Limit*	*Algebraic Technique*
$\frac{0}{0}$	$\lim_{x\to -1}\frac{2x^2-2}{x+1} = \lim_{x\to -1} 2(x-1)$ $= -4$	Divide numerator and denominator by $(x+1)$.
$\frac{\infty}{\infty}$	$\lim_{x\to\infty}\frac{3x^2-1}{2x^2+1} = \lim_{x\to\infty}\frac{3-(1/x^2)}{2+(1/x^2)}$ $= \frac{3}{2}$	Divide numerator and denominator by x^2.

Occasionally, you can extend these algebraic techniques to find limits transcendental functions. For instance, the limit

$$\lim_{x\to 0}\frac{e^{2x}-1}{e^x-1}$$

produces the indeterminate form 0/0. Factoring and then dividing produces

$$\lim_{x\to 0}\frac{e^{2x}-1}{e^x-1} = \lim_{x\to 0}\frac{(e^x+1)(e^x-1)}{e^x-1} = \lim_{x\to 0}(e^x+1) = 2.$$

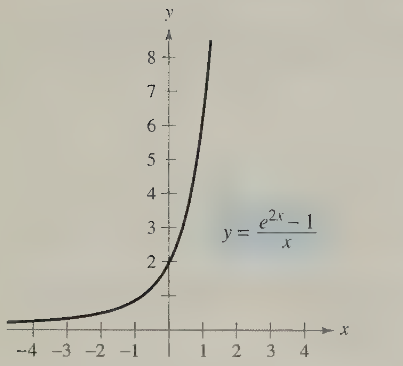
However, not all indeterminate forms can be evaluated by algebraic manipulatio This is particularly true when *both* algebraic and transcendental functions involved. For instance, the limit

$$\lim_{x\to 0}\frac{e^{2x}-1}{x}$$

produces the indeterminate form 0/0. Rewriting the expression to obtain

$$\lim_{x\to 0}\left(\frac{e^{2x}}{x}-\frac{1}{x}\right)$$

merely produces another indeterminate form, ∞ − ∞. Of course, you could use te nology to estimate the limit, as shown in the table and in Figure 7.14. From the ta and the graph, the limit appears to be 2. (This limit will be verified in Example 1.



The limit as x approaches 0 appears to be 2.
Figure 7.14

x	−1	−0.1	−0.01	−0.001	0	0.001	0.01	0.1	1
$\frac{e^{2x}-1}{x}$	0.865	1.813	1.980	1.998	?	2.002	2.020	2.214	6.38

The Granger Collection

Guillaume L'Hôpital (1661–1704)

L'Hôpital's Rule is named after the French nathematician Guillaume François Antoine de L'Hôpital. L'Hôpital is credited with writing the first text on differential calculus (in 1696) in which the rule publicly appeared. It was recently discovered that the rule and its proof were written in a letter from John Bernoulli to L'Hôpital. "… I acknowledge that I owe very much to the bright minds of the Bernoulli brothers. … I have made free use of their discoveries …," said L'Hôpital.

L'Hôpital's Rule

To find the limit illustrated in Figure 7.14, you can use a theorem called **L'Hôpital's Rule.** This theorem states that under certain conditions the limit of the quotient $f(x)/g(x)$ is determined by the limit of the quotient of the derivatives

$$\frac{f'(x)}{g'(x)}.$$

To prove this theorem, you can use a more general result called the **Extended Mean Value Theorem.**

THEOREM 7.3 The Extended Mean Value Theorem

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

NOTE To see why this is called the Extended Mean Value Theorem, consider the special case in which $g(x) = x$. For this case, you obtain the "standard" Mean Value Theorem as presented in Section 3.2.

The Extended Mean Value Theorem and L'Hôpital's Rule are both proved in Appendix B.

THEOREM 7.4 L'Hôpital's Rule

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

NOTE People occasionally use L'Hôpital's Rule incorrectly by applying the Quotient Rule to $f(x)/g(x)$. Be sure you see that the rule involves $f'(x)/g'(x)$, not the derivative of $f(x)/g(x)$.

L'Hôpital's Rule can also be applied to one-sided limits. For instance, if the limit of $f(x)/g(x)$ as x approaches c *from the right* produces the indeterminate form $0/0$, then

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \lim_{x \to c^+} \frac{f'(x)}{g'(x)}$$

provided the limit exists (or is infinite).

OR FURTHER INFORMATION further understand the necessity of e restriction that $g'(x)$ be nonzero for x in (a, b), except possibly at c, e the article "Counterexamples to Hôpital's Rule" by R. P. Boas in *e American Mathematical Monthly.* view this article, go to the website *vw.matharticles.com.*

EXPLORATION

Numerical and Graphical Approaches Use a numerical or a graphical approach to approximate each of the following limits.

a. $\lim\limits_{x\to 0} \dfrac{2^{2x}-1}{x}$

b. $\lim\limits_{x\to 0} \dfrac{3^{2x}-1}{x}$

c. $\lim\limits_{x\to 0} \dfrac{4^{2x}-1}{x}$

d. $\lim\limits_{x\to 0} \dfrac{5^{2x}-1}{x}$

What pattern do you observe? Does an analytic approach have an advantage for these limits? If so, explain your reasoning.

Example 1 Indeterminate Form 0/0

Evaluate $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{x}$.

Solution Because direct substitution results in the indeterminate form $0/0$

$$\begin{tikzcd} & \lim\limits_{x\to 0} (e^{2x}-1) = 0 \\ \lim\limits_{x\to 0} \dfrac{e^{2x}-1}{x} \arrow[ur] \arrow[dr] & \\ & \lim\limits_{x\to 0} x = 0 \end{tikzcd}$$

you can apply L'Hôpital's Rule as follows.

$$\lim_{x\to 0} \frac{e^{2x}-1}{x} = \lim_{x\to 0} \frac{\dfrac{d}{dx}[e^{2x}-1]}{\dfrac{d}{dx}[x]} \qquad \text{Apply L'Hôpital's Rule.}$$

$$= \lim_{x\to 0} \frac{2e^{2x}}{1} \qquad \text{Differentiate numerator and denominator.}$$

$$= 2 \qquad \text{Evaluate the limit.}$$

NOTE In writing the string of equations in Example 1, you actually do not know that the f limit is equal to the second until you have shown that the second limit exists. In other words the second limit had not existed, it would not have been permissible to apply L'Hôpital's R

Another form of L'Hôpital's Rule states that if the limit of $f(x)/g(x)$ a approaches ∞ (or $-\infty$) produces the indeterminate form $0/0$ or ∞/∞, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

Example 2 Indeterminate Form ∞/∞

Evaluate $\lim\limits_{x\to\infty} \dfrac{\ln x}{x}$.

Solution Because direct substitution results in the indeterminate form ∞/∞, can apply L'Hôpital's Rule to obtain

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\dfrac{d}{dx}[\ln x]}{\dfrac{d}{dx}[x]} \qquad \text{Apply L'Hôpital's Rule.}$$

$$= \lim_{x\to\infty} \frac{1}{x} \qquad \text{Differentiate numerator and denominator.}$$

$$= 0. \qquad \text{Evaluate the limit.}$$

NOTE Try graphing $y_1 = \ln x$ and $y_2 = x$ in the same viewing window. Which function grows faster as x approaches ∞? How is this observation related to Example 2?

Occasionally it is necessary to apply L'Hôpital's Rule more than once to remove an indeterminate form, as illustrated in Example 3.

Example 3 Applying L'Hôpital's Rule More than Once

Evaluate $\lim_{x\to-\infty} \dfrac{x^2}{e^{-x}}$.

Solution Because direct substitution results in the indeterminate form ∞/∞, you can apply L'Hôpital's Rule.

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \lim_{x\to-\infty} \frac{\dfrac{d}{dx}[x^2]}{\dfrac{d}{dx}[e^{-x}]} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}}$$

This limit yields the indeterminate form $(-\infty)/(-\infty)$, so you can apply L'Hôpital's Rule again to obtain

$$\lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{\dfrac{d}{dx}[2x]}{\dfrac{d}{dx}[-e^{-x}]} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0.$$

In addition to the forms $0/0$ and ∞/∞, there are other indeterminate forms such as $0 \cdot \infty$, 1^∞, ∞^0, 0^0, and $\infty - \infty$. For example, consider the following four limits that lead to the indeterminate form $0 \cdot \infty$.

$$\underbrace{\lim_{x\to0} (x)\left(\frac{1}{x}\right)}_{\text{Limit is 1.}}, \quad \underbrace{\lim_{x\to0} (x)\left(\frac{2}{x}\right)}_{\text{Limit is 2.}}, \quad \underbrace{\lim_{x\to\infty} (x)\left(\frac{1}{e^x}\right)}_{\text{Limit is 0.}}, \quad \underbrace{\lim_{x\to\infty} (e^x)\left(\frac{1}{x}\right)}_{\text{Limit is } \infty.}$$

Because each limit is different, it is clear that the form $0 \cdot \infty$ is indeterminate in the sense that it does not determine the value (or even the existence) of the limit. The following examples indicate methods for evaluating these forms. Basically, you attempt to convert each of these forms to $0/0$ or ∞/∞ so that L'Hôpital's Rule can be applied.

Example 4 Indeterminate Form $0 \cdot \infty$

Evaluate $\lim_{x\to\infty} e^{-x}\sqrt{x}$.

Solution Because direct substitution produces the indeterminate form $0 \cdot \infty$, you should try to rewrite the limit to fit the form $0/0$ or ∞/∞. In this case, you can rewrite the limit to fit the second form.

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x}$$

Now, by L'Hôpital's Rule, you have

$$\lim_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim_{x\to\infty} \frac{1/(2\sqrt{x})}{e^x} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}\,e^x} = 0.$$

If rewriting a limit in one of the forms $0/0$ or ∞/∞ does not seem to work the other form. For instance, in Example 4 you can write the limit as

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{e^{-x}}{x^{-1/2}}$$

which yields the indeterminate form $0/0$. As it happens, applying L'Hôpital's Rul this limit produces

$$\lim_{x\to\infty} \frac{e^{-x}}{x^{-1/2}} = \lim_{x\to\infty} \frac{-e^{-x}}{-1/(2x^{3/2})}$$

which also yields the indeterminate form $0/0$.

The indeterminate forms 1^∞, ∞^0, and 0^0 arise from limits of functions that h variable bases and variable exponents. When we previously encountered this typ function, we used logarithmic differentiation to find the derivative. You can u similar procedure when taking limits, as indicated in the next example.

Example 5 Indeterminate Form 1^∞

Evaluate $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Solution Because direct substitution yields the indeterminate form 1^∞, you proceed as follows. To begin, assume that the limit exists and is equal to y.

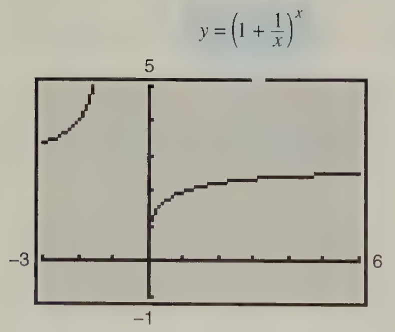
$$y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Taking the natural logarithm of both sides produces

$$\ln y = \ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right].$$

Because the natural logarithmic function is continuous, you can write the followi

$$\begin{aligned}\ln y &= \lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x}\right)\right] && \text{Indeterminate form } \infty\cdot 0\\ &= \lim_{x\to\infty}\left(\frac{\ln[1+(1/x)]}{1/x}\right) && \text{Indeterminate form } 0/0\\ &= \lim_{x\to\infty}\left(\frac{(-1/x^2)\{1/[1+(1/x)]\}}{-1/x^2}\right) && \text{L'Hôpital's Rule}\\ &= \lim_{x\to\infty}\frac{1}{1+(1/x)}\end{aligned}$$



The limit of $[1+(1/x)]^x$ as x approaches infinity is e.
Figure 7.15

Now, because you have shown that $\ln y = 1$, you can conclude that $y = e$ and ob

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

You can use a graphing utility to confirm this result, as shown in Figure 7.15.

L'Hôpital's Rule can also be applied to one-sided limits, as demonstrated in Examples 6 and 7.

Example 6 Indeterminate Form 0^0

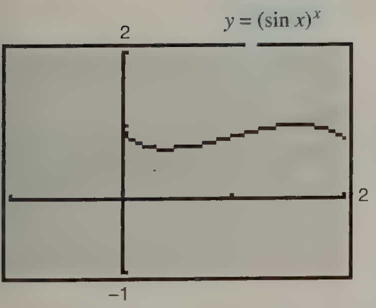
Evaluate $\lim\limits_{x\to 0^+} (\sin x)^x$.

Solution Because direct substitution produces the indeterminate form 0^0, you can proceed as follows. To begin, assume that the limit exists and is equal to y.

$$\begin{aligned}
y &= \lim_{x\to 0^+} (\sin x)^x && \text{Indeterminate form } 0^0\\
\ln y &= \ln\Big[\lim_{x\to 0^+} (\sin x)^x\Big] && \text{Take natural log of both sides.}\\
&= \lim_{x\to 0^+} \big[\ln(\sin x)^x\big] && \text{Continuity}\\
&= \lim_{x\to 0^+} \big[x \ln(\sin x)\big] && \text{Indeterminate form } 0\cdot(-\infty)\\
&= \lim_{x\to 0^+} \frac{\ln(\sin x)}{1/x} && \text{Indeterminate form } -\infty/\infty\\
&= \lim_{x\to 0^+} \frac{\cot x}{-1/x^2} && \text{L'Hôpital's Rule}\\
&= \lim_{x\to 0^+} \frac{-x^2}{\tan x} && \text{Indeterminate form } 0/0\\
&= \lim_{x\to 0^+} \frac{-2x}{\sec^2 x} = 0 && \text{L'Hôpital's Rule}
\end{aligned}$$

Now, because $\ln y = 0$, you can conclude that $y = e^0 = 1$, and it follows that

$$\lim_{x\to 0^+} (\sin x)^x = 1.$$



e limit of $(\sin x)^x$ is 1 as x approaches 0
m the right.
gure 7.16

TECHNOLOGY When evaluating complicated limits such as the one in Example 6, it is helpful to check the reasonableness of the solution with a computer or with a graphing utility. For instance, the calculations in the following table and the graph in Figure 7.16 are consistent with the conclusion that $(\sin x)^x$ approaches 1 as x approaches 0 from the right.

x	1.0	0.1	0.01	0.001	0.0001	0.00001
$(\sin x)^x$	0.8415	0.7942	0.9550	0.9931	0.9991	0.9999

Try using a computer or graphing utility to estimate the following limits.

$$\lim_{x\to 0} (1 - \cos x)^x$$

and

$$\lim_{x\to 0^+} (\tan x)^x$$

Then see if you can verify your estimates analytically.

STUDY TIP In each of the examples presented in this section, L'Hôpital's Rule is used to find a limit that exists. It can also be used to conclude that a limit is infinite. For instance, try using L'Hôpital's Rule to show that

$$\lim_{x\to\infty}\frac{e^x}{x}=\infty.$$

Example 7 Indeterminate Form $\infty - \infty$

Evaluate $\displaystyle\lim_{x\to1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$.

Solution Because direct substitution yields the indeterminate form $\infty - \infty$, should try to rewrite the expression to produce a form to which you can ap L'Hôpital's Rule. In this case, you can combine the two fractions to obtain

$$\lim_{x\to1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to1^+}\left[\frac{x-1-\ln x}{(x-1)\ln x}\right].$$

Now, because direct substitution produces the indeterminate form $0/0$, you can ap L'Hôpital's Rule to obtain

$$\begin{aligned}\lim_{x\to1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)&=\lim_{x\to1^+}\frac{\frac{d}{dx}[x-1-\ln x]}{\frac{d}{dx}[(x-1)\ln x]}\\&=\lim_{x\to1^+}\left[\frac{1-(1/x)}{(x-1)(1/x)+\ln x}\right]\\&=\lim_{x\to1^+}\left(\frac{x-1}{x-1+x\ln x}\right).\end{aligned}$$

This limit also yields the indeterminate form $0/0$, so you can apply L'Hôpital's R again to obtain

$$\begin{aligned}\lim_{x\to1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)&=\lim_{x\to1^+}\left[\frac{1}{1+x(1/x)+\ln x}\right]\\&=\frac{1}{2}.\end{aligned}$$

We have identified the forms $0/0$, ∞/∞, $\infty-\infty$, $0\cdot\infty$, 0^0, 1^∞, and ∞^0 *indeterminate.* There are similar forms that you should recognize as "determinate

$\infty + \infty \to \infty$	Limit is positive infinity.
$-\infty - \infty \to -\infty$	Limit is negative infinity.
$0^\infty \to 0$	Limit is zero.
$0^{-\infty} \to \infty$	Limit is positive infinity.

(You are asked to verify two of these in Exercises 95 and 96.)

As a final comment, we remind you that L'Hôpital's Rule can be applied onl quotients leading to the indeterminate forms $0/0$ and ∞/∞. For instance, the foll ing application of L'Hôpital's Rule is *incorrect*.

$$\lim_{x\to0}\frac{e^x}{x} = \lim_{x\to0}\frac{e^x}{1}=1$$

Incorrect use of L'Hôpital's Rule

The reason this application is incorrect is that, even though the limit of the den inator is 0, the limit of the numerator is 1, which means that the hypotheses L'Hôpital's Rule have not been satisfied.

XERCISES FOR SECTION 7.7

merical and Graphical Analysis **In Exercises 1–4, complete table and use the result to estimate the limit. Use a graphing ity to graph the function to support your result.**

$\lim\limits_{x \to 0} \dfrac{\sin 5x}{\sin 2x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

$\lim\limits_{x \to 0} \dfrac{1 - e^x}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

$\lim\limits_{x \to \infty} x^5 e^{-x/100}$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$						

$\lim\limits_{x \to \infty} \dfrac{6x}{\sqrt{3x^2 - 2x}}$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$						

Exercises 5–10, evaluate the limit (a) using techniques from apters 1 and 3 and (b) using L'Hôpital's Rule.

$\lim\limits_{x \to 3} \dfrac{2(x-3)}{x^2 - 9}$

6. $\lim\limits_{x \to -1} \dfrac{2x^2 - x - 3}{x + 1}$

$\lim\limits_{x \to 3} \dfrac{\sqrt{x+1} - 2}{x - 3}$

8. $\lim\limits_{x \to 0} \dfrac{\sin 4x}{2x}$

$\lim\limits_{x \to \infty} \dfrac{5x^2 - 3x + 1}{3x^2 - 5}$

10. $\lim\limits_{x \to \infty} \dfrac{2x + 1}{4x^2 + x}$

Exercises 11–36, evaluate the limit, using L'Hôpital's Rule if essary. (In Exercise 17, *n* is a positive integer.)

$\lim\limits_{x \to 2} \dfrac{x^2 - x - 2}{x - 2}$

12. $\lim\limits_{x \to -1} \dfrac{x^2 - x - 2}{x + 1}$

$\lim\limits_{x \to 0} \dfrac{\sqrt{4 - x^2} - 2}{x}$

14. $\lim\limits_{x \to 2^-} \dfrac{\sqrt{4 - x^2}}{x - 2}$

$\lim\limits_{x \to 0} \dfrac{e^x - (1 - x)}{x}$

16. $\lim\limits_{x \to 0^+} \dfrac{e^x - (1 + x)}{x^3}$

$\lim\limits_{x \to 0^+} \dfrac{e^x - (1 + x)}{x^n}$

18. $\lim\limits_{x \to 1} \dfrac{\ln x^2}{x^2 - 1}$

$\lim\limits_{x \to 0} \dfrac{\sin 2x}{\sin 3x}$

20. $\lim\limits_{x \to 0} \dfrac{\sin ax}{\sin bx}$

21. $\lim\limits_{x \to 0} \dfrac{\arcsin x}{x}$

22. $\lim\limits_{x \to 1} \dfrac{\arctan x - (\pi/4)}{x - 1}$

23. $\lim\limits_{x \to \infty} \dfrac{3x^2 - 2x + 1}{2x^2 + 3}$

24. $\lim\limits_{x \to \infty} \dfrac{x - 1}{x^2 + 2x + 3}$

25. $\lim\limits_{x \to \infty} \dfrac{x^2 + 2x + 3}{x - 1}$

26. $\lim\limits_{x \to \infty} \dfrac{x^3}{x + 2}$

27. $\lim\limits_{x \to \infty} \dfrac{x^3}{e^{x/2}}$

28. $\lim\limits_{x \to \infty} \dfrac{x^2}{e^x}$

29. $\lim\limits_{x \to \infty} \dfrac{x}{\sqrt{x^2 + 1}}$

30. $\lim\limits_{x \to \infty} \dfrac{x^2}{\sqrt{x^2 + 1}}$

31. $\lim\limits_{x \to \infty} \dfrac{\cos x}{x}$

32. $\lim\limits_{x \to \infty} \dfrac{\sin x}{x - \pi}$

33. $\lim\limits_{x \to \infty} \dfrac{\ln x}{x^2}$

34. $\lim\limits_{x \to \infty} \dfrac{\ln x^4}{x^3}$

35. $\lim\limits_{x \to \infty} \dfrac{e^x}{x^2}$

36. $\lim\limits_{x \to \infty} \dfrac{e^{x/2}}{x}$

In Exercises 37–54, (a) describe the type of indeterminate form (if any) that is obtained by direct substitution. (b) Evaluate the limit, using L'Hôpital's Rule if necessary. (c) Use a graphing utility to graph the function and verify the result in part (b). (For a geometric approach to Exercise 37, see the article "A Geometric Proof of $\lim\limits_{d \to 0^+} (-d \ln d) = 0$" by John H. Mathews in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.)

37. $\lim\limits_{x \to 0^+} (-x \ln x)$

38. $\lim\limits_{x \to 0^+} x^3 \cot x$

39. $\lim\limits_{x \to \infty} \left(x \sin \dfrac{1}{x}\right)$

40. $\lim\limits_{x \to \infty} x \tan \dfrac{1}{x}$

41. $\lim\limits_{x \to 0^+} x^{1/x}$

42. $\lim\limits_{x \to 0^+} (e^x + x)^{2/x}$

43. $\lim\limits_{x \to \infty} x^{1/x}$

44. $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x$

45. $\lim\limits_{x \to 0^+} (1 + x)^{1/x}$

46. $\lim\limits_{x \to \infty} (1 + x)^{1/x}$

47. $\lim\limits_{x \to 0^+} [3(x)^{x/2}]$

48. $\lim\limits_{x \to 4^+} [3(x - 4)]^{x-4}$

49. $\lim\limits_{x \to 1^+} (\ln x)^{x-1}$

50. $\lim\limits_{x \to 0^+} \left[\cos\left(\dfrac{\pi}{2} - x\right)\right]^x$

51. $\lim\limits_{x \to 2^+} \left(\dfrac{8}{x^2 - 4} - \dfrac{x}{x - 2}\right)$

52. $\lim\limits_{x \to 2^+} \left(\dfrac{1}{x^2 - 4} - \dfrac{\sqrt{x - 1}}{x^2 - 4}\right)$

53. $\lim\limits_{x \to 1^+} \left(\dfrac{3}{\ln x} - \dfrac{2}{x - 1}\right)$

54. $\lim\limits_{x \to 0^+} \left(\dfrac{10}{x} - \dfrac{3}{x^2}\right)$

In Exercises 55–58, use a graphing utility to (a) graph the function and (b) find the required limit (if it exists).

55. $\lim\limits_{x \to 3} \dfrac{x - 3}{\ln(2x - 5)}$

56. $\lim\limits_{x \to 0^+} (\sin x)^x$

57. $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + 5x + 2} - x\right)$

58. $\lim\limits_{x \to \infty} \dfrac{x^3}{e^{2x}}$

Getting at the Concept

59. List six different indeterminate forms.

60. State L'Hôpital's Rule.

61. Find the differentiable functions f and g that satisfy the specified condition such that $\lim_{x\to5} f(x)=0$ and $\lim_{x\to5} g(x)=0$. (*Note:* There are many correct answers.)

(a) $\lim_{x\to5}\frac{f(x)}{g(x)}=10$ (b) $\lim_{x\to5}\frac{f(x)}{g(x)}=0$ (c) $\lim_{x\to5}\frac{f(x)}{g(x)}=\infty$

62. Find differentiable functions f and g such that

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty} g(x)=\infty$$

$$\lim_{x\to\infty}[f(x)-g(x)]=25.$$

(*Note:* There are many correct answers.)

Comparing Functions **In Exercises 63–68, use L'Hôpital's Rule to determine the comparative rates of increase of the functions**

$$f(x)=x^m,\quad g(x)=e^{nx},\quad \text{and}\quad h(x)=(\ln x)^n$$

where $n>0$, $m>0$, and $x\to\infty$.

63. $\lim_{x\to\infty}\frac{x^2}{e^{5x}}$

64. $\lim_{x\to\infty}\frac{x^3}{e^{2x}}$

65. $\lim_{x\to\infty}\frac{(\ln x)^3}{x}$

66. $\lim_{x\to\infty}\frac{(\ln x)^2}{x^3}$

67. $\lim_{x\to\infty}\frac{(\ln x)^n}{x^m}$

68. $\lim_{x\to\infty}\frac{x^m}{e^{nx}}$

69. ***Numerical Approach*** Complete the table to show that x eventually "overpowers" $(\ln x)^4$.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$						

70. ***Numerical Approach*** Complete the table to show that e^x eventually "overpowers" x^5.

x	1	5	10	20	30	40	50	100
$\frac{e^x}{x^5}$								

In Exercises 71–74, find any asymptotes and relative extrema that may exist and use a graphing utility to graph the function. (*Hint:* Some of the limits required in finding asymptotes have been found in preceding exercises.)

71. $y=x^{1/x},\quad x>0$

72. $y=x^x,\quad x>0$

73. $y=2xe^{-x}$

74. $y=\frac{\ln x}{x}$

Think About It **In Exercises 75–78, L'Hôpital's Rule is u incorrectly. Describe the error.**

75. $\lim_{x\to0}\frac{e^{2x}-1}{e^x}=\lim_{x\to0}\frac{2e^{2x}}{e^x}=\lim_{x\to0}2e^x=2$ ✗

76. $\lim_{x\to0}\frac{\sin\pi x-1}{x}=\lim_{x\to0}\frac{\pi\cos\pi x}{1}=\pi$ ✗

77. $\lim_{x\to\infty}x\cos\frac{1}{x}=\lim_{x\to\infty}\frac{\cos(1/x)}{1/x}$ ✗

$$=\lim_{x\to\infty}\frac{[-\sin(1/x)](1/x^2)}{-1/x^2}$$

$$=0$$

78. $\lim_{x\to\infty}\frac{e^{-x}}{1+e^{-x}}=\lim_{x\to\infty}\frac{-e^{-x}}{-e^{-x}}=\lim_{x\to\infty}1=1$ ✗

79. ***Analytical Approach*** Consider $\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}$.

(a) Find the limit analytically without trying to use L'Hôpi Rule.

(b) Show that L'Hôpital's Rule fails.

(c) Use a graphing utility to graph the function and appr mate the limit from the graph. Compare the result with in part (a).

80. ***Compound Interest*** The formula for the amount A i savings account compounded n times per year for t years a interest rate r and an initial deposit of P is

$$A=P\left(1+\frac{r}{n}\right)^{nt}.$$

Use L'Hôpital's Rule to show that the limiting formula as number of compoundings per year becomes infinite is

$$A=Pe^{rt}.$$

81. ***Velocity in a Resisting Medium*** The velocity of an ob falling through a resisting medium such as air or water is

$$v=\frac{32}{k}\left(1-e^{-kt}+\frac{v_0ke^{-kt}}{32}\right)$$

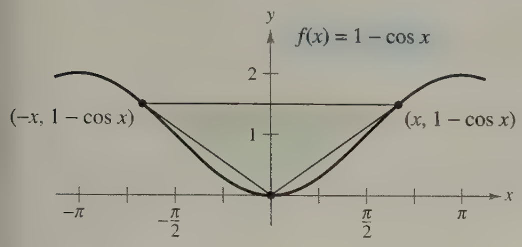
where v_0 is the initial velocity, t is the time in seconds, and the resistance constant of the medium. Use L'Hôpital's Rul find the formula for the velocity of a falling body in a vac by fixing v_0 and t and letting k approach zero. (Assume tha downward direction is positive.)

82. ***The Gamma Function*** The Gamma Function $\Gamma(n)$ is def in terms of the integral of the function

$$f(x)=x^{n-1}e^{-x},\quad n>0.$$

Show that for any fixed value of n, the limit of $f(x)$ approaches infinity is zero.

Area Find the limit, as x approaches 0, of the ratio of the area of the triangle to the total shaded area in the figure.



Use a graphing utility to graph

$$f(x) = \frac{x^k - 1}{k}$$

for $k = 1, 0.1$, and 0.01. Then evaluate the limit

$$\lim_{k\to 0^+} \frac{x^k - 1}{k}.$$

Exercises 85–88, apply the Extended Mean Value Theorem to functions f and g on the indicated interval. Find all values c he interval (a, b) such that

$$\frac{)}{)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Functions	*Interval*
$f(x) = x^3$ $g(x) = x^2 + 1$	$[0, 1]$
$f(x) = \frac{1}{x}$ $g(x) = x^2 - 4$	$[1, 2]$
$f(x) = \sin x$ $g(x) = \cos x$	$\left[0, \frac{\pi}{2}\right]$
$f(x) = \ln x$ $g(x) = x^3$	$[1, 4]$

e or False? **In Exercises 89–92, determine whether the ement is true or false. If it is false, explain why or give an mple that shows it is false.**

$\lim_{x\to 0}\left[\frac{x^2 + x + 1}{x}\right] = \lim_{x\to 0}\left[\frac{2x + 1}{1}\right] = 1$

If $y = e^x/x^2$, then $y' = e^x/2x$.

If $p(x)$ is a polynomial, then $\lim_{x\to\infty} [p(x)/e^x] = 0$.

If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, then $\lim_{x\to\infty} [f(x) - g(x)] = 0$.

93. In Chapter 1 we used a geometric argument (see figure) to prove that

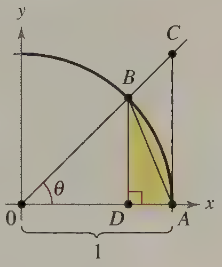
$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

(a) Express the area of the triangle ΔABD in terms of θ.

(b) Express the area of the shaded region in terms of θ.

(c) Express the ratio R of the area of ΔABD to that of the shaded region.

(d) Find $\lim_{\theta\to 0} R$.



94. Sketch the graph of

$$g(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and determine $g'(0)$.

95. Prove that if $f(x) \geq 0$, $\lim_{x\to a} f(x) = 0$, and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x)^{g(x)} = 0$.

96. Prove that if $f(x) \geq 0$, $\lim_{x\to a} f(x) = 0$, and $\lim_{x\to a} g(x) = -\infty$, then $\lim_{x\to a} f(x)^{g(x)} = \infty$.

97. Prove the following generalization of the Mean Value Theorem. If f is twice differentiable on the closed interval $[a, b]$, then

$$f(b) - f(a) = f'(a)(b - a) - \int_a^b f''(t)(t - b)\,dt.$$

98. Show that the indeterminate form 0^0 is not always equal to 1 by evaluating

$$\lim_{x\to 0^+} x^{\ln 2/(1 + \ln x)}.$$

Section 7.8 Improper Integrals

- Evaluate an improper integral that has an infinite limit of integration.
- Evaluate an improper integral that has an infinite discontinuity.

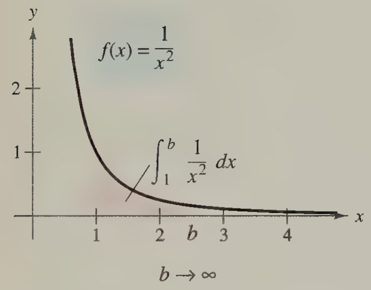
Improper Integrals with Infinite Limits of Integration

The definition of a definite integral

$$\int_a^b f(x)\,dx$$

requires that the interval $[a, b]$ be finite. Furthermore, the Fundamental Theorem Calculus, by which you have been evaluating definite integrals, requires that f continuous on $[a, b]$. In this section you will study a procedure for evaluating integra that do not satisfy these requirements—usually because either one or both of the lim of integration are infinite, or f has a finite number of infinite discontinuities in t interval $[a, b]$. Integrals that possess either property are **improper integrals.** Note th a function f is said to have an **infinite discontinuity** at c if, *from the right or left,*

$$\lim_{x\to c} f(x) = \infty \qquad \text{or} \qquad \lim_{x\to c} f(x) = -\infty.$$



The unbounded region has an area of 1.
Figure 7.17

To get an idea of how to evaluate an improper integral, consider the integral

$$\int_1^b \frac{dx}{x^2} = -\frac{1}{x}\bigg]_1^b = -\frac{1}{b} + 1 = 1 - \frac{1}{b}$$

which can be interpreted as the area of the shaded region shown in Figure 7.17. Taki the limit as $b \to \infty$ produces

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\left(\int_1^b \frac{dx}{x^2}\right) = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1.$$

This improper integral can be interpreted as the area of the *unbounded* region betwe the graph of $f(x) = 1/x^2$ and the x-axis (to the right of $x = 1$).

Definition of Improper Integrals with Infinite Integration Limits

1. If f is continuous on the interval $[a, \infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$
2. If f is continuous on the interval $(-\infty, b]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$
3. If f is continuous on the interval $(-\infty, \infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$
where c is any real number.

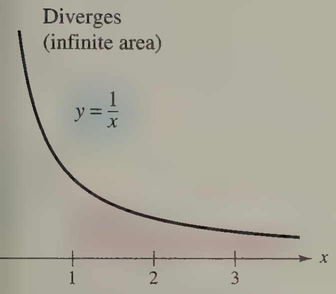
In the first two cases, the improper integral **converges** if the limit exists— otherwise, the improper integral **diverges.** In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

Example 1 An Improper Integral That Diverges

Evaluate $\displaystyle\int_1^{\infty} \frac{dx}{x}$.

Solution

$$\begin{aligned}
\int_1^{\infty} \frac{dx}{x} &= \lim_{b\to\infty} \int_1^{b} \frac{dx}{x} && \text{Take limit as } b \to \infty. \\
&= \lim_{b\to\infty} \Big[\ln x \Big]_1^b && \text{Apply Log Rule.} \\
&= \lim_{b\to\infty} (\ln b - 0) && \text{Apply Fundamental Theorem of Calculus.} \\
&= \infty && \text{Evaluate limit.}
\end{aligned}$$



s unbounded region has an infinite area.
ure 7.18

NOTE Try comparing the regions shown in Figures 7.17 and 7.18. They look similar, yet the region in Figure 7.17 has a finite area of 1 and the region in Figure 7.18 has an infinite area.

Example 2 Improper Integrals That Converge

Evaluate each of the improper integrals.

a. $\displaystyle\int_0^{\infty} e^{-x}\, dx$ **b.** $\displaystyle\int_0^{\infty} \frac{1}{x^2+1}\, dx$

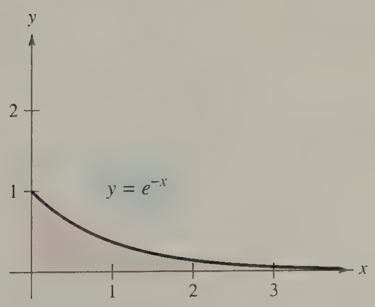
Solution

a.
$$\begin{aligned}
\int_0^{\infty} e^{-x}\, dx &= \lim_{b\to\infty} \int_0^{b} e^{-x}\, dx \\
&= \lim_{b\to\infty} \Big[-e^{-x} \Big]_0^b \\
&= \lim_{b\to\infty} (-e^{-b} + 1) \\
&= 1
\end{aligned}$$

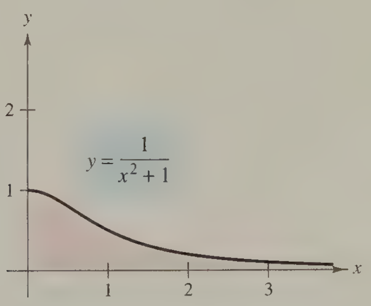
(See Figure 7.19.)

b.
$$\begin{aligned}
\int_0^{\infty} \frac{1}{x^2+1}\, dx &= \lim_{b\to\infty} \int_0^{b} \frac{1}{x^2+1}\, dx \\
&= \lim_{b\to\infty} \Big[\arctan x \Big]_0^b \\
&= \lim_{b\to\infty} \arctan b \\
&= \frac{\pi}{2}
\end{aligned}$$

(See Figure 7.20.)



The area of the unbounded region is 1.
Figure 7.19



The area of the unbounded region is $\pi/2$.
Figure 7.20

In the following example, note how L'Hôpital's Rule can be used to evaluate improper integral.

Example 3 Using L'Hôpital's Rule with an Improper Integral

Evaluate $\int_1^{\infty} (1 - x)e^{-x}\, dx$.

Solution Use integration by parts, with $dv = e^{-x}\, dx$ and $u = (1 - x)$.

$$\begin{aligned}\int (1 - x)e^{-x}\, dx &= -e^{-x}(1 - x) - \int e^{-x}\, dx \\ &= -e^{-x} + xe^{-x} + e^{-x} + C \\ &= xe^{-x} + C\end{aligned}$$

Now, apply the definition of an improper integral.

$$\int_1^{\infty} (1 - x)e^{-x}\, dx = \lim_{b\to\infty} \Big[xe^{-x} \Big]_1^b = \left(\lim_{b\to\infty} \frac{b}{e^b} \right) - \frac{1}{e}$$

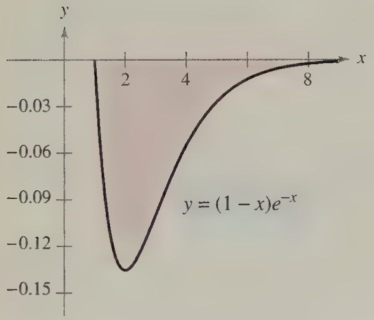
Finally, using L'Hôpital's Rule on the right-hand limit produces

$$\lim_{b\to\infty} \frac{b}{e^b} = \lim_{b\to\infty} \frac{1}{e^b} = 0$$

from which you can conclude that

$$\int_1^{\infty} (1 - x)e^{-x}\, dx = -\frac{1}{e}.$$

(See Figure 7.21.)



The area of the unbounded region is $|-1/e|$.
Figure 7.21

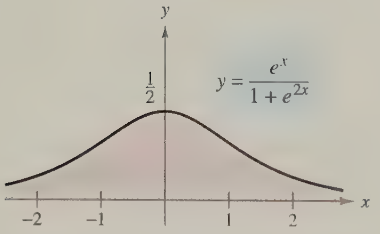
Example 4 Infinite Upper and Lower Limits of Integration

Evaluate $\int_{-\infty}^{\infty} \frac{e^x}{1 + e^{2x}}\, dx$.

Solution Note that the integrand is continuous on $(-\infty, \infty)$. To evaluate the i gral, you can break it into two parts, choosing $c = 0$ as a convenient value.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^x}{1 + e^{2x}}\, dx &= \int_{-\infty}^{0} \frac{e^x}{1 + e^{2x}}\, dx + \int_{0}^{\infty} \frac{e^x}{1 + e^{2x}}\, dx \\ &= \lim_{b\to-\infty} \Big[\arctan e^x \Big]_b^0 + \lim_{b\to\infty} \Big[\arctan e^x \Big]_0^b \\ &= \lim_{b\to-\infty} \left(\frac{\pi}{4} - \arctan e^b \right) + \lim_{b\to\infty} \left(\arctan e^b - \frac{\pi}{4} \right) \\ &= \frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4} \\ &= \frac{\pi}{2}\end{aligned}$$

(See Figure 7.22.)

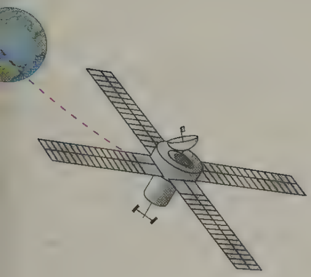


The area of the unbounded region is $\pi/2$.
Figure 7.22

Example 5 Sending a Space Module into Orbit

In Example 3 of Section 6.5, you found that it would require 10,000 mile-tons of work to propel a 15-ton space module to a height of 800 miles above earth. How much work is required to propel the module an unlimited distance away from earth's surface?

Solution At first you might think that an infinite amount of work would be required. But if this were the case, it would be impossible to send rockets into outer space. Because this has been done, the work required must be finite. You can determine the work in the following manner. Using the integral of Example 3, Section 6.5, replace the upper bound of 4800 miles by ∞ and write



e work required to move a space module
nlimited distance away from earth is
roximately 6.336×10^{11} foot-pounds.
ure 7.23

$$\begin{aligned}
W &= \int_{4000}^{\infty} \frac{240{,}000{,}000}{x^2}\,dx \\
&= \lim_{b\to\infty} \left[-\frac{240{,}000{,}000}{x} \right]_{4000}^{b} \\
&= \lim_{b\to\infty} \left(-\frac{240{,}000{,}000}{b} + \frac{240{,}000{,}000}{4000} \right) \\
&= 60{,}000 \text{ mile-tons} \\
&= 6.336 \times 10^{11} \text{ foot-pounds.}
\end{aligned}$$

(See Figure 7.23.)

Improper Integrals with Infinite Discontinuities

The second basic type of improper integral is one that has an infinite discontinuity *at or between* the limits of integration.

Definition of Improper Integrals with Infinite Discontinuities

1. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

2. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

3. If f is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In the first two cases, the improper integral **converges** if the limit exists—otherwise, the improper integral **diverges.** In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

Example 6 An Improper Integral with an Infinite Discontinuity

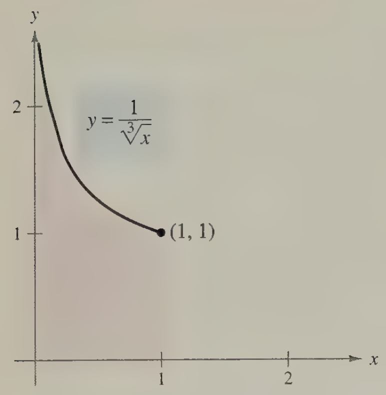
Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$.

Solution The integrand has an infinite discontinuity at $x = 0$, as shown in Figu 7.24. You can evaluate this integral as follows.

$$\int_0^1 x^{-1/3}\,dx = \lim_{b\to 0^+}\left[\frac{x^{2/3}}{2/3}\right]_b^1$$

$$= \lim_{b\to 0^+}\frac{3}{2}(1 - b^{2/3})$$

$$= \frac{3}{2}$$



The area of the unbounded region is 3/2.
Figure 7.24

Example 7 An Improper Integral That Diverges

Evaluate $\displaystyle\int_0^2 \frac{dx}{x^3}$.

Solution Because the integrand has an infinite discontinuity at $x = 0$, you can wri the following.

$$\int_0^2 \frac{dx}{x^3} = \lim_{b\to 0^+}\left[-\frac{1}{2x^2}\right]_b^2 = \lim_{b\to 0^+}\left(-\frac{1}{8} + \frac{1}{2b^2}\right) = \infty$$

So, you can conclude that the improper integral diverges.

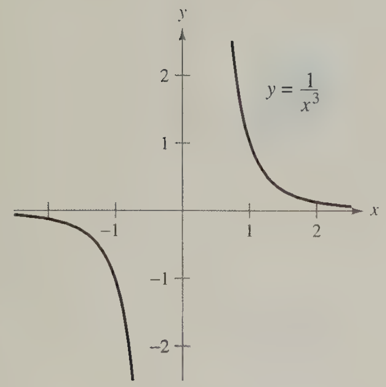
Example 8 An Improper Integral with an Interior Discontinuity

Evaluate $\displaystyle\int_{-1}^2 \frac{dx}{x^3}$.

Solution This integral is improper because the integrand has an infinite discontinui at the interior point $x = 0$, as shown in Figure 7.25. So, you can write the following.

$$\int_{-1}^2 \frac{dx}{x^3} = \int_{-1}^0 \frac{dx}{x^3} + \int_0^2 \frac{dx}{x^3}$$

From Example 7 you know that the second integral diverges. Therefore, the origir improper integral also diverges.



The improper integral $\int_{-1}^{2} 1/x^3\,dx$ diverges.

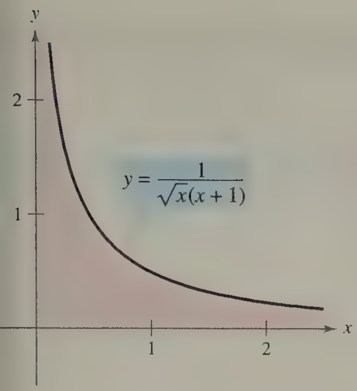
Figure 7.25

NOTE Remember to check for infinite discontinuities at interior points as well as endpoi when determining whether an integral is improper. For instance, if you had not recognized tl the integral in Example 8 was improper, you would have obtained the *incorrect* result

$$\int_{-1}^2 \frac{dx}{x^3} = \left[\frac{-1}{2x^2}\right]_{-1}^2 = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}.$$

Incorrect evaluation

The integral in the next example is improper for *two* reasons. One limit of integration is infinite, and the integrand has an infinite discontinuity at the outer limit of integration, as shown in Figure 7.26.



e area of the unbounded region is π.
gure 7.26

Example 9 A Doubly-Improper Integral

Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$.

Solution To evaluate this integral, split it at a convenient point (say, $x = 1$) and write

$$\begin{aligned}\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)} &= \int_0^{1} \frac{dx}{\sqrt{x}(x+1)} + \int_1^{\infty} \frac{dx}{\sqrt{x}(x+1)}\\ &= \lim_{b\to 0^+}\Big[2 \arctan \sqrt{x}\Big]_b^1 + \lim_{c\to\infty}\Big[2 \arctan \sqrt{x}\Big]_1^c\\ &= 2\left(\frac{\pi}{4}\right) - 0 + 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{4}\right)\\ &= \pi.\end{aligned}$$

Example 10 An Application Involving Arc Length

Use the formula for arc length to show that the circumference of the circle $x^2 + y^2 = 1$ is 2π.

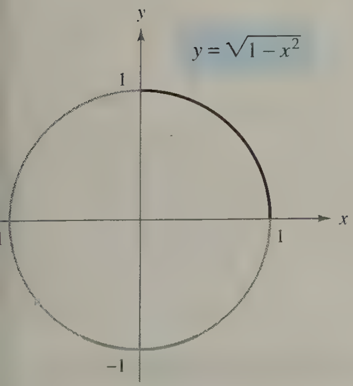
Solution To simplify the work, consider the quarter circle given by $y = \sqrt{1 - x^2}$, where $0 \le x \le 1$. The function y is differentiable for any x in this interval except $x = 1$. Therefore, the arc length of the quarter circle is given by the improper integral

$$\begin{aligned}s &= \int_0^1 \sqrt{1 + (y')^2}\,dx\\ &= \int_0^1 \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx\\ &= \int_0^1 \frac{dx}{\sqrt{1-x^2}}.\end{aligned}$$

This integral is improper because it has an infinite discontinuity at $x = 1$. So, you can write

$$\begin{aligned}s &= \int_0^1 \frac{dx}{\sqrt{1-x^2}}\\ &= \lim_{b\to 1^-}\Big[\arcsin x\Big]_0^b\\ &= \frac{\pi}{2} - 0\\ &= \frac{\pi}{2}.\end{aligned}$$

Finally, multiplying by 4, you can conclude that the circumference of the circle is $4s = 2\pi$, as shown in Figure 7.27.



e circumference of the circle is 2π.
gure 7.27

We conclude this section with a useful theorem describing the convergence divergence of a common type of improper integral. The proof of this theorem is as an exercise (see Exercise 43).

THEOREM 7.5 A Special Type of Improper Integral

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } p \le 1 \end{cases}$$

Example 11 An Application Involving A Solid of Revolution

The solid formed by revolving (about the x-axis) the *unbounded* region lying betwe the graph of $f(x) = 1/x$ and the x-axis ($x \ge 1$) is called **Gabriel's Horn.** (See Fig 7.28.) Show that this solid has a finite volume and an infinite surface area.

FOR FURTHER INFORMATION To further investigate solids that have finite volumes and infinite surface areas, see the article "Supersolids: Solids Having Finite Volume and Infinite Surfaces" by William P. Love in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

Solution Using the disk method and Theorem 7.5, you can determine the volu to be

$$\begin{aligned} V &= \pi \int_1^{\infty} \left(\frac{1}{x}\right)^2 dx \qquad \text{Theorem 7.5, } p = 2 > 1 \\ &= \pi\left(\frac{1}{2-1}\right) \\ &= \pi. \end{aligned}$$

The surface area is given by

$$S = 2\pi \int_1^{\infty} f(x)\sqrt{1 + [f'(x)]^2}\, dx = 2\pi \int_1^{\infty} \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx.$$

Because

$$\sqrt{1 + \frac{1}{x^4}} > 1$$

on the interval $[1, \infty)$, and the improper integral

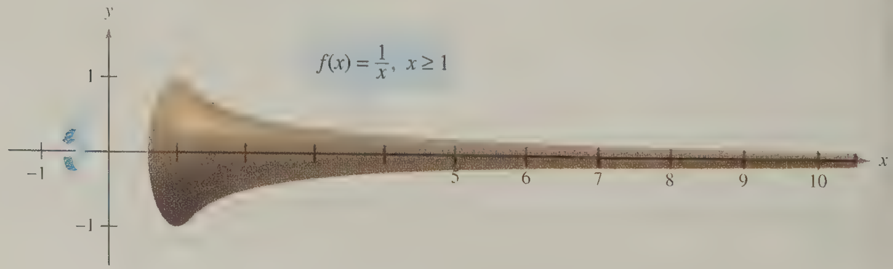
$$\int_1^{\infty} \frac{1}{x}\, dx$$

diverges, you can conclude that the improper integral

$$\int_1^{\infty} \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx$$

also diverges. (See Exercise 46.) So, the surface area is infinite.

FOR FURTHER INFORMATION To learn about another function that has a finite volume and an infinite surface area, see the article "Gabriel's Wedding Cake" by Julian F. Fleron in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

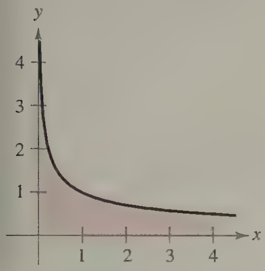


Gabriel's Horn has a finite volume and an infinite surface area.
Figure 7.28

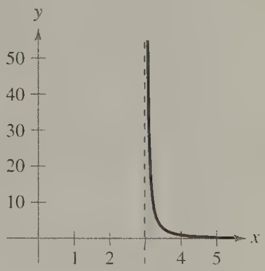
EXERCISES FOR SECTION 7.8

Exercises 1–6, explain why the integral is improper and ermine whether it diverges or converges. Evaluate the egral if it converges.

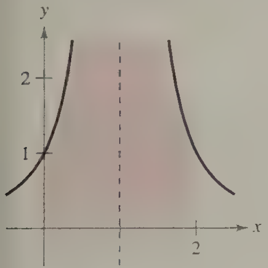
$\int_0^4 \frac{1}{\sqrt{x}}\,dx$



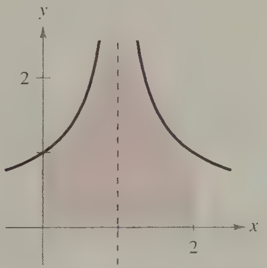
2. $\int_3^4 \frac{1}{(x-3)^{3/2}}\,dx$



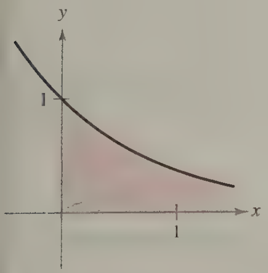
$\int_0^2 \frac{1}{(x-1)^2}\,dx$



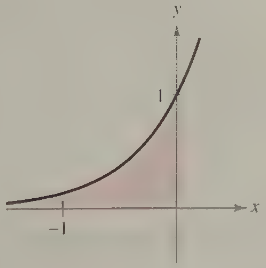
4. $\int_0^2 \frac{1}{(x-1)^{2/3}}\,dx$



$\int_0^\infty e^{-x}\,dx$



6. $\int_{-\infty}^0 e^{2x}\,dx$



iting **In Exercises 7 and 8, explain why the evaluation of the egral is *incorrect*. Use the integration capabilities of a graph- utility to attempt to evaluate the integral. Determine ether the utility gives the correct answer.**

$\int_{-1}^1 \frac{1}{x^2}\,dx = -2$ ✗

$\int_0^\infty e^{-x}\,dx = 0$ ✗

In Exercises 9–26, determine whether the improper integral diverges or converges. Evaluate the integral if it converges.

9.	$\int_1^\infty \frac{1}{x^2}\,dx$	**10.**	$\int_1^\infty \frac{5}{x^3}\,dx$
11.	$\int_1^\infty \frac{3}{\sqrt[3]{x}}\,dx$	**12.**	$\int_1^\infty \frac{4}{\sqrt[4]{x}}\,dx$
13.	$\int_{-\infty}^0 xe^{-2x}\,dx$	**14.**	$\int_0^\infty xe^{-x/2}\,dx$
15.	$\int_0^\infty x^2e^{-x}\,dx$	**16.**	$\int_0^\infty (x-1)e^{-x}\,dx$
17.	$\int_0^\infty e^{-x}\cos x\,dx$	**18.**	$\int_0^\infty e^{-ax}\sin bx\,dx, \quad a > 0$
19.	$\int_4^\infty \frac{1}{x(\ln x)^3}\,dx$	**20.**	$\int_1^\infty \frac{\ln x}{x}\,dx$
21.	$\int_{-\infty}^\infty \frac{2}{4+x^2}\,dx$	**22.**	$\int_0^\infty \frac{x^3}{(x^2+1)^2}\,dx$
23.	$\int_0^\infty \frac{1}{e^x+e^{-x}}\,dx$	**24.**	$\int_0^\infty \frac{e^x}{1+e^x}\,dx$
25.	$\int_0^\infty \cos \pi x\,dx$	**26.**	$\int_0^\infty \sin\frac{x}{2}\,dx$

In Exercises 27–42, determine whether the improper integral diverges or converges. Evaluate the integral if it converges, and check your results with the results obtained by using the integration capabilities of a graphing utility.

27.	$\int_0^1 \frac{1}{x^2}\,dx$	**28.**	$\int_0^4 \frac{8}{x}\,dx$
29.	$\int_0^8 \frac{1}{\sqrt[3]{8-x}}\,dx$	**30.**	$\int_0^6 \frac{4}{\sqrt{6-x}}\,dx$
31.	$\int_0^1 x\ln x\,dx$	**32.**	$\int_0^e \ln x^2\,dx$
33.	$\int_0^{\pi/2} \tan\theta\,d\theta$	**34.**	$\int_0^{\pi/2} \sec\theta\,d\theta$
35.	$\int_2^4 \frac{2}{x\sqrt{x^2-4}}\,dx$	**36.**	$\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$
37.	$\int_2^4 \frac{1}{\sqrt{x^2-4}}\,dx$	**38.**	$\int_0^2 \frac{1}{4-x^2}\,dx$
39.	$\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx$	**40.**	$\int_1^3 \frac{2}{(x-2)^{8/3}}\,dx$
41.	$\int_0^\infty \frac{4}{\sqrt{x}(x+6)}\,dx$	**42.**	$\int_1^\infty \frac{1}{x\ln x}\,dx$

In Exercises 43 and 44, determine all values of *p* for which the improper integral converges.

43. $\int_1^\infty \frac{1}{x^p}\,dx$

44. $\int_0^1 \frac{1}{x^p}\,dx$

45. Use mathematical induction to verify that the following integral converges for any positive integer n.

$$\int_0^{\infty} x^n e^{-x}\, dx$$

46. Given continuous functions f and g such that $0 \le f(x) \le g(x)$ on the interval $[a, \infty)$, prove the following.

(a) If $\int_a^{\infty} g(x)\, dx$ converges, then $\int_a^{\infty} f(x)\, dx$ converges.

(b) If $\int_a^{\infty} f(x)\, dx$ diverges, then $\int_a^{\infty} g(x)\, dx$ diverges.

In Exercises 47–56, use the results of Exercises 43–46 to determine whether the improper integral converges or diverges.

47. $\displaystyle\int_0^1 \frac{1}{x^3}\, dx$ **48.** $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x}}\, dx$

49. $\displaystyle\int_1^{\infty} \frac{1}{x^3}\, dx$ **50.** $\displaystyle\int_0^{\infty} x^4 e^{-x}\, dx$

51. $\displaystyle\int_1^{\infty} \frac{1}{x^2+5}\, dx$ **52.** $\displaystyle\int_2^{\infty} \frac{1}{\sqrt{x-1}}\, dx$

53. $\displaystyle\int_2^{\infty} \frac{1}{\sqrt[3]{x(x-1)}}\, dx$ **54.** $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}(x+1)}\, dx$

55. $\displaystyle\int_0^{\infty} e^{-x^2}\, dx$ **56.** $\displaystyle\int_2^{\infty} \frac{1}{\sqrt{x}\ln x}\, dx$

Getting at the Concept

57. List the different types of improper integrals.

58. Define the terms *converges* and *diverges* when working with improper integrals.

59. Explain why $\displaystyle\int_{-1}^{1} \frac{1}{x^3}\, dx \neq 0$.

60. Give examples of an improper integral with infinite limits that (a) converges and (b) diverges.

Laplace Transforms **Let $f(t)$ be a function defined for all positive values of t. The Laplace Transform of $f(t)$ is defined by**

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

if the improper integral exists. Laplace Transforms are used to solve differential equations. In Exercises 61–68, find the Laplace Transform of the function.

61. $f(t) = 1$ **62.** $f(t) = t$

63. $f(t) = t^2$ **64.** $f(t) = e^{at}$

65. $f(t) = \cos at$ **66.** $f(t) = \sin at$

67. $f(t) = \cosh at$ **68.** $f(t) = \sinh at$

Area and Volume **In Exercises 69 and 70, consider the reg satisfying the inequalities. (a) Find the area of the region. Find the volume of the solid generated by revolving the reg about the x-axis. (c) Find the volume of the solid generated revolving the region about the y-axis.**

69. $y \le e^{-x},\ y \ge 0,\ x \ge 0$ **70.** $y \le 1/x^2,\ y \ge 0,\ x \ge$

71. ***Arc Length*** Sketch the graph of the hypocycloid of f cusps

$$x^{2/3} + y^{2/3} = 4$$

and find its perimeter.

72. ***Surface Area*** The region bounded by

$$(x-2)^2 + y^2 = 1$$

is revolved about the y-axis to form a torus. Find the surf area of the torus.

73. ***The Gamma Function*** The Gamma Function $\Gamma(n)$ is defi by

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\, dx, \quad n > 0.$$

(a) Find $\Gamma(1)$, $\Gamma(2)$, and $\Gamma(3)$.

(b) Use integration by parts to show that $\Gamma(n+1) = n\Gamma(n$

(c) Express $\Gamma(n)$ in terms of factorial notation where n i positive integer.

74. ***Work*** A 5-ton rocket is fired from the surface of earth i outer space.

(a) How much work is required to overcome earth's grav tional force?

(b) How far has the rocket traveled when half the total w has occurred?

Probability **A nonnegative function f is called a *probabi density function* if**

$$\int_{-\infty}^{\infty} f(t)\, dt = 1.$$

The probability that x lies between a and b is given by

$$P(a \le x \le b) = \int_a^b f(t)\, dt.$$

The expected value of x is given by

$$E(x) = \int_{-\infty}^{\infty} t f(t)\, dt.$$

In Exercises 75 and 76, (a) show that the nonnegative funct is a probability density function, (b) find $P(0 \le x \le 4)$, (c) find $E(x)$.

75. $f(t) = \begin{cases} \frac{1}{7}e^{-t/7}, & t \ge 0 \\ 0, & t < 0 \end{cases}$ **76.** $f(t) = \begin{cases} \frac{2}{5}e^{-2t/5}, & t \ge \\ 0, & t < \end{cases}$

pitalized Cost **In Exercises 77 and 78, find the capitalized st C of an asset (a) for $n = 5$ years, (b) for $n = 10$ years, and forever. The capitalized cost is given by**

$$= C_0 + \int_0^n c(t)e^{-rt}\,dt$$

ere C_0 is the original investment, t is the time in years, r is the nual interest rate compounded continuously, and $c(t)$ is the nual cost of maintenance.

. $C_0 = \$650{,}000$

$c(t) = \$25{,}000$

$r = 0.06$

78. $C_0 = \$650{,}000$

$c(t) = \$25{,}000(1 + 0.08t)$

$r = 0.06$

. ***Electromagnetic Theory*** Find the value of the following integral used in electromagnetic theory.

$$P = k\int_1^{\infty} \frac{1}{(a^2 + x^2)^{3/2}}\,dx$$

. ***Writing***

(a) The improper integrals

$$\int_1^{\infty} \frac{1}{x}\,dx \quad \text{and} \quad \int_1^{\infty} \frac{1}{x^2}\,dx$$

diverge and converge, respectively. Describe the essential differences between the integrands that cause one integral to converge and the other to diverge.

(b) Sketch a graph of the function $y = \sin x/x$ over the interval $(1, \infty)$. Use your knowledge of the definite integral to make an inference as to whether or not the integral

$$\int_1^{\infty} \frac{\sin x}{x}\,dx$$

converges. Give reasons for your answer.

(c) Use one iteration of integration by parts on the integral in part (b) to determine its divergence or convergence.

. ***Think About It*** Consider the integral

$$\int_0^3 \frac{10}{x^2 - 2x}\,dx.$$

To determine the convergence or divergence of the integral, how many improper integrals must be analyzed? What must be true of each of these integrals if the given integral converges?

. ***Exploration*** Consider the integral

$$\int_0^{\pi/2} \frac{4}{1 + (\tan x)^n}\,dx$$

where n is a positive integer.

(a) Is the integral improper? Explain.

(b) Use a graphing utility to graph the integrand for $n = 2$, 4, 8, and 12.

(c) Use the graphs to approximate the integral as $n \to \infty$.

(d) Use a computer algebra system to evaluate the integral for the values of n in part (b). Make a conjecture about the value of the integral for any positive integer n. Compare the results with your answer in part (c).

83. Let $I_n = \displaystyle\int_0^{\infty} \frac{x^{2n-1}}{(x^2 + 1)^{n+3}}\,dx, \quad n \geq 1.$

Prove that $I_n = \left(\dfrac{n - 1}{n + 2}\right)I_{n-1}$

and then evaluate each of the following.

(a) $\displaystyle\int_0^{\infty} \frac{x}{(x^2 + 1)^4}\,dx$

(b) $\displaystyle\int_0^{\infty} \frac{x^3}{(x^2 + 1)^5}\,dx$

(c) $\displaystyle\int_0^{\infty} \frac{x^5}{(x^2 + 1)^6}\,dx$

84. ***Normal Probability*** The mean height of American men between 18 and 24 years old is 70 inches, and the standard deviation is 3 inches. If an 18- to 24-year-old man is chosen at random from the population, the probability that he is 6 feet tall or taller is

$$P(72 \leq x < \infty) = \int_{72}^{\infty} \frac{1}{3\sqrt{2\pi}} e^{-(x-70)^2/18}\,dx.$$

(Source: National Center for Health Statistics)

(a) Use a graphing utility to graph the integrand. Use the graphing utility to convince yourself that the area between the x-axis and the integrand is 1.

(b) Use a graphing utility to approximate $P(72 \leq x < \infty)$.

(c) Approximate $0.5 - P(70 \leq x \leq 72)$ using a graphing utility. Use the graph in part (a) to explain why this result is the same as the answer in part (b).

True or False? **In Exercises 85–88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

85. If f is continuous on $[0, \infty)$ and $\displaystyle\lim_{x\to\infty} f(x) = 0$, then $\int_0^{\infty} f(x)\,dx$ converges.

86. If f is continuous on $[0, \infty)$ and $\int_0^{\infty} f(x)\,dx$ diverges, then $\displaystyle\lim_{x\to\infty} f(x) \neq 0$.

87. If f' is continuous on $[0, \infty)$ and $\displaystyle\lim_{x\to\infty} f(x) = 0$, then $\int_0^{\infty} f'(x)\,dx = -f(0)$.

88. If the graph of f is symmetric with respect to the origin or the y-axis, then $\int_0^{\infty} f(x)\,dx$ converges if and only if $\int_{-\infty}^{\infty} f(x)\,dx$ converges.

REVIEW EXERCISES FOR CHAPTER 7

7.1 **In Exercises 1–8, use the basic integration rules to evaluate the integral.**

1. $\int x\sqrt{x^2-1}\,dx$
2. $\int xe^{x^2-1}\,dx$
3. $\int \frac{x}{x^2-1}\,dx$
4. $\int \frac{x}{\sqrt{1-x^2}}\,dx$
5. $\int \frac{\ln(2x)}{x}\,dx$
6. $\int 2x\sqrt{2x-3}\,dx$
7. $\int \frac{16}{\sqrt{16-x^2}}\,dx$
8. $\int \frac{x^4+2x^2+x+1}{(x^2+1)^2}\,dx$

7.2 **In Exercises 9–16, use integration by parts to evaluate the integral.**

9. $\int e^{2x}\sin 3x\,dx$
10. $\int (x^2-1)e^x\,dx$
11. $\int x\sqrt{x-5}\,dx$
12. $\int \arctan 2x\,dx$
13. $\int x^2\sin 2x\,dx$
14. $\int \ln\sqrt{x^2-1}\,dx$
15. $\int x\arcsin 2x\,dx$
16. $\int e^x\arctan e^x\,dx$

7.3 **In Exercises 17–22, evaluate the trigonometric integral.**

17. $\int \cos^3(\pi x-1)\,dx$
18. $\int \sin^2\frac{\pi x}{2}\,dx$
19. $\int \sec^4\frac{x}{2}\,dx$
20. $\int \tan\theta\sec^4\theta\,d\theta$
21. $\int \frac{1}{1-\sin\theta}\,d\theta$
22. $\int \cos 2\theta(\sin\theta+\cos\theta)^2\,d\theta$

7.4 **In Exercises 23–28, use trigonometric substitution to evaluate the integral.**

23. $\int \frac{-12}{x^2\sqrt{4-x^2}}\,dx$
24. $\int \frac{\sqrt{x^2-9}}{x}\,dx, \quad x>3$
25. $\int \frac{x^3}{\sqrt{4+x^2}}\,dx$
26. $\int \sqrt{9-4x^2}\,dx$
27. $\int \sqrt{4-x^2}\,dx$
28. $\int \frac{\sin\theta}{1+2\cos^2\theta}\,d\theta$

In Exercises 29 and 30, evaluate the integral using the indicated methods.

29. $\int \frac{x^3}{\sqrt{4+x^2}}\,dx$
    (a) Trigonometric substitution
    (b) Substitution: $u^2=4+x^2$
    (c) Integration by parts: $dv=\left(x/\sqrt{4+x^2}\right)dx$

30. $\int x\sqrt{4+x}\,dx$
    (a) Trigonometric substitution
    (b) Substitution: $u^2=4+x$
    (c) Substitution: $u=4+x$
    (d) Integration by parts: $dv=\sqrt{4+x}\,dx$

7.5 **In Exercises 31–36, use partial fractions to evaluate t integral.**

31. $\int \frac{x-28}{x^2-x-6}\,dx$
32. $\int \frac{2x^3-5x^2+4x-4}{x^2-x}\,d$
33. $\int \frac{x^2+2x}{x^3-x^2+x-1}\,dx$
34. $\int \frac{4x-2}{3(x-1)^2}\,dx$
35. $\int \frac{x^2}{x^2+2x-15}\,dx$
36. $\int \frac{\sec^2\theta}{\tan\theta(\tan\theta-1)}\,d\theta$

7.6 **In Exercises 37–44, use integration tables to evalua the integral.**

37. $\int \frac{x}{(2+3x)^2}\,dx$
38. $\int \frac{x}{\sqrt{2+3x}}\,dx$
39. $\int \frac{x}{1+\sin x^2}\,dx$
40. $\int \frac{x}{1+e^{x^2}}\,dx$
41. $\int \frac{x}{x^2+4x+8}\,dx$
42. $\int \frac{3}{2x\sqrt{9x^2-1}}\,dx, \quad x >$
43. $\int \frac{1}{\sin\pi x\cos\pi x}\,dx$
44. $\int \frac{1}{1+\tan\pi x}\,dx$

45. Verify the reduction formula

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$$

46. Verify the reduction formula

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx.$$

In Exercises 47–54, evaluate the integral using any method.

47. $\int \theta\sin\theta\cos\theta\,d\theta$
48. $\int \frac{\csc\sqrt{2x}}{\sqrt{x}}\,dx$
49. $\int \frac{x^{1/4}}{1+x^{1/2}}\,dx$
50. $\int \sqrt{1+\sqrt{x}}\,dx$
51. $\int \sqrt{1+\cos x}\,dx$
52. $\int \frac{3x^3+4x}{(x^2+1)^2}\,dx$
53. $\int \cos x\ln(\sin x)\,dx$
54. $\int (\sin\theta+\cos\theta)^2\,d\theta$

Exercises 55–58, solve the differential equation using any thod.

$\frac{dy}{dx} = \frac{9}{x^2 - 9}$ **56.** $\frac{dy}{dx} = \frac{\sqrt{4 - x^2}}{2x}$

$y' = \ln(x^2 + x)$ **58.** $y' = \sqrt{1 - \cos\theta}$

Exercises 59–64, evaluate the definite integral using any thod. Use a graphing utility to verify your result.

$\int_2^{\sqrt{5}} x(x^2 - 4)^{3/2}\,dx$ **60.** $\int_0^1 \frac{x}{(x - 2)(x - 4)}\,dx$

$\int_1^4 \frac{\ln x}{x}\,dx$ **62.** $\int_0^2 xe^{3x}\,dx$

$\int_0^{\pi} x \sin x\,dx$ **64.** $\int_0^3 \frac{x}{\sqrt{1 + x}}\,dx$

ea **In Exercises 65 and 66, find the area of the region bound- by the graphs of the equations.**

$y = x\sqrt{4 - x}, \quad y = 0$

$y = \frac{1}{25 - x^2}, \quad y = 0, x = 0, x = 4$

Exercises 67 and 68, find the centroid of the region bounded the graphs of the equations.

$y = \sqrt{1 - x^2}, \quad y = 0$

$(x - 1)^2 + y^2 = 1, \quad (x - 4)^2 + y^2 = 4$

c Length **In Exercises 69 and 70, approximate to two deci- l places the arc length of the curve over the given interval.**

Function	*Interval*
$y = \sin x$	$[0, \pi]$
$y = \sin^2 x$	$[0, \pi]$

7 **In Exercises 71–78, use L'Hôpital's Rule to evaluate limit.**

$\lim_{x\to 1} \frac{(\ln x)^2}{x - 1}$ **72.** $\lim_{x\to 0} \frac{\sin \pi x}{\sin 2\pi x}$

$\lim_{x\to\infty} \frac{e^{2x}}{x^2}$ **74.** $\lim_{x\to\infty} xe^{-x^2}$

$\lim_{x\to\infty} (\ln x)^{2/x}$ **76.** $\lim_{x\to 1^+} (x - 1)^{\ln x}$

$\lim_{n\to\infty} 1000\left(1 + \frac{0.09}{n}\right)^n$ **78.** $\lim_{x\to 1^+} \left(\frac{2}{\ln x} - \frac{2}{x - 1}\right)$

8 **In Exercises 79–82, determine whether the improper egral converges or diverges. Evaluate the integral if it con- rges.**

$\int_0^{16} \frac{1}{\sqrt[4]{x}}\,dx$ **80.** $\int_0^1 \frac{6}{x - 1}\,dx$

$\int_1^{\infty} x^2 \ln x\,dx$ **82.** $\int_0^{\infty} \frac{e^{-1/x}}{x^2}\,dx$

83. ***Present Value*** The board of directors of a corporation is calculating the price to pay for a business that is forecast to yield a continuous flow of profit of \$500,000 per year. If money will earn a nominal rate of 5% per year compounded continuously, what is the present value of the business

(a) for 20 years?

(b) forever (in perpetuity)?

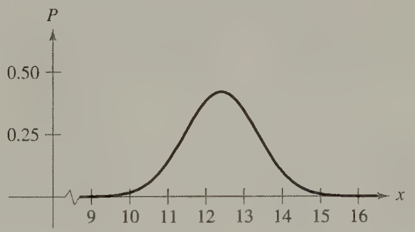
(*Note:* The present value for t_0 years is $\int_0^{t_0} 500{,}000e^{-0.05t}\,dt$.)

84. ***Volume*** Find the volume of the solid generated by revolving the region bounded by the graphs of $y = xe^{-x}$, $y = 0$, and $x = 0$ about the x-axis.

85. ***Probability*** The average lengths (from beak to tail) of different species of warblers in the eastern United States are approximately normally distributed with a mean of 12.9 centimeters and a standard deviation of 0.95 centimeter (see figure). The probability that a randomly selected warbler has a length between a and b centimeters is

$$P(a \le x \le b) = \frac{1}{0.95\sqrt{2\pi}} \int_a^b e^{-(x-12.9)^2/2(0.95)^2}\,dx.$$

Use a graphing utility to approximate the probability that a randomly selected warbler has a length of (a) 13 centimeters or greater and (b) 15 centimeters or greater. *(Source: Peterson's Field Guide: Eastern Birds)*



86. Using the inequality

$$\frac{1}{x^5} + \frac{1}{x^{10}} + \frac{1}{x^{15}} < \frac{1}{x^5 - 1} < \frac{1}{x^5} + \frac{1}{x^{10}} + \frac{2}{x^{15}}$$

for $x \ge 2$, approximate $\int_2^{\infty} \frac{1}{x^5 - 1}\,dx$.

P.S. Problem Solving

1. (a) Evaluate the integrals

$$\int_{-1}^{1} (1 - x^2)\, dx \quad \text{and} \quad \int_{-1}^{1} (1 - x^2)^2\, dx.$$

(b) Use Wallis's Formulas to prove that

$$\int_{-1}^{1} (1 - x^2)^n\, dx = \frac{2^{2n+1}(n!)^2}{(2n + 1)!}$$

for all positive integers n.

2. (a) Evaluate the integrals

$$\int_{0}^{1} \ln x\, dx \quad \text{and} \quad \int_{0}^{1} (\ln x)^2\, dx.$$

(b) Prove that

$$\int_{0}^{1} (\ln x)^n\, dx = (-1)^n n!$$

for all positive integers n.

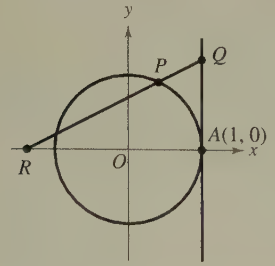
3. Find the value of the positive constant c such that

$$\lim_{x\to\infty} \left(\frac{x + c}{x - c}\right)^x = 9.$$

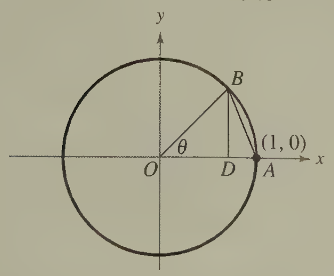
4. Find the value of the positive constant c such that

$$\lim_{x\to\infty} \left(\frac{x - c}{x + c}\right)^x = \frac{1}{4}.$$

5. In the figure, the line $x = 1$ is tangent to the unit circle at A. The length of segment QA equals the length of the circular arc $\overset{\frown}{PA}$. Show that the length of segment OR approaches 2 as P approaches A.



6. In the figure, the segment BD is the height of triangle $\triangle OAB$. Let R be the ratio of the area of $\triangle DAB$ to that of the shaded region formed by deleting $\triangle OAB$ from the circular sector subtended by angle θ. Find $\lim\limits_{\theta\to 0^+} R$.



7. Consider the problem of finding the area of the region bounc by the curve

$$y = \frac{x^2}{[x^2 + 9]^{3/2}},$$

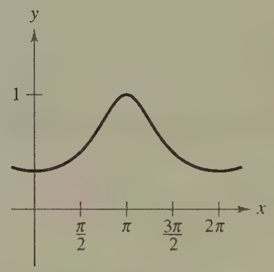
the x-axis and $x = 4$.

(a) Use a graphing utility to graph the region and approxim its area.

(b) Use an appropriate trigonometric substitution to find t exact area.

(c) Use the substitution $x = 3 \sinh u$ to find the exact area a verify that you obtain the same answer as in part (b).

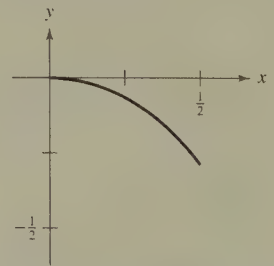
8. Use the substitution $u = \tan \dfrac{x}{2}$ to find the area of the shadec region under the graph of $y = \dfrac{1}{2 + \cos x}$, $0 \le x \le \pi/2$.



9. Find the arc length of the graph of the function

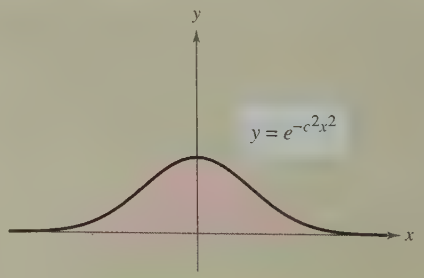
$$y = \ln(1 - x^2)$$

on the interval $0 \le x \le \frac{1}{2}$.



10. Find the centroid of the region above the x-axis and bounc above by the curve $y = e^{-c^2x^2}$, where c is a positive constan

$\left(\textit{Hint:}\ \text{Show that} \displaystyle\int_{0}^{\infty} e^{-c^2x^2}\, dx = \frac{1}{c}\int_{0}^{\infty} e^{-x^2}\, dx.\right)$



P.S.

Some elementary functions, such as $f(x) = \sin(x^2)$, do not have antiderivatives that are elementary functions. Joseph Liouville proved that

$$\int \frac{e^x}{x}\,dx$$

does not have an elementary antiderivative. Use this fact to prove that

$$\int \frac{1}{\ln x}\,dx$$

is not elementary.

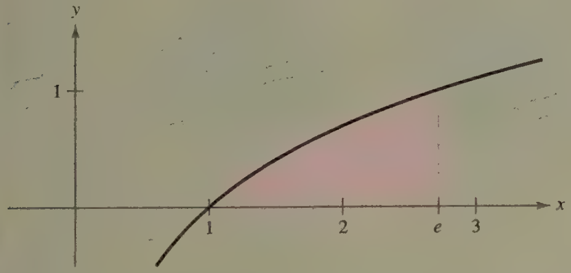
(a) Let $y = f^{-1}(x)$ be the inverse of f. Use integration by parts to derive the formula

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int f(y)\,dy.$$

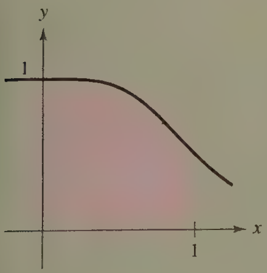
(b) Use the formula in part (a) to evaluate the integral

$$\int \arcsin x\,dx.$$

(c) Use the formula in part (a) to find the area under the graph of $y = \ln x$, $1 \le x \le e$.



Factor the polynomial $p(x) = x^4 + 1$ and then find the area under the graph of $y = \dfrac{1}{x^4 + 1}$, $0 \le x \le 1$.



(a) Use the substitution $u = \dfrac{\pi}{2} - x$ to evaluate the integral

$$\int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}\,dx.$$

(b) Let n be a positive integer. Evaluate the integral

$$\int_0^{\pi/2} \frac{\sin^n x}{\cos^n x + \sin^n x}\,dx.$$

15. Use a graphing utility to estimate each limit. Then calculate each limit using L'Hôpital's Rule. What can you conclude about the indeterminate form $0 \cdot \infty$?

(a) $\displaystyle\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right)$

(b) $\displaystyle\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right)$

(c) $\displaystyle\lim_{x\to 0^+}\left[\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right)\right]$

16. Suppose the denominator of a rational function can be factored into distinct linear factors

$$D(x) = (x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

for a positive integer n and distinct real numbers $c_1, c_2, \ldots, c_n$. If N is a polynomial of degree less than n, show that

$$\frac{N(x)}{D(x)} = \frac{P_1}{x - c_1} + \frac{P_2}{x - c_2} + \cdot \cdot \cdot + \frac{P_n}{x - c_n}$$

where $P_k = N(c_k)/D'(c_k)$ for $k = 1, 2, \ldots, n$. Note that this is the partial fraction decomposition of $N(x)/D(x)$.

17. Use the results of Exercise 16 to find the partial fraction decomposition of

$$\frac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x}.$$

18. The velocity (in feet per second) of a rocket whose initial mass (including fuel) is m is

$$v = gt + u \ln \frac{m}{m - rt}, \quad t < \frac{m}{r}$$

where u is the expulsion speed of the fuel, r is the rate at which the fuel is consumed, and $g = -32$ feet per second per second is the acceleration due to gravity. Find the position equation for a rocket for which $m = 50{,}000$ pounds, $u = 12{,}000$ feet per second, and $r = 400$ pounds per second. What is the height of the rocket when $t = 100$ seconds? (Assume that the rocket was fired from ground level and is moving straight up.)

19. Suppose that $f(a) = f(b) = g(a) = g(b) = 0$ and the second derivatives of f and g are continuous on the closed interval $[a, b]$. Prove that

$$\int_a^b f(x)g''(x)\,dx = \int_a^b f''(x)g(x)\,dx.$$

20. Suppose that $f(a) = f(b) = 0$ and the second derivatives of f exist on the closed interval $[a, b]$. Prove that

$$\int_a^b (x - a)(x - b)f''(x)\,dx = 2\int_a^b f(x)\,dx.$$

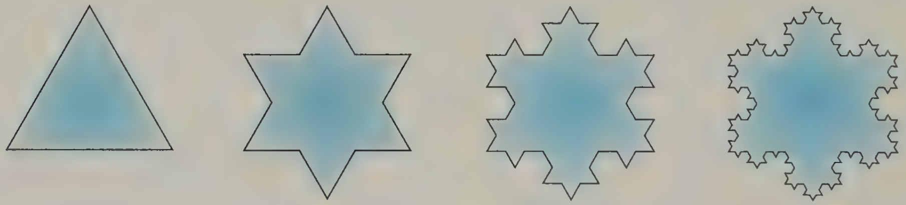
The Koch Snowflake: Infinite Perimeter?

> *Why is geometry often described as "cold" and "dry"? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightening travel in a straight line. ... Nature exhibits not simply a higher degree but an altogether different level of complexity.*
>
> Benoit Mandelbrot (1924–)

To meet the challenge of creating a geometry capable of describing nature, Mandelbrot developed *fractal* geometry. Fractal sets come in diverse forms. Some are curves, others are disconnected "dust" and still others are such odd forms that there are no existing geometric terms to describe them.

One of the "classic" fractals is the Koch snowflake, named after the Swedish mathematician Helge von Koch (1870–1924). It is sometimes classified as a "coastline curve" because of the way a coastline appears increasingly more complex with magnification. To describe the Koch snowflake, Mandelbrot coined the term *teragon*, which translates literally from the Greek words for "monster curve."

The construction of the Koch snowflake begins with an equilateral triangle whose sides are one unit long. In the first iteration, a triangle with sides one-third unit long is added in the center of each side of the original. In the second iteration, a triangle with sides one-ninth unit long is added in the center of each side. Successive iterations continue this process—without stopping.



QUESTIONS

1. Write a formula that describes the side length of the triangles that will be added in the nth iteration.
2. Make a table of the perimeter of the original triangle and of the teragon in the first three iterations, as shown above. Write an expression describing the perimeter of the teragon after the nth iteration. What do you expect will happen to the perimeter as n approaches infinity?
3. Make a table of the area of the teragon in the first four iterations. Write an expression describing the area after the nth iteration. What do you expect will happen to the area as n approaches infinity?
4. Is it possible for a closed and bounded region in the plane to have a finite area and an infinite perimeter? Explain your reasoning.

The concepts presented here will be explored further in this chapter. For an extension of this application, see Lab 11 in the lab series that accompanies this text at college.hmco.com.

Integration Tables

C

Forms Involving u^n

1. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$

2. $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$

Forms Involving $a + bu$

3. $\displaystyle\int \frac{u}{a+bu}\,du = \frac{1}{b^2}\bigl(bu - a\ln|a+bu|\bigr) + C$

4. $\displaystyle\int \frac{u}{(a+bu)^2}\,du = \frac{1}{b^2}\left(\frac{a}{a+bu} + \ln|a+bu|\right) + C$

5. $\displaystyle\int \frac{u}{(a+bu)^n}\,du = \frac{1}{b^2}\left[\frac{-1}{(n-2)(a+bu)^{n-2}} + \frac{a}{(n-1)(a+bu)^{n-1}}\right] + C,\ n \neq 1, 2$

6. $\displaystyle\int \frac{u^2}{a+bu}\,du = \frac{1}{b^3}\left[-\frac{bu}{2}(2a-bu) + a^2\ln|a+bu|\right] + C$

7. $\displaystyle\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

8. $\displaystyle\int \frac{u^2}{(a+bu)^3}\,du = \frac{1}{b^3}\left[\frac{2a}{a+bu} - \frac{a^2}{2(a+bu)^2} + \ln|a+bu|\right] + C$

9. $\displaystyle\int \frac{u^2}{(a+bu)^n}\,du = \frac{1}{b^3}\left[\frac{-1}{(n-3)(a+bu)^{n-3}} + \frac{2a}{(n-2)(a+bu)^{n-2}} - \frac{a^2}{(n-1)(a+bu)^{n-1}}\right] + C,\ n \neq 1, 2, 3$

10. $\displaystyle\int \frac{1}{u(a+bu)}\,du = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

11. $\displaystyle\int \frac{1}{u(a+bu)^2}\,du = \frac{1}{a}\left(\frac{1}{a+bu} + \frac{1}{a}\ln\left|\frac{u}{a+bu}\right|\right) + C$

12. $\displaystyle\int \frac{1}{u^2(a+bu)}\,du = -\frac{1}{a}\left(\frac{1}{u} + \frac{b}{a}\ln\left|\frac{u}{a+bu}\right|\right) + C$

3. $\displaystyle\int \frac{1}{u^2(a+bu)^2}\,du = -\frac{1}{a^2}\left[\frac{a+2bu}{u(a+bu)} + \frac{2b}{a}\ln\left|\frac{u}{a+bu}\right|\right] + C$

Forms Involving $a + bu + cu^2,\ b^2 \neq 4ac$

14. $$\int \frac{1}{a + bu + cu^2}\,du = \begin{cases} \dfrac{2}{\sqrt{4ac - b^2}} \arctan \dfrac{2cu + b}{\sqrt{4ac - b^2}} + C, & b^2 < 4ac \\[2ex] \dfrac{1}{\sqrt{b^2 - 4ac}} \ln \left| \dfrac{2cu + b - \sqrt{b^2 - 4ac}}{2cu + b + \sqrt{b^2 - 4ac}} \right| + C, & b^2 > 4ac \end{cases}$$

15. $$\int \frac{u}{a + bu + cu^2}\,du = \frac{1}{2c}\left(\ln|a + bu + cu^2| - b \int \frac{1}{a + bu + cu^2}\,du\right)$$

Forms Involving $\sqrt{a + bu}$

16. $$\int u^n \sqrt{a + bu}\,du = \frac{2}{b(2n + 3)}\left[u^n(a + bu)^{3/2} - na \int u^{n-1}\sqrt{a + bu}\,du\right]$$

17. $$\int \frac{1}{u\sqrt{a + bu}}\,du = \begin{cases} \dfrac{1}{\sqrt{a}} \ln \left| \dfrac{\sqrt{a + bu} - \sqrt{a}}{\sqrt{a + bu} + \sqrt{a}} \right| + C, & a > 0 \\[2ex] \dfrac{2}{\sqrt{-a}} \arctan \sqrt{\dfrac{a + bu}{-a}} + C, & a < 0 \end{cases}$$

18. $$\int \frac{1}{u^n\sqrt{a + bu}}\,du = \frac{-1}{a(n - 1)}\left[\frac{\sqrt{a + bu}}{u^{n-1}} + \frac{(2n - 3)b}{2} \int \frac{1}{u^{n-1}\sqrt{a + bu}}\,du\right],\quad n \neq 1$$

19. $$\int \frac{\sqrt{a + bu}}{u}\,du = 2\sqrt{a + bu} + a \int \frac{1}{u\sqrt{a + bu}}\,du$$

20. $$\int \frac{\sqrt{a + bu}}{u^n}\,du = \frac{-1}{a(n - 1)}\left[\frac{(a + bu)^{3/2}}{u^{n-1}} + \frac{(2n - 5)b}{2} \int \frac{\sqrt{a + bu}}{u^{n-1}}\,du\right],\quad n \neq 1$$

21. $$\int \frac{u}{\sqrt{a + bu}}\,du = \frac{-2(2a - bu)}{3b^2}\sqrt{a + bu} + C$$

22. $$\int \frac{u^n}{\sqrt{a + bu}}\,du = \frac{2}{(2n + 1)b}\left(u^n\sqrt{a + bu} - na \int \frac{u^{n-1}}{\sqrt{a + bu}}\,du\right)$$

Forms Involving $a^2 \pm u^2,\ a > 0$

23. $$\int \frac{1}{a^2 + u^2}\,du = \frac{1}{a} \arctan \frac{u}{a} + C$$

24. $$\int \frac{1}{u^2 - a^2}\,du = -\int \frac{1}{a^2 - u^2}\,du = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right| + C$$

25. $$\int \frac{1}{(a^2 \pm u^2)^n}\,du = \frac{1}{2a^2(n - 1)}\left[\frac{u}{(a^2 \pm u^2)^{n-1}} + (2n - 3) \int \frac{1}{(a^2 \pm u^2)^{n-1}}\,du\right],\quad n \neq 1$$

Forms Involving $\sqrt{u^2 \pm a^2},\ a > 0$

26. $$\int \sqrt{u^2 \pm a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2 \pm a^2} \pm a^2 \ln\left|u + \sqrt{u^2 \pm a^2}\right|\right) + C$$

27. $$\int u^2\sqrt{u^2 \pm a^2}\,du = \frac{1}{8}\left[u(2u^2 \pm a^2)\sqrt{u^2 \pm a^2} - a^4 \ln\left|u + \sqrt{u^2 \pm a^2}\right|\right] + C$$

28. $$\int \frac{\sqrt{u^2 + a^2}}{u}\,du = \sqrt{u^2 + a^2} - a \ln \left| \frac{a + \sqrt{u^2 + a^2}}{u} \right| + C$$

9. $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\,\operatorname{arcsec}\frac{|u|}{a} + C$

0. $\displaystyle\int \frac{\sqrt{u^2\pm a^2}}{u^2}\,du = \frac{-\sqrt{u^2\pm a^2}}{u} + \ln\left|u+\sqrt{u^2\pm a^2}\right| + C$

1. $\displaystyle\int \frac{1}{\sqrt{u^2\pm a^2}}\,du = \ln\left|u+\sqrt{u^2\pm a^2}\right| + C$

2. $\displaystyle\int \frac{1}{u\sqrt{u^2+a^2}}\,du = \frac{-1}{a}\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C$

3. $\displaystyle\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

4. $\displaystyle\int \frac{u^2}{\sqrt{u^2\pm a^2}}\,du = \frac{1}{2}\left(u\sqrt{u^2\pm a^2} \mp a^2\ln\left|u+\sqrt{u^2\pm a^2}\right|\right) + C$

5. $\displaystyle\int \frac{1}{u^2\sqrt{u^2\pm a^2}}\,du = \mp\frac{\sqrt{u^2\pm a^2}}{a^2u} + C$

6. $\displaystyle\int \frac{1}{(u^2\pm a^2)^{3/2}}\,du = \frac{\pm u}{a^2\sqrt{u^2\pm a^2}} + C$

orms Involving $\sqrt{a^2-u^2},\ a > 0$

7. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{1}{2}\left(u\sqrt{a^2-u^2} + a^2\arcsin\frac{u}{a}\right) + C$

8. $\displaystyle\int u^2\sqrt{a^2-u^2}\,du = \frac{1}{8}\left[u(2u^2-a^2)\sqrt{a^2-u^2} + a^4\arcsin\frac{u}{a}\right] + C$

9. $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

0. $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = \frac{-\sqrt{a^2-u^2}}{u} - \arcsin\frac{u}{a} + C$

1. $\displaystyle\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\frac{u}{a} + C$

2. $\displaystyle\int \frac{1}{u\sqrt{a^2-u^2}}\,du = \frac{-1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

3. $\displaystyle\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = \frac{1}{2}\left(-u\sqrt{a^2-u^2} + a^2\arcsin\frac{u}{a}\right) + C$

4. $\displaystyle\int \frac{1}{u^2\sqrt{a^2-u^2}}\,du = \frac{-\sqrt{a^2-u^2}}{a^2u} + C$

5. $\displaystyle\int \frac{1}{(a^2-u^2)^{3/2}}\,du = \frac{u}{a^2\sqrt{a^2-u^2}} + C$

Forms Involving sin *u* or cos *u*

46. $\int \sin u \, du = -\cos u + C$

47. $\int \cos u \, du = \sin u + C$

48. $\int \sin^2 u \, du = \frac{1}{2}(u - \sin u \cos u) + C$

49. $\int \cos^2 u \, du = \frac{1}{2}(u + \sin u \cos u) + C$

50. $\int \sin^n u \, du = -\frac{\sin^{n-1} u \cos u}{n} + \frac{n-1}{n} \int \sin^{n-2} u \, du$

51. $\int \cos^n u \, du = \frac{\cos^{n-1} u \sin u}{n} + \frac{n-1}{n} \int \cos^{n-2} u \, du$

52. $\int u \sin u \, du = \sin u - u \cos u + C$

53. $\int u \cos u \, du = \cos u + u \sin u + C$

54. $\int u^n \sin u \, du = -u^n \cos u + n \int u^{n-1} \cos u \, du$

55. $\int u^n \cos u \, du = u^n \sin u - n \int u^{n-1} \sin u \, du$

56. $\int \frac{1}{1 \pm \sin u} \, du = \tan u \mp \sec u + C$

57. $\int \frac{1}{1 \pm \cos u} \, du = -\cot u \pm \csc u + C$

58. $\int \frac{1}{\sin u \cos u} \, du = \ln|\tan u| + C$

Forms Involving tan *u*, cot *u*, sec *u*, csc *u*

59. $\int \tan u \, du = -\ln|\cos u| + C$

60. $\int \cot u \, du = \ln|\sin u| + C$

61. $\int \sec u \, du = \ln|\sec u + \tan u| + C$

62. $\int \csc u \, du = \ln|\csc u - \cot u| + C$

3. $\int \tan^2 u\, du = -u + \tan u + C$

4. $\int \cot^2 u\, du = -u - \cot u + C$

5. $\int \sec^2 u\, du = \tan u + C$

6. $\int \csc^2 u\, du = -\cot u + C$

7. $\int \tan^n u\, du = \dfrac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u\, du, \quad n \neq 1$

8. $\int \cot^n u\, du = -\dfrac{\cot^{n-1} u}{n-1} - \int (\cot^{n-2} u)\, du, \quad n \neq 1$

9. $\int \sec^n u\, du = \dfrac{\sec^{n-2} u \tan u}{n-1} + \dfrac{n-2}{n-1} \int \sec^{n-2} u\, du, \quad n \neq 1$

0. $\int \csc^n u\, du = -\dfrac{\csc^{n-2} u \cot u}{n-1} + \dfrac{n-2}{n-1} \int \csc^{n-2} u\, du, \quad n \neq 1$

1. $\int \dfrac{1}{1 \pm \tan u}\, du = \dfrac{1}{2}\left(u \pm \ln|\cos u \pm \sin u|\right) + C$

2. $\int \dfrac{1}{1 \pm \cot u}\, du = \dfrac{1}{2}\left(u \mp \ln|\sin u \pm \cos u|\right) + C$

3. $\int \dfrac{1}{1 \pm \sec u}\, du = u + \cot u \mp \csc u + C$

4. $\int \dfrac{1}{1 \pm \csc u}\, du = u - \tan u \pm \sec u + C$

orms Involving Inverse Trigonometric Functions

5. $\int \arcsin u\, du = u \arcsin u + \sqrt{1 - u^2} + C$

6. $\int \arccos u\, du = u \arccos u - \sqrt{1 - u^2} + C$

7. $\int \arctan u\, du = u \arctan u - \ln\sqrt{1 + u^2} + C$

8. $\int \operatorname{arccot} u\, du = u \operatorname{arccot} u + \ln\sqrt{1 + u^2} + C$

9. $\int \operatorname{arcsec} u\, du = u \operatorname{arcsec} u - \ln\left|u + \sqrt{u^2 - 1}\right| + C$

). $\int \operatorname{arccsc} u\, du = u \operatorname{arccsc} u + \ln\left|u + \sqrt{u^2 - 1}\right| + C$

Forms Involving e^u

81. $\displaystyle\int e^u\,du = e^u + C$

82. $\displaystyle\int ue^u\,du = (u - 1)e^u + C$

83. $\displaystyle\int u^n e^u\,du = u^n e^u - n\int u^{n-1}e^u\,du$

84. $\displaystyle\int \frac{1}{1 + e^u}\,du = u - \ln(1 + e^u) + C$

85. $\displaystyle\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2 + b^2}(a\sin bu - b\cos bu) + C$

86. $\displaystyle\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2 + b^2}(a\cos bu + b\sin bu) + C$

Forms Involving $\ln u$

87. $\displaystyle\int \ln u\,du = u(-1 + \ln u) + C$

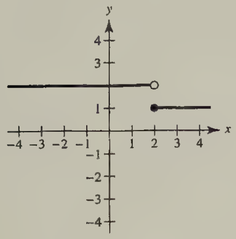
88. $\displaystyle\int u\ln u\,du = \frac{u^2}{4}(-1 + 2\ln u) + C$

89. $\displaystyle\int u^n\ln u\,du = \frac{u^{n+1}}{(n + 1)^2}[-1 + (n + 1)\ln u] + C,\ n \neq -1$

90. $\displaystyle\int (\ln u)^2\,du = u\,[2 - 2\ln u + (\ln u)^2] + C$

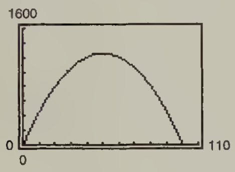
91. $\displaystyle\int (\ln u)^n\,du = u(\ln u)^n - n\int (\ln u)^{n-1}\,du$

(f) $-H(x-2)+2=\begin{cases}1, & x \geq 2\\ 2, & x < 2\end{cases}$



5. (a) $A(x)=x\left(\dfrac{100-x}{2}\right)$; Domain: $(0, 100)$

(b)



Dimensions 50 m × 25 m yield maximum area of 1250 square meters.

(c) 50 m × 25 m; Area = 1250 square meters

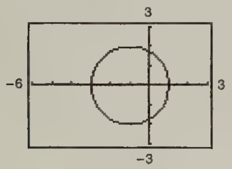
7. $T(x)=\dfrac{2\sqrt{4+x^2}+\sqrt{(3-x)^2+1}}{4}$

9. (a) 5, less (b) 3, greater (c) 4.1, less (d) $4+h$

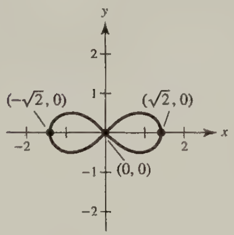
(e) 4; Answers will vary.

11. (a) $x=1, x=-3$

(b) $(x+1)^2+y^2=4$



13. Answers will vary.



Chapter 1

Section 1.1 (page 47)

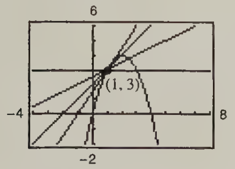
1. Precalculus: 300 feet

3. Calculus: Slope of the tangent line at $x=2$ is 0.16.

5. Precalculus: $\frac{15}{2}$ square units

7. Precalculus: 24 cubic units

9. (a)



(b) The graphs of y_2 approach the tangent line to y_1 at $x=1$.

(c) 2; Use numbers increasingly closer to zero such as 0.2, 0.01, 0.001, ...

11. (a) 5.66 (b) 6.11

(c) Increase the number of line segments.

Section 1.2 (page 54)

1.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.3448	0.3344	0.3334	0.3332	0.3322	0.3226

$\lim_{x\to 2}\dfrac{x-2}{x^2-x-2}\approx 0.3333$ $\left(\text{Actual limit is } \dfrac{1}{3}.\right)$

3.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	0.2911	0.2889	0.2887	0.2887	0.2884	0.2863

$\lim_{x\to 0}\dfrac{\sqrt{x+3}-\sqrt{3}}{x}\approx 0.2887$ $\left(\text{Actual limit is } \dfrac{1}{2\sqrt{3}}.\right)$

5.

x	2.9	2.99	2.999
$f(x)$	−0.0641	−0.0627	−0.0625

x	3.001	3.01	3.1
$f(x)$	−0.0625	−0.0623	−0.0610

$\lim_{x\to 3}\dfrac{[1/(x+1)]-(1/4)}{x-3}\approx -0.0625$ $\left(\text{Actual limit is } -\dfrac{1}{16}.\right)$

7.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$\lim_{x\to 0}\dfrac{\sin x}{x}\approx 1.0000$ (Actual limit is 1.)

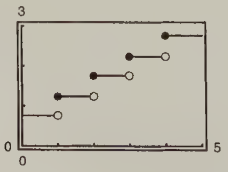
9. 1 **11.** 2

13. Limit does not exist. The function approaches 1 from the right side of 5 but it approaches −1 from the left side of 5.

15. Limit does not exist. The function increases without bound as x approaches $\dfrac{\pi}{2}$ from the left and decreases without bound as x approaches $\dfrac{\pi}{2}$ from the right.

17. Limit does not exist. The function oscillates between 1 and −1 as x approaches 0.

19. (a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	1.75	2.25	2.25	2.25	2.25	2.25	2.25

$\lim_{t\to 3.5} C(t)=2.25$

(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	1.25	1.75	1.75	1.75	2.25	2.25	2.25

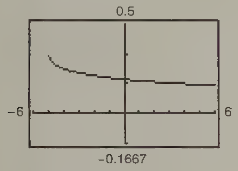
The limit does not exist, because the limits from the right and left are not equal.

21. $\delta = \frac{1}{11} \approx 0.91$ **23.** $L = 8$. Let $\delta = \frac{0.01}{3} \approx 0.0033$.

25. $L = 1$. Assume $1 < x < 3$ and let $\delta = \frac{0.01}{5} = 0.002$.

27. 5 **29.** -3 **31.** 3 **33.** 0 **35.** 4 **37.** 2

39. Answers will vary.

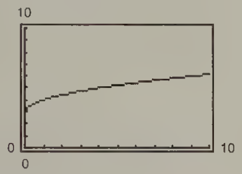


$\lim_{x \to 4} f(x) = \frac{1}{6}$

Domain: $[-5, 4) \cup (4, \infty)$

The graph has a hole at $x = 4$.

41. Answers will vary.



$\lim_{x \to 9} f(x) = 6$

Domain: $[0, 9) \cup (9, \infty)$

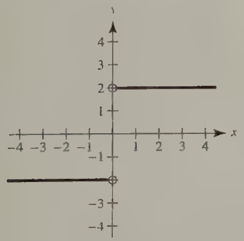
The graph has a hole at $x = 9$.

43. Answers will vary. Sample answer: As x approaches 8 from either side, $f(x)$ becomes arbitrarily close to 25.

45. Examples will vary.

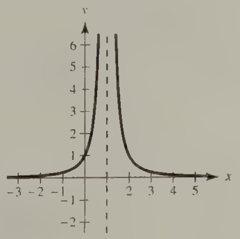
Type 1: $f(x)$ approaches a different number from the right of c than it approaches from the left.

$\lim_{x \to 0} \frac{|2x|}{x}$



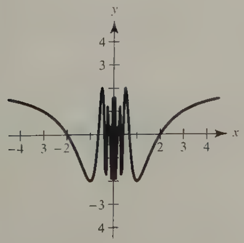
Type 2: $f(x)$ increases or decreases without bound as x approaches c.

$\lim_{x \to 1} \left(\frac{1}{x-1}\right)^2$



Type 3: $f(x)$ oscillates between two fixed values as x approaches c.

$\lim_{x \to 0} \left(2 \cos\left(\frac{\pi}{x}\right)\right)$

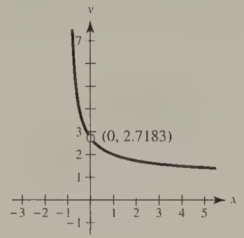


47.

x	-0.001	-0.0001	-0.00001
$f(x)$	2.7196	2.7184	2.7183

x	0.00001	0.0001	0.001
$f(x)$	2.7183	2.7181	2.7169

$\lim_{x \to 0} f(x) \approx 2.7183$



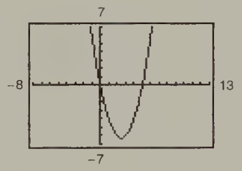
49. False: the existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as $x \to c$.

51. False: see Exercise 11.

53. Answers will vary. **55.** Proof **57.** Proof

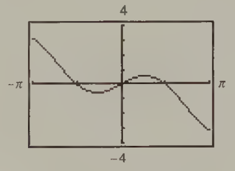
Section 1.3 (page 65)

1.



(a) 0 (b) 6

3.



(a) 0 (b) ≈ 0.52 or $\frac{\pi}{6}$

5. 16

7. -1 **9.** 0 **11.** 7 **13.** $\frac{1}{2}$

15. $-\frac{2}{5}$ **17.** $\frac{35}{3}$ **19.** 2 **21.** 1

23. (a) 4 (b) 64 (c) 64 **25.** (a) 3 (b) 2 (c) 2

27. 1 **29.** $-\frac{1}{2}$ **31.** 1 **33.** $\frac{1}{2}$ **35.** -1

37. (a) 15 (b) 5 (c) 6 (d) $\frac{2}{3}$

39. (a) 64 (b) 2 (c) 12 (d) 8

41. (a) 1 (b) 3

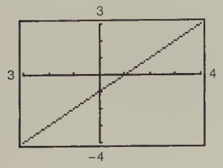
$g(x) = \frac{-2x^2 + x}{x}$ and $f(x) = -2x + 1$ agree except at $x = 0$.

43. (a) 2 (b) 0

$g(x) = \frac{x^3 - x}{x - 1}$ and $f(x) = x^2 + x$ agree except at $x = 1$.

45. -2

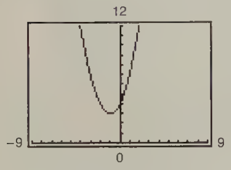
$f(x) = \dfrac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$ agree except at $x = -1$.



The graph has a hole at $x = -1$.

47. 12

$f(x) = \dfrac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

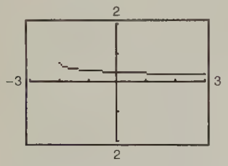


The graph has a hole at $x = 2$.

49. $\frac{1}{10}$ **51.** $\frac{5}{6}$ **53.** $\frac{\sqrt{5}}{10}$ **55.** $\frac{1}{6}$

57. $-\frac{1}{9}$ **59.** 2 **61.** $2x - 2$

63.



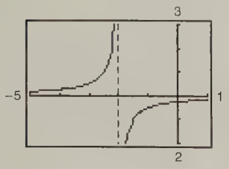
The graph has a hole at $x = 0$.

Answers will vary. Example:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	0.354	0.353	0.349

$$\lim_{x \to 0} \frac{\sqrt{x + 2} - \sqrt{2}}{x} \approx 0.354 \quad \left(\text{Actual limit is } \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.\right)$$

65.



The graph has a hole at $x = 0$.

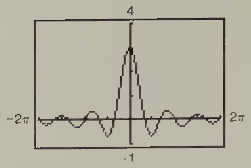
Answers will vary. Example:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-0.263	-0.251	-0.250	-0.250	-0.249	-0.238

$$\lim_{x \to 0} \frac{[1/(2 + x)] - (1/2)}{x} \approx -0.250 \quad \left(\text{Actual limit is } -\frac{1}{4}.\right)$$

67. $\frac{1}{5}$ **69.** 0 **71.** 0 **73.** 0 **75.** 1 **77.** $\frac{3}{2}$

79.



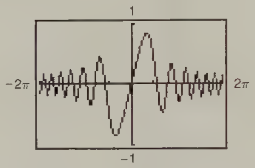
The graph has a hole at $t = 0$.

Answers will vary. Example:

t	-0.1	-0.01	0	0.01	0.1
$f(t)$	2.96	2.9996	?	2.9996	2.96

$$\lim_{t \to 0} \frac{\sin 3t}{t} = 3$$

81.



The graph has a hole at $x = 0$.

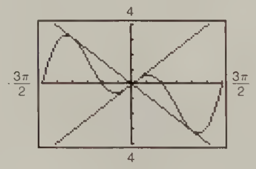
Answers will vary. Example:

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.1	-0.01	-0.001	?	0.001	0.01	0.1

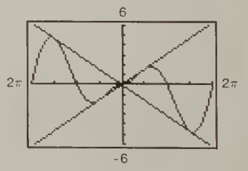
$$\lim_{x \to 0} \frac{\sin x^2}{x} = 0$$

83. 2 **85.** $-\frac{4}{x^2}$ **87.** 4

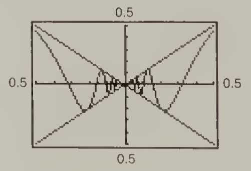
89. 0



91. 0



93. 0

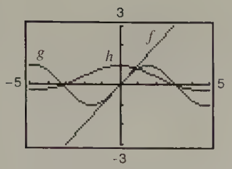


The graph has a hole at $x = 0$.

95. f and g agree at all but one point if c is a real number such that $f(x) = g(x)$ for all $x \neq c$.

97. An indeterminate form is obtained when evaluating a limit using direct substitution produces a meaningless fractional form, such as $\frac{0}{0}$.

99.



The magnitudes of $f(x)$ and $g(x)$ are approximately equal when x is "close to" 0. Therefore, their ratio is approximately 1.

101. 160 feet per second **103.** -29.4 meters per second

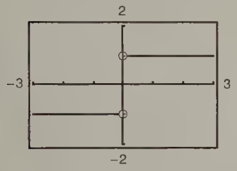
105. Let $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$.

$\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist.

However, $\lim_{x\to 0} [f(x) + g(x)] = \lim_{x\to 0} \left[\frac{1}{x} + \left(-\frac{1}{x}\right)\right] = \lim_{x\to 0} 0 = 0$ and therefore does exist.

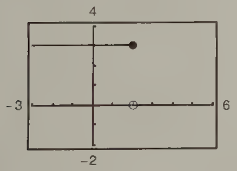
107. Proof **109.** Proof **111.** Proof

113. False. The limit does not exist because the function approaches 1 from the right side of 0 and approaches -1 from the left side of 0. (See graph below.)



115. True. Theorem 1.7

117. False. The limit does not exist because $f(x)$ approaches 3 from the left side of 2 and approaches 0 from the right side of 2. (See graph below.)



119. Let $f(x) = \begin{cases} 4, & \text{if } x \geq 0 \\ -4, & \text{if } x < 0 \end{cases}$

$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} 4 = 4$

$\lim_{x\to 0} f(x)$ does not exist because for $x < 0$, $f(x) = -4$ and for $x \geq 0$, $f(x) = 4$.

121. $\lim_{x\to 0} f(x)$ does not exist because $f(x)$ oscillates between two fixed values as x approaches 0.

$\lim_{x\to 0} g(x) = 0$ because, as x gets increasingly closer to 0, the values of $g(x)$ become increasingly closer to 0.

123. (a) $\frac{1}{2}$

(b) Because $\frac{1 - \cos x}{x^2} \approx \frac{1}{2}$, it follows that

$1 - \cos x \approx \frac{1}{2}x^2$

$\cos x \approx 1 - \frac{1}{2}x^2$ when $x \approx 0$.

(c) 0.995

(d) Calculator: $\cos(0.1) \approx .9950$

Section 1.4 (page 76)

1. (a) 1 (b) 1 (c) 1

$f(x)$ is continuous on $(-\infty, \infty)$.

3. (a) 0 (b) 0 (c) 0

Discontinuity at $x = 3$

5. (a) 2 (b) -2 (c) Limit does not exist.

Discontinuity at $x = 4$

7. $\frac{1}{10}$

9. Limit does not exist. The function decreases without bound as x approaches -3 from the left.

11. -1 **13.** $-\frac{1}{x^2}$ **15.** $\frac{5}{2}$ **17.** 2

19. Limit does not exist. The function decreases without bound as x approaches π from the left and increases without bound as x approaches π from the right.

21. 4

23. Limit does not exist. The function approaches 5 from the left side of 3 but approaches 6 from the right side of 3.

25. Discontinuous at $x = -2$ and $x = 2$

27. Discontinuous at every integer

29. Continuous on $[-5, 5]$ **31.** Continuous on $[-1, 4]$

33. Continuous for all real x **35.** Continuous for all real x

37. Nonremovable discontinuity at $x = 1$

Removable discontinuity at $x = 0$

39. Continuous for all real x

41. Removable discontinuity at $x = -2$

Nonremovable discontinuity at $x = 5$

43. Nonremovable discontinuity at $x = -2$

45. Continuous for all real x

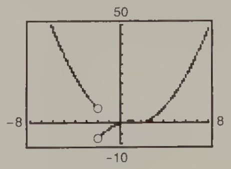
47. Nonremovable discontinuity at $x = 2$

49. Continuous for all real x

51. Nonremovable discontinuities at integer multiples of $\frac{\pi}{2}$

53. Nonremovable discontinuity at each integer

55.



$\lim_{x\to 0^+} f(x) = 0$

$\lim_{x\to 0} f(x) = 0$

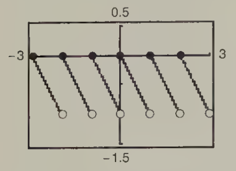
Discontinuity at $x = -2$

57. $a = 2$ **59.** $a = -1$, $b = 1$

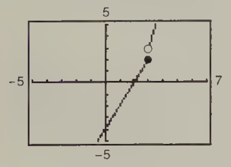
61. Continuous for all real x

63. Nonremovable discontinuities at $x = 1$ and $x = -1$

65. Nonremovable discontinuity at each integer



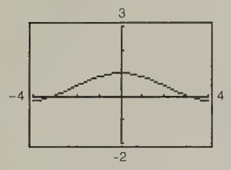
67. Discontinuous at $x = 3$



69. Continuous on $(-\infty, \infty)$

71. $f(x)$ is continuous on the open intervals $\ldots (-6, -2), (-2, 2), (2, 6), \ldots.$

73.



The graph has a hole at $x = 0$.

The graph appears continuous but the function is not continuous on $[-4, 4]$.

It is not obvious from the graph that the function has a discontinuity at $x = 0$.

75. Because $f(x)$ is continuous on the interval $[1, 2]$ and $f(1) = 2.0625$ and $f(2) = -4$, by the Intermediate Value Theorem there exists a real number c in $[1, 2]$ such that $f(c) = 0$.

77. Because $f(x)$ is continuous on the interval $[0, \pi]$ and $f(0) = -3$ and $f(\pi) \approx 8.87$, by the Intermediate Value Theorem there exists a real number c in $[0, \pi]$ such that $f(c) = 0$.

79. 0.68, 0.6823 **81.** 0.56, 0.5636

83. $f(3) = 11$ **85.** $f(2) = 4$

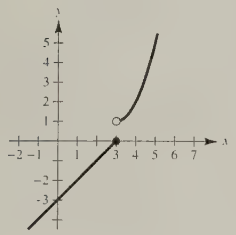
87. (a) The limit does not exist at $x = c$.

(b) The function is not defined at $x = c$.

(c) The limit exists, but it is not equal to the value of the function at $x = c$.

(d) The limit does not exist at $x = c$.

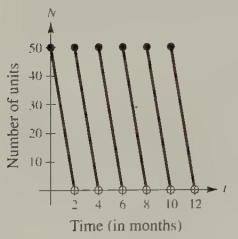
89.



Not continuous because $\lim_{x \to 3} f(x)$ does not exist.

91. $g(x) = f(x)$ where x is an integer, but $g(x) = f(x) + 1$ elsewhere.

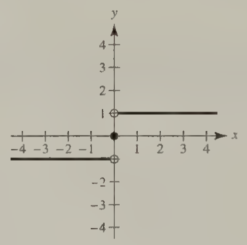
93. The function is discontinuous at every even positive integer. The company must replenish every two months.



95. Because $V(1) = \frac{4}{3}\pi$, $V(5) = 523.6$, and V is continuous, there is at least one real number r, $1 \le r \le 5$, such that $V(r) = 275$.

97. If c is an element of the real numbers, then $\lim_{x \to c} f(x)$ does not exist since there are both rational and irrational numbers arbitrarily close to c. Therefore, f is not continuous at c.

99.



(a) -1

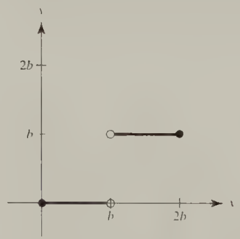
(b) 1

(c) Limit does not exist.

101. True

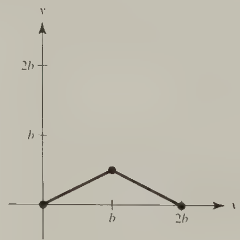
103. False. $f(x)$ is not defined at $x = 1$.

105. (a) $f(x) = \begin{cases} 0, & 0 \le x < b \\ b, & b < x \le 2b \end{cases}$



$f(x)$ is not continuous. There is a discontinuity at $x = b$.

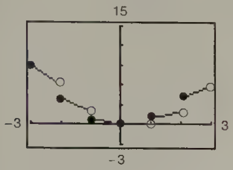
(b) $g(x) = \begin{cases} \frac{1}{2}x, & 0 \le x \le b \\ b - \frac{1}{2}x, & b \le x \le 2b \end{cases}$



$g(x)$ is continuous on $[0, 2b]$ because $g(x)$ is continuous on $[0, b]$ and on $[b, 2b]$, and $\lim_{x \to b} g(x) = g(b)$.

107. Domain: $[-c^2, 0) \cup (0, \infty)$; Let $f(0) = \dfrac{1}{2c}$.

109. $h(x)$ has a nonremovable discontinuity at every integer except 0.



Section 1.5 (page 85)

1. $\displaystyle\lim_{x\to -2^+} 2\left|\frac{x}{x^2-4}\right| = \infty$ $\quad\displaystyle\lim_{x\to -2^-} 2\left|\frac{x}{x^2-4}\right| = \infty$

3. $\displaystyle\lim_{x\to -2^+} \tan\frac{\pi x}{4} = -\infty$ $\quad\displaystyle\lim_{x\to -2^-} \tan\frac{\pi x}{4} = \infty$

5.

x	-3.5	-3.1	-3.01	-3.001
$f(x)$	0.31	1.64	16.6	167

x	-2.999	-2.99	-2.9	-2.5
$f(x)$	-167	-16.6	-1.7	-0.36

$\displaystyle\lim_{x\to -3^+} f(x) = -\infty$ $\quad\displaystyle\lim_{x\to -3^-} f(x) = \infty$

7.

x	-3.5	-3.1	-3.01	-3.001
$f(x)$	3.8	16	151	1501

x	-2.999	-2.99	-2.9	-2.5
$f(x)$	-1499	-149	-14	-2.3

$\displaystyle\lim_{x\to -3^+} f(x) = -\infty$ $\quad\displaystyle\lim_{x\to -3^-} f(x) = \infty$

9. $x = 0$ **11.** $x = 2,\ x = -1$ **13.** $x = \pm 2$

15. No vertical asymptote

17. $x = \dfrac{\pi}{4} + \dfrac{n\pi}{2}$, n is an integer. **19.** $t = 0$

21. $x = -2,\ x = 1$ **23.** No vertical asymptote

25. No vertical asymptote **27.** $t = n\pi$, n is a nonzero integer.

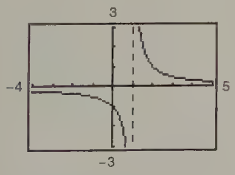
29. Removable discontinuity at $x = -1$

31. Vertical asymptote at $x = -1$

33. $-\infty$ **35.** ∞ **37.** $\frac{4}{5}$ **39.** $\frac{1}{2}$

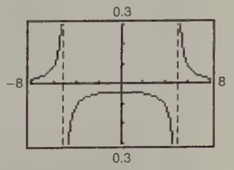
41. $-\infty$ **43.** ∞ **45.** 0 **47.** Does not exist

49.



$\displaystyle\lim_{x\to 1^+} f(x) = \infty$

51.

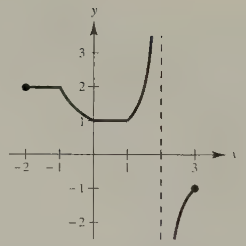


$\displaystyle\lim_{x\to 5^-} f(x) = -\infty$

53. Answers will vary.

55. Answers will vary. Example: $f(x) = \dfrac{x-3}{x^2-4x-12}$

57.



59. ∞

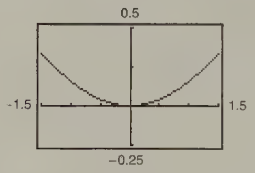
61. (a) \$176 million (b) \$528 million (c) \$1584 million
(d) ∞; As the percentage of drugs seized increases and approaches 100%, the cost to the government increases without bound.

63. (a) $\frac{7}{12}$ foot per second (b) $\frac{3}{2}$ feet per second (c) ∞

65. (a)

x	1	0.5	0.2	0.1
$f(x)$	0.1585	0.0411	0.0067	0.0017

x	0.01	0.001	0.0001
$f(x)$	1.7×10^{-5}	1.7×10^{-7}	1.7×10^{-9}

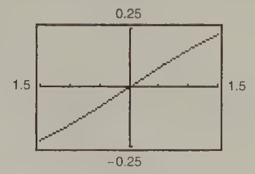


The graph has a hole at $x = 0$.

(b)

x	1	0.5	0.2	0.1
$f(x)$	0.1585	0.0823	0.0333	0.0167

x	0.01	0.001	0.0001
$f(x)$	0.0017	1.7×10^{-4}	1.7×10^{-5}

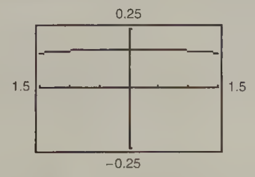


The graph has a hole at $x = 0$.

(c)

x	1	0.5	0.2	0.1
$f(x)$	0.1585	0.1646	0.1663	0.1666

x	0.01	0.001	0.0001
$f(x)$	0.1667	0.1667	0.1667

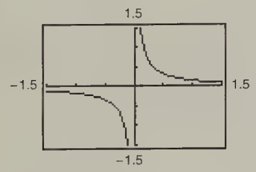


The graph has a hole at $x = 0$.

(d)

x	1	0.5	0.2	0.1
$f(x)$	0.1585	0.3292	0.8317	1.666

x	0.01	0.001	0.0001
$f(x)$	16.67	166.7	1667



The value of the limit when the power on x in the denominator is greater than 3 is ∞.

67. (a) 850 revolutions per minute

(b) Reverse direction

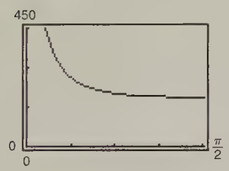
(c) $L = 60 \cot \phi + 30(\pi + 2\phi)$

Domain: $\left(0, \frac{\pi}{2}\right)$

(d)

ϕ	0.3	0.6	0.9	1.2	1.5
L	306.2	217.9	195.9	189.6	188.5

(e)



(f) $60\pi \approx 188.5$

(g) ∞

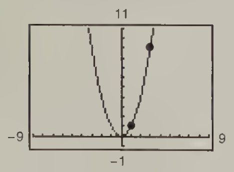
69. False: let $f(x) = \frac{1}{x^2 + 1}$.

71. False: let $f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

73. Proof **75.** Proof

Review Exercises for Chapter 1 (page 88)

1. Calculus



Estimate: 8.261

3.

x	−0.1	−0.01	−0.001	0.001
$f(x)$	−1.0526	−1.0050	−1.0005	−0.9995

x	0.01	0.1
$f(x)$	−0.9950	−0.9524

The estimate of the limit of $f(x)$, as x approaches zero, is −1.00.

5. (a) −2 (b) −3

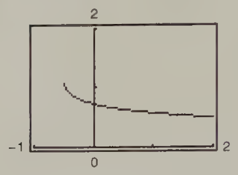
7. 2; Proof **9.** 1; Proof **11.** $\sqrt{6} \approx 2.45$ **13.** $-\frac{1}{4}$

15. $\frac{1}{4}$ **17.** −1 **19.** 75 **21.** 0 **23.** $\frac{\sqrt{3}}{2}$ **25.** $-\frac{1}{2}$

27. (a)

x	1.1	1.01	1.001	1.0001
$f(x)$	0.5680	0.5764	0.5773	0.5773

(b)



The graph has a hole at $x = 1$.

(c) $\lim_{x\to 1^+} \frac{\sqrt{2x+1} - \sqrt{3}}{x - 1} \approx 0.577$ $\left(\text{Actual limit is } \frac{\sqrt{3}}{3}.\right)$

29. −39.2 meters per second **31.** −1 **33.** 0

35. Limit does not exist. The limit as t approaches 1 from the left is 2 whereas the limit as t approaches 1 from the right is 1.

37. Nonremovable discontinuity at each integer

Continuous on $(k, k + 1)$ for all integers k

39. Removable discontinuity at $x = 1$

Continuous on $(-\infty, 1) \cup (1, \infty)$

41. Nonremovable discontinuity at $x = 2$

Continuous on $(-\infty, 2) \cup (2, \infty)$

43. Nonremovable discontinuity at $x = -1$

Continuous on $(-\infty, -1) \cup (-1, \infty)$

45. Nonremovable discontinuity at each even integer

Continuous on $(2k, 2k + 2)$ for all integers k

47. $c = -\frac{1}{2}$ **49.** Proof

51. (a) −4 (b) 4 (c) Limit does not exist.

53. $x = 0$ **55.** $x = 10$ **57.** $-\infty$ **59.** $\frac{1}{3}$

61. $-\infty$ **63.** $-\infty$ **65.** $\frac{4}{5}$ **67.** ∞

69. (a) \$14,117.65 (b) \$80,000.00 (c) \$720,000.00 (d) ∞

P.S. Problem Solving (page 90)

1. (a) Perimeter $\triangle PAO = 1 + \sqrt{(x^2 - 1)^2 + x^2} + \sqrt{x^4 + x^2}$

Perimeter $\triangle PBO = 1 + \sqrt{x^4 + (x - 1)^2} + \sqrt{x^4 + x^2}$

(b)

x	4	2	1
Perimeter $\triangle PAO$	33.0166	9.0777	3.4142
Perimeter $\triangle PBO$	33.7712	9.5952	3.4142
$r(x)$	0.9777	0.9461	1.0000

x	0.1	0.01
Perimeter $\triangle PAO$	2.0955	2.0100
Perimeter $\triangle PBO$	2.0006	2.0000
$r(x)$	1.0475	1.005

(c) 1

3. (a) Area (hexagon) $= \frac{3\sqrt{3}}{2} \approx 2.5981$

Area (circle) $= \pi \approx 3.1416$

Area (circle) − Area (hexagon) ≈ 0.5435

(b) $A_n = \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)$

(c)

n	6	12	24	48	96
A_n	2.5981	3.0000	3.1058	3.1326	3.1394

(d) 3.1416 or π

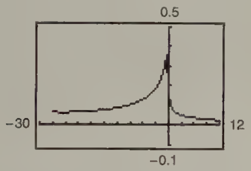
5. (a) $m = -\frac{12}{5}$ (b) $y = \frac{5}{12}x - \frac{169}{12}$

(c) $m_x = \dfrac{-\sqrt{169 - x^2} + 12}{x - 5}$

(d) $\frac{5}{12}$; It is the same as the slope of the tangent line found in (b).

7. (a) Domain: $[-27, 1) \cup (1, \infty)$

(b)

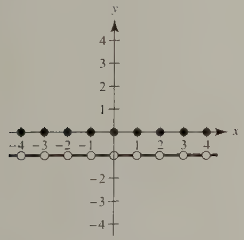


The graph has a hole at $x = 1$.

(c) $\frac{1}{14}$ (d) $\frac{1}{12}$

9. (a) g_1, g_4 (b) g_1 (c) g_1, g_3, g_4

11.



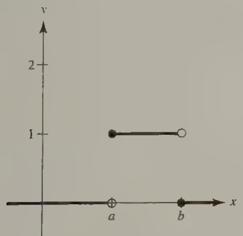
The graph jumps at every integer.

(a) $f(1) = 0,\ f(0) = 0,\ f\left(\frac{1}{2}\right) = -1,\ f(-2.7) = -1$

(b) $\lim\limits_{x\to 1^-} f(x) = -1,\ \lim\limits_{x\to 1^+} f(x) = -1,\ \lim\limits_{x\to 1/2} f(x) = -1$

(c) There is a discontinuity at each integer.

13. (a)



(b) (i) $\lim\limits_{x\to a^+} P_{a,b}(x) = 1$ (ii) $\lim\limits_{x\to a^-} P_{a,b}(x) = 0$

(iii) $\lim\limits_{x\to b^+} P_{a,b}(x) = 0$ (iv) $\lim\limits_{x\to b^-} P_{a,b}(x) = 1$

(c) Continuous for all positive real numbers except a and b

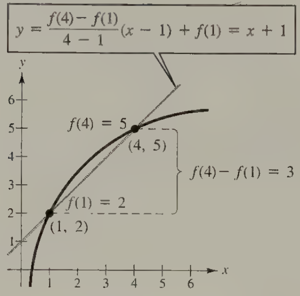
(d) The area under the curve gives a value of 1.

Chapter 2

Section 2.1 (page 101)

1. (a) $m = 0$ (b) $m = -3$

3.



5. $m = -2$ **7.** $m = 2$ **9.** $m = 3$

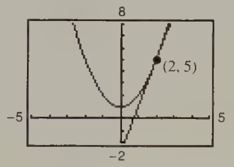
11. $f'(x) = 0$ **13.** $f'(x) = -5$ **15.** $h'(s) = \frac{2}{3}$

17. $f'(x) = 4x + 1$ **19.** $f'(x) = 3x^2 - 12$

21. $f'(x) = \dfrac{-1}{(x - 1)^2}$ **23.** $f'(x) = \dfrac{1}{2\sqrt{x + 1}}$

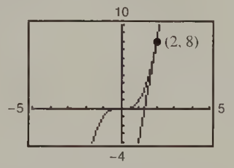
25. (a) Tangent line: $y = 4x - 3$

(b)



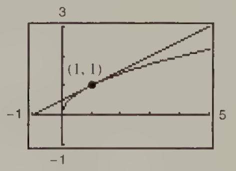
27. (a) Tangent line: $y = 12x - 16$

(b)



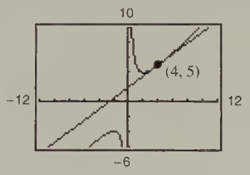
29. (a) Tangent line: $y = \frac{1}{2}x + \frac{1}{2}$

(b)



31. (a) Tangent line: $y = \frac{3}{4}x + 2$

(b)



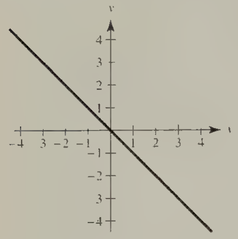
33. $y = 3x - 2;\ y = 3x + 2$

35. $y = -\frac{1}{2}x + \frac{3}{2}$

37. $g(5) = 2$; $g'(5) = -\frac{1}{2}$

39. b **40.** d **41.** a **42.** c

43. Answers will vary. Sample answer: $y = -x$



45. (a) $f'(-c) = 3$ (b) $f'(-c) = -3$

47. $y = 2x + 1$; $y = -2x + 9$

49. (a) -3

(b) 0

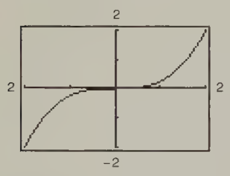
(c) The graph is moving downward to the right when $x = 1$.

(d) The graph is moving upward to the right when $x = -4$.

(e) Positive. Because $g'(x) > 0$ on $[3, 6]$, the graph of g is moving upward to the right.

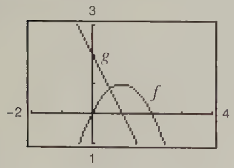
(f) No. Knowing only $g'(2)$ is not sufficient information. $g'(2)$ remains the same for any vertical translation of g.

51.



x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	-2	$-\frac{27}{32}$	$-\frac{1}{4}$	$-\frac{1}{32}$	0	$\frac{1}{32}$	$\frac{1}{4}$	$\frac{27}{32}$	2
$f'(x)$	3	$\frac{27}{16}$	$\frac{3}{4}$	$\frac{3}{16}$	0	$\frac{3}{16}$	$\frac{3}{4}$	$\frac{27}{16}$	3

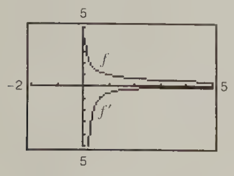
53.



$g(x) \approx f'(x)$

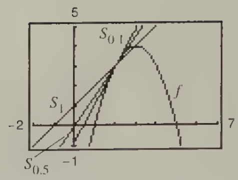
55. $f(2) = 4$; $f(2.1) = 3.99$; $f'(2) \approx -0.1$; Exact $f'(2) = 0$

57.



As x approaches infinity, the graph of f approaches a line of slope 0. Thus $f'(x)$ approaches 0.

59. (a)



(b) The graphs of S for decreasing values of Δx are secant lines approaching the tangent line to the graph of f at the point $(2, f(2))$.

61. 4 **63.** 4

65. $g(x)$ is not differentiable at $x = 0$.

67. $f(x)$ is not differentiable at $x = 6$.

69. $h(x)$ is not differentiable at $x = -5$.

71. $(-\infty, -3) \cup (-3, \infty)$ **73.** $(-\infty, -1) \cup (-1, \infty)$

75. $(-\infty, 3) \cup (3, \infty)$ **77.** $(1, \infty)$ **79.** $(-\infty, 0) \cup (0, \infty)$

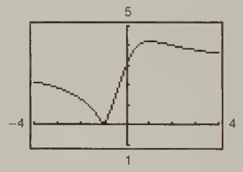
81. The derivative from the left is -1 and from the right is 1, so f is not differentiable at $x = 1$.

83. The derivative from both the right and left is 0, so $f'(1) = 0$.

85. f is differentiable at $x = 2$.

87. (a) $d = \dfrac{3|m + 1|}{\sqrt{m^2 + 1}}$

(b)



Not differentiable at $m = -1$

89. False. It is $\lim\limits_{\Delta x \to 0} \dfrac{f(2 + \Delta x) - f(2)}{\Delta x}$.

91. False. For example: $f(x) = |x|$. The derivative from the left and the derivative from the right both exist but are not equal.

93. Proof

Section 2.2 (page 113)

1. (a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) 2 (d) 3

3. 0 **5.** $6x^5$ **7.** $-\dfrac{7}{x^8}$ **9.** $\dfrac{1}{5x^{4/5}}$ **11.** 1

13. $-4t + 3$ **15.** $2x + 12x^2$ **17.** $3t^2 - 2$

19. $\dfrac{\pi}{2} \cos \theta + \sin \theta$ **21.** $2x + \dfrac{1}{2} \sin x$ **23.** $-\dfrac{1}{x^2} - 3 \cos x$

	Function	*Rewrite*	*Derivative*	*Simplify*
25.	$y = \dfrac{5}{2x^2}$	$y = \dfrac{5}{2}x^{-2}$	$y' = -5x^{-3}$	$y' = -\dfrac{5}{x^3}$
27.	$y = \dfrac{3}{(2x)^3}$	$y = \dfrac{3}{8}x^{-3}$	$y' = -\dfrac{9}{8}x^{-4}$	$y' = -\dfrac{9}{8x^4}$
29.	$y = \dfrac{\sqrt{x}}{x}$	$y = x^{-1/2}$	$y' = -\dfrac{1}{2}x^{-3/2}$	$y' = -\dfrac{1}{2x^{3/2}}$

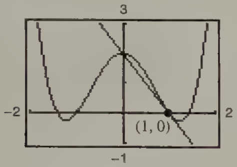
31. -6 **33.** 0 **35.** 4 **37.** 3 **39.** $2x + \dfrac{6}{x^3}$

1. $2t + \frac{12}{t^4}$ **43.** $\frac{x^3 - 8}{x^3}$ **45.** $3x^2 + 1$

7. $\frac{1}{2\sqrt{x}} - \frac{2}{x^{2/3}}$ **49.** $\frac{4}{5s^{1/5}} - \frac{2}{3s^{1/3}}$ **51.** $\frac{3}{\sqrt{x}} - 5 \sin x$

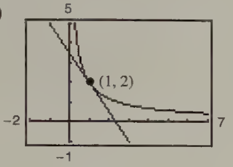
3. (a) $2x + y - 2 = 0$

(b)



55. (a) $3x + 2y - 7 = 0$

(b)

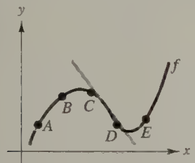


7. $(0, 2), (-2, -14), (2, -14)$ **59.** No horizontal tangents

1. (π, π) **63.** $k = 2,\ k = -10$ **65.** $k = 3$

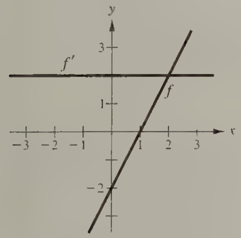
7. (a) A and B (b) Greater

(c)



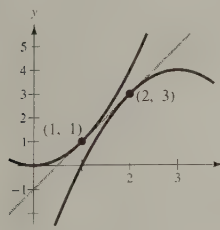
9. $g'(x) = f'(x)$

1.

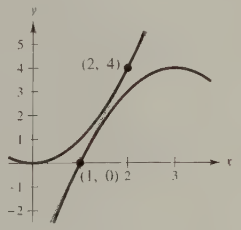


The rate of change of f is constant and therefore f' is a constant function.

3. $y = 2x - 1$

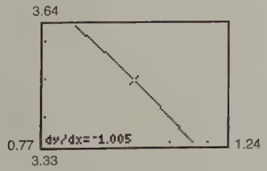


$y = 4x - 4$



5. $x - 4y + 4 = 0$

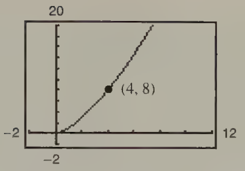
7.



$f'(1)$ appears to be close to -1.

$f'(1) = -1$

79. (a)

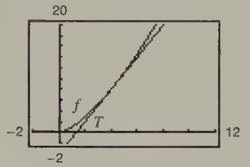


$(3.9, 7.7019)$, $S(x) = 2.981x - 3.924$

(b) $T(x) = 3(x - 4) + 8 = 3x - 4$

The slope (and equation) of the secant line approaches that of the tangent line at $(4, 8)$ as you choose points closer to $(4, 8)$.

(c) It becomes less accurate.



(d)

Δx	-3	-2	-1	-0.5	-0.1	0
$f(4 + \Delta x)$	1	2.828	5.196	6.458	7.702	8
$T(4 + \Delta x)$	-1	2	5	6.5	7.7	8

Δx	0.1	0.5	1	2	3
$f(4 + \Delta x)$	8.302	9.546	11.180	14.697	18.520
$T(4 + \Delta x)$	8.3	9.5	11	14	17

81. False: let $f(x) = x$ and $g(x) = x + 1$.

83. False: $dy/dx = 0$. **85.** True

87. Average rate: 2

Instantaneous rates:

$f'(1) = 2$

$f'(2) = 2$

89. Average rate: $\frac{1}{2}$

Instantaneous rates:

$f'(1) = 1$

$f'(2) = \frac{1}{4}$

91. (a) $s(t) = -16t^2 + 1362$

$v(t) = -32t$

(b) -48 feet per second

(c) $s'(1) = -32$ feet per second

$s'(2) = -64$ feet per second

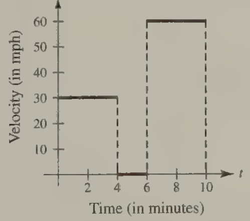
(d) $t = \frac{\sqrt{1362}}{4} \approx 9.226$ seconds

(e) -295.242 feet per second

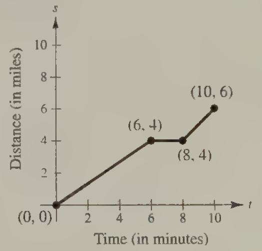
93. $v(5) = 71$ meters per second

$v(10) = 22$ meters per second

95.



97.

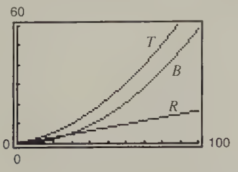


99. (a) $R(v) = 0.167v - 0.02$

(b) $B(v) = 0.006v^2 - 0.024v + 0.460$

(c) $T(v) = 0.006v^2 + 0.143v + 0.440$

(d)



(e) $T'(v) = 0.012v + 0.143$

$T'(40) = 0.623$

$T'(80) = 1.103$

$T'(100) = 1.343$

(f) Stopping distance increases at an increasing rate.

101. 8 square meters per meter change in s

103. $-\$1.91, -\$1.93$

105. (a) The rate of change of gallons of gasoline sold when the price is \$1.479

(b) In general, the rate of change when $p = 1.479$ should be negative.

107. $y = 2x^2 - 3x + 1$ **109.** $y = -9x,\ y = -\frac{9}{4}x - \frac{27}{4}$

111. $a = \frac{1}{3},\ b = -\frac{4}{3}$ **113.** Proof

Section 2.3 (page 124)

1. $2(2x^3 - 3x^2 + x - 1)$ **3.** $\dfrac{7t^2 + 4}{3t^{2/3}}$

5. $x^2(3\cos x - x\sin x)$ **7.** $\dfrac{1 - x^2}{(x^2 + 1)^2}$

9. $\dfrac{1 - 8x^3}{3x^{2/3}(x^3 + 1)^2}$ **11.** $\dfrac{x\cos x - 2\sin x}{x^3}$

13. $f'(x) = (x^3 - 3x)(4x + 3) + (2x^2 + 3x + 5)(3x^2 - 3)$

$= 10x^4 + 12x^3 - 3x^2 - 18x - 15$

$f'(0) = -15$

15. $f'(x) = \dfrac{x^2 - 6x + 4}{(x - 3)^2}$ **17.** $f'(x) = \cos x - x\sin x$

$f'(1) = -\dfrac{1}{4}$ $f'\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{8}(4 - \pi)$

	Function	*Rewrite*	*Differentiate*	*Simplify*
19.	$y = \dfrac{x^2 + 2x}{3}$	$y = \dfrac{1}{3}(x^2 + 2x)$	$y' = \dfrac{1}{3}(2x + 2)$	$y' = \dfrac{2(x + 1)}{3}$
21.	$y = \dfrac{7}{3x^3}$	$y = \dfrac{7}{3}x^{-3}$	$y' = -7x^{-4}$	$y' = -\dfrac{7}{x^4}$
23.	$y = \dfrac{4x^{3/2}}{x}$	$y = 4x^{1/2},$ $x > 0$	$y' = 2x^{-1/2}$	$y' = \dfrac{2}{\sqrt{x}},$ $x > 0$

25. $\dfrac{(x^2 - 1)(-2 - 2x) - (3 - 2x - x^2)(2x)}{(x^2 - 1)^2} = \dfrac{2}{(x + 1)^2},\ x \neq 1$

27. $1 - \dfrac{12}{(x + 3)^2} = \dfrac{x^2 + 6x - 3}{(x + 3)^2}$

29. $\dfrac{\sqrt{x}(2) - (2x + 5)\dfrac{1}{2\sqrt{x}}}{x} = \dfrac{2x - 5}{2x^{3/2}}$

31. $6s^2(s^3 - 2)$ **33.** $-\dfrac{2x^2 - 2x + 3}{x^2(x - 3)^2}$

35. $(3x^3 + 4x)[(x - 5)\cdot 1 + (x + 1)\cdot 1]$

$+ [(x - 5)(x + 1)](9x^2 + 4)$

$= 15x^4 - 48x^3 - 33x^2 - 32x - 20$

37. $\dfrac{(x^2 - c^2)(2x) - (x^2 + c^2)(2x)}{(x^2 - c^2)^2} = -\dfrac{4xc^2}{(x^2 - c^2)^2}$

39. $t(t\cos t + 2\sin t)$ **41.** $-\dfrac{t\sin t + \cos t}{t^2}$

43. $-1 + \sec^2 x = \tan^2 x$ **45.** $\dfrac{1}{4t^{3/4}} + 8\sec t\tan t$

47. $\dfrac{-6\cos^2 x + 6\sin x - 6\sin^2 x}{4\cos^2 x} = \dfrac{3}{2}(-1 + \tan x\sec x - \tan^2 x)$

$= \dfrac{3}{2}\sec x(\tan x - \sec x)$

49. $\csc x\cot x - \cos x = \cos x\cot^2 x$ **51.** $x(x\sec^2 x + 2\tan x)$

53. $2x\cos x + 2\sin x - x^2\sin x + 2x\cos x$

$= 4x\cos x + (2 - x^2)\sin x$

55. $\left(\dfrac{x + 1}{x + 2}\right)(2) + (2x - 5)\left[\dfrac{(x + 2)(1) - (x + 1)(1)}{(x + 2)^2}\right]$

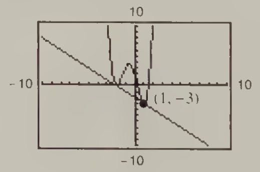
$= \dfrac{2x^2 + 8x - 1}{(x + 2)^2}$

57. $\dfrac{1 - \sin\theta + \theta\cos\theta}{(1 - \sin\theta)^2}$ **59.** $y' = \dfrac{-2\csc x\cot x}{(1 - \csc x)^2},\quad -4\sqrt{3}$

61. $h'(t) = \dfrac{\sec t(t\tan t - 1)}{t^2},\quad \dfrac{1}{\pi^2}$

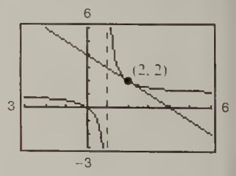
63. (a) $y = -x - 2$

(b)



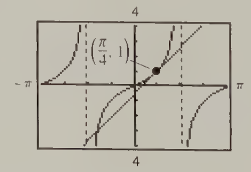
65. (a) $y = -x + 4$

(b)



67. (a) $4x - 2y - \pi + 2 = 0$

(b)



69. $(0, 0), (2, 4)$ **71.** $f(x) + 2 = g(x)$

73. $n = 1, f'(x) = x\cos x + \sin x$

$n = 2, f'(x) = x^2\cos x + 2x\sin x$

$n = 3, f'(x) = x^3\cos x + 3x^2\sin x$

$n = 4, f'(x) = x^4\cos x + 4x^3\sin x$

$f'(x) = x^n\cos x + nx^{n-1}\sin x$

75. $\frac{6t+1}{2\sqrt{t}}$ square centimeters per second

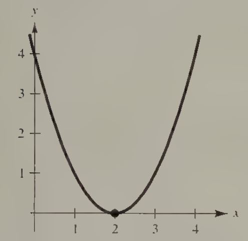
77. (a) $-\$38.13$ (b) $-\$10.37$ (c) $-\$3.80$

The costs decrease with increasing order size.

79. 31.55 bacteria per hour **81.** Proof **83.** $\frac{3}{\sqrt{x}}$

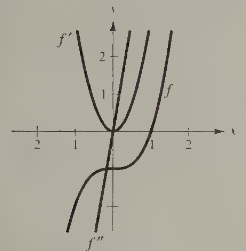
85. $\frac{2}{(x-1)^3}$ **87.** $-3\sin x$ **89.** $2x$ **91.** $\frac{1}{\sqrt{x}}$

93. Answers will vary. For example: $(x-2)^2$



95. 0 **97.** -10

99.



101. $v(3) = 27$ meters per second

$a(3) = -6$ meters per second per second

The speed of the object is decreasing, but the rate of that decrease is increasing.

103. (a) 2.4 ft/sec^2 (b) 1.2 ft/sec^2 (c) 0.5 ft/sec^2

105. (a) $f''(x) = g(x)h''(x) + 2g'(x)h'(x) + g''(x)h(x)$

$$f'''(x) = g(x)h'''(x) + 3g'(x)h''(x) + 3g''(x)h'(x) + g'''(x)h(x)$$

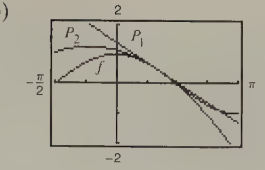
$$f^{(4)}(x) = g(x)h^{(4)}(x) + 4g'(x)h'''(x) + 6g''(x)h''(x) + 4g'''(x)h'(x) + g^{(4)}(x)h(x)$$

(b) $$f^{(n)}(x) = g(x)h^{(n)}(x) + \frac{n!}{1!(n-1)!}g'(x)h^{(n-1)}(x) + \frac{n!}{2!(n-2)!}g''(x)h^{(n-2)}(x) + \cdots + \frac{n!}{(n-1)!1!}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$$

107. (a) $P_1(x) = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$

$$P_2(x) = -\frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$$

(b)



(c) P_2

(d) P_1 and P_2 become less accurate as you move farther from $x = a$.

109. False: $dy/dx = f(x)g'(x) + g(x)f'(x)$ **111.** True

113. True **115.** $f'(x) = 2|x|$; $f''(0)$ does not exist.

Section 2.4 (page 133)

	$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1.	$y = (6x-5)^4$	$u = 6x - 5$	$y = u^4$
3.	$y = \sqrt{x^2-1}$	$u = x^2 - 1$	$y = \sqrt{u}$
5.	$y = \csc^3 x$	$u = \csc x$	$y = u^3$

7. $6(2x-7)^2$ **9.** $-108(4-9x)^3$

11. $\frac{2}{3}(9-x^2)^{-1/3}(-2x) = -\frac{4x}{3(9-x^2)^{1/3}}$

13. $\frac{1}{2}(1-t)^{-1/2}(-1) = -\frac{1}{2\sqrt{1-t}}$

15. $\frac{1}{3}(9x^2+4)^{-2/3}(18x) = \frac{6x}{(9x^2+4)^{2/3}}$

17. $\frac{1}{2}(4-x^2)^{-3/4}(-2x) = \frac{-x}{\sqrt[4]{(4-x^2)^3}}$

19. $-\frac{1}{(x-2)^2}$ **21.** $-2(t-3)^{-3}(1) = -\frac{2}{(t-3)^3}$

23. $-\frac{1}{2(x+2)^{3/2}}$

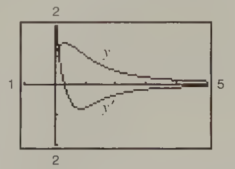
25. $x^2[4(x-2)^3(1)] + (x-2)^4(2x) = 2x(x-2)^3(3x-2)$

27. $x\left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x) + (1-x^2)^{1/2}(1) = \frac{1-2x^2}{\sqrt{1-x^2}}$

29. $\frac{(x^2+1)^{1/2}(1) - x(1/2)(x^2+1)^{-1/2}(2x)}{x^2+1} = \frac{1}{(x^2+1)^{3/2}}$

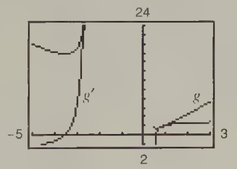
31. $\frac{-2(x+5)(x^2+10x-2)}{(x^2+2)^3}$ **33.** $\frac{-9(2v-1)^2}{(v+1)^4}$

35. $\frac{1-3x^2-4x^{3/2}}{2\sqrt{x}(x^2+1)^2}$



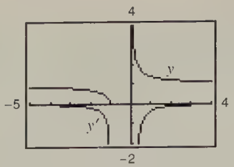
The zero of y' corresponds to the point on the graph of the function where the tangent line is horizontal.

37. $\frac{3t(t^2+3t-2)}{(t^2+2t-1)^{3/2}}$



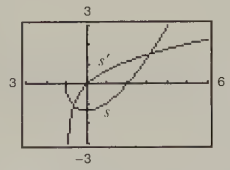
The zeros of $g'(t)$ correspond to the points on the graph of the function where the tangent line is horizontal.

39. $-\dfrac{\sqrt{\dfrac{x+1}{x}}}{2x(x+1)}$



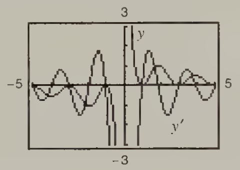
y' has no zeros.

41. $\dfrac{t}{\sqrt{1+t}}$



The zero of $s'(t)$ corresponds to the point on the graph of the function where the tangent line is horizontal.

43. $-\dfrac{\pi x \sin \pi x + \cos \pi x + 1}{x^2}$



The zeros of y' correspond to the points on the graph of the function where the tangent lines are horizontal.

45. (a) 1 (b) 2; The slope of $\sin ax$ at the origin is a.

47. $-3 \sin 3x$ **49.** $12 \sec^2 4x$ **51.** $2\pi^2 x \cos(\pi x)^2$

53. $2\cos(4x)$ **55.** $\dfrac{-1-\cos^2 x}{\sin^3 x}$

57. $8 \sec^2 x \tan x = \dfrac{8 \sin x}{\cos^3 x}$ **59.** $\sin 2\theta \cos 2\theta = \dfrac{1}{2} \sin 4\theta$

61. $\dfrac{6\pi \sin(\pi t - 1)}{\cos^3(\pi t - 1)}$ **63.** $\dfrac{1}{2\sqrt{x}} + 2x\cos(2x)^2$

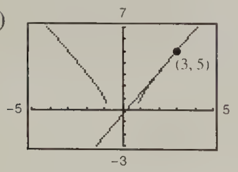
65. $-\sin x \cos(\cos x)$ **67.** $s'(t) = \dfrac{t+1}{\sqrt{t^2+2t+8}}, \dfrac{3}{4}$

69. $f'(x) = \dfrac{-9x^2}{(x^3-4)^2}, -\dfrac{9}{25}$ **71.** $f'(t) = \dfrac{-5}{(t-1)^2}, -5$

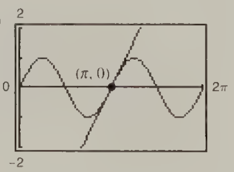
73. $y' = -6\sec^3(2x)\tan(2x), 0$

75. (a) $9x - 5y - 2 = 0$ **77.** (a) $2x - y - 2\pi = 0$

(b)

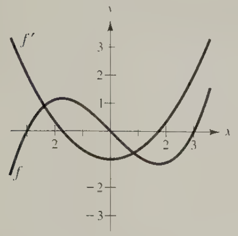


(b)



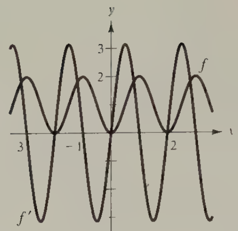
79. $12(5x^2-1)(x^2-1)$ **81.** $2(\cos x^2 - 2x^2 \sin x^2)$

83.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

85.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

87. The rate of change of g will be three times as fast as the rate of change of f.

89. (a) 24 (b) Not possible because $g'(h(5))$ is not known.
(c) $\frac{4}{3}$ (d) 162

91. (a) 1.461 (b) -1.016

93. 0.2 radian, 1.45 radians per second **95.** 0.04224

97. (a) $x = -1.637t^3 + 19.31t^2 - 0.5t - 1$

(b) $\dfrac{dC}{dt} = -294.66t^2 + 2317.2t - 30$

(c) Because x, the number of units produced in t hours, is not a linear function, and therefore the cost with respect to time t is not linear.

99. (a) $f'(x) = \beta \cos \beta x$
$f''(x) = -\beta^2 \sin \beta x$
$f'''(x) = -\beta^3 \cos \beta x$
$f^{(4)}(x) = \beta^4 \sin \beta x$

(b) $f''(x) + \beta^2 f(x) = -\beta^2 \sin \beta x + \beta^2 \sin \beta x = 0$

(c) $f^{(2k)}(x) = (-1)^k \beta^{2k} \sin \beta x$
$f^{(2k-1)}(x) = (-1)^{k+1} \beta^{2k-1} \cos \beta x$

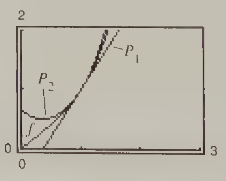
101. (a) 0 (b) $\dfrac{5}{8}$ **103.** Proof **105.** $\dfrac{2(2x-3)}{|2x-3|}, x \neq \dfrac{3}{2}$

107. $-|x| \sin x + \dfrac{x}{|x|} \cos x, \quad x \neq 0$

109. (a) $P_1(x) = \dfrac{\pi}{2}(x-1) + 1$

$P_2(x) = \dfrac{\pi^2}{8}(x-1)^2 + \dfrac{\pi}{2}(x-1) + 1$

(b)



(c) P_2

(d) P_1 and P_2 become less accurate as you move farther from $x = 1$.

111. False. $y' = -\frac{1}{2}(1-x)^{-1/2}$ **113.** True

Section 2.5 (page 142)

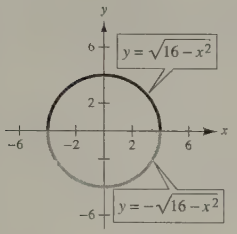
1. $-\dfrac{x}{y}$ **3.** $-\sqrt{\dfrac{y}{x}}$ **5.** $\dfrac{y-3x^2}{2y-x}$ **7.** $\dfrac{1-3x^2y^3}{3x^3y^2-1}$

9. $\dfrac{6xy-3x^2-2y^2}{4xy-3x^2}$ **11.** $\dfrac{\cos x}{4 \sin 2y}$

3. $\dfrac{\cos x - \tan y - 1}{x \sec^2 y}$ **15.** $\dfrac{y \cos(xy)}{1 - x \cos(xy)}$

7. (a) $y_1 = \sqrt{16 - x^2}$
$y_2 = -\sqrt{16 - x^2}$

(b)

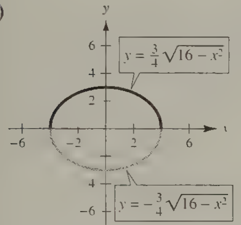


(c) $y' = \mp \dfrac{x}{\sqrt{16 - x^2}} = -\dfrac{x}{y}$

(d) $y' = -\dfrac{x}{y}$

9. (a) $y_1 = \frac{3}{4}\sqrt{16 - x^2}$
$y_2 = -\frac{3}{4}\sqrt{16 - x^2}$

(b)



(c) $y' = \mp \dfrac{3x}{4\sqrt{16 - x^2}} = -\dfrac{9x}{16y}$

(d) $y' = -\dfrac{9x}{16y}$

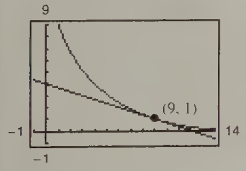
1. $-\dfrac{y}{x}, -\dfrac{1}{4}$ **23.** $\dfrac{8x}{y(x^2 + 4)^2}$, Undefined

5. $-\sqrt[3]{\dfrac{y}{x}}, -\dfrac{1}{2}$ **27.** $-\sin^2(x + y)$ or $-\dfrac{x^2}{x^2 + 1}, 0$

9. $-\dfrac{1}{2}$ **31.** 0 **33.** $\cos^2 y,\ -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}, \dfrac{1}{1 + x^2}$

5. $-\dfrac{36}{y^3}$ **37.** $-\dfrac{16}{y^3}$ **39.** $\dfrac{3x}{4y}$

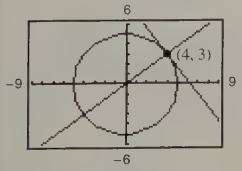
1. $x + 3y - 12 = 0$



3. At (4, 3):

Tangent line: $4x + 3y - 25 = 0$

Normal line: $3x - 4y = 0$

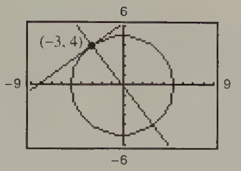


At (−3, 4):

Tangent line: $3x - 4y + 25 = 0$

Normal line: $4x + 3y = 0$

(continued)

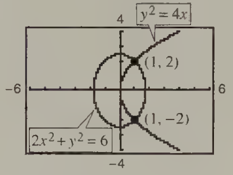


45. $x^2 + y^2 = r^2 \Rightarrow y' = -\dfrac{x}{y} \Rightarrow \dfrac{y}{x}$ = slope of normal line. Then for (x_0, y_0) on the circle, $x_0 \neq 0$, an equation of the normal line is $y = \dfrac{y_0}{x_0}x$, which passes through the origin. If $x_0 = 0$, the normal line is vertical and passes through the origin.

47. Horizontal tangents: (−4, 0), (−4, 10)

Vertical tangents: (0, 5), (−8, 5)

49.



At (1, 2):

Slope of ellipse: −1

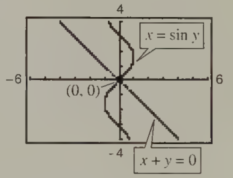
Slope of parabola: 1

At (1, −2):

Slope of ellipse: 1

Slope of parabola: −1

51.

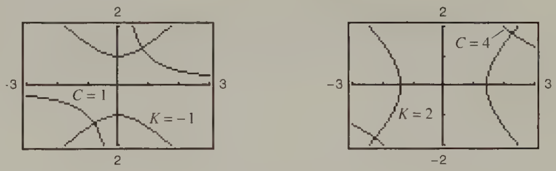


At (0, 0):

Slope of line: −1

Slope of sine curve: 1

53.



Derivatives: $\dfrac{dy}{dx} = -\dfrac{y}{x}, \dfrac{dy}{dx} = \dfrac{x}{y}$

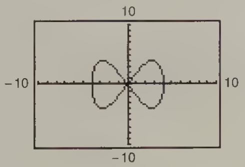
55. (a) $4y\dfrac{dy}{dx} - 12x^3 = 0$ (b) $4y\dfrac{dy}{dt} - 12x^3\dfrac{dx}{dt} = 0$

57. (a) $-\pi \sin \pi y\left(\dfrac{dy}{dx}\right) - 3\pi \cos \pi x = 0$

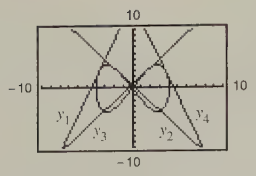
(b) $-\pi \sin \pi y\left(\dfrac{dy}{dt}\right) - 3\pi \cos \pi x\left(\dfrac{dx}{dt}\right) = 0$

59. Answers will vary. In the explicit form of a function, the variable is explicitly written as a function of x. In an implicit equation, the function is only implied by an equation. An example of an implicit function is $x^2 + xy = 5$. In explicit form it would be $y = \dfrac{5 - x^2}{x}$.

61. (a) $x^4 = 4(4x^2 - y^2)$



(b)



(c) $\left(\frac{8\sqrt{7}}{7}, 5\right)$

$$y_1 = \frac{1}{3}\left[\left(\sqrt{7} + 7\right)x + \left(8\sqrt{7} + 23\right)\right]$$

$$y_2 = -\frac{1}{3}\left[\left(-\sqrt{7} + 7\right)x - \left(23 - 8\sqrt{7}\right)\right]$$

$$y_3 = -\frac{1}{3}\left[\left(\sqrt{7} - 7\right)x - \left(23 - 8\sqrt{7}\right)\right]$$

$$y_4 = -\frac{1}{3}\left[\left(\sqrt{7} + 7\right)x - \left(8\sqrt{7} + 23\right)\right]$$

63. Proof

Section 2.6 (page 149)

1. (a) $\frac{3}{4}$ (b) 20 **3.** (a) $-\frac{5}{8}$ (b) $\frac{3}{2}$

5. (a) -4 centimeters per second
(b) 0 centimeter per second
(c) 4 centimeters per second

7. (a) 8 centimeters per second
(b) 4 centimeters per second
(c) 2 centimeters per second

9. (a) Positive (b) Negative

11. In a linear function, if x changes at a constant rate, so does y. However, unless $a = 1$, y does not change at the same rate as x.

13. $\frac{2(2x^3 + 3x)}{\sqrt{x^4 + 3x^2 + 1}}$

15. (a) 36π square centimeters per minute
(b) 144π square centimeters per minute

17. (a) Proof
(b) When $\theta = \frac{\pi}{6}$, $\frac{dA}{dt} = \frac{\sqrt{3}}{8}s^2$.
When $\theta = \frac{\pi}{3}$, $\frac{dA}{dt} = \frac{1}{8}s^2$.
(c) If s and $d\theta/dt$ are constant, dA/dt is proportional to $\cos\theta$.

19. (a) $\frac{2}{9\pi}$ centimeter per minute
(b) $\frac{1}{18\pi}$ centimeter per minute

21. (a) 36 square centimeters per second
(b) 360 square centimeters per second

23. $\frac{8}{405\pi}$ foot per minute

25. (a) 12.5% (b) $\frac{1}{144}$ meter per minute

27. (a) $-\frac{7}{12}$ foot per second; $-\frac{3}{2}$ feet per second; $-\frac{48}{7}$ feet per second
(b) $\frac{527}{24}$ square feet per second (c) $\frac{1}{12}$ radian per second

29. Rate of vertical change: $\frac{1}{5}$ meter per second
Rate of horizontal change: $-\frac{\sqrt{3}}{15}$ meter per second

31. (a) -750 miles per hour (b) 20 minutes

33. $-\frac{28}{\sqrt{10}} \approx -8.85$ feet per second

35. (a) $\frac{25}{3}$ feet per second (b) $\frac{10}{3}$ feet per second

37. (a) 12 seconds
(b) $\frac{1}{2}\sqrt{3}$ meter
(c) $\frac{\sqrt{5}\,\pi}{120}$ meter per second

39. Evaporation rate proportional to

$$S \Rightarrow \frac{dV}{dt} = k(4\pi r^2)$$

$$V = \left(\frac{4}{3}\right)\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

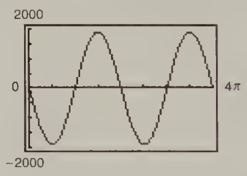
So $k = \frac{dr}{dt}$.

41. $V^{0.3}\left(1.3p\frac{dV}{dt} + V\frac{dp}{dt}\right) = 0$ **43.** $\frac{1}{20}$ radian per second

45. (a) $\frac{1}{2}$ radian per minute (b) $\frac{3}{2}$ radians per minute
(c) 1.87 radians per minute

47. (a) $\frac{dx}{dt} = -600\pi \sin\theta$

(b)



(c) $\theta = 90° + n \cdot 180°$; $\theta = 0° + n \cdot 180°$
(d) -300π centimeters per second;
$-300\sqrt{3}\pi$ centimeters per second

49. $\frac{1}{25}\cos^2\theta$, $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

51. -0.1808 foot per second per second

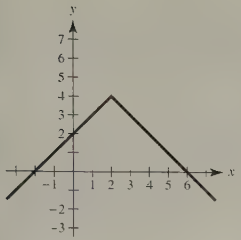
53. (a) $m(s) = -0.881s^2 + 29.10s - 206.2$
(b) $(-1.762s + 29.10)\frac{ds}{dt}$ (c) 2.15 million

Review Exercises for Chapter 2 (page 153)

1. $f'(x) = 2x - 2$ **3.** $f'(x) = \frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2x}$

5. f is differentiable at all $x \neq -1$.

7.



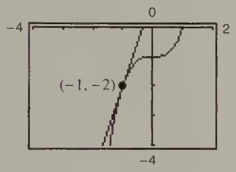
(a) Yes

(b) No, because the derivatives from the left and right are not equal.

9. $-\frac{3}{2}$

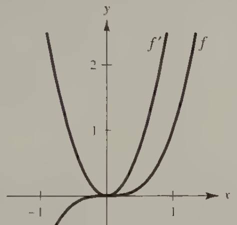
1. (a) $y = 3x + 1$

(b)



13. 8

5. $f'(x) = 2x - 2$



$f' > 0$ where the slopes of tangent lines to the graph of f are positive.

7. 0 **19.** $8x^7$ **21.** $12t^3$ **23.** $3x(x-2)$

5. $\frac{3}{\sqrt{x}} + \frac{1}{x^{2/3}}$ **27.** $-\frac{4}{3t^3}$ **29.** $2 - 3\cos\theta$

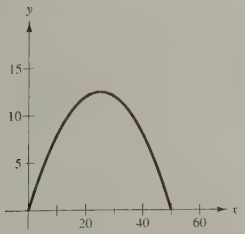
1. $-3\sin\theta - \frac{\cos\theta}{4}$

3. (a) 50 vibrations per second per pound

(b) 33.33 vibrations per second per pound

5. 414.74 meters or 1,354 feet

7. (a)



(b) 50

(c) $x = 25$

(d) $y' = 1 - 0.04x$

x	0	10	25	30	50
y'	1	0.6	0	−0.2	−1

(e) $y'(25) = 0$

39. (a) $x'(t) = 2t - 3$ (b) $(-\infty, 1.5)$ (c) $x = -\frac{1}{4}$ (d) 1

41. $2(6x^3 - 9x^2 + 16x - 7)$ **43.** $\sqrt{x}\cos x + \frac{1}{2\sqrt{x}}\sin x$

45. $2 + \frac{2}{x^3}$ **47.** $-\frac{x^2+1}{(x^2-1)^2}$ **49.** $\frac{6x}{(4-3x^2)^2}$

51. $\frac{2x\cos x + x^2\sin x}{\cos^2 x}$ **53.** $3x^2\sec x\tan x + 6x\sec x$

55. $-x\sec^2 x - \tan x$ **57.** $-x\sin x$ **59.** $6t$

61. $6\sec^2\theta\tan\theta$

63. $y'' + y = -(2\sin x + 3\cos x) + (2\sin x + 3\cos x) = 0$

65. $\frac{-3x^2}{2\sqrt{1-x^3}}$ **67.** $\frac{2(x-3)(-x^2+6x+1)}{(x^2+1)^3}$

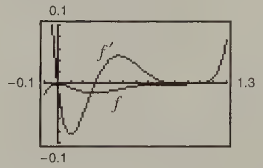
69. $s(s^2-1)^{3/2}(8s^3 - 3s + 25)$ **71.** $-9\sin(3x+1)$

73. $-\csc 2x\cot 2x$ **75.** $\frac{1}{2}(1 - \cos 2x) = \sin^2 x$

77. $\sin^{1/2}x\cos x - \sin^{5/2}x\cos x = \cos^3 x\sqrt{\sin x}$

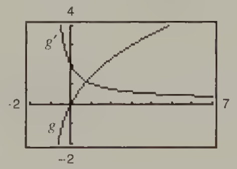
79. $\frac{(x+2)[\pi\cos(\pi x)] - \sin(\pi x)}{(x+2)^2}$

81. $t(t-1)^4(7t-2)$

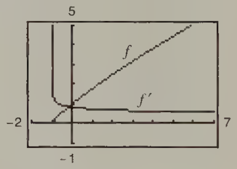


The zeros of f' correspond to the points on the graph of the function where the tangent line is horizontal.

83. $\frac{x+2}{(x+1)^{3/2}}$ **85.** $\frac{5}{6(t+1)^{1/6}}$

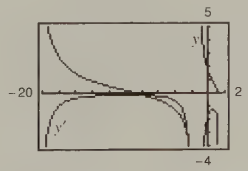


g' is not equal to zero for any x.



f' has no zeros.

87. $-\frac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$



y' has no zeros.

89. $4 - 4\sin 2x$ **91.** $2\csc^2 x\cot x$

93. $\frac{2(t+2)}{(1-t)^4}$ **95.** $18\sec^2(3\theta)\tan(3\theta) + \sin(\theta - 1)$

97. (a) −18.667 degrees per hour

(b) −7.284 degrees per hour

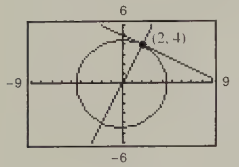
(c) −3.240 degrees per hour

(d) −0.747 degree per hour

99. $-\dfrac{2x+3y}{3(x+y^2)}$ **101.** $\dfrac{2y\sqrt{x}-y\sqrt{y}}{2x\sqrt{y}-x\sqrt{x}}$ **103.** $\dfrac{y\sin x+\sin y}{\cos x - x\cos y}$

105. Tangent line: $x + 2y - 10 = 0$

Normal line: $2x - y = 0$

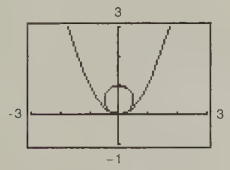


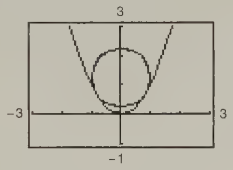
107. (a) $2\sqrt{2}$ units/sec (b) 4 units/sec (c) 8 units/sec

109. $\frac{2}{25}$ meter per minute **111.** −38.34 meters per second

P.S. Problem Solving (page 156)

1. (a) $r = \frac{1}{2}$ (b) Center: $\left(0, \frac{5}{4}\right)$





3. (a) $P_1(x) = 1$

(b) $P_2(x) = 1 - \frac{1}{2}x^2$

(c)

x	−1.0	−0.1	−0.001	0	0.001
cos *x*	0.5403	0.9950	1.000	1	1
$P_2(x)$	0.5000	0.9950	1.000	1	1

x	0.1	1.0
cos *x*	0.9950	0.5403
$P_2(x)$	0.9950	0.5000

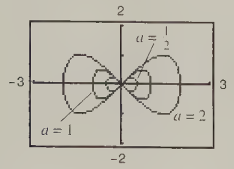
$P_2(x)$ is a good approximation of $f(x) = \cos x$ when x is very close to 0.

(d) $P_3(x) = x - \frac{1}{6}x^3$

5. $p(x) = 2x^3 + 4x^2 - 5$

7. (a) Graph $\begin{cases} y_1 = \dfrac{1}{a}\sqrt{x^2(a^2 - x^2)} \\ y_2 = -\dfrac{1}{a}\sqrt{x^2(a^2-x^2)} \end{cases}$ as separate equations.

(b) Answers will vary.



The intercepts will always be (0, 0), $(a, 0)$, and $(-a, 0)$, and the maximum and minimum y-values appear to be $\pm\frac{1}{2}a$.

(c) $\left(\dfrac{a\sqrt{2}}{2}, \dfrac{a}{2}\right), \left(\dfrac{a\sqrt{2}}{2}, -\dfrac{a}{2}\right), \left(-\dfrac{a\sqrt{2}}{2}, \dfrac{a}{2}\right), \left(-\dfrac{a\sqrt{2}}{2}, -\dfrac{a}{2}\right)$

9. (a) When the man is 90 feet from the light, the tip of his shadov is $112\frac{1}{2}$ feet from the light. The tip of the child's shadow i $111\frac{1}{9}$ feet from the light, so the man's shadow extends 1 feet beyond the child's shadow.

(b) When the man is 60 feet from the light, the tip of his shadov is 75 feet from the light. The tip of the child's shadow is 77 feet from the light, so the child's shadow extends $2\frac{7}{9}$ feet beyond the man's shadow.

(c) $d = 80$ feet

(d) Let x be the distance of the man from the light and s be th distance from the light to the tip of the shadow.

If $0 < x < 80$, $\dfrac{ds}{dt} = -\dfrac{50}{9}$.

If $x > 80$, $\dfrac{ds}{dt} = -\dfrac{25}{4}$.

There is a discontinuity at $x = 80$.

11. Proof. The graph of L is a line passing through the origin (0, 0

13. (a)

$z°$	0.1	0.01	0.0001
$\dfrac{\sin z}{z}$	0.01745241	0.0174532924	0.0174532925

(b) $\dfrac{\pi}{180}$ (c) $\dfrac{\pi}{180}\cos z$ (d) $\dfrac{\pi}{180}C(z)$

(e) Answers will vary.

15. (a) j would be the rate of change of acceleration/deceleration.

(b) $j = 0$. Deceleration is constant, so there is no change i deceleration.

Chapter 3

Section 3.1 (page 165)

1. $f'(0) = 0$ **3.** $f'(3) = 0$

5. $f'(-2)$ is undefined. **7.** 2, absolute maximum

9. 1, absolute maximum; 2, absolute minimum; 3, absolute maximum

11. $x = 0$, $x = 2$ **13.** $t = \dfrac{8}{3}$ **15.** $x = \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$

17. Minimum: (2, 2)

Maximum: (−1, 8)

19. Minimum: (0, 0) and (3,

Maximum: $\left(\frac{3}{2}, \frac{9}{4}\right)$

21. Minimum: $\left(-1, -\frac{5}{2}\right)$

Maximum: (2, 2)

23. Minimum: (0, 0)

Maximum: (−1, 5)

25. Minimum: (0, 0)

Maximum: $\left(-1, \frac{1}{4}\right)$ and $\left(1, \frac{1}{4}\right)$

27. Minimum: (1, −1)

Maximum: $\left(0, -\frac{1}{2}\right)$

29. Minimum: $\left(\dfrac{1}{6}, \dfrac{\sqrt{3}}{2}\right)$

Maximum: (0, 1)

31. Minimum: (2, 3)

Maximum: $\left(1, \sqrt{2} + 3\right)$

33. (a) Minimum: (0, −3);

Maximum: (2, 1)

(b) Minimum: (0, −3)

(c) Maximum: (2, 1)

(d) No extrema

35. (a) Minimum: (1, −1);

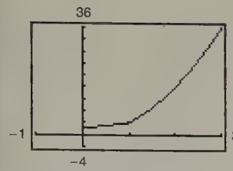
Maximum: (−1, 3)

(b) Minimum: (3, 3)

(c) Minimum: (1, −1)

(d) Minimum: (1, −1)

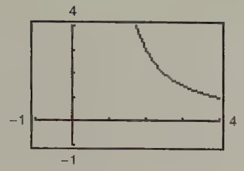
7.



Minimum: (0, 2)

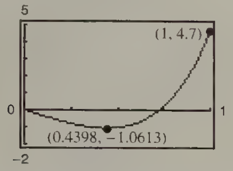
Maximum: (3, 36)

39.



Minimum: (4, 1)

1. (a)



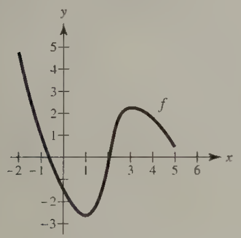
(b) Minimum: (0.4398, −1.0613)

3. Maximum: $\left|f''\left(\sqrt[3]{-10+\sqrt{108}}\right)\right| = f''\left(\sqrt{3}-1\right) \approx 1.47$

5. Maximum: $\left|f^{(4)}(0)\right| = \frac{56}{81}$

7. Because f is continuous on $\left[0, \frac{\pi}{4}\right]$, but not continuous on $[0, \pi]$.

9. Answers will vary. Example:



1. (a) Yes (b) No **53.** (a) No (b) Yes

5. Maximum: $P(12) = 72$

No. P is decreasing for $I \geq 12$.

7. 0.9553 radian

9. (a) $y = \frac{3}{40{,}000}x^2 - \frac{3}{200}x + \frac{75}{4}$

(b)

x	−500	−400	−300	−200	−100	0
d	0	0.75	3	6.75	12	18.75

x	100	200	300	400	500
d	12	6.75	3	0.75	0

(c) Lowest point ≈ (100, 18); No

1. True **63.** True

ection 3.2 (page 172)

1. $f(0) = f(2) = 0$; f is not differentiable on $(0, 2)$.

3. $(2, 0), (-1, 0)$; $f'\left(\frac{1}{2}\right) = 0$ **5.** $(0, 0), (-4, 0)$; $f'\left(-\frac{8}{3}\right) = 0$

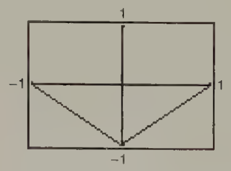
7. $f'(1) = 0$ **9.** $f'\left(\frac{6-\sqrt{3}}{3}\right) = 0$; $f'\left(\frac{6+\sqrt{3}}{3}\right) = 0$

1. Not differentiable at $x = 0$ **13.** $f'\left(-2+\sqrt{5}\right) = 0$

5. $f'\left(\frac{\pi}{2}\right) = 0$; $f'\left(\frac{3\pi}{2}\right) = 0$ **17.** $f'(0.249) \approx 0$

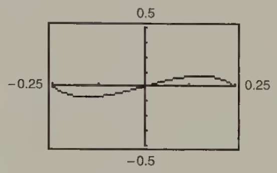
19. Not continuous on $[0, \pi]$

21.



Rolle's Theorem does not apply.

23.

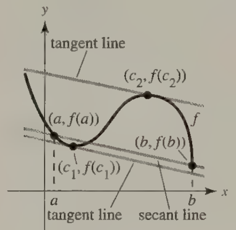


$f'(\pm 0.1533) = 0$

25. (a) $f(1) = f(2) = 64$

(b) Velocity = 0 for some t in $(-1, 2)$; $t = \frac{3}{2}$ seconds

27.



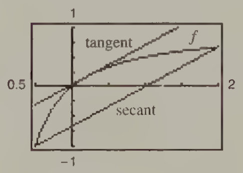
29. The function is discontinuous on [0, 6].

31. $f'\left(-\frac{1}{2}\right) = -1$ **33.** $f'\left(\frac{8}{27}\right) = 1$

35. $f'\left(-\frac{1}{4}\right) = -\frac{1}{3}$ **37.** $f'\left(\frac{\pi}{2}\right) = 0$

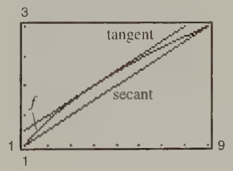
39. Secant line: $2x - 3y - 2 = 0$

Tangent line: $c = \frac{-2+\sqrt{6}}{2}$, $2x - 3y + 5 - 2\sqrt{6} = 0$



41. Secant line: $x - 4y + 3 = 0$

Tangent line: $c = 4$, $x - 4y + 4 = 0$



43. (a) −14.7 meters per second (b) 1.5 seconds

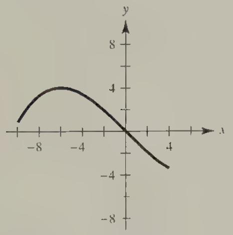
45. No. Let $f(x) = x^2$ on $[-1, 2]$.

47. By the Mean Value Theorem, there is a time when the speed of the plane must equal the average speed of 454.5 miles per hour. The speed was 400 miles per hour when the plane was accelerating to 454.5 miles per hour and decelerating from 454.5 miles per hour.

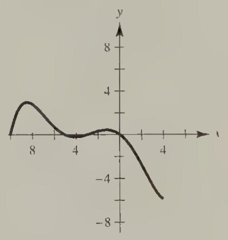
49. (a) f is continuous and changes signs in $[-10, 4]$ (Intermediate Value Theorem).

(b) There exist real numbers a and b such that $-10 < a < b < 4$ and $f(a) = f(b) = 2$. Therefore, f' has a zero in the interval by Rolle's Theorem.

(c)

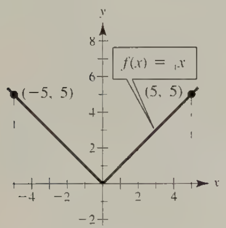


(d)



(e) No, by Theorem 2.1.

51.



53. False. f is not continuous on $[-1, 1]$.

55. True **57.** Proof **59.** Proof **61.** Proof

Section 3.3 (page 181)

1. Increasing on $(3, \infty)$; Decreasing on $(-\infty, 3)$

3. Increasing on $(-\infty, -2)$ and $(2, \infty)$; Decreasing on $(-2, 2)$

5. Increasing on $(-\infty, 0)$; Decreasing on $(0, \infty)$

7. Increasing on $(1, \infty)$; Decreasing on $(-\infty, 1)$

9. Increasing on $\left(-2\sqrt{2}, 2\sqrt{2}\right)$
Decreasing on $\left(-4, -2\sqrt{2}\right), \left(2\sqrt{2}, 4\right)$

11. Critical number: $x = 3$
Increasing on $(3, \infty)$
Decreasing on $(-\infty, 3)$
Relative minimum: $(3, -9)$

13. Critical number: $x = 1$
Increasing on $(-\infty, 1)$
Decreasing on $(1, \infty)$
Relative maximum: $(1, 5)$

15. Critical numbers: $x = -2, 1$
Increasing on $(-\infty, -2)$ and $(1, \infty)$
Decreasing on $(-2, 1)$
Relative maximum: $(-2, 20)$
Relative minimum: $(1, -7)$

17. Critical numbers: $x = 0, 2$
Increasing on $(0, 2)$
Decreasing on $(-\infty, 0), (2, \infty)$
Relative maximum: $(2, 4)$
Relative minimum: $(0, 0)$

19. Critical numbers: $x = -1, 1$
Increasing on $(-\infty, -1)$ and $(1, \infty)$
Decreasing on $(-1, 1)$
Relative maximum: $\left(-1, \frac{4}{5}\right)$
Relative minimum: $\left(1, -\frac{4}{5}\right)$

21. Critical number: $x = 0$
Increasing on $(-\infty, \infty)$
No relative extrema

23. Critical number: $x = 1$
Increasing on $(1, \infty)$
Decreasing on $(-\infty, 1)$
Relative minimum: $(1, 0)$

25. Critical number: $x = 5$
Increasing on $(-\infty, 5)$
Increasing on $(-\infty, 5)$
Relative maximum: $(5, 5)$

27. Critical numbers: $x = -1, 1$
Discontinuity: $x = 0$
Increasing on $(-\infty, -1)$ and $(1, \infty)$
Decreasing on $(-1, 0)$ and $(0, 1)$
Relative maximum: $(-1, -2)$
Relative minimum: $(1, 2)$

29. Critical number: $x = 0$
Discontinuities: $x = -3, 3$
Increasing on $(-\infty, -3)$ and $(-3, 0)$
Decreasing on $(0, 3)$ and $(3, \infty)$
Relative maximum: $(0, 0)$

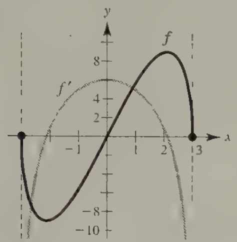
31. Critical numbers: $x = -3, 1$
Discontinuity: $x = -1$
Increasing on $(-\infty, -3)$ and $(1, \infty)$
Decreasing on $(-3, -1)$ and $(-1, 1)$
Relative maximum: $(-3, -8)$
Relative minimum: $(1, 0)$

33. Critical numbers: $x = \frac{\pi}{6}, \frac{5\pi}{6}$
Increasing on $\left(0, \frac{\pi}{6}\right), \left(\frac{5\pi}{6}, 2\pi\right)$
Decreasing on $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$
Relative maximum: $\left(\frac{\pi}{6}, \frac{\left[\pi + 6\sqrt{3}\right]}{12}\right)$
Relative minimum: $\left(\frac{5\pi}{6}, \frac{\left[5\pi - 6\sqrt{3}\right]}{12}\right)$

35. Critical numbers: $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$
Increasing on $\left(0, \frac{\pi}{2}\right), \left(\frac{7\pi}{6}, \frac{3\pi}{2}\right), \left(\frac{11\pi}{6}, 2\pi\right)$
Decreasing on $\left(\frac{\pi}{2}, \frac{7\pi}{6}\right), \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$
Relative maxima: $\left(\frac{\pi}{2}, 2\right), \left(\frac{3\pi}{2}, 0\right)$
Relative minima: $\left(\frac{7\pi}{6}, -\frac{1}{4}\right), \left(\frac{11\pi}{6}, -\frac{1}{4}\right)$

37. (a) $f'(x) = \dfrac{2(9 - 2x^2)}{\sqrt{9 - x^2}}$

(b)



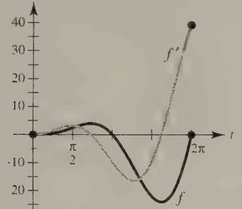
(c) $x = \pm\dfrac{3\sqrt{2}}{2}$

(d) $f' > 0$ on $\left(-\dfrac{3\sqrt{2}}{2}, \dfrac{3\sqrt{2}}{2}\right)$

$f' < 0$ on $\left(-3, -\dfrac{3\sqrt{2}}{2}\right), \left(\dfrac{3\sqrt{2}}{2}, 3\right)$

39. (a) $f'(t) = t(t \cos t + 2 \sin t)$

(b)



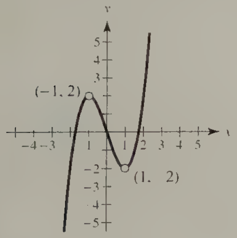
(c) Critical numbers: $t = 2.2889, 5.0870$

(d) $f' > 0$ on $(0, 2.2889), (5.0870, 2\pi)$

$f' < 0$ on $(2.2889, 5.0870)$

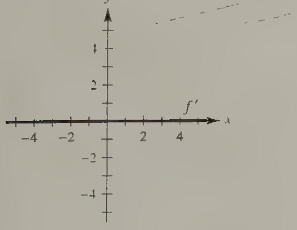
41. $f(x)$ is symmetric with respect to the origin.

Zeros: $(0, 0), (\pm\sqrt{3}, 0)$

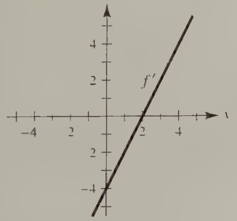


$g(x)$ is continuous on $(-\infty, \infty)$ and $f(x)$ has holes at $x = 1$ and $x = -1$.

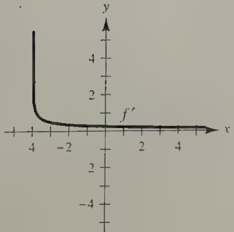
43.



45.

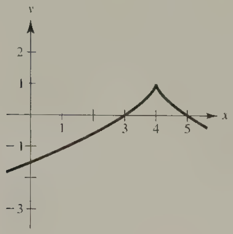


47.

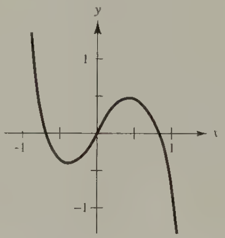


49. $g'(0) < 0$ **51.** $g'(-6) < 0$ **53.** $g'(0) > 0$

55.



57.



Minimum at the approximate critical number $x = -0.40$

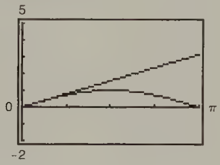
Maximum at the approximate critical number $x = 0.48$

59. (a)

x	0.5	1	1.5	2	2.5	3
$f(x)$	0.5	1	1.5	2	2.5	3
$g(x)$	0.48	0.84	1.00	0.91	0.60	0.14

$f(x) > g(x)$

(b)



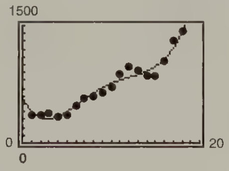
$f(x) > g(x)$

(c) Proof

61. $r = \dfrac{2R}{3}$ **63.** Maximum when $R_2 = R_1$

65. (a) $B = 0.11980t^4 - 4.4879t^3 + 56.991t^2 - 223.02t + 580.0$

(b)



(c) $(2.8, 311.2)$

67. (a) 3

(b) $a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 0$

$a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0 = 2$

$3a_3(0)^2 + 2a_2(0) + a_1 = 0$

$3a_3(2)^2 + 2a_2(2) + a_1 = 0$

(c) $f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2$

69. (a) 4

(b) $a_4(0)^4 + a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 0$

$a_4(2)^4 + a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0 = 4$

$a_4(4)^4 + a_3(4)^3 + a_2(4)^2 + a_1(4) + a_0 = 0$

$4a_4(0)^3 + 3a_3(0)^2 + 2a_2(0) + a_1 = 0$

$4a_4(2)^3 + 3a_3(2)^2 + 2a_2(2) + a_1 = 0$

(c) $f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2$

71. True **73.** False. Let $f(x) = x^3$.

75. False. Let $f(x) = x^3$. There is a critical number at $x = 0$, but not a relative extremum.

77. Proof **79.** Proof

Section 3.4 (page 189)

1. Concave upward: $(-\infty, \infty)$

3. Concave upward: $(-\infty, -2), (2, \infty)$

Concave downward: $(-2, 2)$

5. Concave upward: $(-\infty, -1), (1, \infty)$

Concave downward: $(-1, 1)$

7. Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

9. Concave upward: $\left(-\frac{\pi}{2}, 0\right)$

Concave downward: $\left(0, \frac{\pi}{2}\right)$

11. Point of inflection: $(2, 8)$

Concave downward: $(-\infty, 2)$

Concave upward: $(2, \infty)$

13. Points of inflection: $\left(\pm\frac{2}{\sqrt{3}}, -\frac{20}{9}\right)$

Concave upward: $\left(-\infty, -\frac{2}{\sqrt{3}}\right), \left(\frac{2}{\sqrt{3}}, \infty\right)$

Concave downward: $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

15. Points of inflection: $(2, -16), (4, 0)$

Concave upward: $(-\infty, 2), (4, \infty)$

Concave downward: $(2, 4)$

17. Concave upward: $(-3, \infty)$

19. Points of inflection: $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), (0, 0), \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$

Concave upward: $(-\sqrt{3}, 0), (\sqrt{3}, \infty)$

Concave downward: $(-\infty, -\sqrt{3}), (0, \sqrt{3})$

21. Point of inflection: $(2\pi, 0)$

Concave upward: $(2\pi, 4\pi)$

Concave downward: $(0, 2\pi)$

23. Concave upward: $(0, \pi), (2\pi, 3\pi)$

Concave downward: $(\pi, 2\pi), (3\pi, 4\pi)$

25. Points of inflection: $(\pi, 0), (1.823, 1.452), (4.46, -1.452)$

Concave upward: $(1.823, \pi), (4.46, 2\pi)$

Concave downward: $(0, 1.823), (\pi, 4.46)$

27. Relative minimum: $(3, -25)$

29. Relative minimum: $(5, 0)$

31. Relative maximum: $(0, 3)$

Relative minimum: $(2, -1)$

33. Relative maximum: $(2.4, 268.74)$

Relative minimum: $(0, 0)$

35. Relative minimum: $(0, -3)$

37. Relative maximum: $(-2, -4)$

Relative minimum: $(2, 4)$

39. No relative extrema, because f is nonincreasing.

41. (a) $f'(x) = 0.2x(x - 3)^2(5x - 6)$

$f''(x) = 0.4(x - 3)(10x^2 - 24x + 9)$

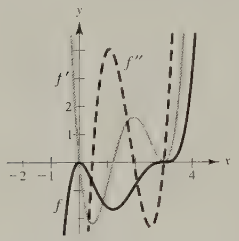
(b) Relative maximum: $(0, 0)$

Relative minimum: $(1.2, -1.6796)$

Points of inflection: $(0.4652, -0.7048)$,

$(1.9348, -0.9048), (3, 0)$

(c)



f is increasing when f' is positive, and decreasing when f' is negative. f is concave upward when f'' is positive, and concave downward when f'' is negative.

43. (a) $f'(x) = \cos x - \cos 3x + \cos 5x$

$f''(x) = -\sin x + 3 \sin 3x - 5 \sin 5x$

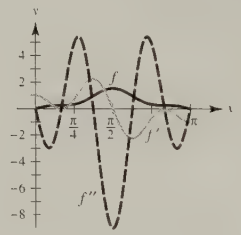
(b) Relative maximum: $(\pi/2, 1.53333)$

Points of inflection:

$(0.5236, 0.2667), (1.1731, 0.9637)$,

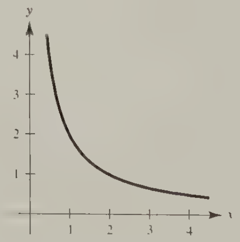
$(1.9685, 0.9637), (2.6180, 0.2667)$

(c)

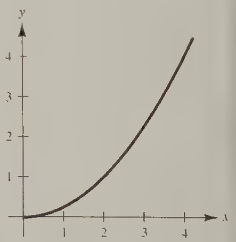


f is increasing when f' is positive, and decreasing when f' is negative. f is concave upward when f'' is positive, and concave downward when f'' is negative.

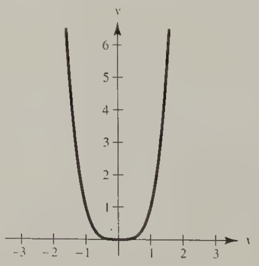
45. (a)



(b)



47. Answers will vary. Example: $f(x) = x^4$ $f''(0) = 0$, but $(0, 0)$ is not point of inflection.



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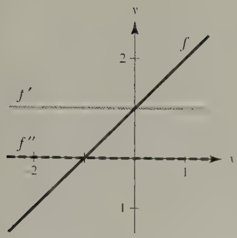

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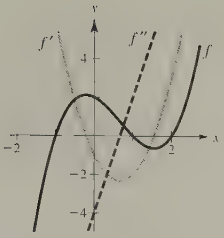

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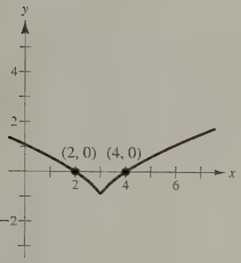

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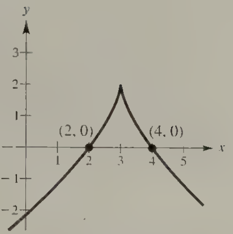
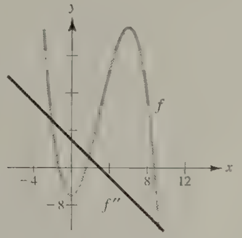
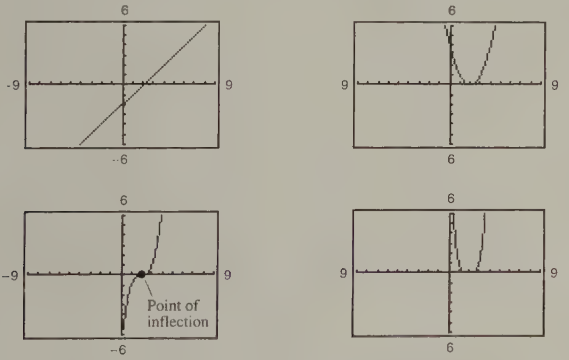


57. Example:



59. (a) $f(x) = (x - 2)^n$ has a point of inflection at (2, 0) if n is odd and $n \geq 3$.



(b) Proof

61. $f(x) = \frac{1}{2}x^3 - 6x^2 + \frac{45}{2}x - 24$

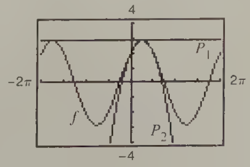
63. (a) $f(x) = -\frac{1}{32}x^3 + \frac{3}{16}x^2$ (b) Two miles from touchdown

65. $x = \left(\frac{15 - \sqrt{33}}{16}\right)L \approx 0.578L$ **67.** $x = 100$ units

69. $t = \sqrt{\frac{8}{3}} \approx 1.633$ years

71. $P_1(x) = 2\sqrt{2}$

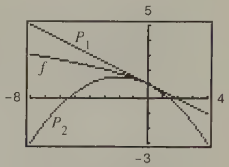
$P_2(x) = 2\sqrt{2} - \sqrt{2}\left(x - \frac{\pi}{4}\right)^2$



The values of f, P_1, and P_2 and their first derivatives are equal when $x = \pi/4$. The approximations worsen as you move away from $x = \pi/4$.

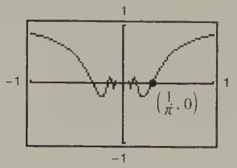
73. $P_1(x) = 1 - \frac{x}{2}$

$P_2(x) = 1 - \frac{x}{2} - \frac{x^2}{8}$



The values of f, P_1, and P_2 and their first derivatives are equal when $x = 0$. The approximations worsen as you move away from $x = 0$.

75.



77. Proof **79.** True

81. False. The maximum value is $\sqrt{13} \approx 3.60555$.

83. False. f is concave upward at $x = c$ if $f''(c) > 0$.

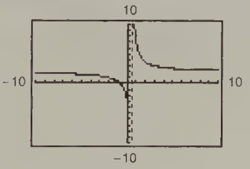
Section 3.5 (page 199)

1. f **2.** c **3.** d **4.** a **5.** b **6.** e

7.

x	10^0	10^1	10^2	10^3	10^4
$f(x)$	7	2.2632	2.0251	2.0025	2.0003

x	10^5	10^6
$f(x)$	2.0000	2.0000

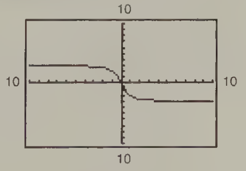


$\lim_{x\to\infty} \frac{4x + 3}{2x - 1} = 2$

9.

x	10^0	10^1	10^2	10^3	10^4
$f(x)$	−2	−2.9814	−2.9998	−3.0000	−3.0000

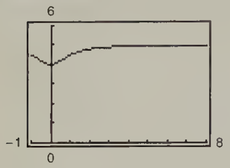
x	10^5	10^6
$f(x)$	−3.0000	−3.0000



$\lim_{x\to\infty} \frac{-6x}{\sqrt{4x^2 + 5}} = -3$

11.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	4.5000	4.9901	4.9999	5.0000	5.0000	5.0000	5.0000



$$\lim_{x\to\infty}\left(5-\frac{1}{x^2+1}\right)=5$$

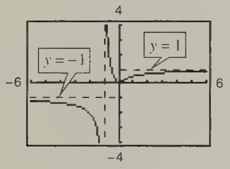
13. (a) ∞ (b) 5 (c) 0 **15.** (a) 0 (b) 1 (c) ∞

17. (a) 0 (b) $-\frac{2}{3}$ (c) $-\infty$

19. $\frac{2}{3}$ **21.** 0 **23.** $-\infty$

25. -1 **27.** -2 **29.** 0 **31.** 0

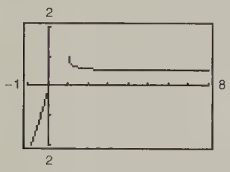
33.



35. 1 **37.** 0 **39.** $-\frac{1}{2}$

41.

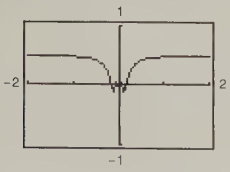
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1.000	0.513	0.501	0.500	0.500	0.500	0.500



$$\lim_{x\to\infty}\left[x-\sqrt{x(x-1)}\right]=\frac{1}{2}$$

43.

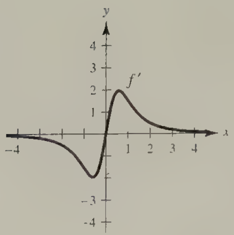
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	0.479	0.500	0.500	0.500	0.500	0.500	0.500



The graph has a hole at $x = 0$.

$$\lim_{x\to\infty} x\sin\frac{1}{2x}=\frac{1}{2}$$

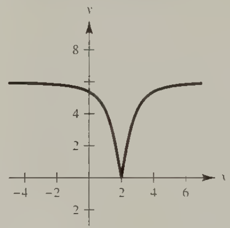
45. (a)



(b) $\lim_{x\to\infty} f(x) = 3,\ \lim_{x\to\infty} f'(x) = 0$

(c) $y = 3$ is a horizontal asymptote. The rate of increase of the function approaches 0 as the graph approaches $y = 3$.

47. Yes. For example, let $f(x)=\dfrac{6|x-2|}{\sqrt{(x-2)^2+1}}$.



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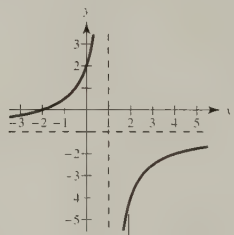

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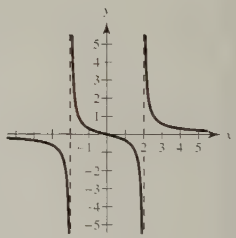

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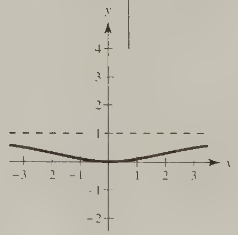

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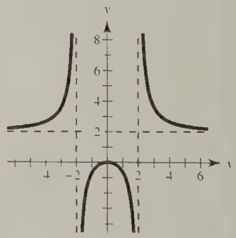

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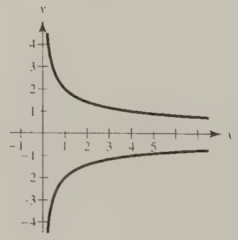

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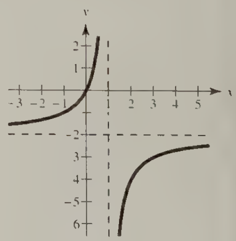

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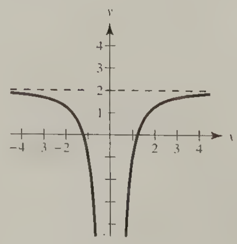

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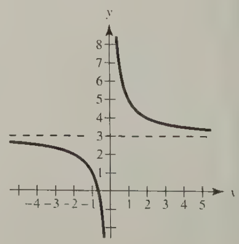

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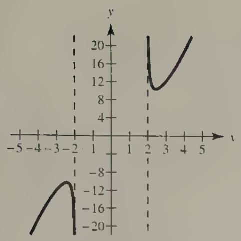

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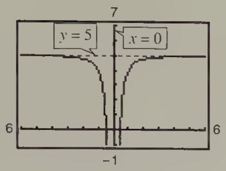

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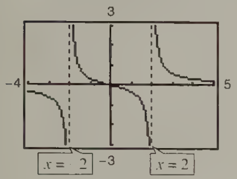

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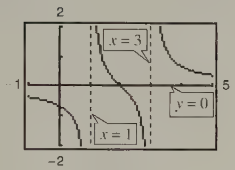
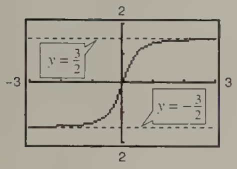


73.



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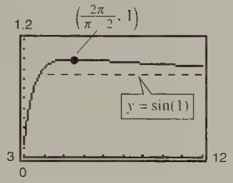
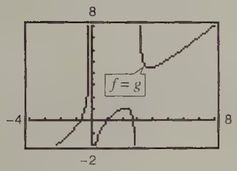
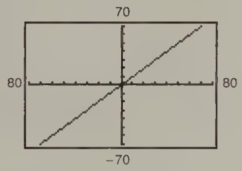


77. (a)



(c)



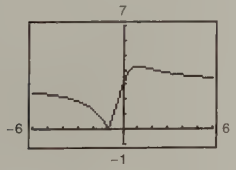
(b) Proof

The slant asymptote $y = x$

79. $\frac{1}{2}$

81. (a) $d = \dfrac{|3m + 3|}{\sqrt{m^2 + 1}}$

(b)

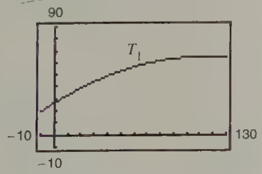


(c) $\lim_{m\to\infty} d(m) = 3$; $\lim_{m\to-\infty} d(m) = 3$

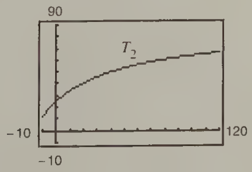
As $m \to \infty$, the line approaches the vertical line $x = 0$. Therefore, the distance approaches 3.

83. (a) $T_1 = -0.003t^2 + 0.68t + 26.6$

(b)



(c)



(d) $T_1(0) \approx 26.6°$, $T_2(0) \approx 25.0°$ (e) 86

(f) The limiting temperature is 86°.
No. T_1 has no horizontal asymptote.

85. Answers will vary. See "Guidelines for Finding Limits at Infinity of Rational Functions" on page 195. Examples:

(a) $\lim_{x\to\infty} \left(\dfrac{5 - 2x}{3x^2 - 4}\right) = 0$ since the degree of the numerator is less than the degree of the denominator.

(b) $\lim_{x\to\infty} \left(\dfrac{2x - 1}{3x + 2}\right) = \dfrac{2}{3}$ since the degree of the numerator is equal to the degree of the denominator.

(c) $\lim_{x\to\infty} \left(\dfrac{x^2 + 2}{x - 1}\right) = \infty$ since the degree of the numerator is greater than the degree of the denominator.

87. False. Let $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$.

$f'(x) > 0$ for all real numbers.

Section 3.6 (page 208)

1. d **2.** c **3.** a **4.** b

5. (a) $f'(x) = 0$ for $x = \pm 2$

$f'(x) > 0$ for $(-\infty, -2), (2, \infty)$

$f'(x) < 0$ for $(-2, 2)$

(b) $f''(x) = 0$ for $x = 0$

$f''(x) > 0$ for $(0, \infty)$

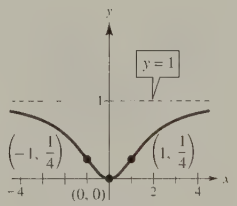
$f''(x) < 0$ for $(-\infty, 0)$

(c) $(0, \infty)$

(d) f' is minimum for $x = 0$.

f is decreasing at the fastest rate.

7.



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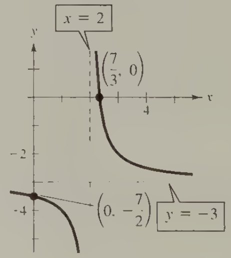

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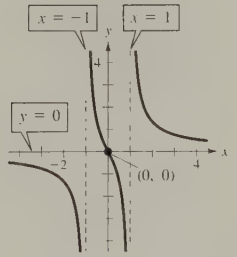

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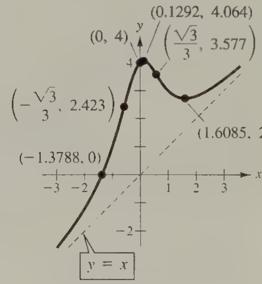

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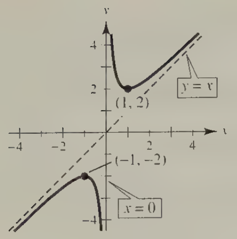

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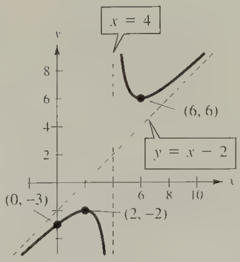

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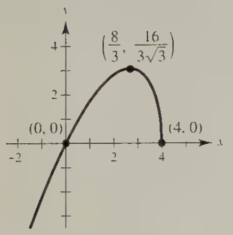

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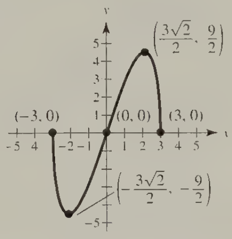

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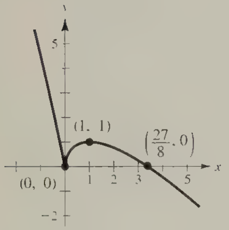

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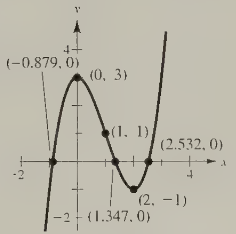

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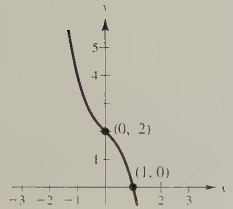

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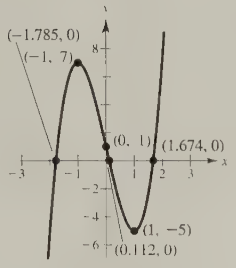
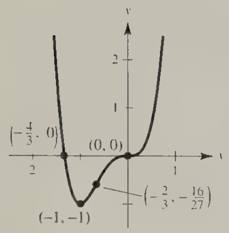


31.



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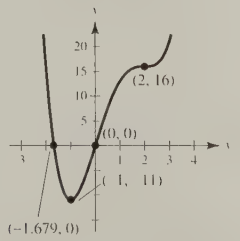

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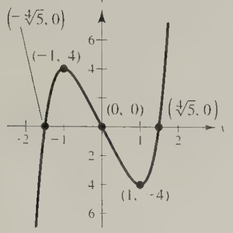

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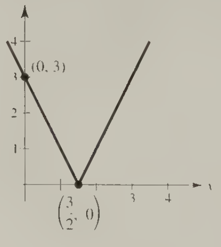

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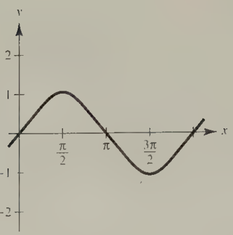

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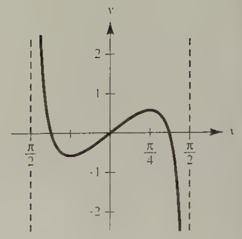
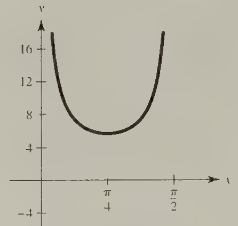


43.



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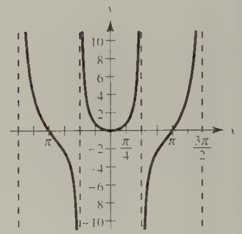

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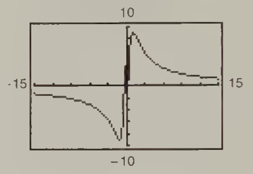


Minimum: $(-1.10, -9.05)$

Maximum: $(1.10, 9.05)$

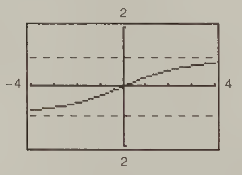
Points of inflection:

$(-1.84, -7.86)$, $(1.84, 7.86)$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$

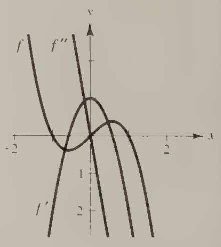
49.



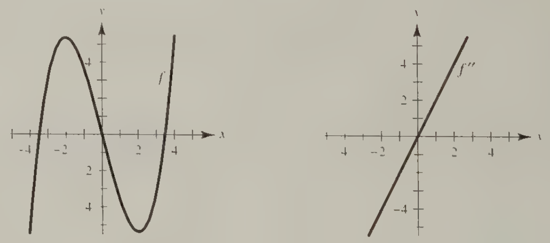
Point of inflection: $(0, 0)$

Horizontal asymptotes: $y = \pm 1$

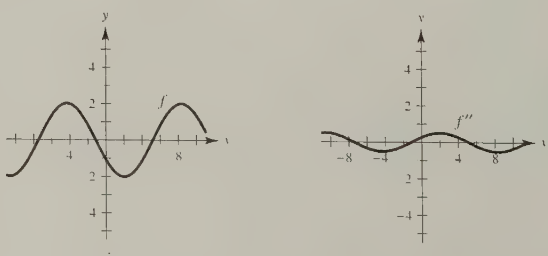
51.



53.

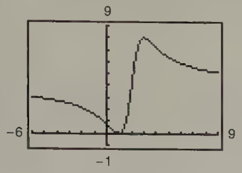


55.



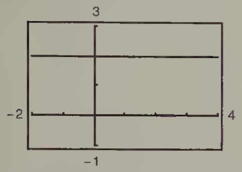
57. f is decreasing on $(2, 8)$ and therefore $f(3) > f(5)$.

59.



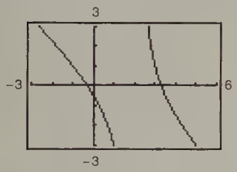
The graph crosses the horizontal asymptote $y = 4$. The graph of f does not cross its vertical asymptote $x = c$ because $f(c)$ does not exist.

61.



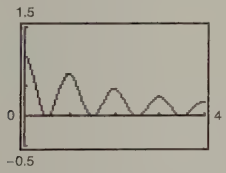
The graph has a hole at $x = 3$. The rational function is not reduced to lowest terms.

63.



The graph appears to approach the line $y = -x + 1$, which is the slant asymptote.

65. (a)



The graph has a hole at $x = 0$ and at $x = 4$.

Visual approximate critical numbers: $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}$

(b) $f'(x) = \dfrac{-\cos^2(\pi x)}{(x^2 + 1)^{3/2}} - \dfrac{2\pi \sin(\pi x) \cos(\pi x)}{\sqrt{x^2 + 1}}$

Approximate critical numbers: $\frac{1}{2}, 0.97, \frac{3}{2}, 1.98, \frac{5}{2}, 2.98, \frac{7}{2}$

The critical numbers where maxima occur appear to be integers in part (a), but approximating them using f' you see that they are not integers.

67. Answers will vary. Example: $y = \dfrac{1}{x - 5}$

69. Answers will vary. Example: $y = \dfrac{3x^2 - 13x - 9}{x - 5}$

71. (a) Rate of change of f changes as a varies. If the sign of a is changed, the graph is reflected through the x-axis.

(b) The locations of the vertical asymptote and the minimum (if $a > 0$) or maximum (if $a < 0$) are changed.

73. (a) If n is even, f is symmetric with respect to the y-axis.

If n is odd, f is symmetric with respect to the origin.

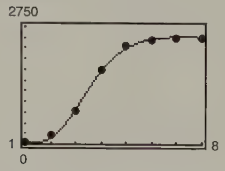
(b) $n = 0, 1, 2, 3$ (c) $n = 4$

(d) When $n = 5$, the slant asymptote is $y = 3x$.

(e)

n	0	1	2	3	4	5
M	1	2	3	2	1	0
N	2	3	4	5	2	3

75. (a)



(b) 2434

(c) The number of bacteria reached its maximum early on the seventh day.

(d) The rate of increase in the number of bacteria was greatest approximately in the middle of the third day.

(e) $\dfrac{13{,}250}{7}$

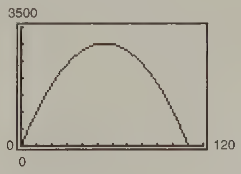
Section 3.7 (page 216)

1. (a) and (b)

First Number, x	*Second Number*	*Product P*
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$
70	$110 - 70$	$70(110 - 70) = 2800$
80	$110 - 80$	$80(110 - 80) = 2400$
90	$110 - 90$	$90(110 - 90) = 1800$
100	$110 - 100$	$100(110 - 100) = 1000$

(c) $P = x(110 - x)$

(d)



(e) 55 and 55

3. $\sqrt{192}$ and $\sqrt{192}$ **5.** 1 and 1 **7.** $l = w = 25$ meters

9. $l = w = 8$ feet **11.** $\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$ **13.** $(1, 1)$

15. $x = \dfrac{Q_0}{2}$ **17.** 600 × 300 meters

19. (a) Proof

(b) $V_1 = 99$ cubic inches

$V_2 = 125$ cubic inches

$V_3 = 117$ cubic inches

(c) 5 × 5 × 5 inches

21. (a) $V = x(s - 2x)^2, 0 < x < \frac{s}{2}$

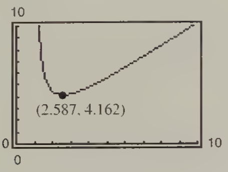
Maximum: $V\left(\frac{s}{6}\right) = \frac{2s^3}{27}$

(b) Increased by a factor of 8

23. Rectangular portion: $\frac{16}{\pi + 4} \times \frac{32}{\pi + 4}$ feet

25. (a) $L = \sqrt{x^2 + 4 + \frac{8}{x-1} + \frac{4}{(x-1)^2}}, \quad x > 1$

(b)



Minimum when $x \approx 2.587$

(c) (0, 0), (2, 0), (0, 4)

27. Width: $\frac{5\sqrt{2}}{2}$; Length: $5\sqrt{2}$

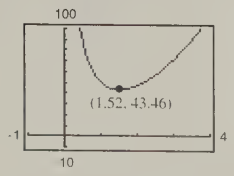
29. Dimensions of page: $(2 + \sqrt{30})$ inches × $(2 + \sqrt{30})$ inches

31. (a) and (b)

Radius, r	*Height*	*Surface Area, S*
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \frac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \frac{22}{\pi(0.8)^2}\right] \approx 59.0$
1.0	$\frac{22}{\pi(1.0)^2}$	$2\pi(1.0)\left[1.0 + \frac{22}{\pi(1.0)^2}\right] \approx 50.3$
1.2	$\frac{22}{\pi(1.2)^2}$	$2\pi(1.2)\left[1.2 + \frac{22}{\pi(1.2)^2}\right] \approx 45.7$
1.4	$\frac{22}{\pi(1.4)^2}$	$2\pi(1.4)\left[1.4 + \frac{22}{\pi(1.4)^2}\right] \approx 43.7$
1.6	$\frac{22}{\pi(1.6)^2}$	$2\pi(1.6)\left[1.6 + \frac{22}{\pi(1.6)^2}\right] \approx 43.6$
1.8	$\frac{22}{\pi(1.8)^2}$	$2\pi(1.8)\left[1.8 + \frac{22}{\pi(1.8)^2}\right] \approx 44.8$
2.0	$\frac{22}{\pi(2.0)^2}$	$2\pi(2.0)\left[2.0 + \frac{22}{\pi(2.0)^2}\right] \approx 47.1$

(c) $S = 2\pi r\left(r + \frac{22}{\pi r^2}\right)$

(d)



(e) $r = \sqrt[3]{\frac{11}{\pi}}, h = 2r$

Minimum area of 43.46 square inches when $r = 1.52$

33. 18 × 18 × 36 inches

35. $\frac{32\pi r^3}{81}$

37. Answers will vary. If area is expressed as a function of eithe length or width, the feasible domain is the interval (0, 10). N dimensions will yield a minimum area because the second derivative on this open interval is always negative.

39. $r = \sqrt[3]{\frac{9}{\pi}} \approx 1.42$ cm

41. Side of square: $\frac{10\sqrt{3}}{9 + 4\sqrt{3}}$

Side of triangle: $\frac{30}{9 + 4\sqrt{3}}$

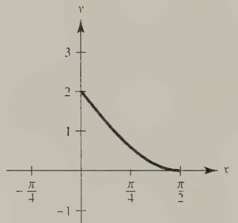
43. $w = 8\sqrt{3}$ inches, $h = 8\sqrt{6}$ inches **45.** $\theta = \frac{\pi}{4}$

47. $h = \sqrt{2}$ feet

49. One mile from the nearest point on the coast

51. Proof

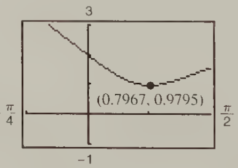
53.



(a) Origin to y-intercept: 2

Origin to x-intercept: $\frac{\pi}{2}$

(b) $d = \sqrt{x^2 + (2 - 2\sin x)^2}$



(c) Minimum distance is 0.9795 when $x \approx 0.7967$.

55. $F = \frac{kW}{\sqrt{k^2 + 1}}$; $\theta = \arctan k$

57. (a)

Base 1	Base 2	Altitude	Area
8	8 + 16 cos 10°	8 sin 10°	≈22.1
8	8 + 16 cos 20°	8 sin 20°	≈42.5
8	8 + 16 cos 30°	8 sin 30°	≈59.7
8	8 + 16 cos 40°	8 sin 40°	≈72.7
8	8 + 16 cos 50°	8 sin 50°	≈80.5
8	8 + 16 cos 60°	8 sin 60°	≈83.1

(b)

Base 1	Base 2	Altitude	Area
8	8 + 16 cos 10°	8 sin 10°	≈22.1
8	8 + 16 cos 20°	8 sin 20°	≈42.5
8	8 + 16 cos 30°	8 sin 30°	≈59.7
8	8 + 16 cos 40°	8 sin 40°	≈72.7
8	8 + 16 cos 50°	8 sin 50°	≈80.5
8	8 + 16 cos 60°	8 sin 60°	≈83.1
8	8 + 16 cos 70°	8 sin 70°	≈80.7
8	8 + 16 cos 80°	8 sin 80°	≈74.0
8	8 + 16 cos 90°	8 sin 90°	≈64.0

The maximum cross-sectional area is approximately 83.1 square feet.

(c) $A = (a + b)\frac{h}{2}$

$= [8 + (8 + 16\cos\theta)]\frac{8\sin\theta}{2}$

$= 64(1 + \cos\theta)\sin\theta,\ 0° < \theta < 90°$

(d) $\frac{dA}{d\theta} = 64(1 + \cos\theta)\cos\theta + (-64\sin\theta)\sin\theta$

$= 64(\cos\theta + \cos^2\theta - \sin^2\theta)$

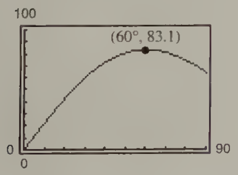
$= 64(2\cos^2\theta + \cos\theta - 1)$

$= 64(2\cos\theta - 1)(\cos\theta + 1)$

$= 0$ when $\theta = 60°, 180°, 300°$

The maximum occurs when 0 = 60°.

(e)



9. 4045 units

1. $y = \frac{64}{141}x;\ S_1 = 6.1$ miles

3. $y = \frac{3}{10}x;\ S_3 = 4.50$ miles

ection 3.8 (page 226)

1.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.7000	−0.1100	3.4000	−0.0324	1.7324
2	1.7324	0.0012	3.4648	0.0003	1.7321

3.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3	0.1411	−0.9900	−0.1425	3.1425
2	3.1425	−0.0009	−1.0000	0.0009	3.1416

5. 0.682 **7.** 1.146, 7.854 **9.** −1.442

11. 0.900, 1.100, 1.900 **13.** −0.489 **15.** 0.569

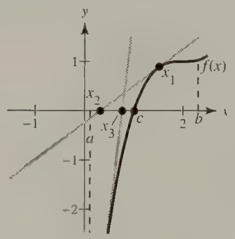
17. 4.493 **19.** $x_{i+1} = \frac{x_i^2 + a}{2x_i}$ **21.** 2.646

23. 1.565 **25.** 3.141 **27.** $f'(x_1) = 0$

29. $2 = x_1 = x_3 = \ldots$

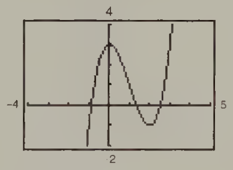
$1 = x_2 = x_4 = \ldots$

31. If f is a function continuous on $[a, b]$ and differentiable on (a, b), where $c \in [a, b]$ and $f(c) = 0$, Newton's Method uses tangent lines to approximate c. First, estimate an initial x_1 close to c. (See graph.) Then determine x_2 by $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. Calculate a third estimate by $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$. Continue this process until $|x_n - x_{n+1}|$ is within the desired accuracy and let x_{n+1} be the final approximation of c.



33. 0.74

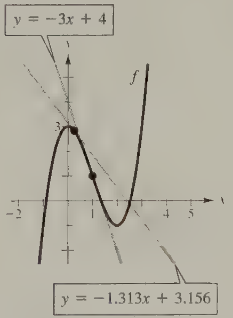
35. (a)



(b) 1.347

(c) 2.532

(d)

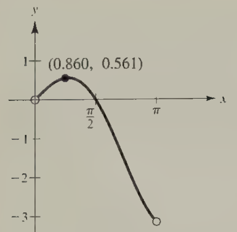


x-intercept of $y = -3x + 4$ is $\frac{4}{3}$.
x-intercept of $y = -1.313x + 3.156$ is approximately 2.404.

(e) If the initial estimate $x = x_1$ is not sufficiently close to the desired zero of a function, the x-intercept of the corresponding tangent line to the function may approximate a second zero of the function.

37. Proof

39. 0.860



41. (1.939, 0.240) **43.** $x \approx 1.563$ miles **45.** \$384,356

47. False: let $f(x) = \dfrac{x^2 - 1}{x - 1}$. **49.** True **51.** $x \approx 11.803$

Section 3.9 (page 233)

1. $T(x) = 4x - 4$

x	1.9	1.99	2	2.01	2.1
$f(x)$	3.610	3.960	4	4.040	4.410
$T(x)$	3.600	3.960	4	4.040	4.400

3. $T(x) = 80x - 128$

x	1.9	1.99	2	2.01	2.1
$f(x)$	24.761	31.208	32	32.808	40.841
$T(x)$	24.000	31.200	32	32.800	40.000

5. $T(x) = (\cos 2)(x - 2) + \sin 2$

x	1.9	1.99	2	2.01	2.1
$f(x)$	0.946	0.913	0.909	0.905	0.863
$T(x)$	0.951	0.913	0.909	0.905	0.868

7. $\Delta y = 0.6305$; $dy = 0.6000$ **9.** $\Delta y = -0.039$; $dy = -0.040$

11. $6x\,dx$ **13.** $-\dfrac{3}{(2x-1)^2}\,dx$ **15.** $\dfrac{1 - 2x^2}{\sqrt{1 - x^2}}\,dx$

17. $(2 + 2\cot x + 2\cot^3 x)\,dx$ **19.** $-\pi \sin\left(\dfrac{6\pi x - 1}{2}\right) dx$

21. (a) 0.9 (b) 1.04 **23.** (a) 1.05 (b) 0.98

25. (a) 8.035 (b) 7.95 **27.** (a) 8 (b) 8

29. $\pm\frac{3}{8}$ square inch **31.** $\pm 7\pi$ square inches

33. (a) $\frac{2}{3}\%$ (b) 1.25%

35. (a) $\pm 2.88\pi$ cubic inches (b) $\pm 0.96\pi$ square inches
(c) 1%, $\frac{2}{3}\%$

37. 80π cubic centimeters

39. (a) $\frac{1}{4}\%$ (b) 216 seconds = 3.6 minutes

41. (a) 0.87% (b) 2.16% **43.** 4961 feet

45. $f(x) = \sqrt{x}$, $dy = \dfrac{1}{2\sqrt{x}}\,dx$

$$f(99.4) \approx \sqrt{100} + \frac{1}{2\sqrt{100}}(-0.6) = 9.97$$

Calculator: 9.97

47. $f(x) = \sqrt[4]{x}$, $dy = \dfrac{1}{4x^{3/4}}\,dx$

$$f(624) \approx \sqrt[4]{625} + \frac{1}{4(625)^{3/4}}(-1) = 4.998$$

Calculator: 4.998

49. $f(x) = \sqrt{x}$; $dy = \dfrac{1}{2\sqrt{x}}\,dx$

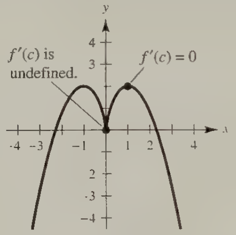
$$f(4.02) \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(0.02) = 2 + \frac{1}{4}(0.02)$$

51. The value of dy becomes closer to the value of Δy as Δ decreases.

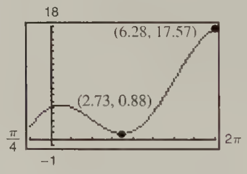
53. True **55.** True

Review Exercises for Chapter 3 (page 235)

1. Let f be defined at c. If $f'(c) = 0$ or if f' is undefined at c, the c is a critical number of f.

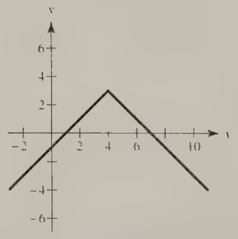


3. Maximum: $(2\pi, 17.57)$
Minimum: $(2.73, 0.88)$



5. $f'\left(\frac{1}{3}\right) = 0$

7. (a)



(b) f is not differentiable at $x = 4$.

9. $f'\left(\dfrac{2744}{729}\right) = \dfrac{3}{7}$ **11.** $f'(0) = 1$ **13.** $c = \dfrac{x_1 + x_2}{2}$

15. Critical numbers: $x = 1, \frac{7}{3}$
Increasing on $(-\infty, 1), \left(\frac{7}{3}, \infty\right)$
Decreasing on $\left(1, \frac{7}{3}\right)$

17. Critical number: $x = 1$
Increasing on $(1, \infty)$
Decreasing on $(0, 1)$

19. Minimum: $(2, -12)$

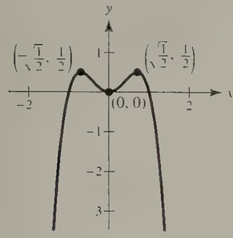
1. (a) $y = \frac{1}{4}$ inch; $v = 4$ inches per second

(b) Proof

(c) Period: $\frac{\pi}{6}$; Frequency: $\frac{6}{\pi}$

3. $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

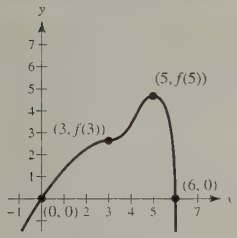
5.



Relative maxima: $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$

Relative minimum: (0, 0)

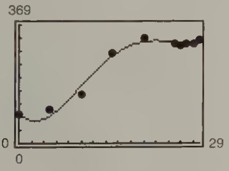
7.



9. Increasing and concave down

1. (a) $D = 0.00340t^4 - 0.2352t^3 + 4.942t^2 - 20.86t + 94.4$

(b)



(c) Maximum occurs in 1991; Minimum occurs in 1972.

(d) 1979

3. $\frac{2}{3}$ **35.** 0

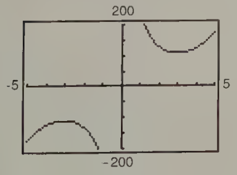
7. Vertical asymptote: $x = 4$

Horizontal asymptote: $y = 2$

9. Vertical asymptote: $x = 0$

Horizontal asymptote: $y = -2$

1.

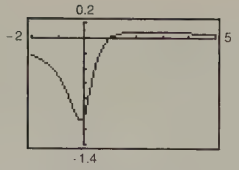


Vertical asymptote: $x = 0$

Relative minimum: (3, 108)

Relative maximum: $(-3, -108)$

43.

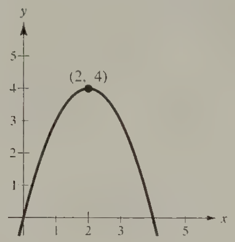


Horizontal asymptote: $y = 0$

Relative minimum: $(-0.155, -1.077)$

Relative maximum: (2.155, 0.077)

45.



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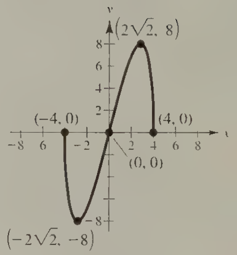

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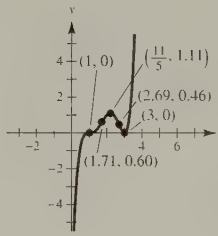

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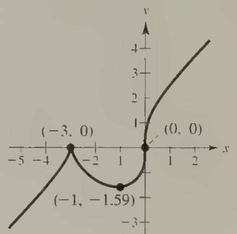

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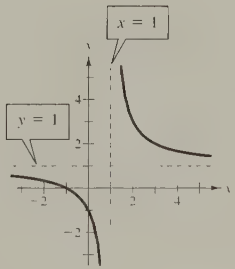

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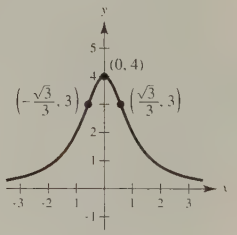

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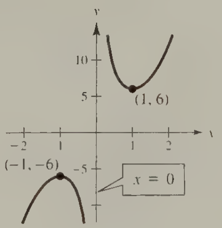

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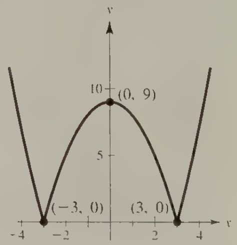
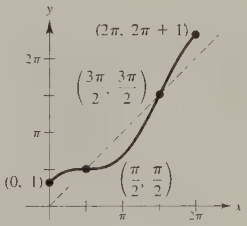


61.



63. Maximum: (1, 3)

Minimum: (1, 1)

65. $t \approx 4.92 \approx$ 4:55 P.M.; $d \approx 64$ kilometers

67. (0, 0), (5, 0), (0, 10) **69.** Proof **71.** 14.05 feet

73. $3(3^{2/3} + 2^{2/3})^{3/2} \approx 21.07$ feet

75. $v \approx 54.77$ miles per hour

77. $-1.532, -0.347, 1.879$ **79.** $-1.164, 1.453$

81. $dy = (1 - \cos x + x \sin x)\,dx$

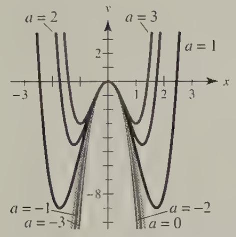
83. $dS = \pm 1.8\pi$ square centimeters, $\dfrac{dS}{S} \times 100 \approx \pm 0.56\%$

$dV = \pm 8.1\pi$ cubic centimeters, $\dfrac{dV}{V} \times 100 \approx \pm 0.83\%$

P.S. Problem Solving (page 238)

1. Proof

3. (a)



Relative maximum for all a at $(0, 0)$

Two relative minima for $a = 1, 2, 3$

(b) $p = ax^4 - 6x^2$

$p' = 4ax^3 - 12x$ has critical points at $x = 0$ and $x = \pm\sqrt{3/a},\ a > 0$

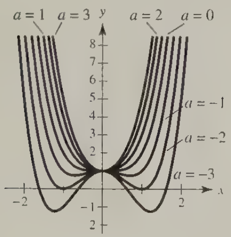
$p'' = 12ax^2 - 12,\ p''(0) = -12$

Therefore, by the Second Derivative Test, p has a relative maximum for all a at $x = 0$.

(c) $p''\left(\pm\sqrt{3/a}\right) = 24$. Therefore, by the Second Derivative Test, p has a relative minimum when $x = \pm\sqrt{3/a},\ a > 0$.

(d) Relative extrema of p occur at $x = 0,\ \pm\sqrt{3/a},\ a > 0$. If $x = 0$, $p(x) = 0$ and $(0, 0)$ also lies on the graph of $y = -3x^2$. If $x = \pm\sqrt{3/a}$, $p(x) = -9/a$ and $\left(\pm\sqrt{3/a}, -9/a\right)$ also lies on the graph of $y = -3x^2$.

5. Choices of a may vary.



(a) One relative minimum at $(0, 1)$ for $a \ge 0$

(b) One relative maximum at $(0, 1)$ for $a < 0$

(c) Two relative minima for $a < 0$ when $x = \pm\sqrt{-\dfrac{a}{2}}$

(d) If $a < 0$, there are three critical points; if $a \ge 0$, there is only one critical point.

7. All c where c is a real number

9. Proof **11.** $\phi \approx 42.1°$ or 0.736 radians

13. $\theta = \dfrac{\pi}{2} + 2n\pi$ and $\theta = \dfrac{3\pi}{2} + 2n\pi$, where n is an integer.

15. Rectangle: $\frac{3}{2} \times 2$

Circle: $r = 1$

Semicircle: $r = \frac{12}{7}$

Calculus was helpful for the rectangle.

17. Greatest slope at $\left(-\dfrac{\sqrt{3}}{3}, \dfrac{3}{4}\right)$; Least slope at $\left(\dfrac{\sqrt{3}}{3}, \dfrac{3}{4}\right)$

19. (a) Proof (b) Proof

Chapter 4

Section 4.1 (page 249)

1. Proof **3.** Proof **5.** $y = t^3 + C$

7. $y = \frac{2}{5}x^{5/2} + C$

	Original Integral	*Rewrite*	*Integrate*	*Simplify*
9.	$\int \sqrt[3]{x}\,dx$	$\int x^{1/3}\,dx$	$\dfrac{x^{4/3}}{4/3} + C$	$\dfrac{3}{4}x^{4/3} + C$
11.	$\int \dfrac{1}{x\sqrt{x}}\,dx$	$\int x^{-3/2}\,dx$	$\dfrac{x^{-1/2}}{-1/2} + C$	$-\dfrac{2}{\sqrt{x}} + C$
13.	$\int \dfrac{1}{2x^3}\,dx$	$\dfrac{1}{2}\int x^{-3}\,dx$	$\dfrac{1}{2}\left(\dfrac{x^{-2}}{-2}\right) + C$	$-\dfrac{1}{4x^2} + C$

15. $\frac{1}{2}x^2 + 3x + C$ **17.** $x^2 - x^3 + C$ **19.** $\frac{1}{4}x^4 + 2x + C$

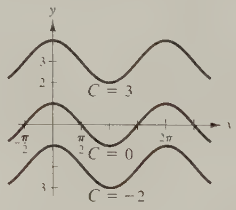
21. $\dfrac{2}{5}x^{5/2} + x^2 + x + C$ **23.** $\dfrac{3}{5}x^{5/3} + C$ **25.** $-\dfrac{1}{2x^2} + C$

27. $\frac{2}{15}x^{1/2}(3x^2 + 5x + 15) + C$ **29.** $x^3 + \frac{1}{2}x^2 - 2x + C$

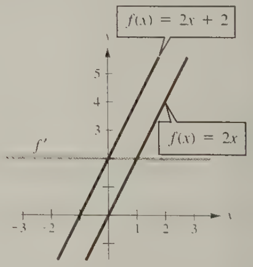
31. $\frac{2}{7}y^{7/2} + C$ **33.** $x + C$ **35.** $-2\cos x + 3\sin x + C$

37. $t + \csc t + C$ **39.** $\tan\theta + \cos\theta + C$ **41.** $\tan y + C$

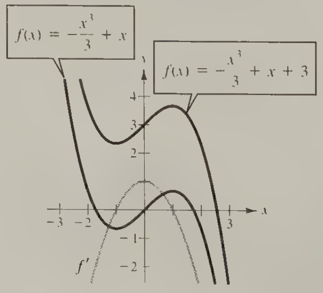
43.



45. Answers will vary. Example

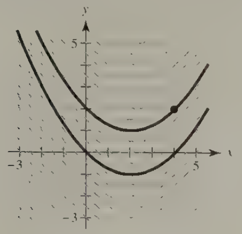


47. Answers will vary. Example:

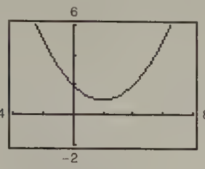


49. $y = x^2 - x + 1$ **51.** $y = \sin x + 4$

53. (a) Answers will vary.
Example:



(b) $y = \frac{1}{4}x^2 - x + 2$



55. $f(x) = 2x^2 + 6$ **57.** $h(t) = 2t^4 + 5t - 11$

59. $f(x) = x^2 + x + 4$ **61.** $f(x) = -4\sqrt{x} + 3x$

63. (a) $h(t) = \frac{3}{4}t^2 + 5t + 12$ (b) 69 centimeters

65. (a) -1; $f'(4)$ represents the slope of f at $x = 4$.

(b) No. The slope of the tangent lines are greater than 2 on $[0, 2]$. Therefore, f must increase more than four units on $[0, 2]$.

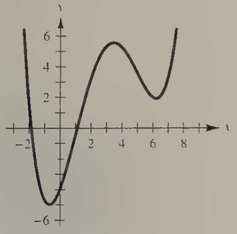
(c) No. The function is decreasing on $[4, 5]$.

(d) 3.5; $f'(3.5) \approx 0$

(e) Concave upward: $(-\infty, 1)$, $(5, \infty)$
Concave downward: $(1, 5)$
Points of inflection at $x \approx 1$ and $x \approx 5$

(f) 3

(g)



67. 62.25 feet **69.** $v_0 \approx 187.617$ feet per second

71. $v(t) = -9.8t + C_1 = -9.8t + v_0$
$f(t) = -4.9t^2 + v_0t + C_2 = -4.9t^2 + v_0t + s_0$

73. 7.1 meters

75. 320 meters; -32 meters per second

77. (a) $v(t) = 3t^2 - 12t + 9$; $a(t) = 6t - 12$
(b) $(0, 1)$, $(3, 5)$ (c) -3

79. $a(t) = \dfrac{-1}{2t^{3/2}}$; $s(t) = 2\sqrt{t} + 2$

81. (a) 1.18 meters per second per second (b) 190 meters

83. (a) 300 feet (b) 60 feet per second $\approx$ 41 miles per hour

85. (a)

t	0	5	10	15	20	25	30
v_1	0	3.67	10.27	23.47	42.53	66.00	95.33
v_2	0	30.80	55.73	74.80	88.00	93.87	95.33

(b) $v_1(t) = 0.1068t^2 - 0.042t + 0.37$
$v_2(t) = -0.1208t^2 + 6.799t - 0.07$

(c) Distance of car 1 $\approx$ 953 feet
Distance of car 2 $\approx$ 1970 feet
Car 2 traveled farther because it accelerated faster for about the first 15 seconds.

87. 7.45 feet per second per second

89. True **91.** True

93. False. Let $f(x) = x$ and $g(x) = x + 1$.

95. $f(x) = \begin{cases} x + 2, & 0 \le x < 2 \\ \frac{3}{2}x^2 - 2, & 2 \le x < 5 \end{cases}$

f is not differentiable at $x = 2$ because the left- and right-hand derivatives at $x = 2$ do not agree.

Section 4.2 (page 261)

1. 35 **3.** $\dfrac{158}{85}$ **5.** $4c$ **7.** $\displaystyle\sum_{i=1}^{9} \frac{1}{3i}$ **9.** $\displaystyle\sum_{j=1}^{8} \left[5\left(\frac{j}{8}\right) + 3\right]$

11. $\displaystyle\frac{2}{n}\sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right)\right]$ **13.** $\displaystyle\frac{3}{n}\sum_{i=1}^{n} \left[2\left(1 + \frac{3i}{n}\right)^2\right]$

15. 420 **17.** 2470 **19.** 12,040 **21.** 2930

23. The area of the shaded region falls between 12.5 square units and 16.5 square units.

25. The area of the shaded region falls between 7 square units and 11 square units.

27. $A \approx S \approx 0.768$
$A \approx s \approx 0.518$

29. $A \approx S \approx 0.746$
$A \approx s \approx 0.646$

31. $\frac{81}{4}$ **33.** 9

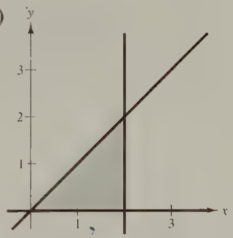
35. $\dfrac{n + 2}{n}$
$n = 10$; $S = 1.2$
$n = 100$; $S = 1.02$
$n = 1000$; $S = 1.002$
$n = 10{,}000$; $S = 1.0002$

37. $\dfrac{2(n + 1)(n - 1)}{n^2}$
$n = 10$; $S = 1.98$
$n = 100$; $S = 1.9998$
$n = 1000$; $S = 1.999998$
$n = 10{,}000$; $S = 1.99999998$

39. $\displaystyle\lim_{n\to\infty} \left[8\left(\frac{n^2 + n}{n^2}\right)\right] = 8$ **41.** $\displaystyle\lim_{n\to\infty} \frac{1}{6}\left(\frac{2n^3 - 3n^2 + n}{n^3}\right) = \frac{1}{3}$

43. $\displaystyle\lim_{n\to\infty} \left(\frac{3n + 1}{n}\right) = 3$

45. (a)



(b) $\Delta x = \dfrac{2 - 0}{n} = \dfrac{2}{n}$

(c) $\displaystyle s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x = \sum_{i=1}^{n} \left[(i - 1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$

(d) $\displaystyle S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$

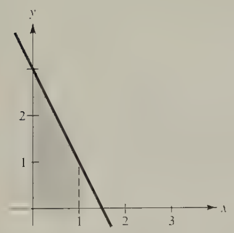
(e)

n	5	10	50	100
$s(n)$	1.6	1.8	1.96	1.98
$S(n)$	2.4	2.2	2.04	2.02

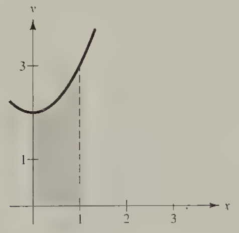
(f) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left[(i - 1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = 2$

$\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = 2$

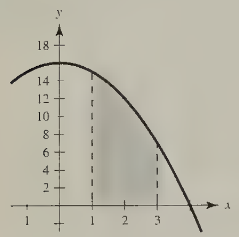
47. $A = 2$



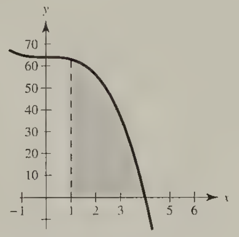
49. $A = \frac{7}{3}$



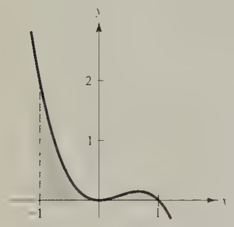
51. $A = \frac{70}{3}$



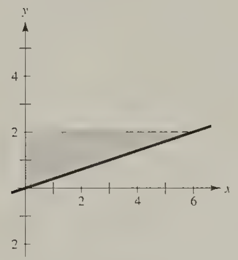
53. $A = \frac{513}{4}$



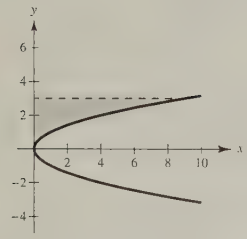
55. $A = \frac{2}{3}$



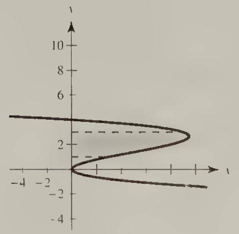
57. $A = 6$



59. $A = 9$



61. $A = \frac{44}{3}$



63. $\frac{69}{8}$ **65.** 0.345

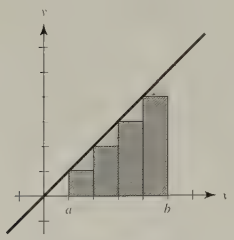
67.

n	4	8	12	16	20
Approximate area	5.3838	5.3523	5.3439	5.3403	5.3384

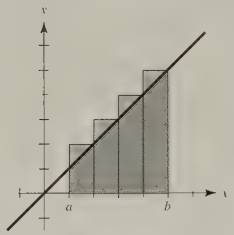
69.

n	4	8	12	16	20
Approximate area	2.2223	2.2387	2.2418	2.2430	2.2435

71. We can use the line $y = x$ bounded by $x = a$ and $x = b$. The sum of the areas of the inscribed rectangles in the figure below is the lower sum.

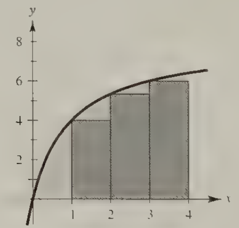


The sum of the areas of the circumscribed rectangles in the figure below is the upper sum.



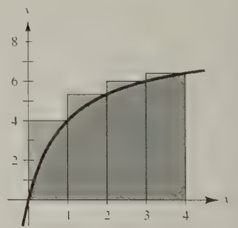
The rectangles in the first graph do not contain all of the area of the region, and the rectangles in the second graph cover more than the area of the region. The exact value of the area lies between these two sums.

73. (a)



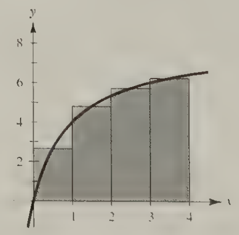
$s(4) = \frac{46}{3}$

(b)



$S(4) = \frac{326}{15}$

(c)



$M(4) = \frac{6112}{315}$

(d) Proof

(e)

n	4	8	20	100	200
$s(n)$	15.333	17.368	18.459	18.995	19.060
$S(n)$	21.733	20.568	19.739	19.251	19.188
$M(n)$	19.403	19.201	19.137	19.125	19.125

(f) f is an increasing function.

75. b **77.** True **79.** Answers will vary.

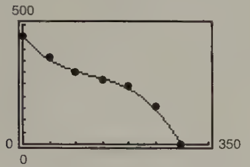
81. Suppose there are n rows in the figure. The stars on the left total $1 + 2 + \cdots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total. So,

$2[1 + 2 + \cdots + n] = n(n + 1)$

$1 + 2 + \cdots + n = \dfrac{n(n + 1)}{2}$.

83. (a) $y = (-4.09 \times 10^{-5})x^3 + 0.016x^2 - 2.67x + 452.9$

(b)



(c) 76,897 square feet

Section 4.3 (page 272)

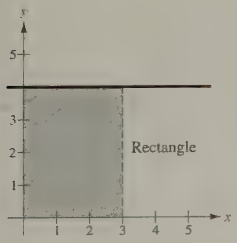
1. $2\sqrt{3} \approx 3.464$ **3.** 36 **5.** 0 **7.** $\frac{10}{3}$

9. $\displaystyle\int_{-1}^{5} (3x + 10)\,dx$ **11.** $\displaystyle\int_{0}^{3} \sqrt{x^2 + 4}\,dx$

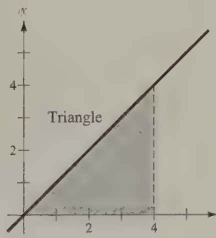
13. $\displaystyle\int_{0}^{5} 3\,dx$ **15.** $\displaystyle\int_{-4}^{4} (4 - |x|)\,dx$ **17.** $\displaystyle\int_{-2}^{2} (4 - x^2)\,dx$

19. $\displaystyle\int_{0}^{\pi} \sin x\,dx$ **21.** $\displaystyle\int_{0}^{2} y^3\,dy$

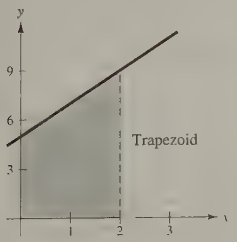
23. $A = 12$



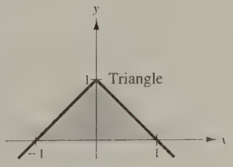
25. $A = 8$



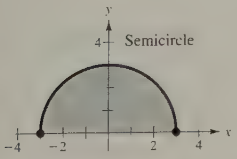
27. $A = 14$



29. $A = 1$



31. $A = \dfrac{9\pi}{2}$



33. −6 **35.** 24 **37.** −10 **39.** 16

41. (a) 13 (b) −10 (c) 0 (d) 30

43. (a) 8 (b) −12 (c) −4 (d) 30

45. (a) $-\pi$ (b) 4 (c) $-(1 + 2\pi)$ (d) $3 - 2\pi$ (e) $5 + 2\pi$ (f) $23 - 2\pi$

47. $\displaystyle\sum_{i=1}^{n} f(x_i)\,\Delta x > \int_{1}^{5} f(x)\,dx$ **49.** $\displaystyle\sum_{i=1}^{n} f(x_i)\,\Delta x < \int_{1}^{5} f(x)\,dx$

51. No. There is a discontinuity at $x = 4$.

53. a **55.** d

57.

n	4	8	12	16	20
$L(n)$	3.6830	3.9956	4.0707	4.1016	4.1177
$M(n)$	4.3082	4.2076	4.1838	4.1740	4.1690
$R(n)$	3.6830	3.9956	4.0707	4.1016	4.1177

59.

n	4	8	12	16	20
$L(n)$	0.5890	0.6872	0.7199	0.7363	0.7461
$M(n)$	0.7854	0.7854	0.7854	0.7854	0.7854
$R(n)$	0.9817	0.8836	0.8508	0.8345	0.8247

61. True **63.** True

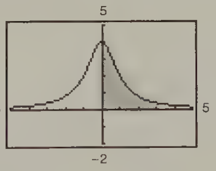
65. False: $\displaystyle\int_{0}^{2} (-x)\,dx = -2$ **67.** 272

69. No. No matter how small the subintervals, the number of both rational and irrational numbers within each subinterval is infinite and $f(c_i) = 0$ or $f(c_i) = 1$.

71. $\frac{1}{3}$

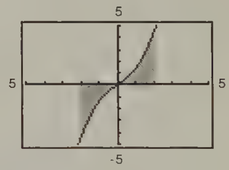
Section 4.4 (page 284)

1.



Positive

3.



Zero

5. 1 **7.** $-\frac{5}{2}$ **9.** $-\frac{10}{3}$ **11.** $\frac{1}{3}$ **13.** $\frac{1}{2}$ **15.** $\frac{2}{3}$

17. −4 **19.** $-\frac{1}{18}$ **21.** $-\frac{27}{20}$ **23.** $\frac{9}{2}$ **25.** $\frac{23}{3}$

27. $\pi + 2$ **29.** $\dfrac{2\sqrt{3}}{3}$ **31.** 0

33. $\displaystyle\int_{0}^{3} 10{,}000(t - 6)\,dt = -\$135{,}000$ **35.** $\frac{1}{6}$

37. $\dfrac{12\sqrt{3}}{5}$ **39.** 1 **41.** 10 **43.** 6

45. 0.4380, 1.7908 **47.** $\pm\arccos \dfrac{\sqrt{\pi}}{2} \approx \pm 0.4817$

49. Average value $= \dfrac{8}{3}$

$x = \pm\dfrac{2\sqrt{3}}{3} \approx \pm 1.155$

51. Average value $= \dfrac{2}{\pi}$

$x \approx 0.690, x \approx 2.451$

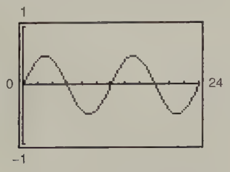
53. The Fundamental Theorem of Calculus states that if a function f is continuous on $[a, b]$ and F is an antiderivative of f on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

55. -1.5 **57.** 6.5 **59.** 15.5

61. (a) $F(x) = 500 \sec^2 x$ (b) 827 newtons

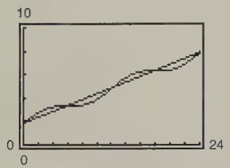
63. ≈ 0.5318 liter

65. (a)



The average value of $f(t)$ over the interval $0 \le t \le 24$ is represented by $\int_0^{24} 0.5 \sin\left(\frac{\pi t}{6}\right) = 0.$

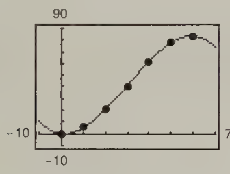
(b)



Even though the average value of $f(t) = 0$, the trend represented by g increases over the interval $0 \le t \le 24$ as does $S(t)$.

67. (a) $v = -0.00086t^3 + 0.0782t^2 - 0.208t + 0.10$

(b)



(c) 2475.6 meters

69. $F(x) = \frac{1}{2}x^2 - 5x$

$F(2) = -8$

$F(5) = -12\frac{1}{2}$

$F(8) = -8$

71. $F(x) = -\frac{10}{x} + 10$

$F(2) = 5$

$F(5) = 8$

$F(8) = 8\frac{3}{4}$

73. $F(x) = \sin x - \sin 1$

$F(2) = \sin 2 - \sin 1 \approx 0.0678$

$F(5) = \sin 5 - \sin 1 \approx -1.8004$

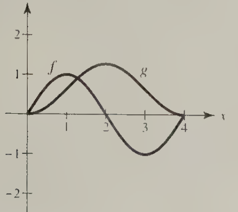
$F(8) = \sin 8 - \sin 1 \approx 0.1479$

75. $\frac{1}{2}x^2 + 2x$ **77.** $\frac{3}{4}x^{4/3} - 12$ **79.** $\tan x - 1$

81. $x^2 - 2x$ **83.** $\sqrt{x^4 + 1}$ **85.** $x \cos x$ **87.** 8

89. $\cos x\sqrt{\sin x}$ **91.** $3x^2 \sin x^6$

93.



An extremum of g occurs at $x = 2$.

95. (a) $C(x) = 1000(12x^{5/4} + 125)$

(b) $C(1) = \$137{,}000$

$C(5) = \$214{,}721$

$C(10) = \$338{,}394$

97. True

99. False: $f(x) = x^{-2}$ has a nonremovable discontinuity at $x = 0$.

101. $f'(x) = \frac{1}{(1/x)^2 + 1}\left(-\frac{1}{x^2}\right) + \frac{1}{x^2 + 1} = 0$

Since $f'(x) = 0$, $f(x)$ is constant.

103. 28 units **105.** 2 units

Section 4.5 (page 297)

	$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
1.	$\int (5x^2 + 1)^2(10x)\,dx$	$5x^2 + 1$	$10x\,dx$
3.	$\int \frac{x}{\sqrt{x^2 + 1}}\,dx$	$x^2 + 1$	$2x\,dx$
5.	$\int \tan^2 x \sec^2 x\,dx$	$\tan x$	$\sec^2 x\,dx$

7. $\frac{(1 + 2x)^5}{5} + C$ **9.** $\frac{2}{3}(9 - x^2)^{3/2} + C$

11. $\frac{(x^4 + 3)^3}{12} + C$ **13.** $\frac{(x^3 - 1)^5}{15} + C$

15. $\frac{(t^2 + 2)^{3/2}}{3} + C$ **17.** $-\frac{15}{8}(1 - x^2)^{4/3} + C$

19. $\frac{1}{4(1 - x^2)^2} + C$ **21.** $-\frac{1}{3(1 + x^3)} + C$

23. $-\sqrt{1 - x^2} + C$ **25.** $-\frac{1}{4}\left(1 + \frac{1}{t}\right)^4 + C$

27. $\sqrt{2x} + C$

29. $\frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C = \frac{2}{5}\sqrt{x}(x^2 + 5x + 35) + C$

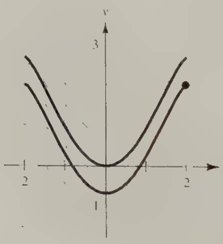
31. $\frac{1}{4}t^4 - t^2 + C$

33. $6y^{3/2} - \frac{2}{5}y^{5/2} + C = \frac{2}{5}y^{3/2}(15 - y) + C$

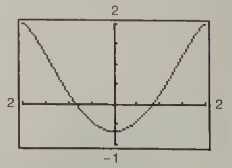
35. $2x^2 - 4\sqrt{16 - x^2} + C$ **37.** $-\frac{1}{2(x^2 + 2x - 3)} + C$

39. (a) Answers will vary.

Example:



(b) $y = -\frac{1}{3}(4 - x^2)^{3/2} + 2$



41. $-\cos(\pi x) + C$ **43.** $-\frac{1}{2}\cos 2x + C$ **45.** $-\sin\frac{1}{\theta} + C$

47. $\frac{1}{4}\sin^2 2x + C_1$ or $-\frac{1}{4}\cos^2 2x + C_2$ or $-\frac{1}{8}\cos 4x + C_3$

49. $\frac{1}{5}\tan^5 x + C$ **51.** $\frac{1}{2}\tan^2 x + C$ or $\frac{1}{2}\sec^2 x + C_1$

53. $-\cot x - x + C$ **55.** $f(x) = 2\sin\frac{x}{2} + 3$

57. $\frac{2}{15}(x + 2)^{3/2}(3x - 4) + C$

59. $-\frac{2}{105}(1 - x)^{3/2}(15x^2 + 12x + 8) + C$

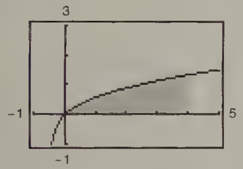
61. $\frac{\sqrt{2x - 1}}{15}(3x^2 + 2x - 13) + C$

63. $-x - 1 - 2\sqrt{x + 1} + C$ or $-\left(x + 2\sqrt{x + 1}\right) + C_1$

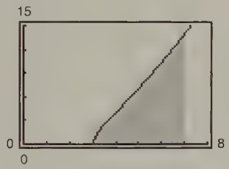
65. 0 **67.** $12 - \frac{8\sqrt{2}}{9}$ **69.** 2 **71.** $\frac{1}{2}$ **73.** $\frac{4}{15}$

75. $\frac{3\sqrt{3}}{4}$ **77.** $\frac{1209}{28}$ **79.** 4 **81.** $2(\sqrt{3} - 1)$

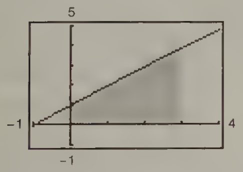
83. $\frac{10}{3}$



85. $\frac{144}{5}$



87. 7.38



89. $\frac{1}{6}(2x - 1)^3 + C_1 = \frac{4}{3}x^3 - 2x^2 + x - \frac{1}{6} + C_1$
or $\frac{4}{3}x^3 - 2x^2 + x + C_2$
Answers differ by a constant: $C_2 = C_1 - \frac{1}{6}$

91. $\frac{272}{15}$ **93.** 0

95. (a) $\frac{8}{3}$ (b) $\frac{16}{3}$ (c) $-\frac{8}{3}$ (d) 8

97. $2\int_0^4 (6x^2 - 3)\,dx = 232$

99. Answers will vary. See "Guidelines for Making a Change of Variables" on page 292.

101. It is an odd function.

103. $V(t) = \frac{200{,}000}{t + 1} + 300{,}000$
\$340,000

05. (a) 102.352 thousand units (b) 102.352 thousand units
(c) 74.5 thousand units

07. (a) 1.273 amperes (b) 1.382 amperes
(c) 0 amperes

09. False. $\int (2x + 1)^2\,dx = \frac{1}{6}(2x + 1)^3 + C$

11. True **113.** True **115.** Proof

Section 4.6 (page 305)

	Trapezoidal	*Simpson's*	*Exact*
1.	2.7500	2.6667	2.6667
3.	4.2500	4.0000	4.0000
5.	4.0625	4.0000	4.0000
7.	12.6640	12.6667	12.6667
9.	0.1676	0.1667	0.1667

	Trapezoidal	*Simpson's*	*Graphing utility*
11.	3.2833	3.2396	3.2413
13.	0.3415	0.3720	0.3927
15.	0.9567	0.9778	0.9775
17.	0.0891	0.0888	0.0891
19.	0.1940	0.1860	0.1858

21. The Trapezoidal Rule will yield a result greater than $\int_a^b f(x)\,dx$ if f is concave upward on $[a, b]$ because the graph of f will lie within the trapezoids.

23. (a) 0.500 (b) 0.000 **25.** (a) $n = 366$ (b) $n = 26$

27. (a) $n = 130$ (b) $n = 12$

29. (a) $n = 643$ (b) $n = 48$ **31.** Proof

33.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	12.7771	15.3965	18.4340	15.6055	15.4845
8	14.0868	15.4480	16.9152	15.5010	15.4662
10	14.3569	15.4544	16.6197	15.4883	15.4658
12	14.5386	15.4578	16.4242	15.4814	15.4657
16	14.7674	15.4613	16.1816	15.4745	15.4657
20	14.9056	15.4628	16.0370	15.4713	15.4657

35.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	2.8163	3.5456	3.7256	3.2709	3.3996
8	3.1809	3.5053	3.6356	3.4083	3.4541
10	3.2478	3.4990	3.6115	3.4296	3.4624
12	3.2909	3.4952	3.5940	3.4425	3.4674
16	3.3431	3.4910	3.5704	3.4568	3.4730
20	3.3734	3.4888	3.5552	3.4643	3.4759

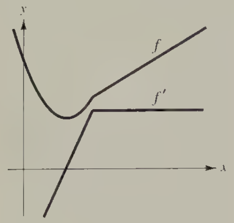
37. 0.701

39. 10,233.58 foot-pounds **41.** 3.1416

43. 89,250 square meters **45.** 2.477

Review Exercises for Chapter 4 (page 307)

1.



3. $\frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$

5. $\frac{1}{2}x^2 - \frac{1}{x} + C$ **7.** $2x^2 + 3\cos x + C$ **9.** $y = 2 - x^2$

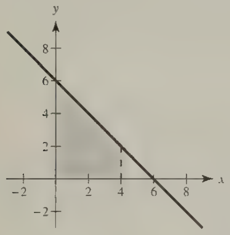
11. 240 feet per second

13. (a) 3 seconds (b) 144 feet (c) $\frac{3}{2}$ seconds (d) 108 feet

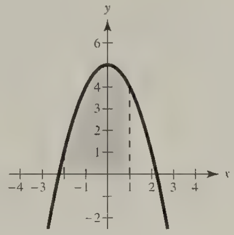
15. (a) $\sum_{i=1}^{10} (2i - 1)$ (b) $\sum_{i=1}^{n} i^3$ (c) $\sum_{i=1}^{10} (4i + 2)$

17. $9.038 < (\text{Area of Region}) < 13.038$

19. $A = 16$

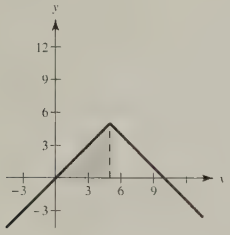


21. $A = 12$



23. $\frac{27}{2}$ **25.** $\int_4^6 (2x - 3)\,dx$

27. $A = \frac{25}{2}$

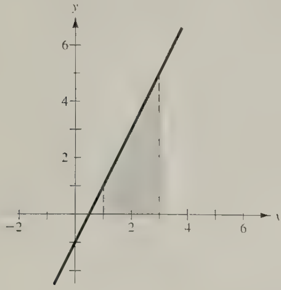


29. (a) 13 (b) 7 (c) 11 (d) 50

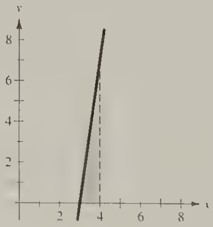
31. c **33.** 16 **35.** 0 **37.** $\frac{422}{5}$

39. $\frac{\sqrt{2} + 2}{2}$

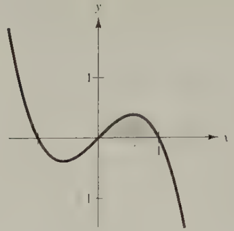
41. $A = 6$



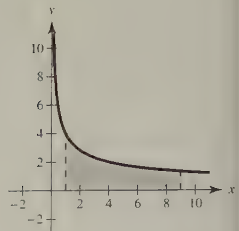
43. $A = \frac{10}{3}$



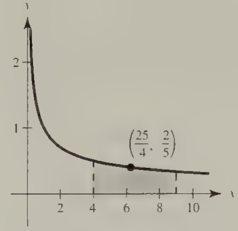
45. $A = \frac{1}{4}$



47. $A = 16$



49. Average value $= \frac{2}{5}$, $x = \frac{25}{4}$



51. $x^2\sqrt{1 + x^3}$ **53.** $x^2 + 3x + 2$

55. $\frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x + C$ **57.** $\frac{2}{3}\sqrt{x^3 + 3} + C$

59. $-\frac{1}{30}(1 - 3x^2)^5 + C = \frac{(3x^2 - 1)^5}{30} + C$

61. $\frac{1}{4}\sin^4 x + C$ **63.** $2\sqrt{1 - \cos\theta} + C$

65. $\frac{\tan^{n+1} x}{n + 1} + C,\quad n \neq -1$ **67.** $\frac{1}{3\pi}(1 + \sec \pi x)^3 + C$

69. $-\frac{9}{4}$ **71.** 2 **73.** $\frac{28\pi}{15}$ **75.** 2

77. (a) 0.353 (b) 58.6%

79. (a) $\frac{24{,}300}{M}$ (b) $\frac{27{,}300}{M}$

81. Trapezoidal Rule: 0.257
Simpson's Rule: 0.254
Graphing Utility: 0.254

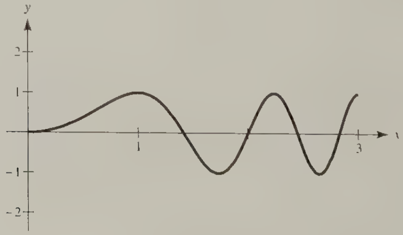
83. Trapezoidal Rule: 0.637
Simpson's Rule: 0.685
Graphing Utility: 0.704

85. (a) $R(n), I, T(n), L(n)$
(b) $S(4) = \frac{1}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)] \approx 5.42$

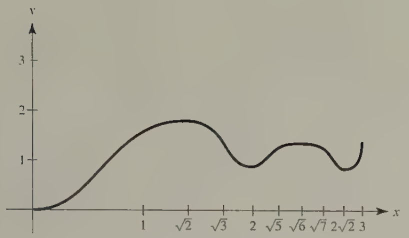
P.S. Problem Solving (page 310)

1. (a) $L(1) = 0$ (b) $L'(x) = \frac{1}{x}$, $L'(1) = 1$
(c) $x \approx 2.718$ (d) Proof

3. (a)



(b)

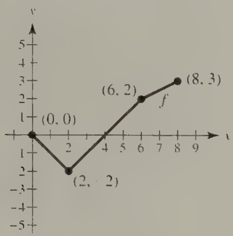


(c) Relative maxima at $x = \sqrt{2}, \sqrt{6}$

Relative minima at $x = 2, 2\sqrt{2}$

(d) Points of inflection at $x = 1, \sqrt{3}, \sqrt{5}, \sqrt{7}$

5. (a)



(b)

x	0	1	2	3	4	5	6	7	8
$F(x)$	0	$-\frac{1}{2}$	-2	$-\frac{7}{2}$	-4	$-\frac{7}{2}$	-2	$\frac{1}{4}$	3

(c) $x = 4, 8$ (d) $x = 2$

7. (a) 1.6758; Error of approximation ≈ 0.0071

(b) $\frac{3}{2}$ (c) Proof

9. Proof

1. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^5 \left(\frac{1}{n}\right) = \frac{1}{6}$ **13.** $1 \le \int_0^1 \sqrt{1 + x^4}\,dx \le \sqrt{2}$

5. Proof **17.** 100,000 pounds

Chapter 5

Section 5.1 (page 321)

1.

x	0.5	1.5	2	2.5	3
$\int_1^x (1/t)\,dt$	-0.6932	0.4055	0.6932	0.9163	1.0987

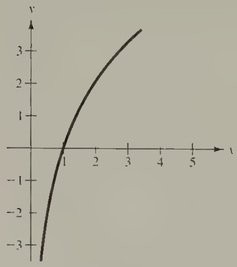
x	3.5	4
$\int_1^x (1/t)\,dt$	1.2529	1.3865

3. (a) 3.8067 (b) $\ln 45 = \int_1^{45} \frac{1}{t}\,dt \approx 3.8067$

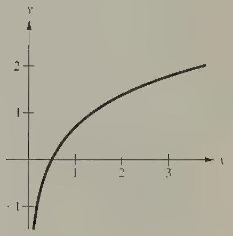
5. (a) -0.2231 (b) $\ln 0.8 = \int_1^{0.8} \frac{1}{t}\,dt \approx -0.2231$

7. b **8.** d **9.** a **10.** c

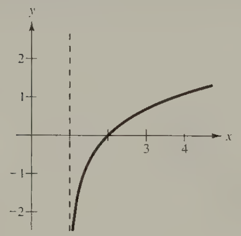
11. Domain: $x > 0$



13. Domain: $x > 0$



15. Domain: $x > 1$



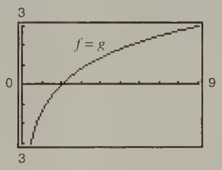
17. (a) 1.7917 (b) -0.4055 (c) 4.3944 (d) 0.5493

19. $\ln 2 - \ln 3$ **21.** $\ln x + \ln y - \ln z$ **23.** $\frac{1}{3}\ln(a^2 + 1)$

25. $3[\ln(x + 1) + \ln(x - 1) - 3\ln x]$ **27.** $\ln z + 2\ln(z - 1)$

29. $\ln \frac{x - 2}{x + 2}$ **31.** $\ln \sqrt[3]{\frac{x(x + 3)^2}{x^2 - 1}}$ **33.** $\ln \frac{9}{\sqrt{x^2 + 1}}$

35.



37. $-\infty$

39. $\ln 4$ **41.** 3 **43.** 2 **45.** $\frac{2}{x}$ **47.** $\frac{4(\ln x)^3}{x}$

49. $\frac{2x^2 - 1}{x(x^2 - 1)}$ **51.** $\frac{1 - x^2}{x(x^2 + 1)}$ **53.** $\frac{1 - 2\ln t}{t^3}$

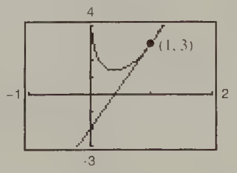
55. $\frac{2}{x\ln x^2} = \frac{1}{x\ln x}$ **57.** $\frac{1}{1 - x^2}$ **59.** $\frac{-4}{x(x^2 + 4)}$

61. $\frac{\sqrt{x^2 + 1}}{x^2}$ **63.** $\cot x$ **65.** $-\tan x + \frac{\sin x}{\cos x - 1}$

67. $\frac{3\cos x}{(\sin x - 1)(\sin x + 2)}$ **69.** $\frac{2}{x}(\sin 2x + x\cos 2x \ln x^2)$

71. (a) $5x - y - 2 = 0$

(b)



73. $\frac{2xy}{3 - 2y^2}$ **75.** $xy'' + y' = x\left(\frac{-2}{x^2}\right) + \frac{2}{x} = 0$

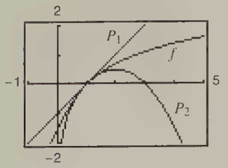
77. Relative minimum: $\left(1, \frac{1}{2}\right)$

79. Relative minimum: $(e^{-1}, -e^{-1})$

81. Relative minimum: (e, e)

Point of inflection: $\left(e^2, \dfrac{e^2}{2}\right)$

83. $P_1 = x - 1$; $P_2 = x - 1 - \frac{1}{2}(x - 1)^2$



The values of f, P_1, and P_2 and their first derivatives agree at $x = 1$.

85. $x \approx 0.567$ **87.** $\dfrac{2x^2 - 1}{\sqrt{x^2 - 1}}$

89. $\dfrac{3x^3 - 15x^2 + 8x}{2(x - 1)^3\sqrt{3x - 2}}$ **91.** $\dfrac{(2x^2 + 2x - 1)\sqrt{x - 1}}{(x + 1)^{3/2}}$

93. The domain of the natural logarithmic function is $(0, \infty)$ and the range is $(-\infty, \infty)$. The function is continuous, increasing, and one-to-one and its graph is concave downward. In addition, if a and b are positive numbers and n is rational, then $\ln(1) = 0$, $\ln(a \cdot b) = \ln a + \ln b$, $\ln(a^n) = n \ln a$, and $\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b$.

95. Using properties of logarithms, $\ln e^x$ can be rewritten as $x \ln e$. Then, since $\ln e = 1$ by the definition of e, $x \ln e = x(1) = x$.

97. (a) Rolle's Theorem does not apply because $f(1) \neq f(3)$.

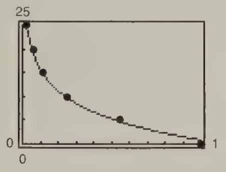
(b) Yes. $f'(2) = 0$ and $2 \in [1, 3]$.

99. $\beta = 160 + 10 \log_{10} I$; $\beta = 60$ decibels

101. (a) $h = 0$ is not in the domain of the function.

(b) $h = 0.86 - 6.447 \ln p$

(c)



(d) 2.7 kilometers

(e) 0.15 atmosphere

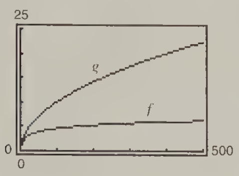
(f) $h = 5: \dfrac{dp}{dh} = -0.085$

$h = 20: \dfrac{dp}{dh} = -0.009$

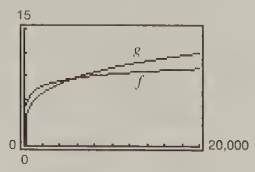
As the altitude increases, the pressure decreases at a slower rate.

103. For large values of x, g increases at a faster rate than f in both cases. The natural logarithmic function increases very slowly for large values of x.

(a)



(b)



105. False: $\ln x + \ln 25 = \ln 25x$.

Section 5.2 (page 330)

1. $5 \ln|x| + C$ **3.** $\ln|x + 1| + C$

5. $-\frac{1}{2}\ln|3 - 2x| + C$ **7.** $\ln\sqrt{x^2 + 1} + C$

9. $\dfrac{x^2}{2} - \ln(x^4) + C$ **11.** $\dfrac{1}{3}\ln|x^3 + 3x^2 + 9x| + C$

13. $\dfrac{x^2}{2} - 4x + 6\ln|x + 1| + C$ **15.** $\dfrac{x^3}{3} + 5\ln|x - 3| + C$

17. $\dfrac{x^3}{3} - 2x + \ln\sqrt{x^2 + 2} + C$ **19.** $\dfrac{1}{3}(\ln x)^3 + C$

21. $2\sqrt{x + 1} + C$ **23.** $2\ln|x - 1| - \dfrac{2}{x - 1} + C$

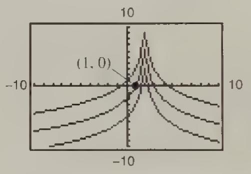
25. $\sqrt{2x} - \ln\left|1 + \sqrt{2x}\right| + C$

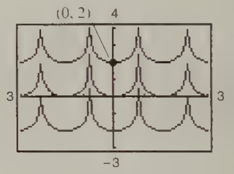
27. $x + 6\sqrt{x} + 18\ln\left|\sqrt{x} - 3\right| + C$ **29.** $\ln|\sin\theta| + C$

31. $-\frac{1}{2}\ln|\csc 2x + \cot 2x| + C$ **33.** $\ln|1 + \sin t| + C$

35. $\ln|\sec x - 1| + C$

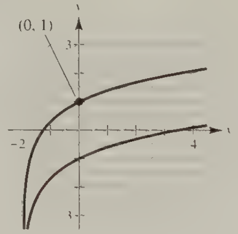
37. $y = -3\ln|2 - x| + C$ **39.** $y = -\frac{1}{2}\ln|\cos 2\theta| + C$



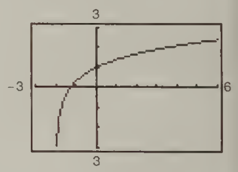


The graph has a hole at $x = 2$.

41. (a)



(b) $y = \ln\left|\dfrac{x + 2}{2}\right| + 1$



43. $\frac{5}{3}\ln 13 \approx 4.275$ **45.** $\frac{7}{3}$ **47.** $-\ln 3 \approx -1.099$

49. $\ln\left|\dfrac{2 - \sin 2}{1 - \sin 1}\right| \approx 1.929$

51. $-\ln|\cos x| + C = \ln\left|\dfrac{1}{\cos x}\right| + C = \ln|\sec x| + C$

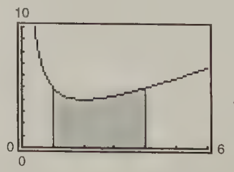
53. $\ln|\sec x + \tan x| + C = \ln\left|\dfrac{\sec^2 x - \tan^2 x}{\sec x - \tan x}\right| + C$

$= -\ln|\sec x - \tan x| + C$

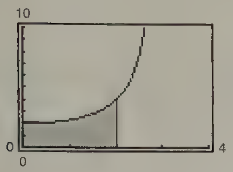
55. $2\left[\sqrt{x} - \ln\left(1 + \sqrt{x}\right)\right] + C$ **57.** $-\sin(1 - x) + C$

59. $\ln\left(\sqrt{2} + 1\right) - \dfrac{\sqrt{2}}{2} \approx 0.174$ **61.** $\dfrac{1}{x}$ **63.** 0 **65.** d

67. $\frac{15}{2} + 8\ln 2 \approx 13.045$



9. $\frac{12}{\pi}\left[2 \ln\left(\sqrt{3} + 1\right) - \ln 2\right] \approx 5.03$



1. Power Rule **73.** Log rule

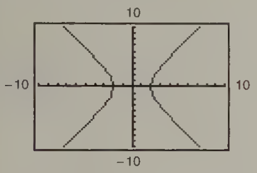
5. Use long division to rewrite the integrand.

7. 1 **79.** $\frac{1}{2(e - 1)} \approx 0.291$

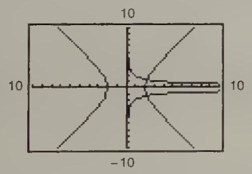
1. $P(t) = 1000(12 \ln|1 + 0.25t| + 1)$; $P(3) \approx 7715$

3. \$168.27

5. (a)



(b) Answers will vary. Example: $y^2 = e^{-\ln x + \ln 4} = \frac{4}{x}$



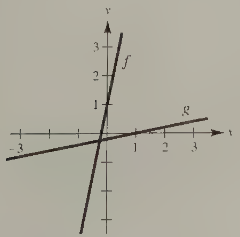
7. False. $\frac{1}{2}(\ln x) = \ln x^{1/2}$ **89.** True

ection 5.3 (page 338)

1. (a) $f(g(x)) = 5\left(\frac{x - 1}{5}\right) + 1 = x$

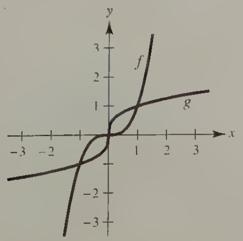
$g(f(x)) = \frac{(5x + 1) - 1}{5} = x$

(b)



3. (a) $f(g(x)) = \left(\sqrt[3]{x}\right)^3 = x$; $g(f(x)) = \sqrt[3]{x^3} = x$

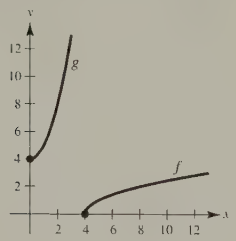
(b)



5. (a) $f(g(x)) = \sqrt{x^2 + 4 - 4} = x$;

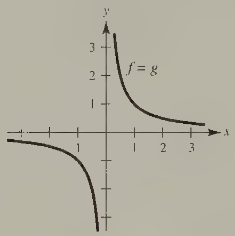
$g(f(x)) = \left(\sqrt{x - 4}\right)^2 + 4 = x$

(b)



7. (a) $f(g(x)) = \frac{1}{1/x} = x$; $g(f(x)) = \frac{1}{1/x} = x$

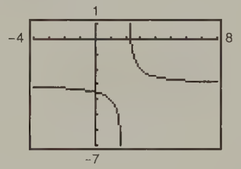
(b)



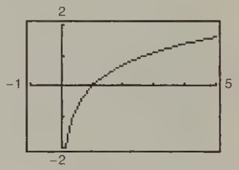
9. c **10.** b **11.** a **12.** d

13. Inverse exists. **15.** Inverse does not exist.

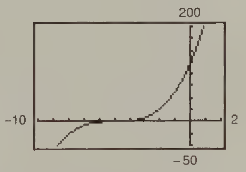
17. One-to-one



19. One-to-one



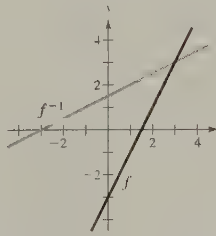
21. One-to-one



23. Inverse exists. **25.** Inverse does not exist.

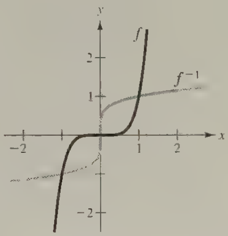
27. Inverse exists.

29. $f^{-1}(x) = \frac{x + 3}{2}$



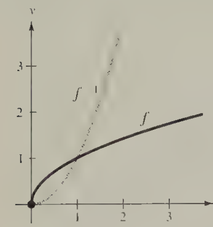
f and f^{-1} are symmetric about $y = x$.

31. $f^{-1}(x) = x^{1/5}$



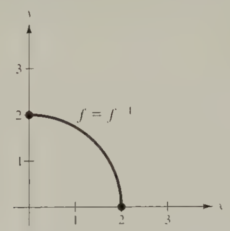
f and f^{-1} are symmetric about $y = x$.

33. $f^{-1}(x) = x^2,\ x \geq 0$



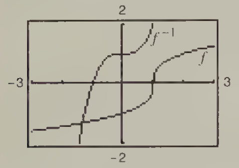
f and f^{-1} are symmetric about $y = x$.

35. $f^{-1}(x) = \sqrt{4 - x^2},\ 0 \leq x \leq 2$



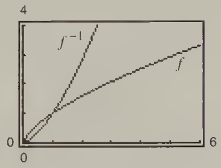
f and f^{-1} are symmetric about $y = x$.

37. $f^{-1}(x) = x^3 + 1$



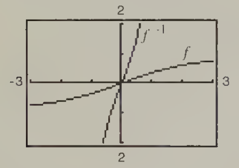
f and f^{-1} are symmetric about $y = x$.

39. $f^{-1}(x) = x^{3/2},\ x \geq 0$



f and f^{-1} are symmetric about $y = x$.

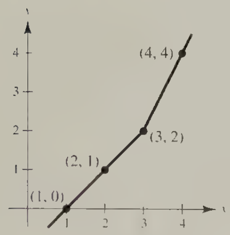
41. $f^{-1}(x) = \dfrac{\sqrt{7}x}{\sqrt{1 - x^2}},\ -1 < x < 1$



f and f^{-1} are symmetric about $y = x$.

43.

x	1	2	3	4
$f^{-1}(x)$	0	1	2	4



45. (a) Proof

(b) $y = \frac{20}{7}(80 - x)$

x: total cost

y: number of pounds of the less expensive commodity

(c) $[62.5, 80]$

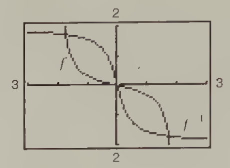
(d) 20 pounds

47. $f'(x) = 2(x - 4) > 0$ on $(4, \infty)$

49. $f'(x) = -\dfrac{8}{x^3} < 0$ on $(0, \infty)$

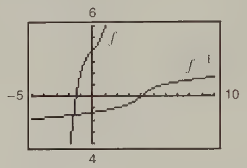
51. $f'(x) = -\sin x < 0$ on $(0, \pi)$

53. $f^{-1}(x) = \begin{cases} \dfrac{1 - \sqrt{1 + 16x^2}}{2x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$



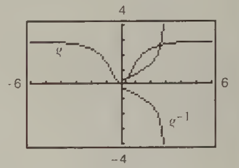
The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

55. (a) and (b)



(c) f is one-to-one and has an inverse function.

57. (a) and (b)



(c) g is not one-to-one and does not have an invers function.

59. One-to-one

$f^{-1}(x) = x^2 + 2,\ x \geq 0$

61. One-to-one

$f^{-1}(x) = 2 - x,\ x \geq 0$

63. $f^{-1}(x) = \sqrt{x} + 3,\ x \geq 0$

(Answer is not unique.)

65. $f^{-1}(x) = x - 3,\ x \geq 0$

(Answer is not unique.)

67. Inverse exists. Volume is an increasing function, therefor one-to-one. The inverse function gives the time t correspondin to the volume V.

69. Inverse does not exist. **71.** $\frac{1}{5}$ **73.** $\frac{2\sqrt{3}}{3}$ **75.** $\frac{1}{13}$

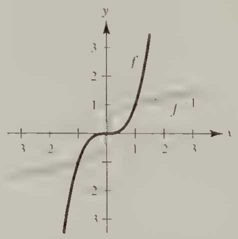
77. (a) Domain of f: $(-\infty, \infty)$

Domain of f^{-1}: $(-\infty, \infty)$

(b) Range of f: $(-\infty, \infty)$

Range of f^{-1}: $(-\infty, \infty$

(c)



(d) $f'\left(\frac{1}{2}\right) = \frac{3}{4},\ (f^{-1})'\left(\frac{1}{8}\right) = \frac{4}{3}$

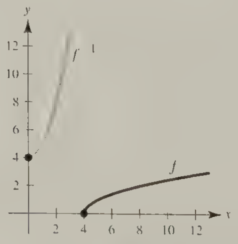
79. (a) Domain of f: $[4, \infty)$

Domain of f^{-1}: $[0, \infty)$

(b) Range of f: $[0, \infty)$

Range of f^{-1}: $[4, \infty)$

(c)



(d) $f'(5) = \frac{1}{2},\ (f^{-1})'(1) = 2$

81. $-\frac{1}{11}$ **83.** 32 **85.** 600 **87.** $(g^{-1} \circ f^{-1})(x) = \dfrac{x+1}{2}$

89. $(f \circ g)^{-1}(x) = \dfrac{x+1}{2}$

91. Let $y = f(x)$ be one-to-one. Solve for x as a function of y. Interchange x and y to get $y = f^{-1}(x)$. Let the domain of f^{-1} be the range of f. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Example: $f(x) = x^3$

$$\begin{aligned} y &= x^3 \\ x &= \sqrt[3]{y} \\ y &= \sqrt[3]{x} \\ f^{-1}(x) &= \sqrt[3]{x} \end{aligned}$$

93. Answers will vary. Example: $y = x^4 - 2x^3$

95. Many x-values yield the same y-value.

For example, $f(\pi) = 0 = f(0)$.

The graph is not continuous at $x = \dfrac{(2n-1)\pi}{2}$, where n is an integer.

97. Proof **99.** Proof

101. False. Let $f(x) = x^2$. **103.** True

105. No. Let $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1 - x, & 1x \le 2 \end{cases}$. **107.** $\sqrt{17}$

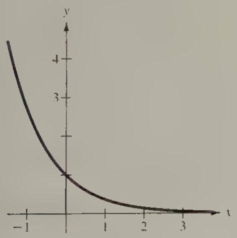
Section 5.4 (page 347)

1. $\ln 1 = 0$ **3.** $e^{0.6931\ldots} = 2$ **5.** $x = 4$

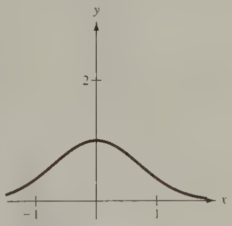
7. $x \approx 2.485$ **9.** $x = 0$ **11.** $x \approx 0.511$

13. $x \approx 7.389$ **15.** $x \approx 10.389$ **17.** $x \approx 5.389$

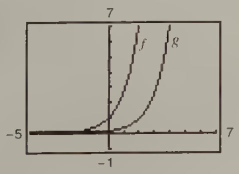
19.



21.

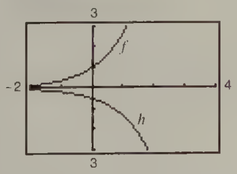


23. (a)



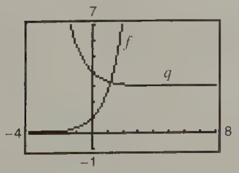
Translation 2 units to the right

(b)



Reflection in the x-axis and a vertical shrink

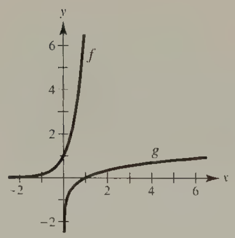
(c)



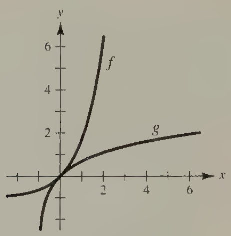
Reflection in the y-axis and a translation 3 units upward

25. c **26.** d **27.** a **28.** b

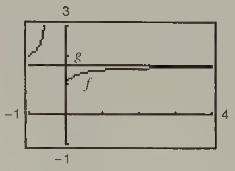
29.



31.



33.



$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = e^{0.5}$

35. $2.7182805 < e$ **37.** (a) 3 (b) -3

39. $2e^{2x}$ **41.** $2(x-1)e^{-2x+x^2}$ **43.** $\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

45. $3(e^{-t} + e^{t})^2(e^{t} - e^{-t})$ **47.** $2x$ **49.** $\dfrac{2e^{2x}}{1 + e^{2x}}$

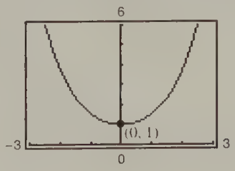
51. $\dfrac{-2(e^{x} - e^{-x})}{(e^{x} + e^{-x})^2}$ **53.** x^2e^{x} **55.** $e^{-x}\left(\dfrac{1}{x} - \ln x\right)$

57. $2e^{x}\cos x$ **59.** $\dfrac{10 - e^{y}}{xe^{y} + 3}$ **61.** $3(6x + 5)e^{-3x}$

63. $y'' - 2y' + 3y = 0$

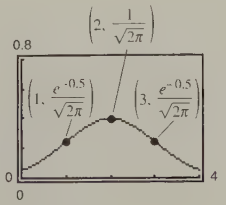
$$\begin{aligned} & e^{x}\left[-\cos\sqrt{2}x - \sin\sqrt{2}x - 2\sqrt{2}\sin\sqrt{2}x + 2\sqrt{2}\cos\sqrt{2}x\right] - \\ & 2e^{x}\left[-\sqrt{2}\sin\sqrt{2}x + \sqrt{2}\cos\sqrt{2}x + \cos\sqrt{2}x + \sin\sqrt{2}x\right] + \\ & 3e^{x}\left[\cos\sqrt{2}x + \sin\sqrt{2}x\right] \\ & = 0 \\ & 0 = 0 \end{aligned}$$

65. Relative minimum: $(0, 1)$



67. Relative maximum: $\left(2, \dfrac{1}{\sqrt{2\pi}}\right)$

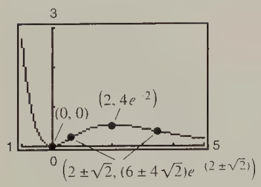
Points of inflection: $\left(1, \dfrac{e^{-0.5}}{\sqrt{2\pi}}\right), \left(3, \dfrac{e^{\ 0.5}}{\sqrt{2\pi}}\right)$



69. Relative minimum: $(0, 0)$

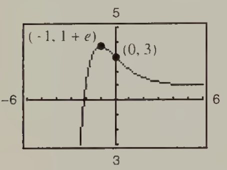
Relative maximum: $(2, 4e^{-2})$

Points of inflection: $\left(2 \pm \sqrt{2}, \left(6 \pm 4\sqrt{2}\right)e^{-(2 \pm \sqrt{2})}\right)$



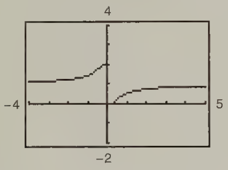
71. Relative maximum: $(-1, 1 + e)$

Point of inflection: $(0, 3)$



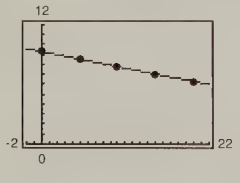
73. $A = \sqrt{2}e^{-1/2}$ **75.** Proof **77.** 0.567

79. (a)

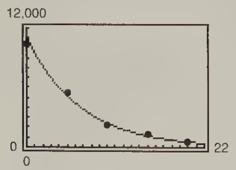


(b) When x increases without bound, $1/x$ approaches zero and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $\frac{2}{1+1} = 1$. Thus, $f(x)$ has a horizontal asymptote at $y = 1$. As x approaches zero from the right, $1/x$ approaches ∞, $e^{1/x}$ approaches ∞, and $f(x)$ approaches 0. As x approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches 0, and $f(x)$ approaches 2. The limit does not exist, because the limit from the left does not equal the limit from the right. Therefore, $x = 0$ is a nonremovable discontinuity.

81. (a) $\ln P = -0.1499h + 9.3018$ (b) $P = 10{,}957.7e^{-0.1499h}$



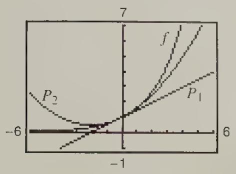
(c)



(d) $h = 5$: -776

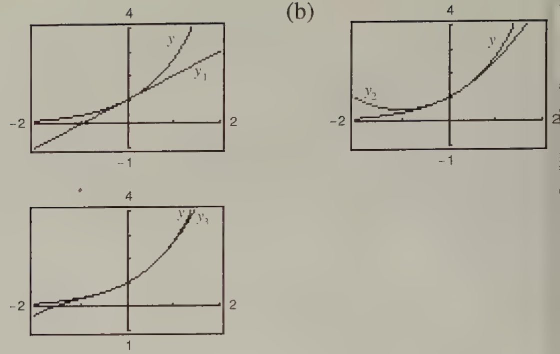
$h = 18$: -111

83. $P_1 = 1 + \frac{x}{2}$; $P_2 = 1 + \frac{x}{2} + \frac{x^2}{8}$



The values of f, P_1, and P_2 and their first derivatives agree at $x = 0$.

85. (a) (b) (c)



87. $e^{5x} + C$ **89.** $\frac{e^2 - 1}{2e^2}$

91. $-\frac{1}{2}e^{-x^2} + C$ **93.** $2e^{\sqrt{x}} + C$

95. $x - \ln(e^x + 1) + C_1$ or $-\ln(1 + e^{-x}) + C_2$

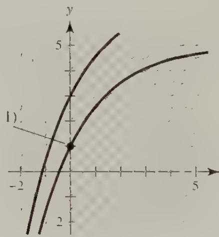
97. $\frac{e}{3}(e^2 - 1)$ **99.** $-\frac{2}{3}(1 - e^x)^{3/2} + C$

101. $\ln|e^x - e^{-x}| + C$ **103.** $-\frac{5}{2}e^{-2x} + e^{-x} + C$

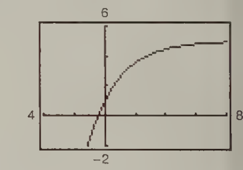
105. $\frac{1}{\pi}e^{\sin \pi x} + C$ **107.** $\ln|\cos e^{-x}| + C$

109. $\frac{1}{2a}e^{ax^2} + C$ **111.** $f(x) = \frac{1}{2}(e^x + e^{-x})$

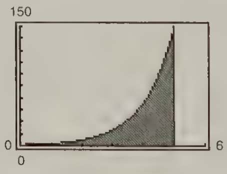
113. (a)



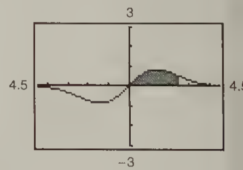
(b) $y = -4e^{-x/2} + 5$



115. $e^5 - 1 \approx 147.413$



117. $2(1 - e^{-3/2}) \approx 1.554$



119. (a)

$$\begin{aligned} f(x) &= e^x \\ f(u - v) &= e^{u-v} \\ &= \frac{e^u}{e^v} \\ &= \frac{f(u)}{f(v)} \end{aligned}$$

(b)

$$\begin{aligned} f(x) &= e^x \\ f(kx) &= e^{xk} \\ &= [f(x)]^k \end{aligned}$$

121. The probability that a given battery will last between months and 60 months is approximately 47.72%.

123. $\int_0^x e^t\,dt \geq \int_0^x 1\,dt$; $e^x - 1 \geq x$; $e^x > x + 1$ for $x \geq 0$

125. $f(x) = e^x$

The domain of $f(x)$ is $(-\infty, \infty)$ and the range of $f(x)$ is $(0, \infty)$. $f(x)$ is continuous, increasing, one-to-one, and concave upward on its entire domain.

$\lim\limits_{x \to -\infty} e^x = 0$ and $\lim\limits_{x \to \infty} e^x = \infty$

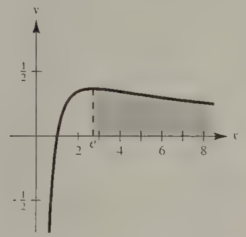
127. $f(x) = e^x = f'(x)$

129. $e^{-x} > 0$ implies $\int_0^2 e^{-x}\,dx > 0$

131. (a) $f'(x) = \dfrac{1 - \ln x}{x^2} = 0$ when $x = e$.

On $(0, e)$, $f'(x) > 0 \Rightarrow f$ is increasing.

On (e, ∞), $f'(x) < 0 \Rightarrow f$ is decreasing.



(b) For $e \le A < B$, we have:

$$\frac{\ln A}{A} > \frac{\ln B}{B}$$

$$B \ln A > A \ln B$$

$$\ln A^B > \ln B^A$$

$$A^B > B^A.$$

(c) Since $e < \pi$, from part (b) we have $e^\pi > \pi^e$.

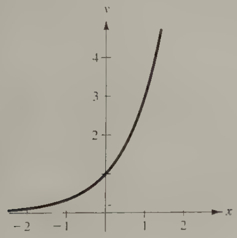
Section 5.5 (page 357)

1. $y(t) = \left(\frac{1}{2}\right)^{t/3}, \frac{1}{4}$ **3.** $y(t) = \left(\frac{1}{2}\right)^{t/7}, \left(\frac{1}{2}\right)^{10/7} \approx 0.371$

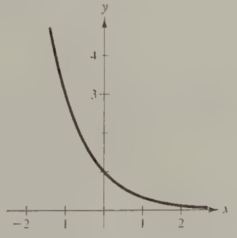
5. -3 **7.** 0 **9.** (a) $\log_2 8 = 3$ (b) $\log_3(1/3) = -1$

11. (a) $10^{-2} = 0.01$ (b) $\left(\frac{1}{2}\right)^{-3} = 8$

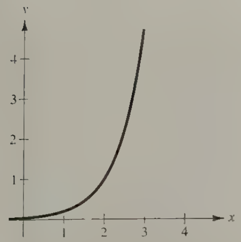
13.



15.



17.



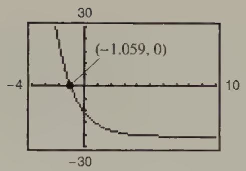
19. (a) $x = 3$ (b) $x = -1$

21. (a) $x = \frac{1}{3}$ (b) $x = \frac{1}{16}$ **23.** (a) $x = -1, 2$ (b) $x = \frac{1}{3}$

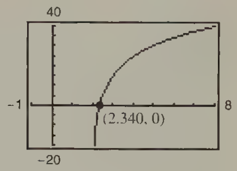
25. 1.965 **27.** -6.288 **29.** 12.253

31. 33.000 **33.** ± 11.845

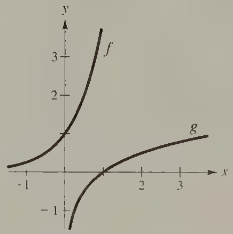
35.



37.



39.



41. $(\ln 4)4^x$ **43.** $(\ln 5)5^{x-2}$

45. $t2^t(t \ln 2 + 2)$ **47.** $-2^{-\theta}[(\ln 2)\cos \pi\theta + \pi \sin \pi\theta]$

49. $\dfrac{1}{x(\ln 3)}$ **51.** $\dfrac{x - 2}{(\ln 2)x(x - 1)}$ **53.** $\dfrac{x}{(\ln 5)(x^2 - 1)}$

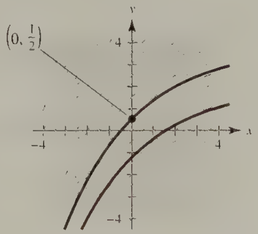
55. $\dfrac{5}{(\ln 2)t^2}(1 - \ln t)$ **57.** $2(1 - \ln x)x^{(2/x)-2}$

59. $(x - 2)^{x+1}\left[\dfrac{x + 1}{x - 2} + \ln(x - 2)\right]$

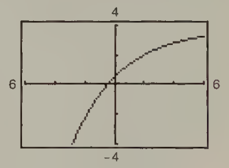
61. $\dfrac{3^x}{\ln 3} + C$ **63.** $\dfrac{7}{\ln 4}$ **65.** $-\dfrac{1}{2 \ln 5}(5^{x^2}) + C$

67. $\dfrac{\ln(3^{2x} + 1)}{2 \ln 3} + C$

69. (a)



(b) $y = \dfrac{3(1 - 0.4^{x/3})}{\ln 2.5} + \dfrac{1}{2}$



71. Answers will vary. Example: Growth and decay problems

73. (a) False. $y = a^x \Rightarrow 0 = a' \Rightarrow a = 0$, but exponential functions are not defined for $a = 0$.

(b) True: $y = \log_2 x$

(c) True: $2^y = x$

(d) False. (1, 0), (2, 1), and (8, 3) are not collinear.

75. $g(x) = x^x$, $k(x) = 2^x$, $h(x) = x^2$, $f(x) = \log_2 x$

77. (a) $40.64 (b) $C'(1) \approx 0.051P$, $C'(8) \approx 0.072P$

(c) ln 1.05

79.

n	1	2	4	12
A	$1410.60	$1414.78	$1416.91	$1418.34

n	365	Continuous
A	$1419.04	$1419.07

81.

n	1	2	4	12
A	\$4321.94	\$4399.79	\$4440.21	\$4467.74

n	365	Continuous
A	\$4481.23	\$4481.69

83.

t	1	10	20	30
P	\$95,122.94	\$60,653.07	\$36,787.94	\$22,313.02

t	40	50
P	\$13,533.53	\$8208.50

85.

t	1	10	20	30
P	\$95,132.82	\$60,716.10	\$36,864.45	\$22,382.66

t	40	50
P	\$13,589.88	\$8251.24

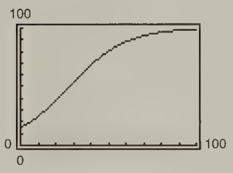
87. c

89. (a) 6.7 million cubic feet per acre

(b) $t = 20$: $\dfrac{dV}{dt} = 0.073$

$t = 60$: $\dfrac{dV}{dt} = 0.040$

91. (a)



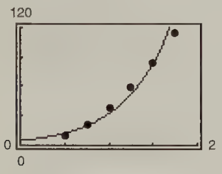
(b) 16.7%

(c) $x \approx 38.8$ or 38,800 egg masses

(d) $x \approx 2.78$ or 27,800 egg masses

93. (a) $B = 4.75(6.774)^d$

(b)

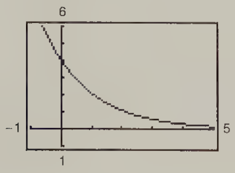


(c) When $d = 0.8$, the rate of growth is 41.99.

When $d = 1.5$, the rate of growth is 160.21.

95. (a) 5.67

(b)



(c) $f(t) = g(t) = h(t)$. No, because the definite integrals of two functions over a given interval may be equal even though the functions are not equal.

97. \$15,039.61 **99.** $y = 1200(0.6^t)$

101. False: e is an irrational number.

103. True **105.** True **107.** Proof

Section 5.6 (page 366)

1. $y = \dfrac{x^2}{2} + 2x + C$ **3.** $y = Ce^x - 2$

5. $y^2 - 5x^2 = C$ **7.** $y = Ce^{(2x^{3/2})/3}$ **9.** $y = C(1 + x^2)$

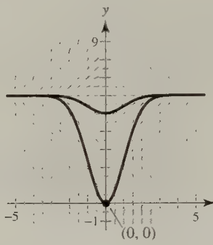
11. $\dfrac{dQ}{dt} = \dfrac{k}{t^2}$

$Q = -\dfrac{k}{t} + C$

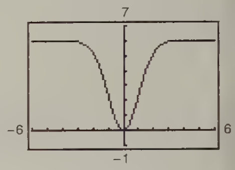
13. $\dfrac{dN}{ds} = k(250 - s)$

$N = -\dfrac{k}{2}(250 - s)^2 + C$

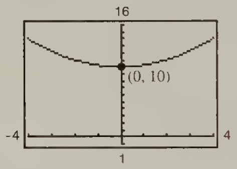
15. (a)



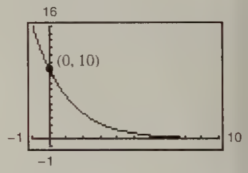
(b) $y = 6 - 6e^{-x^2/2}$



17. $y = \frac{1}{4}t^2 + 10$



19. $y = 10e^{-t/2}$



21. $\dfrac{dy}{dx} = ky$

$y = 4e^{0.3054x}$

$y(6) \approx 25$

23. $\dfrac{dy}{dt} = kV$

$V = 20{,}000e^{-0.1175t}$

$V(6) \approx 9882$

25. $y = \frac{1}{2}e^{0.4605t}$ **27.** $y = 0.6687e^{0.4024t}$

29. A differential equation in x and y is an equation that involve x, y, and derivatives of y.

Example: $y' = \dfrac{3x}{y}$

31. Quadrants I and III; dy/dx is positive when both x and y ar positive (Quadrant I) or when both x and y are negativ (Quadrant III).

33. Amount after 1000 years: 6.52 grams

Amount after 10,000 years: 0.14 gram

35. Initial quantity: 36.07 grams

Amount after 1000 years: 23.65 grams

37. Amount after 1000 years: 4.43 grams

Amount after 10,000 years: 1.49 grams

39. Initial quantity: 2.16 grams

Amount after 10,000 years: 1.63 grams

41. 95.81%

43. Time to double: 11.55 years

Amount after 10 years: \$1822.12

45. Annual rate: 8.94%

Amount after 10 years: \$1833.67

47. Annual rate: 9.50%

Time to double: 7.30 years

49. \$112,087.09 **51.** \$30,688.87

53. (a) 10.24 years (b) 9.93 years (c) 9.90 years (d) 9.90 years

55. (a) 8.50 years (b) 8.18 years (c) 8.16 years (d) 8.15 years

57. 7.43 million **59.** 6.83 million

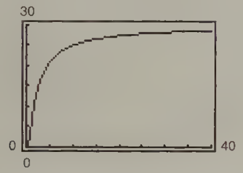
61. When $k > 0$, the population is increasing. When $k < 0$, the population is decreasing.

63. 527.06 millimeters of mercury

65. (a) $N \approx 30(1 - e^{-0.0502t})$ (b) 36 days

67. (a) $S \approx 30e^{-1.7918/t}$ (b) 20,965 units

(c)



69. 2014 ($t = 16$)

71. (a) 20 decibels (b) 70 decibels (c) 95 decibels (d) 120 decibels

73. (a) $10^{8.3} \approx 199{,}526{,}231.5$ (b) 10^R (c) $\dfrac{1}{I \ln 10}$

75. False. The rate of growth $\dfrac{dy}{dx}$ is proportional to y.

77. True

Section 5.7 (page 377)

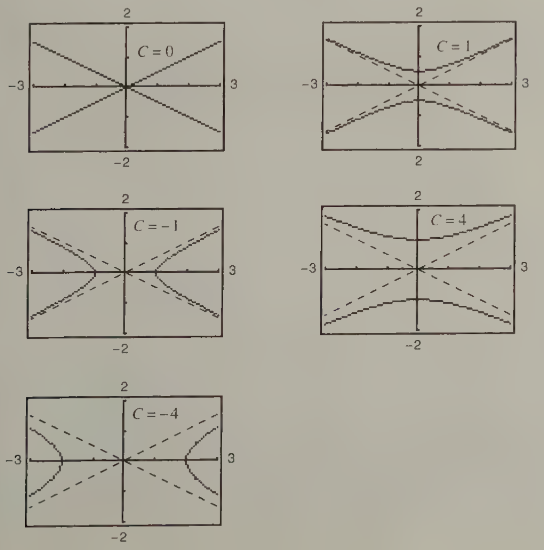
1. Proof **3.** Proof **5.** Proof **7.** Not a solution

9. Solution **11.** Solution **13.** Not a solution

15. Solution **17.** Not a solution **19.** $k = 0.07$

21. $4y^2 = x^3$

23.



25. $y = 3e^{-2x}$ **27.** $y = 2 \sin 3x - \frac{1}{3} \cos 3x$

29. $y = -2x + \frac{1}{2}x^3$ **31.** $y = x^3 + C$

33. $y = \frac{1}{2} \ln(1 + x^2) + C$

35. $y = x - \ln x^2 + C$ **37.** $y = -\frac{1}{2} \cos 2x + C$

39. $y = \frac{2}{5}(x - 3)^{5/2} + 2(x - 3)^{3/2} + C$ **41.** $y = \frac{1}{2}e^{x^2} + C$

43. $y^2 - x^2 = C$ **45.** $r = Ce^{0.05s}$ **47.** $y = C(x + 2)^3$

49. $y^2 = C - 2 \cos x$ **51.** $y = -\frac{1}{4}\sqrt{1 - 4x^2} + C$

53. $y = Ce^{(\ln x)^2/2}$ **55.** $y^2 = 2e^x + 14$ **57.** $y = e^{-(x^2+2x)/2}$

59. $y^2 = 4x^2 + 3$ **61.** $u = e^{(1-\cos v^2)/2}$ **63.** $P = P_0e^{kt}$

65. $9x^2 + 16y^2 = 25$ **67.** $f(x) = Ce^{-x/2}$

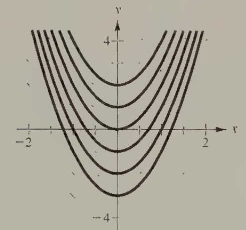
69. Homogeneous of degree 3 **71.** Homogeneous of degree 3

73. Not homogeneous **75.** Homogeneous of degree 0

77. $|x| = C(x - y)^2$ **79.** $|y^2 + 2xy - x^2| = C$

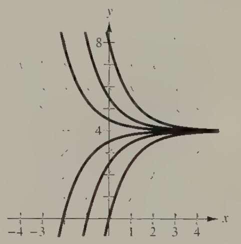
81. $y = Ce^{-x^2/2y^2}$ **83.** $e^{y/x} = 1 + \ln x^2$ **85.** $x = e^{\sin(y/x)}$

87.



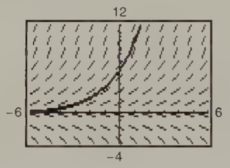
$y = \frac{1}{2}x^2 + C$

89.

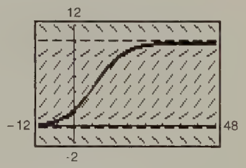


$y = 4 + Ce^{-x}$

91.



93.



95. 98.9% of the original amount

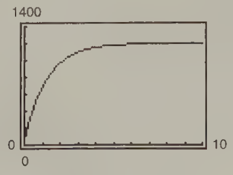
97. (a) $\dfrac{dy}{dx} = k(y - 4)$ (b) a (c) Proof

98. (a) $\dfrac{dy}{dx} = k(x - 4)$ (b) b (c) Proof

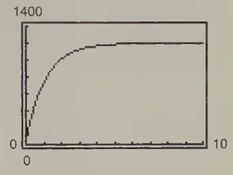
99. (a) $\dfrac{dy}{dx} = ky(y - 4)$ (b) c (c) Proof

100. (a) $\dfrac{dy}{dx} = ky^2$ (b) d (c) Proof

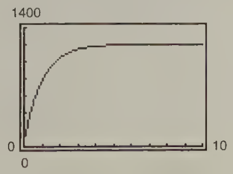
101. (a) $w = 1200 - 1140e^{-0.8t}$



$w = 1200 - 1140e^{-0.9t}$



$w = 1200 - 1140e^{-t}$



(b) 1.31 years; 1.16 years; 1.05 years

(c) 1200 pounds

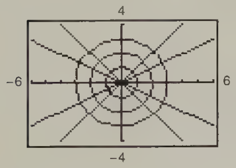
103. (a) $\dfrac{dv}{dt} = k(W - v);\ v = 20(1 - e^{-0.2877t})$

(b) $s = 20t + 69.5(e^{-0.2877t} - 1)$

105. Circles: $x^2 + y^2 = C$

Lines: $y = Kx$

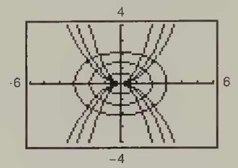
Graphs will vary.



107. Parabolas: $x^2 = Cy$

Ellipses: $x^2 + 2y^2 = K$

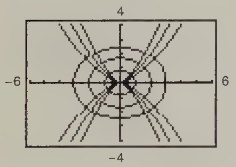
Graphs will vary.



109. Curves: $y^2 = Cx^3$

Ellipses: $2x^2 + 3y^2 = K$

Graphs will vary.



111. The general solution is a family of curves that satisfies the differential equation. A particular solution is one member of the family that satisfies given conditions.

Example: $(x^3 + y^3)\,dx - (xy^2)\,dy = 0$

113. A homogeneous differential equation is an equation of the form $M(x, y)\,dx + N(x, y)\,dy = 0$, where M and N are homogeneous functions of the same degree.

115. False: $y = x^3$ is a solution to $xy' - 3y = 0$, but $y = x^3 + 1$ is not a solution.

117. False: $f(tx, ty) \neq t^n f(x, y)$.

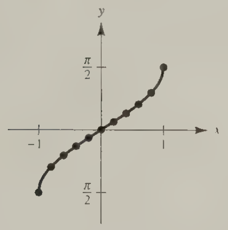
Section 5.8 (page 386)

1. (a)

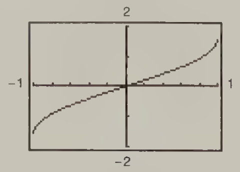
x	-1	-0.8	-0.6	-0.4	-0.2	0
y	-1.57	-0.93	-0.64	-0.41	-0.20	0

x	0.2	0.4	0.6	0.8	1
y	0.20	0.41	0.64	0.93	1.57

(b)



(c)



(d) Intercept: $(0, 0)$; Symmetry: origin

3. False: the range of $y = \arccos x$ is $[0, \pi]$.

5. $\dfrac{\pi}{6}$

7. $\dfrac{\pi}{3}$

9. $\dfrac{\pi}{6}$

11. $-\dfrac{\pi}{4}$

13. 2.50

15. $\arccos\left(\dfrac{1}{1.269}\right) \approx 0.66$

17. (a) $\frac{3}{5}$ (b) $\frac{5}{3}$

19. (a) $-\sqrt{3}$ (b) $-\frac{13}{5}$

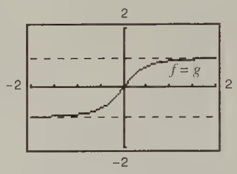
21. $\sqrt{1 - 4x^2}$

23. $\dfrac{\sqrt{x^2 - 1}}{|x|}$

25. $\dfrac{\sqrt{x^2 - 9}}{3}$

27. $\dfrac{\sqrt{x^2 + 2}}{x}$

29.



g is the algebraic form of f.

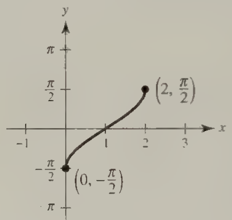
Horizontal asymptotes: $y = -1,\ y = 1$

31. $x = \frac{1}{3}[\sin(\frac{1}{2}) + \pi] \approx 1.207$

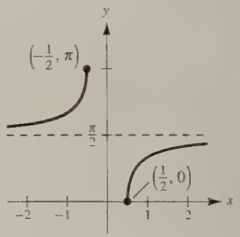
33. $x = \frac{1}{3}$

35. Proof

37.



39.



41. $\dfrac{2}{\sqrt{2x - x^2}}$

43. $-\dfrac{3}{\sqrt{4 - x^2}}$

45. $\dfrac{a}{a^2 + x^2}$

47. $\dfrac{3x - \sqrt{1 - 9x^2}\arcsin 3x}{x^2\sqrt{1 - 9x^2}}$

49. $-\dfrac{t}{\sqrt{1 - t^2}}$

51. $\arccos x$

53. $\dfrac{1}{1 - x^4}$

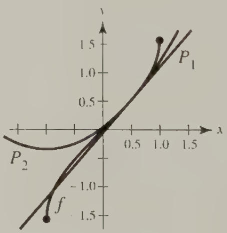
55. $\arcsin x$

57. $\dfrac{x^2}{\sqrt{16 - x^2}}$

59. $\dfrac{2}{(1 + x^2)^2}$

61. $P_1(x) = \dfrac{\pi}{6} + \dfrac{2\sqrt{3}}{3}\left(x - \dfrac{1}{2}\right)$

$P_2(x) = \dfrac{\pi}{6} + \dfrac{2\sqrt{3}}{3}\left(x - \dfrac{1}{2}\right) + \dfrac{2\sqrt{3}}{9}\left(x - \dfrac{1}{2}\right)^2$



63. Relative maximum: $(1.272, -0.606)$

Relative minimum: $(-1.272, 3.747)$

65. Relative minimum: $(2, 2.214)$

67. If the domains were not restricted, then the trigonometric fun tions would not be one-to-one and hence would have no invers

69. If $x > 0$, $y = \operatorname{arccot} x = \arctan \frac{1}{x}$; If $x < 0$, $y = \arctan \frac{1}{x} + \pi$.

71. (a) $\theta = \operatorname{arccot} \frac{x}{5}$

(b) $x = 10$: 16 radians per hour

$x = 3$: 58.824 radians per hour

73. (a) $h(t) = -16t^2 + 256$

$t = 4$ seconds

(b) $t = 1$: -0.0520 radian per second

$t = 2$: -0.1116 radian per second

75. Proof **77.** $k \leq -1$ or $k \geq 1$ **79.** True **81.** True

Section 5.9 (page 393)

1. $5 \arcsin \frac{x}{3} + C$ **3.** $\frac{\pi}{18}$ **5.** $\frac{7}{4} \arctan \frac{x}{4} + C$ **7.** $\frac{\pi}{6}$

9. $\operatorname{arcsec}|2x| + C$ **11.** $\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2 + 1) + C$

13. $\arcsin(x + 1) + C$ **15.** $\frac{1}{2}\arcsin t^2 + C$

17. $\frac{\pi^2}{32} \approx 0.308$ **19.** $\frac{\sqrt{3} - 2}{2} \approx -0.134$

21. $\frac{1}{4} \arctan \frac{e^{2x}}{2} + C$ **23.** $\frac{\pi}{4}$ **25.** $2 \arcsin \sqrt{x} + C$

27. $\ln\sqrt{x^2 + 1} - 3 \arctan x + C$

29. $8 \arcsin\left(\frac{x - 3}{3}\right) - \sqrt{6x - x^2} + C$

31. $\frac{\pi}{2}$ **33.** $\ln|x^2 + 6x + 13| - 3 \arctan\left(\frac{x + 3}{2}\right) + C$

35. $\arcsin\left(\frac{x + 2}{2}\right) + C$ **37.** $-\sqrt{-x^2 - 4x} + C$

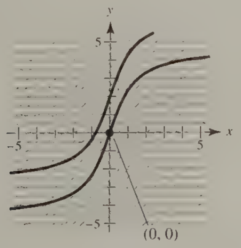
39. $4 - 2\sqrt{3} + \frac{\pi}{6} \approx 1.059$ **41.** $\frac{1}{2}\arctan(x^2 + 1) + C$

43. $2\sqrt{e^t - 3} - 2\sqrt{3} \arctan\left(\frac{\sqrt{e^t - 3}}{\sqrt{3}}\right) + C$

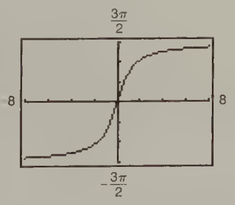
45. A trinomial of the form $x^2 + 2bx + b^2$

47. a and b **49.** a, b, and c

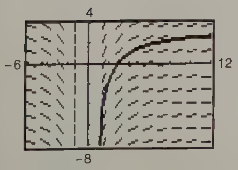
51. (a)



(b) $y = 3 \arctan x$



53.



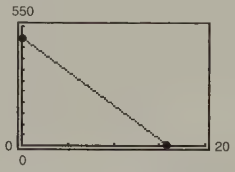
55. $\frac{\pi}{8}$ **57.** c

59. (a) $\int_0^1 \frac{4}{1 + x^2}\,dx = \left[4 \arctan x\right]_0^1 = 4 \arctan 1 - 4 \arctan 0 = \pi$

(b) 3.1415918 (c) 3.1415927

61. (a)–(c) Proof

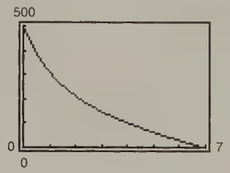
63. (a) $v(t) = -32t + 500$



(b) $s(t) = -16t^2 + 500t$; 3906.25 feet

(c) $v(t) = \sqrt{\frac{32}{k}} \tan\left[\arctan\left(500\sqrt{\frac{k}{32}}\right) - \sqrt{32k}\,t\right]$

(d)



6.86 seconds

(e) 1088 feet

(f) When air resistance is taken into account, the maximum height of the object is not as great.

Section 5.10 (page 403)

1. (a) 10.018 (b) -0.964 **3.** (a) $\frac{4}{3}$ (b) $\frac{13}{12}$

5. (a) 1.317 (b) 0.962

7. Proof **9.** Proof **11.** Proof

13. $\cosh x = \frac{\sqrt{13}}{2}$ **15.** $-2x\cosh(1 - x^2)$ **17.** $\coth x$

$\tanh x = \frac{3\sqrt{13}}{13}$

$\operatorname{csch} x = \frac{2}{3}$

$\operatorname{sech} x = \frac{2\sqrt{13}}{13}$

$\coth x = \frac{\sqrt{13}}{3}$

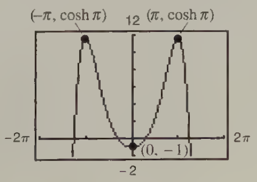
19. $\operatorname{csch} x$ **21.** $\sinh^2 x$ **23.** $\operatorname{sech} t$

25. $\frac{y}{x}[\cosh x + x(\sinh x)\ln x] = \frac{x^{\cosh x}}{x}[\cosh x + x(\sinh x)\ln x]$

27. $-2(\cosh x - \sinh x)^2 = -2e^{-2x}$

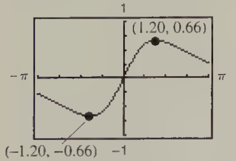
29. Relative maxima: $(\pm\pi, \cosh \pi)$

Relative minimum: $(0, -1)$



31. Relative maximum: $(1.20, 0.66)$

Relative minimum: $(-1.20, -0.66)$



33. $y = a \sinh x$

$y' = a \cosh x$

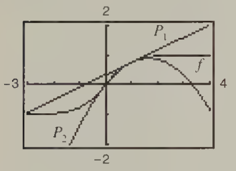
$y'' = a \sinh x$

$y''' = a \cosh x$

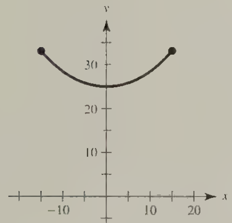
Therefore, $y''' - y' = 0$.

35. $P_1(x) = 0.76 + 0.42(x - 1)$

$P_2(x) = 0.76 + 0.42(x - 1) - 0.32(x - 1)^2$



37. (a)



(b) 33.146 units; 25 units (c) $m = \sinh(1) \approx 1.175$

39. $-\frac{1}{2}\cosh(1 - 2x) + C$ **41.** $\frac{1}{3}\cosh^3(x - 1) + C$

43. $\ln|\sinh x| + C$ **45.** $-\coth\frac{x^2}{2} + C$ **47.** $\operatorname{csch}\frac{1}{x} + C$

49. $\frac{1}{5}\ln 3$ **51.** $\frac{\pi}{4}$ **53.** $\frac{1}{2}\arctan x^2 + C$ **55.** $\frac{3}{\sqrt{9x^2 - 1}}$

57. $|\sec x|$ **59.** $2 \sec 2x$ **61.** $2 \sinh^{-1}(2x)$

63. See "Definition of the Hyperbolic Functions" on page 395.

65. $-\frac{\sqrt{a^2 - x^2}}{x}$

67. $-\operatorname{csch}^{-1}(e^x) + C = -\ln\left(\frac{1 + \sqrt{1 + e^{2x}}}{e^x}\right) + C$

69. $2\sinh^{-1}\sqrt{x} + C = 2\ln\left(\sqrt{x} + \sqrt{1 + x}\right) + C$

71. $\frac{1}{4}\ln\left|\frac{x - 4}{x}\right| + C$ **73.** $\frac{1}{2\sqrt{6}}\ln\left|\frac{\sqrt{2}(x + 1) + \sqrt{3}}{\sqrt{2}(x + 1) - \sqrt{3}}\right| + C$

75. $\frac{1}{4}\arcsin\left(\frac{4x - 1}{9}\right) + C$

77. $-\frac{x^2}{2} - 4x - \frac{10}{3}\ln\left|\frac{x - 5}{x + 1}\right| + C$

79. $8\arctan(e^2) - 2\pi \approx 5.207$ **81.** $\frac{5}{2}\ln\left(\sqrt{17} + 4\right) \approx 5.237$

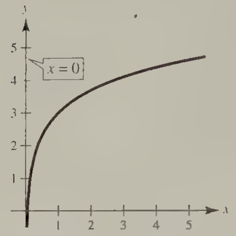
83. $\frac{52}{31}$ kilograms

85. If k were increased, the time of descent would increase.

87. Proof **89.** Proof **91.** Proof

Review Exercises for Chapter 5 (page 405)

1. Vertical asymptote: $x = 0$



3. $\frac{1}{5}[\ln(2x + 1) + \ln(2x - 1) - \ln(4x^2 + 1)]$

5. $\ln\left(\frac{3\sqrt[3]{4 - x^2}}{x}\right)$ **7.** $e^4 - 1 \approx 53.598$

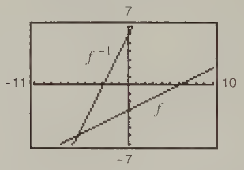
9. $\frac{1}{2x}$ **11.** $\frac{1 + 2\ln x}{2\sqrt{\ln x}}$ **13.** $\frac{x}{(a + bx)^2}$ **15.** $\frac{1}{x(a + bx)}$

17. $\frac{1}{7}\ln|7x - 2| + C$ **19.** $-\ln|1 + \cos x| + C$

21. $3 + \ln 4$ **23.** $\ln\left(2 + \sqrt{3}\right)$

25. (a) $f^{-1}(x) = 2x + 6$

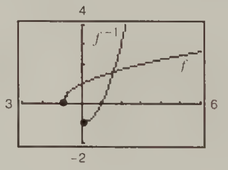
(b)



(c) Proof

27. (a) $f^{-1}(x) = x^2 - 1, \quad x \geq 0$

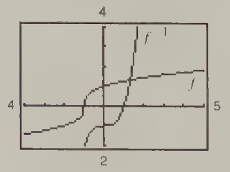
(b)



(c) Proof

29. (a) $f^{-1}(x) = x^3 - 1$

(b)

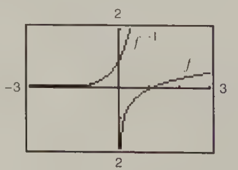


(c) Proof

31. $\frac{1}{3(\sqrt[3]{-3})^2} \approx 0.160$ **33.** $\frac{3}{4}$

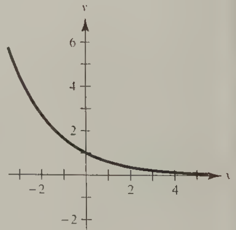
35. (a) $f^{-1}(x) = e^{2x}$

(b)



(c) Proof

37.



39. $-2x$ **41.** $te^t(t+2)$ **43.** $\dfrac{e^{2x}-e^{-2x}}{\sqrt{e^{2x}+e^{-2x}}}$

45. $\dfrac{x(2-x)}{e^x}$ **47.** $-\dfrac{y}{x(2y+\ln x)}$

49. $-\dfrac{1}{6}e^{-3x^2}+C$ **51.** $\dfrac{e^{4x}-3e^{2x}-3}{3e^x}+C$

53. $-\frac{1}{2}e^{1-x^2}+C$ **55.** $\ln|e^x-1|+C$

57. $y = e^x(a\cos 3x + b\sin 3x)$

$y' = e^x[(-3a+b)\sin 3x + (a+3b)\cos 3x]$

$y'' = e^x[(-6a-8b)\sin 3x + (-8a+6b)\cos 3x]$

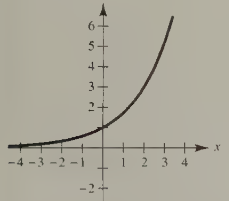
$y'' - 2y' + 10y$

$= e^x\{[(-6a-8b) - 2(-3a+b) + 10b]\sin 3x +$

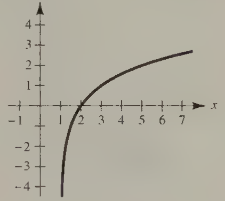
$[(-8a+6b) - 2(a+3b) + 10a]\cos 3x\} = 0$

59. $-\frac{1}{2}(e^{-16}-1) \approx 0.500$

61.



63.



65. $3^{x-1}\ln 3$ **67.** $x^{2x+1}\left(\dfrac{2x+1}{x}+2\ln x\right)$

69. $-\dfrac{1}{\ln 3(2-2x)}$ **71.** $\dfrac{5^{(x+1)^2}}{2\ln 5}+C$

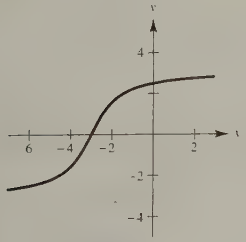
73. (a) ax^{a-1} (b) $(\ln a)a^x$ (c) $x^x(1+\ln x)$ (d) 0

75. ≈$3499.38 **77.** ≈7.79 inches **79.** About 46.2 years

81. $y = \dfrac{x^2}{2} + 3\ln|x| + C$ **83.** $y = Ce^{x^2}$

85. $\dfrac{x}{x^2-y^2} = C$ **87.** Proof; $y = -2x + \dfrac{1}{2}x^3$

89.



91. (a) $\dfrac{1}{2}$ (b) $\dfrac{\sqrt{3}}{2}$

93. $(1-x^2)^{-3/2}$ **95.** $\dfrac{x}{|x|\sqrt{x^2-1}} + \operatorname{arcsec} x$

97. $(\arcsin x)^2$ **99.** $\frac{1}{2}\arctan(e^{2x}) + C$

01. $\frac{1}{2}\arcsin x^2 + C$ **103.** $\ln\sqrt{16+x^2} + C$

05. $\dfrac{1}{4}\left(\arctan\dfrac{x}{2}\right)^2 + C$ **107.** $y = A\sin\left(\sqrt{\dfrac{k}{m}}\,t\right)$

09. $2 - \dfrac{\sinh\sqrt{x}}{2\sqrt{x}}$ **111.** $\dfrac{1}{2}\ln\left(\sqrt{x^4-1}+x^2\right) + C$

P.S. Problem Solving (page 408)

1. $\theta \approx 1.7263$ or 98.9°

3. (a) $(0, \infty)$

(b) Answers will vary.

Example: $e^{\pi/2} \approx 4.8105$ and $e^{5\pi/2} \approx 2575.9705$

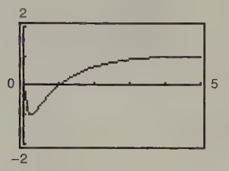
(c) Answers will vary.

Example: $e^{-\pi/2} \approx 0.2079$ and $e^{3\pi/2} \approx 111.3178$

(d) $[-1, 1]$

(e) $f'(x) = \dfrac{\cos(\ln x)}{x}$; Maximum value is 1.

(f)



Limit does not exist.

(g) Limit does not exist.

5. (a) Area of sector $= \dfrac{t}{2}$

(b) $A(t) = \frac{1}{2}\text{ base} \cdot \text{height} - \displaystyle\int_1^{\cosh t} \sqrt{x^2-1}\,dx$

$A(t) = \frac{1}{2}\cosh t \cdot \sinh t - \displaystyle\int_1^{\cosh t} \sqrt{x^2-1}\,dx$

$A(t) = \frac{1}{2}t$

7. Tangent line: $y = \dfrac{1}{a}x + (b-1)$

Passes through $(0, c)$, therefore $c = b - 1$.

Distance between b and c is $b - c = 1$.

9. $2\ln\left(\frac{3}{2}\right) \approx 0.8109$

11. (a) $y = \dfrac{1}{(1-0.01t)^{100}}$; $T = 100$

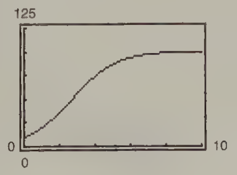
(b) $y = \dfrac{1}{\left[\left(\dfrac{1}{y_0}\right)^e - ket\right]^{1/e}}$; Answers will vary.

13. 22.35°F

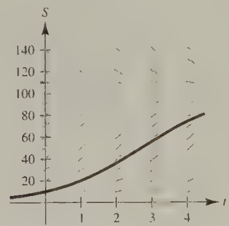
15. (a) $\dfrac{dS}{dt} = kS(L-S)$; $S = \dfrac{100}{1+9e^{-0.8109t}}$

(b) 2.7 months

(c)



(d)



(e) Sales will decrease toward the line $S = L$.

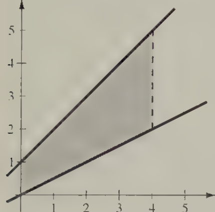
Chapter 6

Section 6.1 (page 418)

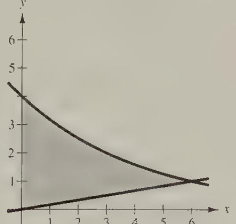
1. $-\int_0^6 (x^2 - 6x)\,dx$ **3.** $\int_0^3 (-2x^2 + 6x)\,dx$

5. $-6\int_0^1 (x^3 - x)\,dx$

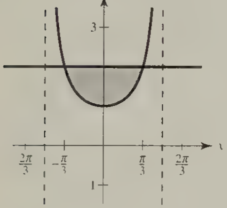
7.



9.

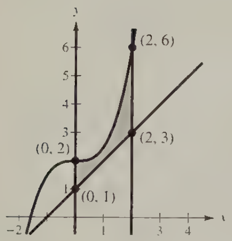


11.

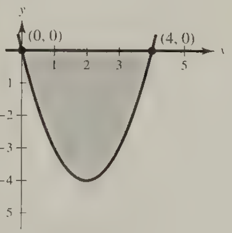


13. d

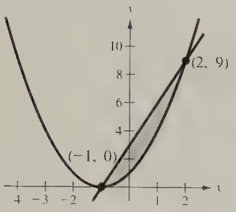
15. 2



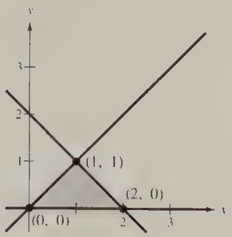
17. $\frac{32}{3}$



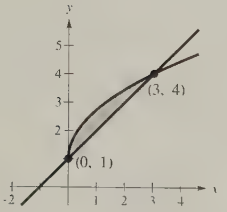
19. $\frac{9}{2}$



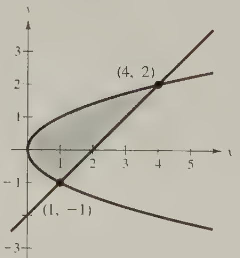
21. 1



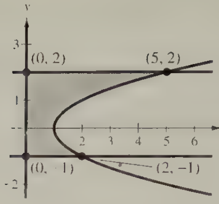
23. $\frac{3}{2}$



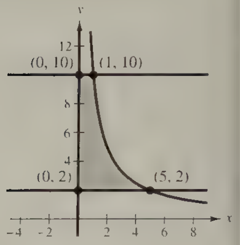
25. $\frac{9}{2}$



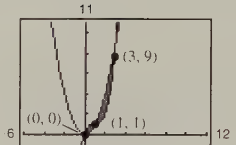
27. 6



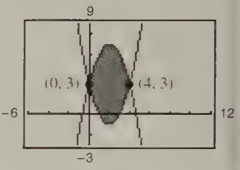
29. 16.094



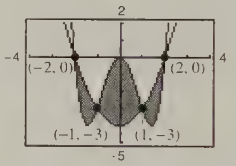
31. $\frac{37}{12}$



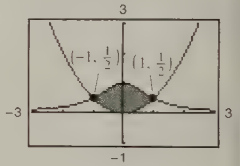
33. $\frac{64}{3}$



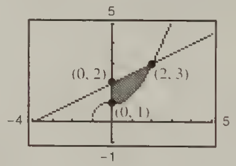
35. 8



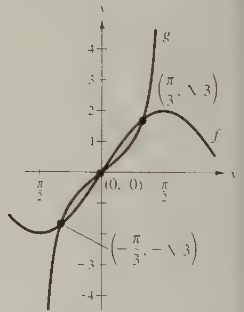
37. $\frac{\pi}{2} - \frac{1}{3} \approx 1.237$



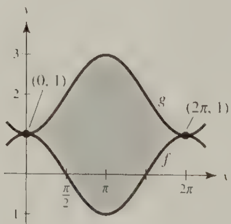
39. ≈ 1.759



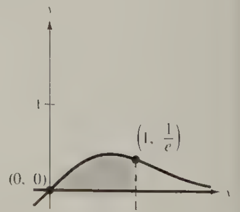
41. $2(1 - \ln 2) \approx 0.614$



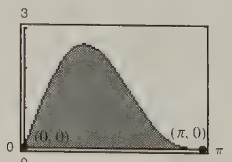
43. $4\pi \approx 12.566$



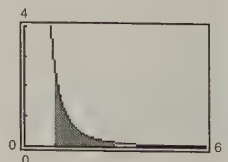
45. $\frac{1}{2}\left(1 - \frac{1}{e}\right) \approx 0.316$



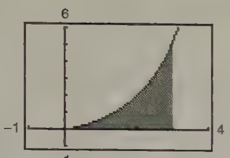
47. 4



49. ≈ 1.323



51. (a)

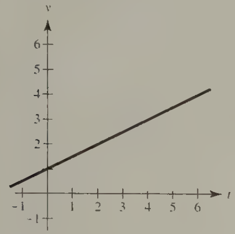


(b) $\int_0^3 \sqrt{\frac{x^3}{4-x}}\,dx$; No

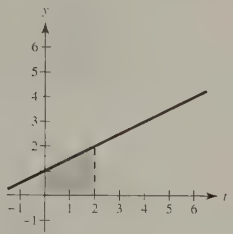
(c) ≈ 4.772

53. $F(x) = \frac{1}{4}x^2 + x$

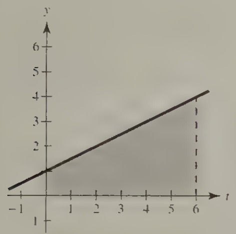
(a) $F(0) = 0$



(b) $F(2) = 3$

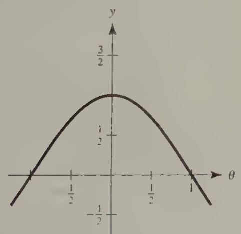


(c) $F(6) = 15$

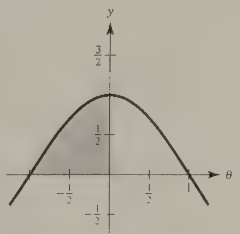


55. $F(\alpha) = \frac{2}{\pi}\left(\sin\frac{\pi\alpha}{2} + 1\right)$

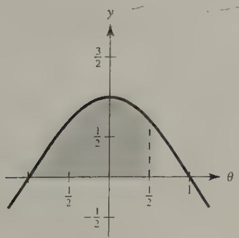
(a) $F(-1) = 0$



(b) $F(0) = \frac{2}{\pi} \approx 0.6366$



(c) $F\left(\frac{1}{2}\right) = \frac{\sqrt{2} + 2}{\pi} \approx 1.0868$



7. $\frac{1}{2}ac$ **59.** $\int_{-2}^{1} [x^3 - (3x - 2)]\,dx = \frac{27}{4}$ **61.** y

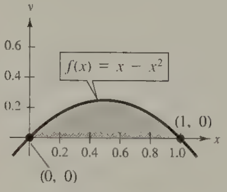
3. Answers will vary. Example: $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$

$$\int_{-1}^{1} [(1 - x^2) - (x^4 - 2x^2 + 1)]\,dx = \frac{4}{15}$$

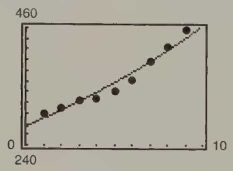
65. Offer 2 is better because the cumulative salary (area under the curve) is greater.

67. $b = 9\left(1 - \frac{1}{\sqrt[3]{4}}\right) \approx 3.330$

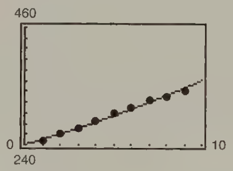
69. $\frac{1}{6}$ **71.** $1.625 billion



73. (a) $y = 275.0675(1.0537)^t$ or $y = 275.0675e^{0.0523t}$



(b) $y = 239.9407(1.0417)^t$ or $y = 239.9407e^{0.0408t}$



(c) $649.5 billion

(d) No. The model for total receipts is increasing at a faster rate than the model for total expenditures. No.

75. $\frac{16}{3}(4\sqrt{2} - 5) \approx 3.5$

77. (a) 6.031 square meters (b) 12.062 cubic meters

(c) 60,310 pounds

79. True

81. False. Let $f(x) = x$ and $g(x) = 2x - x^2$ on the interval $[0, 2]$.

Section 6.2 (page 428)

1. $\pi\int_0^1 (-x + 1)^2\,dx = \frac{\pi}{3}$ **3.** $\pi\int_1^4 (\sqrt{x})^2\,dx = \frac{15\pi}{2}$

5. $\pi\int_0^1 [(x^2)^2 - (x^3)^2]\,dx = \frac{2\pi}{35}$ **7.** $\pi\int_0^4 (\sqrt{y})^2\,dy = 8\pi$

9. $\pi\int_0^1 (y^{3/2})^2\,dy = \frac{\pi}{4}$

11. (a) 8π (b) $\frac{128\pi}{5}$ (c) $\frac{256\pi}{15}$ (d) $\frac{192\pi}{5}$

13. (a) $\frac{32\pi}{3}$ (b) $\frac{64\pi}{3}$

15. 18π **17.** $\pi\left(16\ln 2 - \frac{3}{4}\right) \approx 32.485$

19. $\frac{208\pi}{3}$ **21.** $\frac{384\pi}{5}$ **23.** $\pi\ln 4$ **25.** $\frac{3\pi}{4}$

27. $\frac{\pi}{2}\left(1 - \frac{1}{e^2}\right) \approx 1.358$ **29.** $\frac{277\pi}{3}$ **31.** 8π

33. $\frac{\pi^2}{2} \approx 4.935$ **35.** 1.969 **37.** 49.022 **39.** a

41. (a) See page 422 for the disk method.

(b) Horizontal axis of revolution:

$$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2)\,dx$$

Vertical axis of revolution:

$$V = \pi \int_c^d ([R(y)]^2 - [r(y)]^2)\,dy$$

43. The parabola $y = 4x - x^2$ is a horizontal translation of the parabola $y = 4 - x^2$. Therefore, their volumes are equal.

45. 18π **47.** Proof **49.** $\pi r^2 h\left(1 - \frac{h}{H} + \frac{h^2}{3H^2}\right)$ **51.** $\frac{\pi}{30}$

53. (a) 60π (b) 50π

55. One-fourth: 32.64 feet; Three-fourths: 67.36 feet

57. (a) ii; right circular cylinder of radius r and height h

(b) iv; ellipsoid whose underlying ellipse has the equation

$$\left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$$

(c) iii; sphere of radius r

(d) i; right circular cone of radius r and height h

(e) v; torus of cross-sectional radius r and other radius R

59. (a) $\frac{81}{10}$ (b) $\frac{9}{2}$

61. (a) $\frac{1}{10}$ (b) $\frac{\pi}{80}$ (c) $\frac{\sqrt{3}}{40}$ (d) $\frac{\pi}{20}$

63. Proof **65.** $5\sqrt{1 - 2^{-2/3}} \approx 3.0415$

67. (a) $\frac{2r^3}{3}$ (b) $\frac{2r^3 \tan\theta}{3}$, $\lim\limits_{\theta \to 90^\circ} V = \infty$

Section 6.3 (page 437)

1. $2\pi \int_0^2 x^2\,dx = \frac{16\pi}{3}$ **3.** $2\pi \int_0^4 x\sqrt{x}\,dx = \frac{128\pi}{5}$

5. $2\pi \int_0^2 x^3\,dx = 8\pi$ **7.** $2\pi \int_0^2 x(4x - 2x^2)\,dx = \frac{16\pi}{3}$

9. $2\pi \int_0^2 x(x^2 - 4x + 4)\,dx = \frac{8\pi}{3}$

11. $2\pi \int_0^1 x\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)dx = \sqrt{2\pi}\left(1 - \frac{1}{\sqrt{e}}\right) \approx 0.986$

13. $2\pi \int_0^2 y(2 - y)\,dy = \frac{8\pi}{3}$

15. $2\pi\left[\int_0^{1/2} y\,dy + \int_{1/2}^1 y\left(\frac{1}{y} - 1\right)dy\right] = \frac{\pi}{2}$

17. 16π **19.** 64π

21. (a) $\frac{128\pi}{7}$ (b) $\frac{64\pi}{5}$ (c) $\frac{96\pi}{5}$

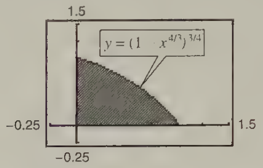
23. (a) $\frac{\pi a^3}{15}$ (b) $\frac{\pi a^3}{15}$ (c) $\frac{4\pi a^3}{15}$

25. $V = 2\pi \int_c^d p(y)h(y)\,dy$ for horizontal axis of revolution

$V = 2\pi \int_a^b p(x)h(x)\,dx$ for vertical axis of revolution

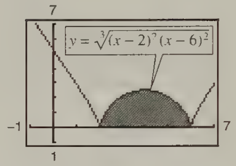
27. Both integrals yield the volume of the solid generated by revol ing the region bounded by the graphs of $y = \sqrt{x - 1}$, $y =$ and $x = 5$ about the x-axis.

29. (a)



(b) 1.506

31. (a)



(b) 187.25

33. d **35.** Diameter $= 2\sqrt{4 - 2\sqrt{3}} \approx 1.464$

37. $4\pi^2$ **39.** Proof

41. (a) ii; right circular cone of radius r and height h

(b) v; torus of cross-sectional radius r and other radius R

(c) iii; sphere of radius r

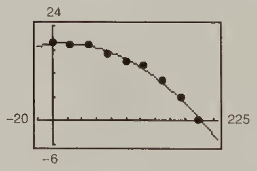
(d) i; right circular cylinder of radius r and height h

(e) iv; ellipsoid whose underlying ellipse has the equation

$$\left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$$

43. (a) 1,366,593 cubic feet

(b) $d = -0.000561x^2 + 0.0189x + 19.39$



(c) 1,343,345 cubic feet

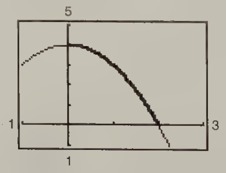
(d) 10,048,221 gallons

Section 6.4 (page 447)

1. 13 **3.** $\frac{2}{3}(2\sqrt{2} - 1) \approx 1.219$

5. $5\sqrt{5} - 2\sqrt{2} \approx 8.352$ **7.** $\frac{33}{16}$ **9.** 1.763

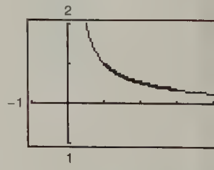
11. (a)



(b) $\int_0^2 \sqrt{1 + 4x^2}\,dx$

(c) ≈ 4.647

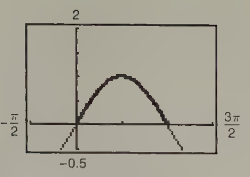
13. (a)



(b) $\int_1^3 \sqrt{1 + \frac{1}{x^4}}\,dx$

(c) ≈ 2.147

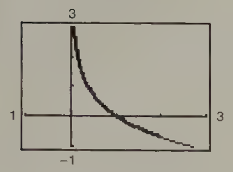
5. (a)



(b) $\int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$

(c) ≈ 3.820

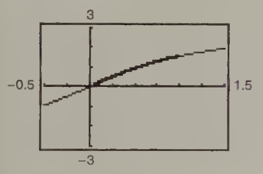
7. (a)



(b) $\int_0^2 \sqrt{1 + e^{-2y}}\, dy = \int_{e^{-2}}^{1} \sqrt{1 + \frac{1}{x^2}}\, dx$

(c) ≈ 2.221

9. (a)



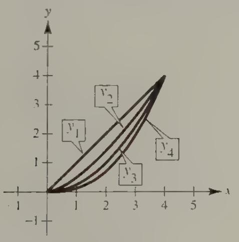
(b) $\int_0^1 \sqrt{1 + \left(\frac{2}{1 + x^2}\right)^2}\, dx$

(c) ≈ 1.871

1. b

3. (a) 64.125 (b) 64.525 (c) 64.666 (d) 64.672

5. (a)



(b) y_1, y_2, y_3, y_4

(c) $s_1 \approx 5.657$; $s_2 \approx 5.759$; $s_3 \approx 5.916$; $s_4 \approx 6.063$

7. Fleeing object: $\frac{2}{3}$ unit

Pursuer: $\frac{1}{2}\int_0^1 \frac{x + 1}{\sqrt{x}}\, dx = \frac{1}{2}\left[\frac{2}{3}x^{3/2} + 2x^{1/2}\right]_0^1$

$= \frac{4}{3} = 2\left(\frac{2}{3}\right)$

. 20[sinh 1 − sinh(−1)] ≈ 47.0 meters

1. 3 arcsin $\frac{2}{3} \approx 2.1892$

3. $2\pi\int_0^3 \frac{1}{3}x^3\sqrt{1 + x^4}\, dx = \frac{\pi}{9}(82\sqrt{82} - 1) \approx 258.85$

5. $2\pi\int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \frac{47\pi}{16}$

7. $2\pi\int_1^8 x\sqrt{1 + \frac{1}{9x^{4/3}}}\, dx = \frac{\pi}{27}(145\sqrt{145} - 10\sqrt{10}) \approx 199.48$

. 14.424

1. A rectifiable curve is a curve with a finite arc length.

. The integral formula for the area of a surface of revolution is derived from the formula for the lateral surface area of the frustum of a right circular cone. The formula is $S = 2\pi rL$ where $r = \frac{1}{2}(r_1 + r_2)$, which is the average radius of the frustum, and L is the length of a line segment on the frustum.

45. Proof **47.** $6\pi(3 - \sqrt{5}) \approx 14.40$

49. Surface area $= \frac{\pi}{27}$ square feet ≈ 16.8 square inches

Amount of glass $= \frac{\pi}{27}\left(\frac{0.015}{12}\right)$ cubic foot

≈ 0.00015 cubic foot

≈ 0.25 cubic inch

51. (a) $y = (1.953 \times 10^{-7})x^4 - (1.804 \times 10^{-4})x^3 + 0.0496x^2 - 4.8323x + 536.927$

(b) 131,734.5 square feet ≈ 3 acres

(c) 794.9 feet

53. (a) $\pi\left(1 - \frac{1}{b}\right)$ (b) $2\pi\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\, dx$

(c) $\lim_{b\to\infty} V = \lim_{b\to\infty} \pi\left(1 - \frac{1}{b}\right) = \pi$

(d) Since $\frac{\sqrt{x^4 + 1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x} > 0$ on $[1, b]$,

we have $\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\, dx > \int_1^b \frac{1}{x}\, dx = \left[\ln x\right]_1^b = \ln b$

and $\lim_{b\to\infty} \ln b \to \infty$. Thus, $\lim_{b\to\infty} 2\pi\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\, dx = \infty$.

55. (a) Area of circle with radius L: $A = \pi L^2$

Area of sector with central angle θ (in radians):

$$S = \frac{\theta}{2\pi}A = \frac{\theta}{2\pi}(\pi L^2) = \frac{1}{2}L^2\theta$$

(b) Let s be the arc length of the sector, which is the circumference of the base of the cone. Here, $s = L\theta = 2\pi r$, and you have

$$S = \frac{1}{2}L^2\theta = \frac{1}{2}L^2\left(\frac{s}{L}\right) = \frac{1}{2}Ls = \frac{1}{2}L(2\pi r) = \pi rL.$$

(c) The lateral surface area of the frustum is the difference between the large cone and the small one.

$S = \pi r_2(L + L_1) - \pi r_1 L_1$

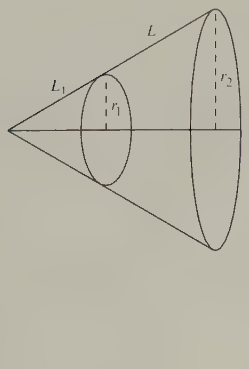
$= \pi r_2 L + \pi L_1(r_2 - r_1)$

By similar triangles, $\frac{L + L_1}{r_2} = \frac{L_1}{r_1} \Rightarrow Lr_1 = L_1(r_2 - r_1)$.

Hence,

$S = \pi r_2 L + \pi L_1(r_2 - r_1) = \pi r_2 L + \pi L r_1$

$= \pi L(r_1 + r_2)$.



Section 6.5 (page 456)

1. 1000 foot-pounds **3.** 448 newton-meters

5. If an object is moved a distance D in the direction of an applied constant force F, then the work W done by the force is defined as $W = FD$.

7. c, d, a, b **9.** 30.625 inch-pounds ≈ 2.55 foot-pounds

11. 8750 newton-centimeters = 87.5 newton-meters

13. 160 inch-pounds ≈ 13.3 foot-pounds

15. 37.125 foot-pounds

17. (a) 487.805 mile-tons ≈ $5.151(10^9)$ foot-pounds

(b) 1395.349 mile-tons ≈ $1.473(10^{10})$ foot-pounds

19. (a) 2.93×10^4 mile-tons ≈ 3.10×10^{11} foot-pounds

(b) 3.38×10^4 mile-tons ≈ 3.57×10^{11} foot-pounds

21. (a) 2496 foot-pounds (b) 9984 foot-pounds

23. $470{,}400\pi$ newton-meters **25.** 2995.2π foot-pounds

27. $20{,}217.6\pi$ foot-pounds **29.** 2457π foot-pounds

31. 337.5 foot-pounds **33.** 300 foot-pounds

35. 168.75 foot-pounds **37.** 7987.5 foot-pounds

39. $2000 \ln \frac{3}{2} \approx 810.93$ foot-pounds **41.** $\frac{3k}{4}$

43. 3249.4 foot-pounds **45.** 10,330.3 foot-pounds

Section 6.6 (page 467)

1. $\bar{x} = -\frac{6}{7}$ **3.** $\bar{x} = 12$ **5.** (a) $\bar{x} = 17$ (b) $\bar{x} = -3$

7. $x = 6$ feet **9.** $(\bar{x}, \bar{y}) = \left(\frac{10}{9}, -\frac{1}{9}\right)$ **11.** $(\bar{x}, \bar{y}) = \left(-\frac{7}{8}, -\frac{7}{16}\right)$

13. $M_x = 4\rho,\ M_y = \frac{64\rho}{5},\quad (\bar{x}, \bar{y}) = \left(\frac{12}{5}, \frac{3}{4}\right)$

15. $M_x = \frac{\rho}{35},\ M_y = \frac{\rho}{20},\quad (\bar{x}, \bar{y}) = \left(\frac{3}{5}, \frac{12}{35}\right)$

17. $M_x = \frac{99\rho}{5},\ M_y = \frac{27\rho}{4},\quad (\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{22}{5}\right)$

19. $M_x = \frac{192\rho}{7},\ M_y = 96\rho,\quad (\bar{x}, \bar{y}) = \left(5, \frac{10}{7}\right)$

21. $M_x = 0,\ M_y = \frac{256\rho}{15},\quad (\bar{x}, \bar{y}) = \left(\frac{8}{5}, 0\right)$

23. $M_x = \frac{27\rho}{4},\ M_y = -\frac{27\rho}{10},\quad (\bar{x}, \bar{y}) = \left(-\frac{3}{5}, \frac{3}{2}\right)$

25. $A = \int_0^1 (x - x^2)\,dx = \frac{1}{6}$

$M_x = \int_0^1 \left(\frac{x + x^2}{2}\right)(x - x^2)\,dx = \frac{1}{15}$

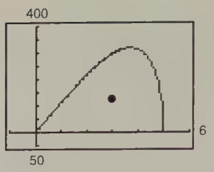
$M_y = \int_0^1 x(x - x^2)\,dx = \frac{1}{12}$

27. $A = \int_0^3 (2x + 4)\,dx = 21$

$M_x = \int_0^3 \left(\frac{2x + 4}{2}\right)(2x + 4)\,dx = 78$

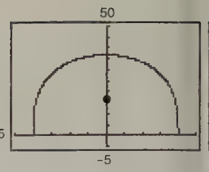
$M_y = \int_0^3 x(2x + 4)\,dx = 36$

29.



$(\bar{x}, \bar{y}) = (3.0, 126.0)$

31.

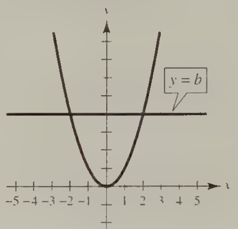


$(\bar{x}, \bar{y}) = (0, 16.2)$

33. $(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3}\right)$ **35.** $(\bar{x}, \bar{y}) = \left(\frac{(a + 2b)c}{3(a + b)}, \frac{a^2 + ab + b^2}{3(a + b)}\right.$

37. $(\bar{x}, \bar{y}) = \left(0, \frac{4b}{3\pi}\right)$

39. (a)



(b) $\bar{x} = 0$ by symmetry

(c) $M_y = \int_{-\sqrt{b}}^{\sqrt{b}} x(b - x^2)\,dx = 0$ because $x(b - x^2)$ is an o function.

(d) $\bar{y} > \frac{b}{2}$ because the area is greater for $y > \frac{b}{2}$.

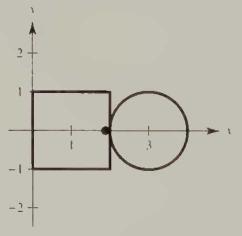
(e) $\bar{y} = \frac{3}{5}b$

41. (a) $(\bar{x}, \bar{y}) = (0, 12.98)$

(b) $y = (-1.02 \times 10^{-5})x^4 - 0.0019x^2 + 29.28$

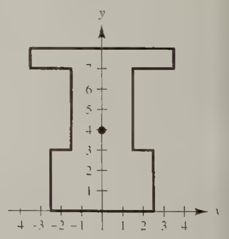
(c) $(\bar{x}, \bar{y}) = (0, 12.85)$

43.



$(\bar{x}, \bar{y}) = \left(\frac{4 + 3\pi}{4 + \pi}, 0\right)$

45.



$(\bar{x}, \bar{y}) = \left(0, \frac{135}{34}\right)$

47. $(\bar{x}, \bar{y}) = \left(\frac{2 + 3\pi}{2 + \pi}, 0\right)$ **49.** $160\pi^2 \approx 1579.14$

51. $\frac{128\pi}{3} \approx 134.04$

53. The center of mass $(\bar{x}, \bar{y})$ is $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$ where:

(a) $m = m_1 + m_2 + \cdots + m_n$ is the total mass of the syste

(b) $M_y = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n$ is the moment ab the y-axis.

(c) $M_x = m_1 y_1 + m_2 y_2 + \cdots + m_n y_n$ is the moment ab the x-axis.

5. (a) $\left(\frac{5}{6}, 2\frac{5}{18}\right)$; The plane region has been translated 2 units up.
(b) $\left(2\frac{5}{6}, \frac{5}{18}\right)$; The plane region has been translated 2 units to the right.
(c) $\left(\frac{5}{6}, -\frac{5}{18}\right)$; The plane region has been reflected across the x-axis.
(d) Not possible

7. $(\bar{x}, \bar{y}) = \left(0, \dfrac{2r}{\pi}\right)$

9. $(\bar{x}, \bar{y}) = \left(\dfrac{n+1}{n+2}, \dfrac{n+1}{4n+2}\right)$; As $n \to \infty$, the region shrinks towards the line segments $y = 0$ for $0 \le x \le 1$ and $x = 1$ for $0 \le y \le 1$; $(\bar{x}, \bar{y}) \to \left(1, \dfrac{1}{4}\right)$.

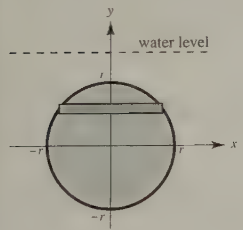
ection 6.7 (page 474)

1. 936 pounds **3.** 748.8 pounds **5.** 1123.2 pounds
7. 748.8 pounds **9.** 1064.96 pounds
1. 117,600 newtons **13.** 2,381,400 newtons
5. 2814 pounds **17.** 6753.6 pounds **19.** 94.5 pounds

1. $h(y) = k - y$

$L(y) = 2\sqrt{r^2 - y^2}$

$$F = w\int_{-r}^{r} (k - y)\sqrt{r^2 - y^2}(2)\,dy$$
$$= w\left[2k\int_{-r}^{r} \sqrt{r^2 - y^2}\,dy + \int_{-r}^{r} \sqrt{r^2 - y^2}(-2y)\,dy\right]$$



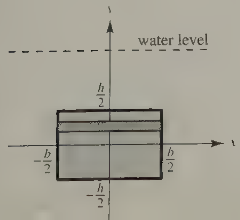
The second integral is zero since its integrand is odd and the limits of integration are symmetric with respect to the origin. The first integral is the area of a semicircle of radius r.

$$F = w\left[(2k)\frac{\pi r^2}{2} + 0\right] = wk\pi r^2$$

3. $h(y) = k - y$

$L(y) = b$

$$F = w\int_{-h/2}^{h/2} (k - y)b\,dy$$
$$= wb\left[ky - \frac{y^2}{2}\right]_{-h/2}^{h/2} = wb(hk) = wkhb$$



25. 960 pounds **27.** 3010.8 pounds **29.** 6448.7 pounds

31. (a) $\dfrac{3\sqrt{2}}{2} \approx 2.12$ feet

(b) The pressure increases with increasing depth.

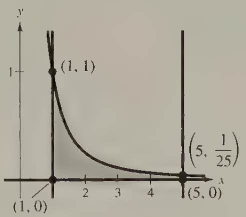
33. The fluid force F of constant weight-density w (per unit of volume) against a submerged vertical plane region from $y = c$ to $y = d$ is

$$F = w \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} h(y_i)L(y_i)\,\Delta y = w\int_{c}^{d} h(y)L(y)\,dy$$

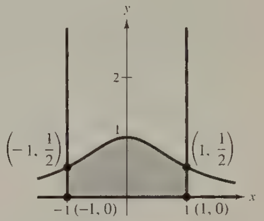
where $h(y)$ is the depth of the fluid at y and $L(y)$ is the horizontal length of the region at y.

Review Exercises for Chapter 6 (page 476)

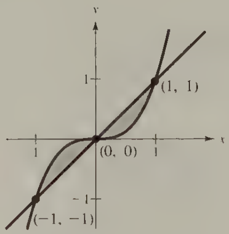
1. $\dfrac{4}{5}$



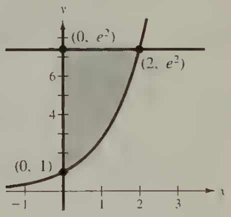
3. $\dfrac{\pi}{2}$



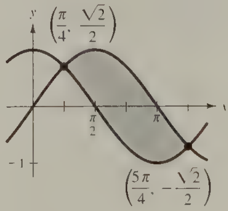
5. $\frac{1}{2}$



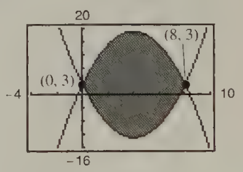
7. $e^2 + 1$



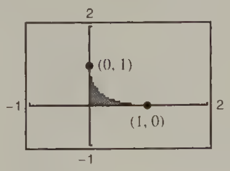
9. $2\sqrt{2}$



11. $\frac{512}{3}$



13. $\frac{1}{6}$



15. $\displaystyle\int_{0}^{2} [0 - (y^2 - 2y)]\,dy = \int_{-1}^{0} 2\sqrt{x + 1}\,dx = \frac{4}{3}$

17. $\displaystyle\int_0^2 \left[1 - \left(1 - \frac{x}{2}\right)\right] dx + \int_2^3 [1 - (x - 2)]\, dx$

$\displaystyle = \int_0^1 [(y + 2) - (2 - 2y)]\, dy = \frac{3}{2}$

19. Job 1. The salary for job 1 is greater than the salary for job 2 for all the years except the first and tenth years.

21. (a) $\dfrac{64\pi}{3}$ (b) $\dfrac{128\pi}{3}$ (c) $\dfrac{64\pi}{3}$ (d) $\dfrac{160\pi}{3}$

23. (a) 64π (b) 48π **25.** $\dfrac{\pi^2}{4}$

27. $\dfrac{4\pi}{3}(20 - 9 \ln 3) \approx 42.359$

29. $\frac{4}{15}$ **31.** 1.958 feet

33. $\frac{8}{15}(1 + 6\sqrt{3}) \approx 6.076$ **35.** 4018.2 feet **37.** 15π

39. 50 inch-pounds $\approx$ 4.167 foot-pounds

41. $104{,}000\pi$ foot-pounds $\approx$ 163.4 foot-tons

43. 250 foot-pounds

45. $a = \dfrac{15}{4}$ **47.** $(\bar{x}, \bar{y}) = \left(\dfrac{a}{5}, \dfrac{a}{5}\right)$

49. $(\bar{x}, \bar{y}) = \left(0, \dfrac{2a^2}{5}\right)$

51. $(\bar{x}, \bar{y}) = \left(\dfrac{2(9\pi + 49)}{3(\pi + 9)}, 0\right)$

53. Let D = surface of liquid; ρ = weight per cubic volume.

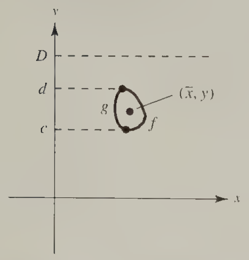
$$F = \rho \int_c^d (D - y)[f(y) - g(y)]\, dy$$

$$= \rho\left[\int_c^d D[f(y) - g(y)]\, dy - \int_c^d y[f(y) - g(y)]\, dy\right]$$

$$= \rho\left[\int_c^d [f(y) - g(y)]\, dy\right]\left[D - \frac{\int_c^d y[f(y) - g(y)]\, dy}{\int_c^d [f(y) - g(y)]\, dy}\right]$$

$$= \rho(\text{area})(D - \bar{y})$$

$$= \rho(\text{area})(\text{depth of centroid})$$



P.S. Problem Solving (page 478)

1. 3 **3.** (a) $4\pi^2$ (b) $2\pi^2 r^2 R$ **5.** $\dfrac{\pi h^3}{6}$

7. (a) Area S is 16 times area R.

(b) Let point A be (a, a^3). The equation of the tangent line to curve $y = x^3$ at A is $y = 3a^2 x - 2a^3$, and point B $(-2a, -8a^3)$. Area R is

$$\int_{-2a}^{a} (x^3 - 3a^2 x + 2a^3)\, dx = \frac{27a^4}{4}.$$

Then, the equation of the tangent line to the cu $y = x^3$ at B is $y = 12a^2 x + 16a^3$, and point C is $(4a, 64$ Area S is

$$\int_{-2a}^{4a} (12a^2 x + 16a^3 - x^3)\, dx = 108a^4.$$

Therefore, area S is 16 times area R.

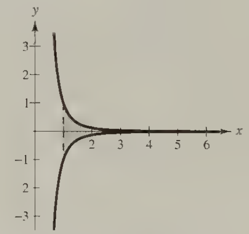
9. (a) $\dfrac{ds}{dx} = \sqrt{1 + [f'(x)]^2}$

(b) $ds = \sqrt{1 + [f'(x)]^2}\, dx$; $(ds)^2 = (dx)^2 + (dy)^2$

(c) $\displaystyle\int_1^x \sqrt{1 + \tfrac{9}{4}t}\, dt$

(d) $s(2) \approx 2.0858$. This is the arc length of the curve.

11.



(a) $\left(\dfrac{12}{7}, 0\right)$

(b) $\left(\dfrac{2b}{b + 1}, 0\right)$

(c) $(2, 0)$

13. (a) 12 (b) 7.5

15. Consumer surplus: 1600; Producer surplus: 400

17. Wall at shallow end: 9984 pounds

Wall at deep end: 39,936 pounds

Side wall: 19,968 + 26,624 = 46,592 pounds

Chapter 7

Section 7.1 (page 486)

1. b **3.** c

5. $\displaystyle\int u^n\, du$

$u = 3x - 2,\ n = 4$

7. $\displaystyle\int \frac{du}{u}$

$u = 1 - 2\sqrt{x}$

9. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}}$

$u = t,\ a = 1$

11. $\displaystyle\int \sin u\, du$

$u = t^2$

13. $\displaystyle\int e^u\, du$

$u = \sin x$

15. $-\frac{1}{5}(-2x + 5)^{5/2} + C$

17. $-\dfrac{5}{4(z - 4)^4} + C$ **19.** $\dfrac{1}{4}(t^3 - 1)^{4/3} + C$

21. $\dfrac{1}{2}v^2 - \dfrac{1}{6(3v - 1)^2} + C$ **23.** $-\dfrac{1}{3}\ln\left|-t^3 + 9t + 1\right| + C$

25. $\frac{1}{2}x^2 + x + \ln|x - 1| + C$ **27.** $\ln(1 + e^x) + C$

9. $\frac{x}{15}(12x^4 + 20x^2 + 15) + C$ **31.** $\frac{1}{4\pi}\sin 2\pi x^2 + C$

3. $-\frac{1}{\pi}\csc \pi x + C$ **35.** $\frac{1}{5}e^{5x} + C$ **37.** $2\ln(1 + e^x) + C$

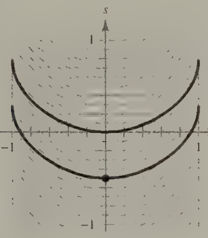
9. $(\ln x)^2 + C$ **41.** $\ln|\sec x(\sec x + \tan x)| + C$

3. $\csc \theta + \cot \theta + C$ **45.** $\frac{3}{2}\ln(z^2 + 9) + \frac{2}{3}\arctan\frac{z}{3} + C$

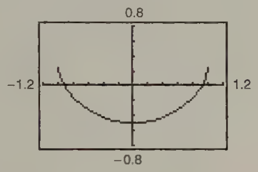
7. $-\frac{1}{2}\arcsin(2t - 1) + C$ **49.** $\frac{1}{2}\ln\left|\cos\frac{2}{t}\right| + C$

1. $3\arcsin\frac{x - 3}{3} + C$ **53.** $\frac{1}{4}\arctan\frac{2x + 1}{8} + C$

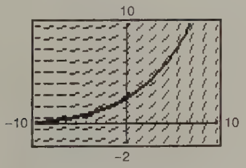
5. (a)



(b) $\frac{1}{2}\arcsin t^2 - \frac{1}{2}$



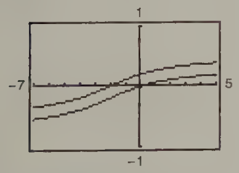
7. $\frac{dy}{dx} = 0.2y$



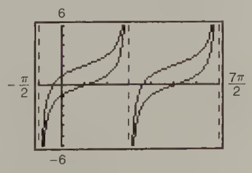
9. $y = \frac{1}{2}e^{2x} + 2e^x + x + C$ **61.** $y = \frac{1}{2}\arctan\frac{\tan x}{2} + C$

3. $\frac{1}{2}$ **65.** $\frac{1}{2}(1 - e^{-1}) \approx 0.316$ **67.** 4 **69.** $\frac{\pi}{18}$

1. $\frac{1}{3}\arctan\left(\frac{x + 2}{3}\right) + C$ **73.** $\tan \theta - \sec \theta + C$



One graph is a vertical translation of the other.



One graph is a vertical translation of the other.

5. Power rule: $\int u^n\,du = \frac{u^{n+1}}{n + 1} + C$; $u = x^2 + 1$, $du = 2x$, $n = 3$

7. Log rule: $\int \frac{du}{u} = \ln|u| + C$; $u = x^2 + 1$, $du = 2x$

9. Using laws of logarithms, $y_1 = e^{x+C_1} = e^x \cdot e^{C_1}$ where e^{C_1} is a constant. Therefore, e^{C_1} can be replaced by C resulting in $y_2 = Ce^x$.

1. $a = \sqrt{2}, b = \frac{\pi}{4}$

$-\frac{1}{\sqrt{2}}\ln\left|\csc\left(x + \frac{\pi}{4}\right) + \cot\left(x + \frac{\pi}{4}\right)\right| + C$

83. a **85.** $\frac{4}{3}$ **87.** $a = \frac{1}{2}$

89. (a) $\pi(1 - e^{-1}) \approx 1.986$

(b) $b = \sqrt{\ln\left(\frac{3\pi}{3\pi - 4}\right)} \approx 0.743$

91. $\frac{2}{\arcsin(4/5)} \approx 2.157$ **93.** 1.0320

Section 7.2 (page 494)

1. b **2.** d **3.** c **4.** a

5. $u = x, dv = e^{2x}\,dx$ **7.** $u = (\ln x)^2, dv = dx$

9. $u = x, dv = \sec^2 x\,dx$ **11.** $-\frac{1}{4e^{2x}}(2x + 1) + C$

13. $e^x(x^3 - 3x^2 + 6x - 6) + C$ **15.** $\frac{1}{3}e^{x^3} + C$

17. $\frac{1}{4}[2(t^2 - 1)\ln|t + 1| - t^2 + 2t] + C$

19. $\frac{(\ln x)^3}{3} + C$ **21.** $\frac{e^{2x}}{4(2x + 1)} + C$ **23.** $(x - 1)^2e^x + C$

25. $\frac{2(x - 1)^{3/2}}{15}(3x + 2) + C$ **27.** $x\sin x + \cos x + C$

29. $(6x - x^3)\cos x + (3x^2 - 6)\sin x + C$

31. $-t\csc t - \ln|\csc t + \cot t| + C$

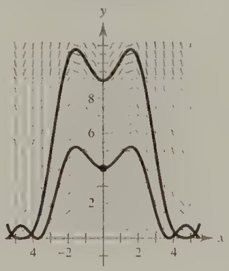
33. $x\arctan x - \frac{1}{2}\ln(1 + x^2) + C$

35. $\frac{1}{5}e^{2x}(2\sin x - \cos x) + C$ **37.** $y = \frac{1}{2}e^{x^2} + C$

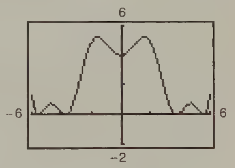
39. $y = \frac{2}{405}(27t^2 - 24t + 32)\sqrt{2 + 3t} + C$

41. $\sin y = x^2 + C$

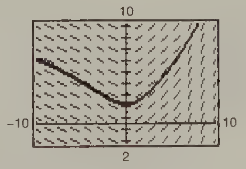
43. (a)



(b) $2\sqrt{y} - \cos x - x\sin x = 3$



45.



47. $4 - \frac{12}{e^2}$ **49.** $\frac{\pi}{2} - 1$ **51.** $\frac{\pi - 3\sqrt{3} + 6}{6} \approx 0.658$

53. $\frac{e[\sin(1) - \cos(1)] + 1}{2} \approx 0.909$ **55.** $\frac{24\ln 2 - 7}{9} \approx 1.071$

57. $8\,\text{arcsec}\,4 + \frac{\sqrt{3}}{2} - \frac{\sqrt{15}}{2} - \frac{2\pi}{3} \approx 7.380$

59. $\frac{e^{2x}}{4}(2x^2 - 2x + 1) + C$

61. $(3x^2 - 6)\sin x - (x^3 - 6x)\cos x + C$

63. $x\tan x + \ln|\cos x| + C$ **65.** Product Rule

67. No **69.** Yes. Let $u = x^2$ and $dv = e^{2x}\,dx$.

71. Yes. Let $u = x$, $dv = \frac{1}{\sqrt{x+1}}\,dx$. (Substitution also works. Let $u = \sqrt{x+1}$.)

73. $-\frac{e^{-4t}}{128}(32t^3 + 24t^2 + 12t + 3) + C$

75. $\frac{1}{13}(2e^{-\pi} + 3) \approx 0.2374$ **77.** $\frac{2}{5}(2x - 3)^{3/2}(x + 1) + C$

79. $\frac{1}{3}\sqrt{4 + x^2}(x^2 - 8) + C$

81. $n = 0$: $x(\ln x - 1) + C$

$n = 1$: $\frac{x^2}{4}(2\ln x - 1) + C$

$n = 2$: $\frac{x^3}{9}(3\ln x - 1) + C$

$n = 3$: $\frac{x^4}{16}(4\ln x - 1) + C$

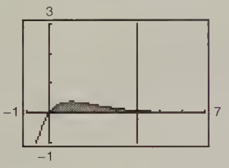
$n = 4$: $\frac{x^5}{25}(5\ln x - 1) + C$

$$\int x^n \ln x\,dx = \frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x - 1] + C$$

83. Proof **85.** Proof **87.** Proof

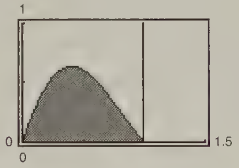
89. $\frac{x^4}{16}(4\ln x - 1) + C$ **91.** $\frac{e^{2x}}{13}(2\cos 3x + 3\sin 3x) + C$

93.



$1 - \frac{5}{e^4} \approx 0.908$

95.



$\frac{\pi}{1 + \pi^2}\left(\frac{1}{e} + 1\right) \approx 0.395$

97. (a) 1 (b) $\pi(e - 2) \approx 2.257$

(c) $\frac{(e^2 + 1)\pi}{2} \approx 13.177$

(d) $\left(\frac{e^2 + 1}{4}, \frac{e - 2}{2}\right) \approx (2.097, 0.359)$

99. $\frac{7}{10\pi}(1 - e^{-4\pi}) \approx 0.223$ **101.** \$931,265

103. Proof **105.** $b_n = \frac{8h}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$

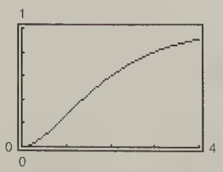
107. Shell: $V = \pi\left[b^2 f(b) - a^2 f(a) - \int_a^b x^2 f'(x)\,dx\right]$

Disk: $V = \pi\left[b^2 f(b) - a^2 f(a) - \int_{f(a)}^{f(b)} [f^{-1}(y)]^2\,dy\right]$

Both methods yield the same volume because $x = f^{-1}(y)$, $f'(x)\,dx = dy$, if $y = f(a)$ then $x = a$, and if $y = f(b)$ then $x = b$.

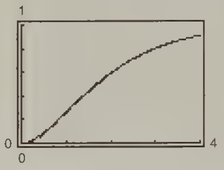
109. (a) $f(x) = -xe^{-x} - e^{-x} + 1$

(b)



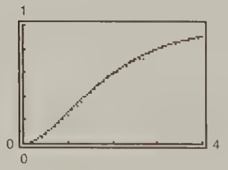
(c) You obtain the following points.

n	x_n	y_n
0	0	0
1	0.05	0
2	0.10	2.378×10^{-3}
3	0.15	0.0069
4	0.20	0.0134
⋮	⋮	⋮
80	4.0	0.9064



(d) You obtain the following points.

n	x_n	y_n
0	0	0
1	0.1	0
2	0.2	0.0090484
3	0.3	0.025423
4	0.4	0.047648
⋮	⋮	⋮
40	4.0	0.9039



(e) $f(4) = 0.9084$

The approximations are tangent line approximations. Th result in (c) is better because Δx is smaller.

Section 7.3 (page 503)

1. (a) $\frac{1}{4}(3 + \cos 4x)$

(b) $2\cos^4 x - 2\cos^2 x + 1$

(c) $1 - 2\sin^2 x\cos^2 x$ (d) $1 - \frac{1}{2}\sin^2 2x$

(e) Four. No; there is often more than one way to rewrite trigonometric expression.

3. $-\frac{1}{4}\cos^4 x + C$ **5.** $\frac{1}{12}\sin^6 2x + C$

7. $-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$

9. $\frac{2}{3}\sin^{3/2}\theta - \frac{2}{7}\sin^{7/2}\theta + C$ **11.** $\frac{1}{12}(6x + \sin 6x) + C$

3. $\dfrac{\alpha - (1/4)\sin 4\alpha}{8} + C$ or $\dfrac{\alpha}{8} - \dfrac{1}{32}\sin 4\alpha + C$

5. $\frac{1}{8}(2x^2 - 2x\sin 2x - \cos 2x) + C$

7. Proof **19.** Proof **21.** $\frac{1}{3}\ln|\sec 3x + \tan 3x| + C$

3. $\frac{1}{15}\tan 5x(3 + \tan^2 5x) + C$

5. $\dfrac{1}{2\pi}\left(\sec \pi x \tan \pi x + \ln|\sec \pi x + \tan \pi x|\right) + C$

7. $\tan^4\left(\dfrac{x}{4}\right) - 2\tan^2\left(\dfrac{x}{4}\right) - 4\ln\left|\cos\dfrac{x}{4}\right| + C$

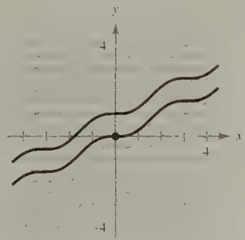
9. $\dfrac{1}{2}\tan^2 x + C$ **31.** $\dfrac{\tan^3 x}{3} + C$ **33.** $\dfrac{\sec^6 4x}{24} + C$

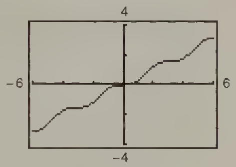
5. $\frac{1}{3}\sec^3 x + C$ **37.** $\ln|\sec x + \tan x| - \sin x + C$

9. $r = \dfrac{1}{32\pi}(12\pi\theta - 8\sin 2\pi\theta + \sin 4\pi\theta) + C$

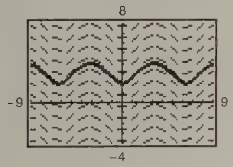
1. $y = \frac{1}{9}\sec^3 3x - \frac{1}{3}\sec 3x + C$

3. (a) (b) $y = \frac{1}{2}x - \frac{1}{4}\sin 2x$





5.



7. $-\frac{1}{10}(\cos 5x + 5\cos x) + C$ **49.** $\frac{1}{8}(2\sin 2\theta - \sin 4\theta) + C$

1. $\frac{1}{4}(\ln|\csc^2 2x| - \cot^2 2x) + C$ **53.** $-\cot\theta - \frac{1}{3}\cot^3\theta + C$

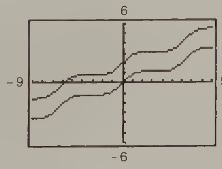
5. $\ln|\csc t - \cot t| + \cos t + C$

7. $\ln|\csc x - \cot x| + \cos x + C$ **59.** $t - 2\tan t + C$

1. π **63.** $\frac{1}{2}(1 - \ln 2)$ **65.** $\ln 2$ **67.** $\frac{4}{3}$

9. $\frac{1}{16}(6x + 8\sin x + \sin 2x) + C$

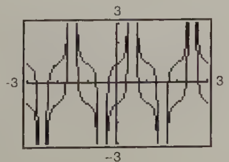
Graphs will vary. Example:



1. $\dfrac{1}{4\pi}\Big[\sec^3 \pi x \tan \pi x +$

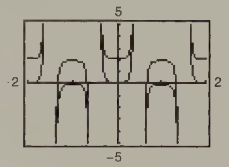
$\dfrac{3}{2}\left(\sec \pi x \tan \pi x + \ln|\sec \pi x + \tan \pi x|\right)\Big] + C$

Graphs will vary. Example:



73. $\dfrac{1}{5\pi}\sec^5 \pi x + C$ **75.** $\dfrac{3\sqrt{2}}{10}$ **77.** $\dfrac{3\pi}{16}$

Graphs will vary.
Example:



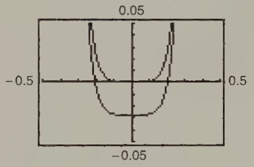
79. (a) Save one sine factor and convert the remaining factors to cosine. Then, expand and integrate.

(b) Save one cosine factor and convert the remaining factors to sine. Then, expand and integrate.

(c) Make repeated use of the power-reducing formulas to convert the integrand to odd powers of the cosine. Then, proceed as in part (b).

81. (a) $\dfrac{\tan^6 3x}{18} + \dfrac{\tan^4 3x}{12} + C_1,\ \dfrac{\sec^6 3x}{18} - \dfrac{\sec^4 3x}{12} + C_2$

(b) (c) Proof



83. $\dfrac{1}{2}$ **85.** (a) $\dfrac{\pi^2}{2}$ (b) $(\bar{x}, \bar{y}) = \left(\dfrac{\pi}{2}, \dfrac{\pi}{8}\right)$

87. Proof **89.** Proof

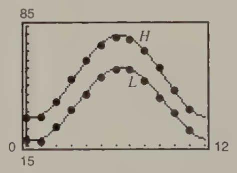
91. $-\frac{1}{15}\cos x(3\sin^4 x + 4\sin^2 x + 8) + C$

93. $\dfrac{5}{6\pi}\tan\dfrac{2\pi x}{5}\left(\sec^2\dfrac{2\pi x}{5} + 2\right) + C$

95. (a) $H(t) = 55.46 - 23.88\cos\dfrac{\pi t}{6} - 3.34\sin\dfrac{\pi t}{6}$

(b) $L(t) = 39.34 - 20.78\cos\dfrac{\pi t}{6} - 4.33\sin\dfrac{\pi t}{6}$

(c) Summer



97. Proof

Section 7.4 (page 512)

1. b **2.** d **3.** a **4.** c **5.** $\dfrac{x}{25\sqrt{25 - x^2}} + C$

7. $5\ln\left|\dfrac{5 - \sqrt{25 - x^2}}{x}\right| + \sqrt{25 - x^2} + C$

9. $\ln\left|x + \sqrt{x^2 - 4}\right| + C$ **11.** $\frac{1}{15}(x^2 - 4)^{3/2}(3x^2 + 8) + C$

13. $\dfrac{1}{3}(1 + x^2)^{3/2} + C$ **15.** $\dfrac{1}{2}\left(\arctan x + \dfrac{x}{1 + x^2}\right) + C$

17. $\frac{1}{2}x\sqrt{4 + 9x^2} + \frac{2}{3}\ln\left|3x + \sqrt{4 + 9x^2}\right| + C$

19. $\sqrt{x^2 + 9} + C$ **21.** $\arcsin\left(\dfrac{x}{4}\right) + C$

23. $4\arcsin\left(\frac{x}{2}\right) + x\sqrt{4 - x^2} + C$ **25.** $\ln\left|x + \sqrt{x^2 - 9}\right| + C$

27. $-\frac{(1 - x^2)^{3/2}}{3x^3} + C$ **29.** $-\frac{1}{3}\ln\left|\frac{\sqrt{4x^2 + 9} + 3}{2x}\right| + C$

31. $\frac{5\sqrt{x^2 + 5}}{x^2 + 5} + C$ **33.** $\frac{1}{3}(1 + e^{2x})^{3/2} + C$

35. $\frac{1}{2}\left(\arcsin e^x + e^x\sqrt{1 - e^{2x}}\right) + C$

37. $\frac{1}{4}\left(\frac{x}{x^2 + 2} + \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}\right) + C$

39. $x\,\mathrm{arcsec}\, 2x - \frac{1}{2}\ln\left|2x + \sqrt{4x^2 - 1}\right| + C$

41. $\arcsin\left(\frac{x - 2}{2}\right) + C$

43. $\sqrt{x^2 + 4x + 8} - 2\ln\left|\sqrt{x^2 + 4x + 8} + (x + 2)\right| + C$

45. (a) and (b) $\sqrt{3} - \frac{\pi}{3} \approx 0.685$

47. (a) and (b) $9\left(2 - \sqrt{2}\right) \approx 5.272$

49. (a) and (b) $-\frac{9}{2}\ln\left(\frac{2\sqrt{7}}{3} - \frac{4\sqrt{3}}{3} - \frac{\sqrt{21}}{3} + \frac{8}{3}\right)$
$+ 9\sqrt{3} - 2\sqrt{7} \approx 12.644$

51. $\frac{1}{2}(x - 15)\sqrt{x^2 + 10x + 9}$
$+ 33\ln\left|\sqrt{x^2 + 10x + 9} + (x + 5)\right| + C$

53. $\frac{1}{2}\left(x\sqrt{x^2 - 1} + \ln\left|x + \sqrt{x^2 - 1}\right|\right) + C$

55. (a) Let $u = a\sin\theta$, $\sqrt{a^2 - u^2} = a\cos\theta$, where
$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

(b) Let $u = a\tan\theta$, $\sqrt{a^2 + u^2} = a\sec\theta$, where
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

(c) Let $u = a\sec\theta$, $\sqrt{u^2 - a^2} = \tan\theta$ if $u > a$ and
$\sqrt{u^2 - a^2} = -\tan\theta$ if $u < -a$, where $0 \le \theta < \frac{\pi}{2}$
or $\frac{\pi}{2} < \theta \le \pi$.

57. πab **59.** $\frac{a^2\pi}{2} - a^2\arcsin\frac{h}{a} - h\sqrt{a^2 - h^2}$ **61.** $6\pi^2$

63. $\ln\left[\frac{5(\sqrt{2} + 1)}{\sqrt{26} + 1}\right] + \sqrt{26} - \sqrt{2} \approx 4.367$

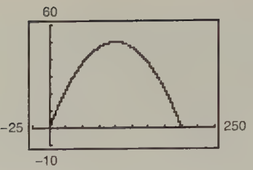
65. Length of one arch of sine curve: $y = \sin x$, $y' = \cos x$

$$L_1 = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$$

Length of one arch of cosine curve: $y = \cos x$, $y' = -\sin x$

$$\begin{aligned} L_2 &= \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x}\, dx \\ &= \int_{-\pi/2}^{\pi/2} \sqrt{1 + \cos^2\left(x - \frac{\pi}{2}\right)}\, dx, \quad u = x - \frac{\pi}{2},\ du = dx \\ &= \int_{-\pi}^{0} \sqrt{1 + \cos^2 u}\, du \\ &= \int_0^{\pi} \sqrt{1 + \cos^2 u}\, du = L_1 \end{aligned}$$

67. (a)



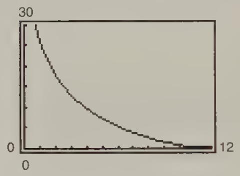
(b) 200 (c) $100\sqrt{2} + 50\ln\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right) \approx 229.559$

69. $(0, 0.422)$ **71.** $\frac{\pi}{32}\left[102\sqrt{2} - \ln\left(3 + 2\sqrt{2}\right)\right] \approx 13.989$

73. (a) 187.2π pounds (b) $62.4\pi d$ pounds

75. (a) $m = \frac{dy}{dx} = \frac{y - \left(y + \sqrt{144 - x^2}\right)}{x - 0}$
$= -\frac{\sqrt{144 - x^2}}{x}$

(b) $y = -12\ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$



(c) $x = 0$ (d) 5.2 meters

77. True

79. False: $\int_0^{\sqrt{3}} \frac{dx}{(1 + x^2)^{3/2}} = \int_0^{\pi/3} \cos\theta\, d\theta$ **81.** Proof

Section 7.5 (page 522)

1. $\frac{A}{x} + \frac{B}{x - 10}$ **3.** $\frac{A}{x} + \frac{Bx + C}{x^2 + 10}$ **5.** $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 10}$

7. $\frac{1}{2}\ln\left|\frac{x - 1}{x + 1}\right| + C$ **9.** $\ln\left|\frac{x - 1}{x + 2}\right| + C$

11. $\frac{3}{2}\ln|2x - 1| - 2\ln|x + 1| + C$

13. $5\ln|x - 2| - \ln|x + 2| - 3\ln|x| + C$

15. $x^2 + \frac{3}{2}\ln|x - 4| - \frac{1}{2}\ln|x + 2| + C$

17. $\frac{1}{x} + \ln|x^4 + x^3| + C$

19. $2\ln|x - 2| - \ln|x| - \frac{3}{x - 2} + C$

21. $\ln\left|\frac{x^2 + 1}{x}\right| + C$

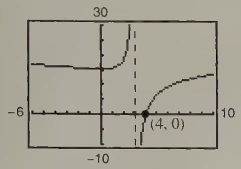
23. $\frac{1}{6}\left[\ln\left|\frac{x - 2}{x + 2}\right| + \sqrt{2}\arctan\left(\frac{x}{\sqrt{2}}\right)\right] + C$

25. $\frac{1}{16}\ln\left|\frac{4x^2 - 1}{4x^2 + 1}\right| + C$

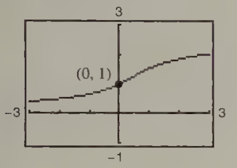
27. $\ln|x + 1| + \sqrt{2}\arctan\left(\frac{x - 1}{\sqrt{2}}\right) + C$

29. $\ln 2$ **31.** $\frac{1}{2}\ln\left(\frac{8}{5}\right) - \frac{\pi}{4} + \arctan 2 \approx 0.557$

33. $y = 3 \ln|x - 3| - \dfrac{9}{x - 3} + 9$



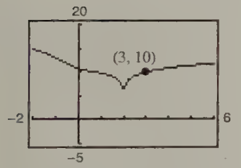
35. $y = \dfrac{\sqrt{2}}{2} \arctan \dfrac{x}{\sqrt{2}} - \dfrac{1}{2(x^2 + 2)} + \dfrac{5}{4}$



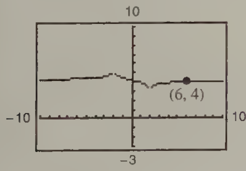
37. $y = \ln|x - 2| + \dfrac{1}{2} \ln|x^2 + x + 1|$

$- \sqrt{3} \arctan\left(\dfrac{2x + 1}{\sqrt{3}}\right) - \dfrac{1}{2} \ln 13$

$+ \sqrt{3} \arctan \dfrac{7}{\sqrt{3}} + 10$



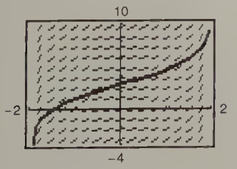
39. $y = \dfrac{1}{4} \ln\left|\dfrac{x - 2}{x + 2}\right| + \dfrac{1}{4} \ln 2 + 4$



41. $\ln\left|\dfrac{\cos x}{\cos x - 1}\right| + C$ **43.** $\ln\left|\dfrac{-1 + \sin x}{2 + \sin x}\right| + C$

45. $\dfrac{1}{5} \ln\left|\dfrac{e^x - 1}{e^x + 4}\right| + C$ **47.** Proof **49.** Proof

51. $y = \dfrac{3}{2} \ln\left|\dfrac{2 + x}{2 - x}\right| + 3$ **53.** First divide x^3 by $(x - 5)$.



55. (a) Log Rule (b) Partial fractions
(c) Inverse Tangent Rule

57. 4.90 or \$490,000 **59.** c

61. $x = \dfrac{n[e^{(n+1)kt} - 1]}{n + e^{(n+1)kt}}$ **63.** $\dfrac{\pi}{8}$

Section 7.6 (page 528)

1. $-\frac{1}{2}x(2 - x) + \ln|1 + x| + C$

3. $\frac{1}{2}\left[e^x\sqrt{e^{2x} + 1} + \ln\left(e^x + \sqrt{e^{2x} + 1}\right)\right] + C$

5. $-\dfrac{\sqrt{1 - x^2}}{x} + C$

7. $\frac{1}{16}(6x - 3 \sin 2x \cos 2x - 2 \sin^3 2x \cos 2x) + C$

9. $-2\left(\cot\sqrt{x} + \csc\sqrt{x}\right) + C$

11. $x - \frac{1}{2}\ln(1 + e^{2x}) + C$ **13.** $\frac{1}{16}x^4(4 \ln x - 1) + C$

15. (a) and (b) $e^x(x^2 - 2x + 2) + C$

17. (a) and (b) $\ln\left|\dfrac{x + 1}{x}\right| - \dfrac{1}{x} + C$ **19.** $\dfrac{1}{2}e^{x^2} + C$

21. $\frac{1}{2}\left\{(x^2 + 1)\operatorname{arcsec}(x^2 + 1) - \ln\left[(x^2 + 1) + \sqrt{x^4 + 2x^2}\right]\right\} + C$

23. $\dfrac{1}{9}x^3(-1 + 3 \ln x) + C$ **25.** $\dfrac{\sqrt{x^2 - 4}}{4x} + C$

27. $\dfrac{2}{9}\left(\ln|1 - 3x| + \dfrac{1}{1 - 3x}\right) + C$

29. $e^x \arccos(e^x) - \sqrt{1 - e^{2x}} + C$

31. $\frac{1}{2}(x^2 + \cot x^2 + \csc x^2) + C$ **33.** $\arctan(\sin x) + C$

35. $\dfrac{\sqrt{2}}{2} \arctan\left(\dfrac{1 + \sin\theta}{\sqrt{2}}\right) + C$ **37.** $-\dfrac{\sqrt{2 + 9x^2}}{2x} + C$

39. $(t^3 - 6t) \sin t + 3(t^2 - 2) \cos t + C$

41. $\frac{1}{4}\left(2 \ln|x| - 3 \ln|3 + 2 \ln|x||\right) + C$

43. $\dfrac{3x - 10}{2(x^2 - 6x + 10)} + \dfrac{3}{2} \arctan(x - 3) + C$

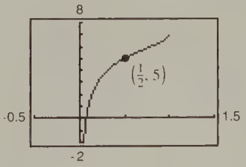
45. $\frac{1}{2} \ln\left|x^2 - 3 + \sqrt{x^4 - 6x^2 + 5}\right| + C$

47. $-\frac{1}{3}\sqrt{4 - x^2}(x^2 + 8) + C$

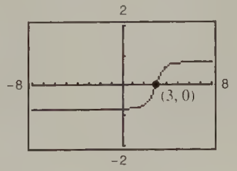
49. $\dfrac{2}{1 + e^x} - \dfrac{1}{2(1 + e^x)^2} + \ln(1 + e^x) + C$

51. Proof **53.** Proof **55.** Proof

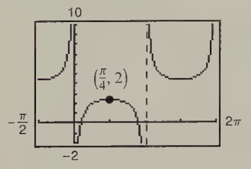
57. $y = -\dfrac{2\sqrt{1 - x}}{\sqrt{x}} + 7$



59. $y = \dfrac{1}{2}\left[\dfrac{x - 3}{x^2 - 6x + 10} + \arctan(x - 3)\right]$



61. $y = -\csc\theta + \sqrt{2} + 2$



63. $\frac{1}{\sqrt{5}}\ln\left|\frac{2\tan(\theta/2) - 3 - \sqrt{5}}{2\tan(\theta/2) - 3 + \sqrt{5}}\right| + C$ **65.** $\ln 2$

67. $\frac{1}{2}\ln(3 - 2\cos\theta) + C$ **69.** $2\sin\sqrt{\theta} + C$ **71.** $\frac{40}{3}$

73. Use Formula 23 and let $a = 1$, $u = e^x$, and $du = e^x\,dx$.

75. Use Formula 81 and let $u = x^2$ and $du = 2x\,dx$.

77. Impossible

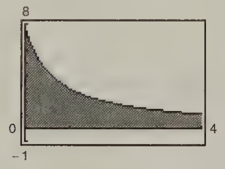
79. Answers will vary. For example: $\int x^3\cos x\,dx$ can be integrated using Formula 55 where $u = x$, $du = dx$, and $n = 3$.

81. 1919.145 foot-pounds

83. (a) $V = 80\ln\left(\sqrt{10} + 3\right) \approx 145.5$ cubic feet
$W = 11{,}840\ln\left(\sqrt{10} + 3\right) \approx 21{,}530.4$ pounds
(b) $(0, 1.19)$

85. (a) $k = \frac{30}{\ln 7} \approx 15.42$

(b)



87. False. Substitutions may first have to be made to rewrite the integral in a form that appears in the table.

Section 7.7 (page 537)

1.

x	−0.1	−0.01	−0.001
$f(x)$	2.4132	2.4991	2.500

x	0.001	0.01	0.1
$f(x)$	2.500	2.4991	2.4132

2.5

3.

x	1	10	10^2
$f(x)$	0.9900	90,483.7	3.7×10^9

x	10^3	10^4	10^5
$f(x)$	4.5×10^{10}	0	0

0

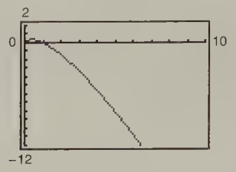
5. $\frac{1}{3}$ **7.** $\frac{1}{4}$ **9.** $\frac{5}{3}$ **11.** 3 **13.** 0 **15.** 2

17. $n = 1$: 0
$n = 2$: $\frac{1}{2}$
$n \geq 3$: ∞

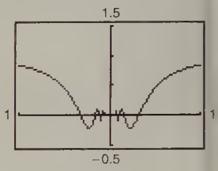
19. $\frac{2}{3}$ **21.** 1 **23.** $\frac{3}{2}$ **25.** ∞

27. 0 **29.** 1 **31.** 0 **33.** 0 **35.** ∞

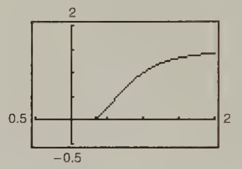
37. (a) $0 \cdot \infty$ (b) 0
(c)



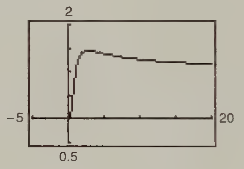
39. (a) $0 \cdot \infty$ (b) 1
(c)



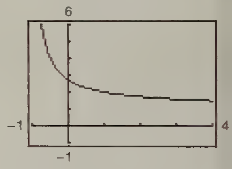
41. (a) Not indeterminate (b) 0
(c)



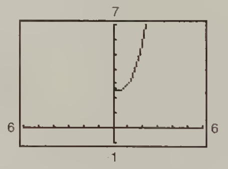
43. (a) ∞^0 (b) 1
(c)



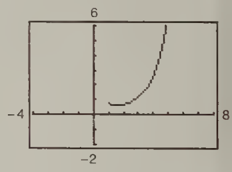
45. (a) 1^∞ (b) e
(c)



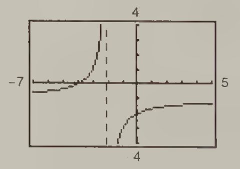
47. (a) 0^0 (b) 3
(c)



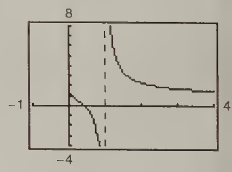
49. (a) 0^0 (b) 1
(c)



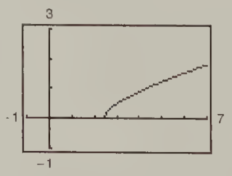
51. (a) $\infty - \infty$ (b) $-\frac{3}{2}$
(c)



53. (a) $\infty - \infty$ (b) ∞
(c)

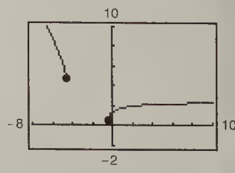


55. (a)



(b) $\frac{1}{2}$

57. (a)



(b) $\frac{5}{2}$

59. $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 1^\infty, 0^0, \infty - \infty$

61. Answers will vary. Examples:
(a) $f(x) = x^2 - 25$, $g(x) = x - 5$
(b) $f(x) = (x - 5)^2$, $g(x) = x^2 - 25$
(c) $f(x) = x^2 - 25$, $g(x) = (x - 5)^3$

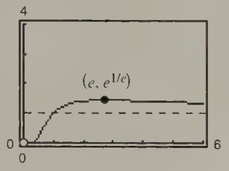
3. 0 **65.** 0 **67.** 0

9.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$	2.811	4.498	0.720	0.036	0.001	0.000

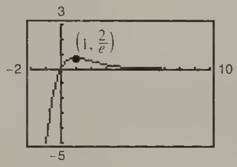
1. Horizontal asymptote: $y = 1$

Relative maximum: $(e, e^{1/e})$



3. Horizontal asymptote: $y = 0$

Relative minimum: $\left(1, \frac{2}{e}\right)$



5. Limit is not of the form 0/0 or ∞/∞.

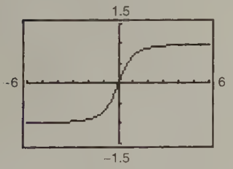
7. Limit is not of the form 0/0 or ∞/∞.

9. (a) 1

(b) $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$

Applying L'Hôpital's Rule twice results in the original limit, so L'Hôpital's Rule fails.

(c)



1

1. $v = 32t + v_0$ **83.** $\frac{3}{4}$ **85.** $c = \frac{2}{3}$ **87.** $c = \frac{\pi}{4}$

9. False: L'Hôpital's Rule does not apply, because $\lim_{x\to 0}(x^2 + x + 1) \neq 0$.

1. True

3. (a) $\frac{1}{2}\sin\theta - \frac{1}{2}\sin\theta\cos\theta$ (b) $\frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta$

(c) $\frac{\sin\theta - \sin\theta\cos\theta}{\theta - \sin\theta\cos\theta}$ (d) $\frac{3}{4}$

5. Proof **97.** Proof

ection 7.8 (page 547)

1. Infinite discontinuity at $x = 0$; 4

3. Infinite discontinuity at $x = 1$; diverges

5. Infinite limit of integration; 1

7. Infinite discontinuity at $x = 0$; diverges

. 1 **11.** Diverges

3. Diverges **15.** 2 **17.** $\frac{1}{2}$ **19.** $\frac{1}{2(\ln 4)^2}$

21. π **23.** $\frac{\pi}{4}$ **25.** Diverges **27.** Diverges

29. 6 **31.** $-\frac{1}{4}$ **33.** Diverges **35.** $\frac{\pi}{3}$

37. $\ln(2 + \sqrt{3})$ **39.** 0 **41.** $\frac{2\pi\sqrt{6}}{3}$ **43.** $p > 1$

45. Proof **47.** Diverges **49.** Converges

51. Converges **53.** Diverges **55.** Converges

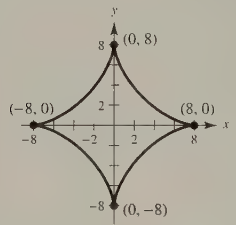
57. An integral with infinite integration limits, an integral with an infinite discontinuity at or between the integration limits

59. The improper integral diverges.

61. $\frac{1}{s}$, $s > 0$ **63.** $\frac{2}{s^3}$, $s > 0$ **65.** $\frac{s}{s^2 + a^2}$, $s > 0$

67. $\frac{s}{s^2 - a^2}$, $s > |a|$ **69.** (a) 1 (b) $\frac{\pi}{2}$ (c) 2π

71.



Perimeter = 48

73. (a) $\Gamma(1) = 1, \Gamma(2) = 1, \Gamma(3) = 2$ (b) Proof

(c) $\Gamma(n) = (n - 1)!$

75. (a) Proof (b) $P = 43.53\%$ (c) $E(x) = 7$

77. (a) $757,992.41 (b) $837,995.15 (c) $1,066,666.67

79. $\frac{k\left(\sqrt{a^2+1} - 1\right)}{a^2\sqrt{a^2+1}}$ **81.** Three. All three must converge.

83. (a) $\frac{1}{6}$ (b) $\frac{1}{24}$ (c) $\frac{1}{60}$

85. False. Let $f(x) = \frac{1}{x+1}$. **87.** True

Review Exercises for Chapter 7 (page 550)

1. $\frac{(x^2-1)^{3/2}}{3} + C$ **3.** $\frac{1}{2}\ln|x^2 - 1| + C$

5. $\frac{(\ln(2x))^2}{2} + C$ **7.** $16 \arcsin\frac{x}{4} + C$

9. $\frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C$

11. $\frac{2}{15}(x - 5)^{3/2}(3x + 10) + C$

13. $-\frac{1}{2}x^2\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$

15. $\frac{1}{16}\left[(8x^2 - 1)\arcsin 2x + 2x\sqrt{1 - 4x^2}\right] + C$

17. $\frac{1}{3\pi}\sin(\pi x - 1)[\cos^2(\pi x - 1) + 2] + C$

19. $\frac{2}{3}\left[\tan^3\left(\frac{x}{2}\right) + 3\tan\left(\frac{x}{2}\right)\right] + C$ **21.** $\tan\theta + \sec\theta + C$

23. $\dfrac{3\sqrt{4-x^2}}{x} + C$ **25.** $\dfrac{1}{3}(x^2+4)^{1/2}(x^2-8) + C$

27. $\dfrac{1}{2}\left(4 \arcsin \dfrac{x}{2} + x\sqrt{4-x^2}\right) + C$

29. (a), (b), and (c) $\frac{1}{3}\sqrt{4+x^2}(x^2-8) + C$

31. $6 \ln|x+2| - 5 \ln|x-3| + C$

33. $\frac{1}{4}[6 \ln|x-1| - \ln(x^2+1) + 6 \arctan x] + C$

35. $x + \frac{9}{8} \ln|x-3| - \frac{25}{8} \ln|x+5| + C$

37. $\dfrac{1}{9}\left(\dfrac{2}{2+3x} + \ln|2+3x|\right) + C$ **39.** $\dfrac{1}{2}[\tan x^2 - \sec x^2] + C$

41. $\dfrac{1}{2} \ln|x^2+4x+8| - \arctan\left(\dfrac{x+2}{x}\right) + C$

43. $\left(\dfrac{1}{\pi}\right) \ln|\tan \pi x| + C$

45. Proof **47.** $\frac{1}{8}(\sin 2\theta - 2\theta \cos 2\theta) + C$

49. $\frac{4}{3}[x^{3/4} - 3x^{1/4} + 3 \arctan(x^{1/4})] + C$

51. $2\sqrt{1-\cos x} + C$ **53.** $\sin x \ln(\sin x) - \sin x + C$

55. $y = \dfrac{3}{2} \ln\left|\dfrac{x-3}{x+3}\right| + C$

57. $y = x \ln|x^2+x| - 2x + \ln|x+1| + C$ **59.** $\frac{1}{5}$

61. $\frac{1}{2}(\ln 4)^2 \approx 0.961$ **63.** π **65.** $\frac{128}{15}$

67. $(\bar{x}, \bar{y}) = \left(0, \dfrac{4}{3\pi}\right)$ **69.** 3.82 **71.** 0 **73.** ∞ **75.** 1

77. $1000e^{0.09} \approx 1094.17$ **79.** Converges; $\frac{32}{3}$ **81.** Diverges

83. (a) \$6,321,205.59 (b) \$10,000,000

85. (a) 0.4581 (b) 0.0135

P.S. Problem Solving (page 552)

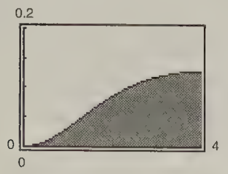
1. (a) $\frac{4}{3}, \frac{16}{15}$ (b) Proof **3.** ln 3

5. Let P be represented by $\left(c, \sqrt{1-c^2}\right)$. Then $S = \displaystyle\int_c^1 \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\, dx = \frac{\pi}{2} - \arcsin c$. Then Q is represented by $\left(1, \dfrac{\pi}{2} - \arcsin c\right)$ and line PQ is represented by $y - \sqrt{1-c^2} = \left(\dfrac{\dfrac{\pi}{2} - \arcsin c - \sqrt{1-c^2}}{1-c}\right)(x-c)$.

Since R is on the x-axis, set $y = 0$. Then simplify and find $\displaystyle\lim_{c \to -1}\left(c - \frac{(1-c)\sqrt{1-c^2}}{\dfrac{\pi}{2} - \arcsin c - \sqrt{1-c^2}}\right)$. This limit is -2 and therefore the length of segment OR is 2.

7. (a) Area ≈ 0.2986 (b) $\ln 3 - \frac{4}{5}$

(c) $\ln 3 - \frac{4}{5}$



9. $\ln 3 - \frac{1}{2} \approx 0.5986$ **11.** Proof

13. $x^4 + 1 = \left(x^2 + \sqrt{2}x + 1\right)\left(x^2 - \sqrt{2}x + 1\right)$

$$A = \frac{\sqrt{2}}{4}\left[\arctan\left(\sqrt{2}+1\right) + \arctan\left(\sqrt{2}-1\right)\right] + \frac{\sqrt{2}}{8}\left[\ln\left(2+\sqrt{2}\right) - \ln\left(2-\sqrt{2}\right)\right]$$

$A \approx 0.8670$

15. (a) ∞ (b) 0 (c) $-\frac{2}{3}$

The indeterminate form $0 \cdot \infty$ does not determine the value the limit or even whether the limit exists.

17. $\dfrac{1/12}{x} + \dfrac{111/140}{x+4} + \dfrac{1/42}{x-3} + \dfrac{1/10}{x-1}$ **19.** Proof

Chapter 8

Section 8.1 (page 564)

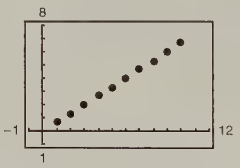
1. 2, 4, 8, 16, 32 **3.** $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}$ **5.** 1, 0, −1, 0,

7. $-1, -\frac{1}{4}, \frac{1}{9}, \frac{1}{16}, -\frac{1}{25}$ **9.** $5, \frac{19}{4}, \frac{43}{9}, \frac{77}{16}, \frac{121}{25}$

11. $3, \frac{9}{2}, \frac{27}{6}, \frac{81}{24}, \frac{243}{120}$ **13.** 3, 4, 6, 10, 18 **15.** 32, 16, 8, 4, 2

17. d **18.** a **19.** c **20.** b

21.



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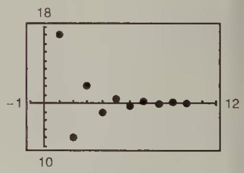

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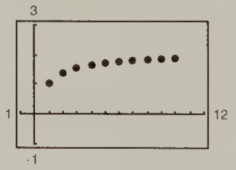


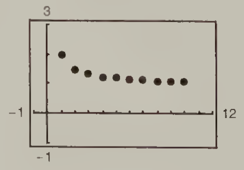
27. 14, 17; add 3 to preceding term

29. $\frac{3}{16}, -\frac{3}{32}$; multiply preceding term by $-\frac{1}{2}$

31. $10 \cdot 9 = 90$ **33.** $n + 1$ **35.** $\dfrac{1}{(2n+1)(2n)}$

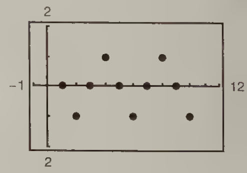
37. 5 **39.** 2 **41.** 0

43.



Converges to 1

45.



Diverges

47. Diverges **49.** Converges to $\frac{3}{2}$

51. Converges to 0 **53.** Converges to 0

55. Converges to 0 **57.** Diverges

59. Converges to 0 **61.** Converges to 0

63. Converges to e^k **65.** Converges to 0

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